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# Disjunctive conic cuts for mixed integer second order cone optimization

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**Abstract:** We investigate the derivation of disjunctive conic cuts for mixed integer second order cone optimization (MISOCO). These conic cuts characterize the convex hull of the intersection of a disjunctive set and the feasible set of a MISOCO problem. We present a full characterization of these inequalities when the disjunctive set considered is defined by parallel hyperplanes.

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## 1 Introduction

The use of valid inequalities in mixed integer linear optimization [1] has proven to be essential for the development of successful solvers. Particularly, we are interested in the disjunctive cuts proposed originally by Balas [2]. The goal of this paper is to present an extension of the disjunctive cuts for mixed integer second order cone optimization (MISOCO) problems. A MISOCO problem has the form

$$\begin{aligned} & \text{minimize: } c^\top x \\ & \text{subject to: } Ax = b \\ & \quad x \in \mathcal{K} \\ & \quad x \in \mathbb{Z}^d \times \mathbb{R}^{\ell-d}, \end{aligned} \tag{MISOCO}$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $c \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ , and the rows of  $A$  are linearly independent. Additionally, we have that  $\mathcal{K} = \mathbb{L}_1^{n_1} \times \cdots \times \mathbb{L}_k^{n_k}$ , where  $\mathbb{L}^{n_i} = \{x^i | x_1^i \geq \|x_{2:n_i}^i\|\}$ ,  $i = 1, \dots, k$  and  $\sum_{i=1}^k n_i = n$ . Here, the notation  $x_{2:n}$  refers to the vector formed by the components 2 to  $n$  of vector  $x$ .

The technique proposed by Balas was generalized for 0-1 mixed integer convex optimization in [3]. Later, the authors in [4] investigated the generation of convex cuts for 0-1 mixed integer conic optimization (MICO) problems and discussed how to extend the Chvátal-Gomory [5] procedure for generating linear cuts for MICO problems. They also discussed the extension of *lift-and-project* techniques for MICO problems. In particular, they showed how to generate linear and convex quadratic valid inequalities using the relaxation obtained by a projection procedure. Drewes [6] use the ideas proposed by [4] and [3] and applies them to MISOCO problems. More recently, close forms for the derivation of conic disjunctive cuts that extend the procedure of Balas for MISOCO problems are presented in [7, 8, 9, 10, 11, 12, 13].

Here, we elaborate over the theory of disjunctive conic cuts (DCCs) developed previously in [8, 9]. The authors in these papers analyzed disjunctive sets given by the intersection of two hyperplanes with the feasible set of the continuous relaxation of (MISOCO). That analysis assumed that the intersection of the hyperplanes and the feasible set of (MISOCO) was bounded. Under that assumption closed form DCCs were provided. However, to have a complete characterization of DCCs it is necessary to consider the cases where the boundedness assumption is not longer true. This paper aims to provide a complete characterization of the DCCs for MISOCO problems when the disjunctive set is defined by two parallel hyperplanes. A different approach for the derivation of these cuts is independently proposed in [12], with two major differences. First, the construction presented here uses an algebraic analysis of quadrics for the derivation of the cuts, while the approach in [12] uses an interpolation technique. Second, the derivation proposed here generalizes to all quadrics that are needed to describe the geometry of the continuous relaxation of (MISOCO) problems. This is not the case in [12], since in that approach the characterization of hyperboloids was not possible due to the involved formulas in the analysis.

This paper is organized as follows. First, in Section 2 we characterize the shapes of the feasible set of a MISOCO problem with a single cone. Then, in Section 3 we describe a detailed general procedure to derive conic cuts for MISOCO problems. Finally, we close the paper with some conclusions and directions for future research in Section 4.

## 2 Quadrics and the feasible set of a MISOCO problem

In this section we analyze the feasible set of (MISOCO) problems when  $k = 1$ . In other words, we consider the set

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid Ax = b, x \in \mathbb{L}^n\}. \tag{1}$$

First we show that the set  $\mathcal{F}$  can be reformulated in terms of a quadric. Quadrics are sets of the form

$$\{w \in \mathbb{R}^\ell \mid w^\top Pw + 2p^\top w + \rho \leq 0\}, \tag{2}$$

where  $P \in \mathbb{R}^{\ell \times \ell}$ ,  $p \in \mathbb{R}^\ell$ , and  $\rho$  is a scalar. Second, we show that a valid DCC for a quadric  $\mathcal{Q}$  representing  $\mathcal{F}$  is also valid for  $\mathcal{F}$ . Finally, we characterize the shapes of the quadrics that can be used to represent  $\mathcal{F}$ .

Recall that the set  $\{x \in \mathbb{R}^n \mid Ax = b\}$  can be reformulated in terms of an orthonormal basis of the null space of  $A$ . Consider a vector  $x^0 \in \mathcal{F}$ , and let  $H_{n \times \ell}$  be a matrix whose columns form an orthonormal basis for the null space of  $A$ , where  $\ell = n - m$ . Then, we have that

$$\{x \in \mathbb{R}^n \mid Ax = b\} = \{x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^\ell, x = x^0 + Hw\}. \quad (3)$$

One can use (3) to rewrite the set  $\mathcal{F}$  in terms of a quadric as follows. First, let  $J \in \mathbb{R}^{n \times n}$  be a diagonal matrix defined as

$$J = \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix},$$

and let us relax the constraint  $x \in \mathbb{L}^n$  to  $x^\top Jx \leq 0$ . Substituting  $x = x^0 + Hw$  in the relaxed constraint one obtains

$$(x^0 + Hw)^\top J(x^0 + Hw) = w^\top H^\top JHw + 2(x^0)^\top JHw + (x^0)^\top Jx^0 \leq 0. \quad (4)$$

Define  $P = H^\top JH$ ,  $p = H^\top Jx^0$ , and  $\rho = (x^0)^\top Jx^0$ . Substituting these  $P$ ,  $p$ , and  $\rho$  in (4) one obtains the quadric

$$\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid w^\top Pw + 2p^\top w + \rho \leq 0\}. \quad (5)$$

Thus, the set  $\mathcal{F}$  can be reformulated in terms of  $\mathcal{Q}$  as follows

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid x = x^0 + Hw, \text{ with } w \in \mathcal{Q}, \text{ and } x_1 \geq 0\}. \quad (6)$$

We can show now that it is possible to derive a cut to exclude a solution  $\hat{x} \in \mathcal{F}$  using the quadric  $\mathcal{Q}$ .

**Lemma 1** *Given a vector  $\hat{x} \in \mathcal{F}$  and a vector  $\hat{w} \in \mathcal{Q}$  such that  $\hat{x} = x^0 + H\hat{w}$ , if a valid cut excludes the vector  $\hat{w}$  from  $\mathcal{Q}$ , then it excludes the vector  $\hat{x}$  from  $\mathcal{F}$ .*

**Proof.** Recall the alternative representation of  $\mathcal{F}$  given in (6). Note that any  $x \in \mathcal{F}$  is a linear combination of  $x^0$  and the columns of  $H$ . Additionally, recall that the columns of  $H$  are linearly independent. Then, the vector  $\hat{w}$  defining  $\hat{x}$  is unique. Therefore, given a cut that excludes  $\hat{w}$  from  $\mathcal{Q}$ , it excludes  $\hat{x}$  from  $\mathcal{F}$ .  $\square$

We can use Lemma 1 to derive disjunctive cuts for problem (MISOCO) as follows. Consider two given hyperplanes  $\mathcal{U} = \{x \in \mathbb{R}^n \mid u^\top x \geq \varphi\}$  and  $\mathcal{V} = \{x \in \mathbb{R}^n \mid u^\top x \leq \varpi\}$ . We want to derive the DCC obtained by convexifying the set

$$\{x \in \mathbb{R}^n \mid Ax = b, x \in \mathbb{L}, x \in \mathcal{U} \cup \mathcal{V}\}. \quad (7)$$

Note that the sets  $\mathcal{U}$  and  $\mathcal{V}$  can be reformulated as

$$\mathcal{U} = \{x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^n, x = x^0 + Hw, u^\top Hw \geq \varphi - u^\top x^0\},$$

and

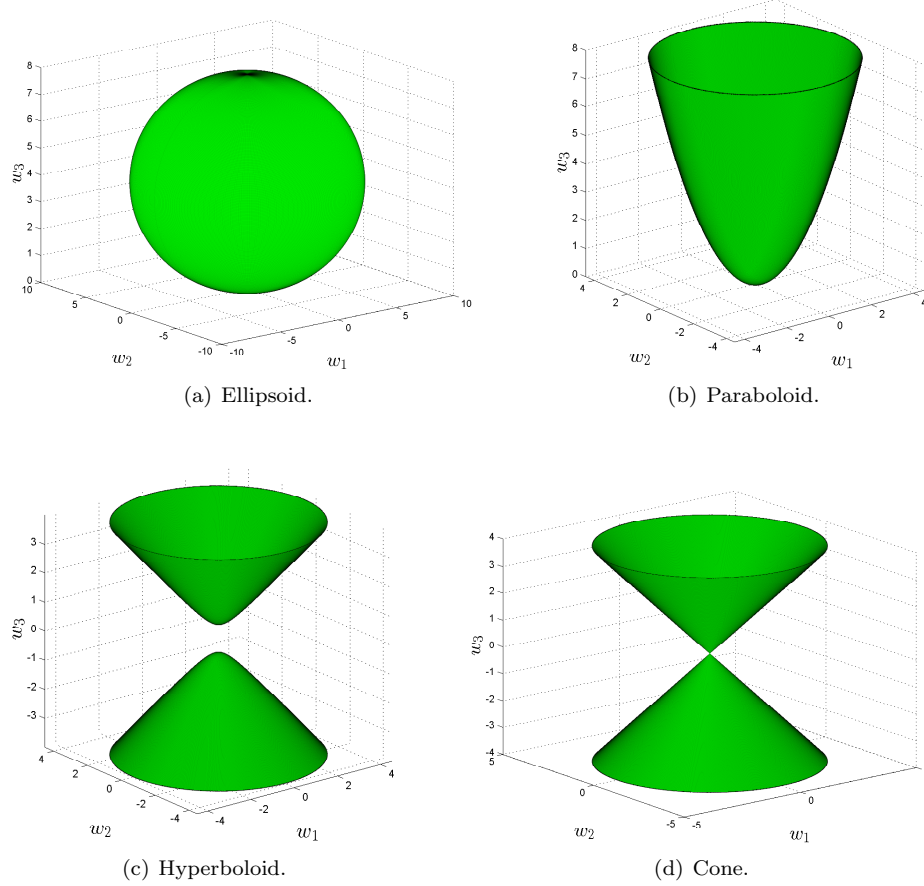
$$\mathcal{V} = \{x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^n, x = x^0 + Hw, u^\top Hw \leq \varpi - u^\top x^0\}.$$

Define  $a = u^\top H$ ,  $\alpha = \varphi - u^\top x^0$  and  $\beta = \varpi - u^\top x^0$ . Now, let  $\mathcal{A} = \{w \in \mathbb{R}^\ell \mid a^\top w \geq \alpha\}$  and  $\mathcal{B} = \{w \in \mathbb{R}^\ell \mid a^\top w \leq \beta\}$ . One can rewrite (7) in terms of the set  $\mathcal{A} \cup \mathcal{B}$  as follows

$$\{x \in \mathbb{R}^n \mid x = x^0 + Hw, \text{ with } w \in \mathcal{Q}, x_1 \geq 0, \text{ and } w \in \mathcal{A} \cup \mathcal{B}\}. \quad (8)$$

Thus, from Lemma 1 we obtain that convexifying the set  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$  one can derive disjunctive cuts for the feasible set of (MISOCO).

Before moving forward with the derivation of cuts for MISOCO problems, we need to analyze the shapes of the set  $\mathcal{Q}$ . The inertia of matrix  $P$  in the representation of  $\mathcal{Q}$  is one of the elements defining its shape. We have the following result about the inertia of  $P$ .

Figure 1: Illustration of the shapes of  $\mathcal{Q}$ .

**Lemma 2 (Belotti et al. [8])** *The matrix  $P$  in the definition of the quadric  $\mathcal{Q}$  has at most one non-positive eigenvalue, and at least  $\ell - 1$  positive eigenvalues.*

Hence, from Lemma 2 we have the following possible shapes for  $\mathcal{Q}$ :

- if  $P \succ 0$ , then  $\mathcal{Q}$  is an ellipsoid, see Figure 1(a) for an illustration;
- if  $P \succeq 0$  and it is singular, then  $\mathcal{Q}$  is:
  - a paraboloid if there is no vector  $w^c \in \mathbb{R}^\ell$  such that  $Pw^c = -p$ , see Figure 1(b) for an illustration;
  - a line if there is a vector  $w^c \in \mathbb{R}^\ell$  such that  $Pw^c = -p$ ;
- if  $P$  is indefinite with one negative eigenvalue (ID1), then  $\mathcal{Q}$  is:
  - a hyperboloid of two sheets if  $p^\top P^{-1}p - \rho < 0$ , see Figure 1(c) for an illustration;
  - a cone if  $p^\top P^{-1}p - \rho = 0$ , see Figure 1(d) for an illustration.

The argument to show that the shapes of  $\mathcal{Q}$  are limited to the cases presented in this list is mainly technical. Hence, to facilitate the flow of the paper its discussion is omitted here and presented in Appendix A.

### 3 Derivation of DCCs for MISOCO problems

In this section we analyze a MISOCO problem where  $\mathcal{K} = \mathbb{L}^n$ . Let  $\mathcal{Q} \subseteq \mathbb{R}^\ell$  be a full dimensional quadric used to reformulate the feasible set  $\mathcal{F}$  of a MISOCO problem, where  $\ell \geq 2$ . Also, let  $\mathcal{A} = \{w \in \mathbb{R}^\ell \mid a^\top w \geq \alpha\}$  and

$\mathcal{B} = \{w \in \mathbb{R}^\ell \mid a^\top w \leq \beta\}$  be two half spaces with  $\alpha \neq \beta$ , and  $\|a\| = 1$ . We denote the boundary hyperplanes of the half-spaces  $\mathcal{A}$  and  $\mathcal{B}$  by  $\mathcal{A}^\#$  and  $\mathcal{B}^\#$ , respectively. In what follows, we provide a full characterization of the DCCs that can be derived by explicit characterization of the set  $\text{conv}(\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B}))$ . From Section 2 we know that in this case we need to consider only the following quadrics: ellipsoids, paraboloids, hyperboloids of two sheets, and cones. Hence, here we provide the derivation of DCCs for these quadrics. Note that if  $\mathcal{A} \cap \mathcal{Q} = \emptyset$ , then  $\text{conv}(\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{B} \cap \mathcal{Q}$ . Similarly, if  $\mathcal{B} \cap \mathcal{Q} = \emptyset$ , then  $\text{conv}(\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{A} \cap \mathcal{Q}$ . Therefore, for the rest of this section we assume that  $\mathcal{A} \cap \mathcal{Q} \neq \emptyset$  and  $\mathcal{B} \cap \mathcal{Q} \neq \emptyset$ . Also, note that if  $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ , then  $\text{conv}(\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{Q}$ . Hence, we also assume in this section that  $\mathcal{A} \cap \mathcal{B} = \emptyset$ . To facilitate the algebra in this section, we may assume w.l.o.g. that  $\beta < \alpha$  and that the quadric  $\mathcal{Q}$  has been normalized using an appropriate linear transformation  $L$ , see Appendix B for details on the normalization of  $\mathcal{Q}$ .

We divide this analysis as follows. In Section 3.1 we recall the definitions of DCCs and disjunctive cylindrical cuts (DCyCs) introduced in [9]. In Section 3.2, we revisit some results from [8] about quadrics. In Section 3.3 we present the derivation when  $\mathcal{A}^\# \cap \mathcal{Q}$  and  $\mathcal{B}^\# \cap \mathcal{Q}$  are bounded. Finally, in Section 3.4 we present the derivation when  $\mathcal{A}^\# \cap \mathcal{Q}$  and  $\mathcal{B}^\# \cap \mathcal{Q}$  are unbounded.

### 3.1 Disjunctive conic cuts

First we need to provide the definitions of DCCs and DCyCs. In this paper, we use the definitions as they were introduced in [9].

**Definition 1 (Belotti et al. [9])** *A full dimensional closed convex cone  $\mathcal{K} \subset \mathbb{R}^n$  with  $\dim(\mathcal{K}) > 1$  is called a disjunctive conic cut (DCC) for the set  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$  and the disjunctive set  $\mathcal{A} \cup \mathcal{B}$  if*

$$\text{conv}(\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{Q} \cap \mathcal{K}.$$

The following proposition gives a sufficient condition for a convex cone  $\mathcal{K}$  to be a disjunctive conic cut for the set  $\mathcal{Q}$  when the sets  $\mathcal{Q} \cap \mathcal{A}^\#$  and  $\mathcal{Q} \cap \mathcal{B}^\#$  are bounded.

**Proposition 1 (Belotti et al. [9])** *Let  $\mathcal{Q}$  and the disjunctive set  $\mathcal{A} \cup \mathcal{B}$  be such that  $(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{Q}$  is empty, and both  $\mathcal{Q} \cap \mathcal{A}^\#$  and  $\mathcal{Q} \cap \mathcal{B}^\#$  are nonempty and bounded. A full dimensional convex cone  $\mathcal{K} \subset \mathbb{R}^\ell$  with  $\dim(\mathcal{K}) > 1$  is the unique DCC for  $\mathcal{Q}$  and the disjunctive set  $\mathcal{A} \cup \mathcal{B}$  if*

$$\mathcal{K} \cap \mathcal{A}^\# = \mathcal{Q} \cap \mathcal{A}^\# \quad \text{and} \quad \mathcal{K} \cap \mathcal{B}^\# = \mathcal{Q} \cap \mathcal{B}^\#. \quad (9)$$

Now, we provide the definition of a DCyC. Here we use the definition of a cylinder as given in Appendix A.

**Definition 2 (Belotti et al. [9])** *A closed convex cylinder  $\mathcal{C} \subset \mathbb{R}^\ell$  is a disjunctive cylindrical cut (DCyC) for the set  $\mathcal{Q}$  and the disjunctive set  $\mathcal{A} \cup \mathcal{B}$  if*

$$\text{conv}(\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{C} \cap \mathcal{Q}.$$

The following proposition gives a sufficient condition for a convex cylinder  $\mathcal{C}$  to be a disjunctive cylindrical cut for the set  $\mathcal{Q}$ .

**Proposition 2** *Let  $\mathcal{Q}$  and the disjunctive set  $\mathcal{A} \cup \mathcal{B}$  be such that  $(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{Q}$  is empty, and both  $\mathcal{Q} \cap \mathcal{A}^\#$  and  $\mathcal{Q} \cap \mathcal{B}^\#$  are nonempty. A convex cylinder  $\mathcal{C} \in \mathbb{R}^\ell$  with a unique direction  $d^0 \in \mathbb{R}^\ell$ , such that  $a^\top d^0 \neq 0$ , is the unique DCyC for  $\mathcal{Q}$  and the disjunctive set  $\mathcal{A} \cup \mathcal{B}$  if*

$$\mathcal{C} \cap \mathcal{A}^\# = \mathcal{Q} \cap \mathcal{A}^\# \quad \text{and} \quad \mathcal{C} \cap \mathcal{B}^\# = \mathcal{Q} \cap \mathcal{B}^\#. \quad (10)$$

Note that this version of Proposition 2 requires the condition for the sets  $\mathcal{Q} \cap \mathcal{A}^\#$  and  $\mathcal{Q} \cap \mathcal{B}^\#$  to be nonempty, but the boundedness requirement is relaxed. The boundedness was in fact a requirement in [9]. However, the proof of this version of Proposition 2 does not significantly differ from the proof in [9], hence the proof of this version is omitted. A version of the proof can be found in [14].



### 3.2 Family of quadrics

For the derivation of the DCCs and DCyCs presented in this section we use the result from [8].

**Theorem 1 (Belotti et al. [8])** *The uni-parametric family of quadrics  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  having the same intersection with  $\mathcal{A}^\#$  and  $\mathcal{B}^\#$  as the quadric  $\mathcal{Q}$  is defined as  $\mathcal{Q}(\tau) = \{w \in \mathbb{R}^n \mid w^\top P(\tau)w + 2p(\tau)^\top w + \rho(\tau) \leq 0\}$ , where*

$$P(\tau) = P + \tau aa^\top, \quad p(\tau) = p - \tau \frac{(\alpha + \beta)}{2} a, \quad \rho(\tau) = \rho + \tau \alpha \beta.$$

From Theorem 1, we have that for any  $\tau \in \mathbb{R}$  the quadric  $\mathcal{Q}(\tau)$  satisfies the conditions  $\mathcal{Q}(\tau) \cap \mathcal{A}^\# = \mathcal{Q} \cap \mathcal{A}^\#$  and  $\mathcal{Q}(\tau) \cap \mathcal{B}^\# = \mathcal{Q} \cap \mathcal{B}^\#$ . Hence, to use the results of Propositions 1 and 2, it remains to show the existence of a convex cone  $\mathcal{K}$  or a convex cylinder  $\mathcal{C}$  in this family of quadrics. Then, we can show that  $\mathcal{K} \cap \mathcal{Q}$  or  $\mathcal{C} \cap \mathcal{Q}$  characterizes the convex hull for  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ . To prove the existence of a convex cone  $\mathcal{K}$  or a convex cylinder  $\mathcal{C}$ , it is necessary to characterize the shapes of the quadrics in the family  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  of Theorem 1.

The matrix  $P(\tau)$  is in general non-singular, except for some specific values of  $\tau$ , which are discussed in Sections 3.3 and 3.4. For the case where  $P(\tau)$  is non-singular, one can rewrite the definition of the quadric  $\mathcal{Q}(\tau)$  as

$$(w + P(\tau)^{-1}p(\tau))^\top P(\tau)(w + P(\tau)^{-1}p(\tau)) \leq p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau). \quad (11)$$

As is mentioned in Section 2, the shapes of the quadrics in the family  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  can be classified using the inertia of  $P(\tau)$  and the term  $p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau)$ . These facts will be used in Sections 3.3 and 3.4.

### 3.3 Derivation of DCCs when both $\mathcal{Q} \cap \mathcal{A}^\#$ and $\mathcal{Q} \cap \mathcal{B}^\#$ are bounded

The analysis of this case is done in [8, 9]. Nevertheless for the sake of completeness, we summarize the main results of that analysis in this section. In [8], the properties of the family of quadrics  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  in Theorem 1 are analyzed when  $\mathcal{Q}$  is an ellipsoid, which implies that the sets  $\mathcal{Q} \cap \mathcal{A}^\#$  and  $\mathcal{Q} \cap \mathcal{B}^\#$  are bounded. It is important to notice that this analysis is also valid when  $\mathcal{Q}$  is a paraboloid, a cone, or a hyperboloid of two sheets, provided that the sets  $\mathcal{Q} \cap \mathcal{A}^\#$  and  $\mathcal{Q} \cap \mathcal{B}^\#$  are bounded. Here, for the sake of keeping the algebra simple, we also work with the assumption that  $\mathcal{Q}$  is an ellipsoid. Now, in [8] it is shown that the term  $p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau)$  can be written as the ratio

$$p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau) = \frac{\tau^2 \frac{(\alpha_1 - \alpha_2)^2}{4} + \tau(1 - \alpha_1 \alpha_2) + 1}{(1 + \tau)}. \quad (12)$$

Let  $\bar{\tau}_1 \leq \bar{\tau}_2$  be the roots of the numerator of (12). A full characterization of the family  $\mathcal{Q}(\tau)$  for  $\tau \in \mathbb{R}$ , depending on the geometry of  $\mathcal{Q}$  and the hyperplanes  $\mathcal{A}^\#$  and  $\mathcal{B}^\#$ , is presented in [8, Theorem 3.4], which we recall here.

**Theorem 2** *The following cases may occur for the shape of  $\mathcal{Q}(\tau)$ :*

1.  $\bar{\tau}_1 < \bar{\tau}_2 < -1$ :  $\mathcal{Q}(-1)$  is a paraboloid, and  $\mathcal{Q}(\bar{\tau}_1)$ ,  $\mathcal{Q}(\bar{\tau}_2)$  are two cones.
2.  $\bar{\tau}_1 = \bar{\tau}_2 < -1$ :  $\mathcal{Q}(-1)$  is a paraboloid and  $\mathcal{Q}(\bar{\tau}_2)$  is a cone.
3.  $\bar{\tau}_1 < \bar{\tau}_2 = -1$ :  $\mathcal{Q}(\bar{\tau}_1)$  is cone and  $\mathcal{Q}(\bar{\tau}_2)$  is a cylinder.
4.  $\bar{\tau}_1 = \bar{\tau}_2 = -1$ :  $\mathcal{Q}(\bar{\tau}_2)$  is a line.

This geometrical analysis is then used to identify the conic cut in the family of Theorem 1, which convexify the intersection of  $\mathcal{Q}$  with a parallel disjunction. To simplify the analysis, we separate the cases of cylinders and cones.

#### 3.3.1 Cylinders

Consider the families  $\{\mathcal{Q}(\tau), \tau \in \mathbb{R}\}$  described in the third and fourth cases in Theorem 2, where  $\mathcal{Q}(\bar{\tau}_2)$  is a cylinder. Recall that

$$\mathcal{Q}(\bar{\tau}_2) = \{w \in \mathbb{R}^n \mid w^\top P(\bar{\tau}_2)w + 2p(\bar{\tau}_2)^\top w + \rho(\bar{\tau}_2) \leq 0\}, \quad (13)$$

where  $Q(\bar{\tau}_2)$  is a positive semidefinite matrix. Hence, the cylinder  $\mathcal{Q}(\bar{\tau}_2)$  is convex, and from Proposition 2 we obtain that  $\mathcal{Q}(\bar{\tau}_2) \cap \mathcal{Q} = \mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ .

### 3.3.2 Cones

Consider the cones described in the first and second cases of Theorem 2. First, note that we need a criterion to identify which of the cones  $\mathcal{Q}(\bar{\tau}_1)$  and  $\mathcal{Q}(\bar{\tau}_2)$  to use. The following criterion is provided in [9].

**Lemma 3 (Belotti et al. [9])** *The quadric  $\mathcal{Q}(\bar{\tau}_2)$  in the families  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  of the cases 1 and 2 of Theorem 2 contains a cone that satisfies Definition 1.*

The proof of Lemma 3 in [9] is based on the following representation of  $\mathcal{Q}(\bar{\tau}_2)$ . Note that  $\mathcal{Q}(\bar{\tau}_2)$  is a double sided cone, hence a non-convex set. However, recall that  $\bar{\tau}_2 < -1$ , and then  $P(\bar{\tau}_2)$  is a symmetric and non-singular matrix that has exactly one negative eigenvalue. Then,  $P(\bar{\tau}_2)$  can be diagonalized as  $P(\bar{\tau}_2) = VD V^\top$ , where  $V \in \mathbb{R}^{\ell \times \ell}$  is an orthogonal matrix and  $D \in \mathbb{R}^{\ell \times \ell}$  is a diagonal matrix having the eigenvalues of  $P(\bar{\tau}_2)$  in its diagonal. We may assume w.l.o.g. that  $D_{1,1} < 0$ , and let  $W = V\tilde{D}^{1/2}$ , where  $\tilde{D}_{l,k} = |D_{l,k}|$ . Also, let  $w^c = -Q(\bar{\tau}_2)p(\bar{\tau}_2)$ . Thus, we may write  $\mathcal{Q}(\bar{\tau}_2)$  in terms of  $W$  as follows

$$\mathcal{Q}(\bar{\tau}_2) = \left\{ w \in \mathbb{R}^n \mid (w - w^c)^\top W_{2:\ell} W_{2:\ell}^\top (w - w^c) \leq (W_1^\top (w - w^c))^2 \right\},$$

where  $W_{2:\ell} \in \mathbb{R}^{\ell \times (\ell-1)}$ , is formed by the columns 2 to  $\ell$  of  $W$ . Now, define the sets  $\mathcal{Q}(\bar{\tau}_2)^+$ ,  $\mathcal{Q}(\bar{\tau}_2)^-$  as follows

$$\mathcal{Q}(\bar{\tau}_2)^+ \equiv \left\{ x \in \mathbb{R}^n \mid \|W_{2:\ell}^\top (w - w^c)\| \leq W_1^\top (w - w^c) \right\}, \quad (14)$$

$$\mathcal{Q}(\bar{\tau}_2)^- \equiv \left\{ x \in \mathbb{R}^n \mid \|W_{2:\ell}^\top (w - w^c)\| \leq -W_1^\top (w - w^c) \right\}, \quad (15)$$

which are two second order cones. It is easy to verify that  $\mathcal{Q}(\bar{\tau}_2) = \mathcal{Q}(\bar{\tau}_2)^+ \cup \mathcal{Q}(\bar{\tau}_2)^-$ . Also, it is clear from (14) and (15) that  $\mathcal{Q}(\bar{\tau}_2)^+$  and  $\mathcal{Q}(\bar{\tau}_2)^-$  are two convex sets.

We now need to decide between the two cones  $\mathcal{Q}(\bar{\tau}_2)^+$  and  $\mathcal{Q}(\bar{\tau}_2)^-$ . For this purpose we can use the sign of  $W_1^\top (-P^{-1}p - w^c)$ . Thus, we choose  $\mathcal{Q}(\bar{\tau}_2)^+$  if  $W_1^\top (-P^{-1}p - w^c) > 0$ , and we choose  $\mathcal{Q}(\bar{\tau}_2)^-$  when  $W_1^\top (-P^{-1}p - w^c) < 0$ . Finally, from Proposition 1 we have that the selected cone gives the convex hull for  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ . Note that if  $W_1^\top (-P^{-1}p - w^c) = 0$  the center of the ellipsoid  $\mathcal{Q}$  coincides with the vertex of the selected cone. In this case the quadric  $\mathcal{Q}$  is a single point, which is a trivial case that does not allow the generation of cuts. This completes the procedure.

Up to this point we have shown that for all the cases in Theorem 2, one can find a cone  $\mathcal{K}$  or a cylinder  $\mathcal{C}$  that satisfies Definitions 1 or 2 respectively. Hence, by combining Theorem 2 with Propositions 1 and 2 one obtains a procedure to find the convex hull of  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ , when the disjunctive set  $\mathcal{A} \cup \mathcal{B}$  is such that both  $\mathcal{Q} \cap \mathcal{A}^=$  and  $\mathcal{Q} \cap \mathcal{B}^=$  are bounded. We need to consider now the cases when both  $\mathcal{Q} \cap \mathcal{A}^=$  and  $\mathcal{Q} \cap \mathcal{B}^=$  are unbounded.

## 3.4 Derivation of DCCs when both $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are unbounded

To complete the derivation of all the DCCs and DCyCs for MISOCO problems we need to consider the case when both intersections  $\mathcal{Q} \cap \mathcal{A}^=$  and  $\mathcal{Q} \cap \mathcal{B}^=$  are unbounded. We analyze first the cases when the cuts are cylinders and then we analyze the cases when the cuts are cones. Note that in this section the quadric  $\mathcal{Q}$  cannot be an ellipsoid. Hence, for the rest of this section we assume that  $\mathcal{Q}$  is a paraboloid, a hyperboloid of two sheets, or a double-sided cone. Appendices C and D provide some results that are used in the proofs of this section.

### 3.4.1 Cylinders

We divide the results for DCyCs in two parts. First, we analyze the case when  $\mathcal{Q}$  is a paraboloid. Second, we analyze the case when  $\mathcal{Q}$  is a hyperboloid of two sheets or a double-sided cone.

Let us assume that  $\mathcal{Q}$  is a paraboloid, which implies that the system  $Px = -p$  has no solution. In this case, we have the following result.

**Lemma 4 (Góez [14])** *If  $\mathcal{Q}$  is a paraboloid and  $a_1 = 0$ , then the quadric  $\mathcal{Q}(-1)$  in the family  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  is a convex cylinder.*

**Proof.** In this case the characteristic polynomial (38) for  $P(\tau)$  simplifies to

$$(1 - \lambda)^{n-2}(\lambda^2 - \lambda(1 + \tau \|a\|^2) + \tau a_1^2) = (1 - \lambda)^{n-2}(\lambda^2 - \lambda(1 + \tau) + \tau a_1^2) = 0.$$

Thus, 1 is an eigenvalue of  $P$  with multiplicity  $\ell - 2$ . The other two eigenvalues are given by the roots of  $\lambda^2 - \lambda(1 + \tau) + \tau a_1^2 = 0$ , which are

$$\frac{(1 + \tau) \pm \sqrt{(1 + \tau)^2 - 4\tau a_1^2}}{2}. \quad (16)$$

Hence, we have that zero is an eigenvalue of the matrix  $P(-1)$ , with multiplicity 2. We perform the proof in three steps. First, we find a basis for the null space of  $P(-1)$ . Then, we find a direction in that space that is orthogonal to  $p(-1)$ . Finally, we show that  $\mathcal{Q}(-1)$  is a convex cylinder in that direction.

Recall that for  $\tau \in \mathbb{R}$  the first row and first column of  $P(-1)$  are zero vectors. Since  $\|a\| = 1$ , and  $a_1 = 0$ , we have that

$$P(-1)a = (\tilde{J} - aa^\top)a = a - a = 0.$$

Thus,  $a$  and  $(1, 0^\top)^\top$  are eigenvectors of  $P(-1)$  associated with the 0 eigenvalue, and form a basis for the null space of  $P(-1)$ . Hence, any vector of the form  $(\gamma, a_{2:\ell}^\top)^\top$ , for all  $\gamma \in \mathbb{R}$ , belongs to the null space of  $P(-1)$ . Define  $\tilde{\gamma}$  as

$$\tilde{\gamma} = \frac{-p_{2:\ell}^\top a_{2:\ell} - \frac{\alpha + \beta}{2}}{p_1},$$

and recall that  $\mathcal{Q}$  being a paraboloid implies that  $p_1 \neq 0$ . The vector  $[\tilde{\gamma}, a_{2:\ell}^\top]^\top$  is orthogonal to  $p(-1)$ , since

$$\begin{aligned} p(-1)^\top \begin{bmatrix} \tilde{\gamma} \\ a_{2:\ell} \end{bmatrix} &= \left( p^\top + \frac{\alpha + \beta}{2} a \right) \begin{bmatrix} -p_{2:\ell}^\top a_{2:\ell} - \frac{\alpha + \beta}{2} \\ p_1 \\ a_{2:\ell} \end{bmatrix} \\ &= -p_{2:\ell}^\top a_{2:\ell} - \frac{\alpha + \beta}{2} + \frac{\alpha + \beta}{2} + p_{2:\ell}^\top a_{2:\ell} = 0. \end{aligned}$$

Let  $\tilde{w} \in \mathbb{R}^\ell$  be a vector such that  $\tilde{w} \in \mathcal{Q}(-1) \cap (\mathcal{A}^\# \cup \mathcal{B}^\#)$ , then we have that

$$\tilde{w}^\top P(-1)\tilde{w} + 2p(-1)^\top \tilde{w} + \rho(-1) \leq 0.$$

Now, let  $\tilde{u}^\top = \tilde{w}^\top + \theta[\tilde{\gamma}, a_{2:\ell}^\top]^\top$  for some  $\theta \in \mathbb{R}$ , then we have that

$$\begin{aligned} \tilde{u}^\top P(-1)\tilde{u} + 2p(-1)^\top \tilde{u} + \rho(-1) &= \tilde{w}^\top P(-1)\tilde{w} + \theta(\tilde{\gamma}, a_{2:\ell}^\top)P(-1)\tilde{w} + \theta^2(\tilde{\gamma}, a_{2:\ell}^\top)P(-1) \begin{bmatrix} \tilde{\gamma} \\ a_{2:\ell} \end{bmatrix} + 2p(-1)^\top \tilde{u} + \rho(-1) \\ &= \tilde{w}^\top P(-1)\tilde{w} + 2p(-1)^\top \tilde{w} + 2\theta p(-1)^\top \begin{bmatrix} \tilde{\gamma} \\ a_{2:\ell} \end{bmatrix} + \rho(-1) \\ &= \tilde{w}^\top P(-1)\tilde{w} + 2p(-1)^\top \tilde{w} + \rho(-1) \leq 0, \end{aligned}$$

where the last inequality follows from the assumption  $\tilde{w} \in \mathcal{Q}(-1) \cap (\mathcal{A}^\# \cup \mathcal{B}^\#)$ . Hence, the distance of a vector  $\tilde{z}$  to the boundary of  $\mathcal{Q}(-1)$  is constant for any  $\theta \in \mathbb{R}$ . Finally, we need to show that any cross section of  $\mathcal{Q}(-1)$  in the direction  $[\tilde{\gamma}, a_{2:\ell}^\top]^\top$  is a convex set. Consider the hyperplane  $[\tilde{\gamma}, a_{2:\ell}^\top]w = \varrho$ , where  $\varrho \in \mathbb{R}$ , and let  $\tilde{P}(-1)$  be the matrix the lower right  $\ell - 1 \times \ell - 1$  sub-matrix of  $P(-1)$ . Then, for a fixed  $\varrho$  we obtain a quadric  $\tilde{\mathcal{Q}}(-1) \in \mathbb{R}^{\ell-1 \times \ell-1}$  defined by the inequality

$$w_{2:n}^\top \tilde{P}(-1)w_{2:n} + 2 \left( p(-1)_{2:n} - \frac{p_1}{\tilde{\gamma}} a_{2:n} \right) w_{2:n} + 2p_1 \varrho + \rho(-1) \leq 0.$$

Note that  $\tilde{P}(-1)$  is a positive semi-definite matrix, thus  $\tilde{\mathcal{Q}}(-1)$  is a convex set. Therefore,  $\mathcal{Q}(-1)$  is a convex cylinder in the direction  $(\tilde{\gamma}, a_{2:\ell}^\top)^\top$ .  $\square$

Recall that the quadric  $\mathcal{Q}$  is assumed to be normalized using the procedure given in Appendix B. Note that in this case, if  $a_1 \neq 0$ , then the intersections  $\mathcal{Q} \cap \mathcal{A}^\pm$  and  $\mathcal{Q} \cap \mathcal{B}^\pm$  are bounded. This shows that the case considered in Lemma 4 is the only case of interest for this section when  $\mathcal{Q}$  is a paraboloid. Additionally, it is shown in the proof of Lemma 4 that the direction of the cylinder  $\mathcal{Q}(-1)$  is given by a vector  $[\gamma \ a_{2:\ell}^\top]^\top$ , for some  $\gamma \in \mathbb{R}$ . Hence, from Lemma 4 we have that the product of the normal vector  $a$  of the hyperplanes  $\mathcal{A}^\pm$  and  $\mathcal{B}^\pm$  with the direction of the cylinder is different from 0. Hence, from Theorem 2 and Lemma 1 we can conclude that  $\mathcal{Q}(-1)$  is a DCyC for Problem (MISOCO).

We analyze now the cases when  $\mathcal{Q}$  is a two-sided cone or a hyperboloid of two sheets. Recall that  $\mathcal{Q}$  is assumed to be normalized, then from Appendix B we know that in these two cases the vector  $p$  in (2) is the zero vector, and  $P_{1,1} = -1$ . The classification of the quadrics in the family  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  is done here using the inertia of the matrix  $P(\tau)$  and the term  $p(\tau)P(\tau)^{-1}p(\tau) - \rho(\tau)$ . The inertia of  $P(\tau)$  can be computed using a close formula. In this case the characteristic polynomial (38) simplifies to

$$(1 - \lambda)^{\ell-2} \left( \lambda^2 - \lambda\tau \|a\|^2 + (\tau a_1^2 - \tau \sum_{i=2}^{\ell} a_i^2 - 1) \right) = (1 - \lambda)^{\ell-2} (\lambda^2 - \lambda\tau + (2\tau a_1^2 - \tau - 1)) = 0.$$

Thus, 1 is an eigenvalue of  $P$  with multiplicity  $\ell - 2$ . The other two eigenvalues are given as the roots of  $\lambda^2 - \lambda\tau + (2\tau a_1^2 - \tau - 1) = 0$ , which are

$$\frac{\tau \pm \sqrt{\tau^2 + 4 + 4\tau(1 - 2a_1^2)}}{2}. \quad (17)$$

Assuming that  $P(\tau)$  is non-singular we have that

$$\begin{aligned} & p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) \\ &= \left( -\tau \frac{\alpha + \beta}{2} a \right)^\top \left( \tilde{J} + \tau a a^\top \right)^{-1} \left( -\tau \frac{\alpha + \beta}{2} a \right) - (\rho + \tau \alpha \beta). \\ &= \frac{\tau^2 (1 - 2a_1^2) \frac{(\alpha - \beta)^2}{4} - \tau(\rho(1 - 2a_1^2) + \alpha\beta) - \rho}{1 + \tau(1 - 2a_1^2)}. \end{aligned} \quad (18)$$

Recall that we characterize the cases when the sets  $\mathcal{Q} \cap \mathcal{A}^\pm$  and  $\mathcal{Q} \cap \mathcal{B}^\pm$  are unbounded. Consider the case when  $\mathcal{Q}$  is a two sided cone, and then from Appendix B we know that  $\rho = 0$ . Note that in this case, if  $a_1^2 \geq \frac{1}{2}$  then  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B}) = \text{conv}(\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B}))$ , hence there is no disjunctive cut in this case. This is true because if  $a_1^2 \geq \frac{1}{2}$ , then either the sets  $\mathcal{Q} \cap \mathcal{A}^\pm$  and  $\mathcal{Q} \cap \mathcal{B}^\pm$  are bounded or the vector  $a$  is parallel to a ray of  $\mathcal{Q}$ . Hence, we need to focus our analysis on the case when  $a_1^2 < \frac{1}{2}$ . Now, since  $\rho = 0$  we have that

$$p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) = \frac{\tau^2 (1 - 2a_1^2) \frac{(\alpha - \beta)^2}{4} - \tau \alpha \beta}{1 + \tau(1 - 2a_1^2)}. \quad (19)$$

Hence, we have that  $p(\tau)^\top P(\tau)^{-1} p(\tau) = 0$  is satisfied when the numerator of (19) is zero, and we obtain the values

$$\bar{\tau}_1 = 2 \left( \frac{\alpha\beta - |\alpha\beta|}{(1 - 2a_1^2)(\alpha - \beta)^2} \right) \quad \text{and} \quad \bar{\tau}_2 = 2 \left( \frac{\alpha\beta + |\alpha\beta|}{(1 - 2a_1^2)(\alpha - \beta)^2} \right). \quad (20)$$

Let  $\bar{\tau}$  be the non zero root of the numerator of (19), then based on the value of  $\bar{\tau}$  we have the following result.

**Theorem 3 (Classification for a two-sided cone, Góez [14])** *If  $\mathcal{Q}$  is a two sided cone, and  $a_1^2 < \frac{1}{2}$ , then the shapes of the quadric  $\mathcal{Q}(\bar{\tau})$  in the family  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  is:*

1. a cone if  $\bar{\tau} > -\frac{1}{(1-2a_1^2)}$ ,
2. a hyperbolic cylinder of two sheets if  $\bar{\tau} = -\frac{1}{(1-2a_1^2)}$ .

To facilitate the flow of the discussion we moved the proof of this result to Appendix D.1.

Consider now the case when  $\mathcal{Q}$  is a hyperboloid of two sheets, and then from Appendix B we know that  $\rho = 1$ . In this case if  $a_1^2 > \frac{1}{2}$  then the sets  $\mathcal{Q} \cap \mathcal{A}^=$  and  $\mathcal{Q} \cap \mathcal{B}^=$  are bounded. Now, if  $a_1^2 = \frac{1}{2}$  we will show later that there is not cylinder in the family  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ . Hence, we are interested in the case when  $a_1^2 < \frac{1}{2}$ . Now, since  $\rho = 1$  we have that

$$p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) = \frac{\tau^2(1 - 2a_1^2) \frac{(\alpha - \beta)^2}{4} - \tau((1 - 2a_1^2) + \alpha\beta) - 1}{1 + \tau(1 - 2a_1^2)}. \quad (21)$$

Let  $\bar{\tau}_1 \leq \bar{\tau}_2$  be the roots of the numerator of (21), then based in this roots we have the following result.

**Theorem 4 (Classification for a hyperboloid of two sheets, Góez [14])** *If  $\mathcal{Q}$  is a hyperboloid of two sheets, and  $a_1^2 < \frac{1}{2}$ , then the shapes of the quadrics  $\mathcal{Q}(\bar{\tau}_1)$  and  $\mathcal{Q}(\bar{\tau}_2)$  may be as follows:*

1. if  $\beta \neq -\alpha$ , then both quadrics are cones,
2. if  $\beta = -\alpha$ , then  $\mathcal{Q}(\bar{\tau}_1)$  is a hyperbolic cylinder of two sheets and  $\mathcal{Q}(\bar{\tau}_2)$  is a cone.

To facilitate the flow of the discussion we moved the proof of this result to Appendix D.2.

We need to analyze the second cases of Theorems 4 and 3 to show that they are DCyCs for Problem (MISOCO) when  $\mathcal{Q}$  is either a cone or a hyperboloid of two sheets, respectively. In these cases the quadric  $\mathcal{Q}(\hat{\tau})$ , where  $\hat{\tau} = -\frac{1}{(1-2a_1^2)}$ , is a hyperbolic cylinder of two sheets, which is a non-convex quadric. Even more, recall that both cases of Theorems 3 and 4 happen only when  $\alpha = -\beta$ , then we have that  $p(\hat{\tau}) = 0$ ,  $\rho(\hat{\tau}) > 0$  and

$$\mathcal{Q}(\hat{\tau}) = \{x \in \mathbb{R}^\ell \mid x^\top P(\hat{\tau})x \leq -\rho(\hat{\tau})\}.$$

Consider the eigenvalue decomposition  $P(\hat{\tau}) = VDV^\top$ , where  $D \in \mathbb{R}^{\ell \times \ell}$  is a diagonal matrix, and  $V \in \mathbb{R}^{\ell \times \ell}$  is non-singular. We may assume w.l.o.g. that  $D_{1,1} = -1$ ,  $D_{2,2} = 0$ , and  $D_{i,i} > 0$ ,  $i \in \{3, \dots, \ell\}$ . Now, let  $W = V\bar{D}^{\frac{1}{2}}$ , where  $\bar{D}$  is a diagonal matrix such that  $\bar{D}_{i,i} = |D_{i,i}|$ . Let  $W_{3:\ell}$  be the matrix that has the last  $\ell - 2$  columns of  $W$ , and  $W_1$  be the first column of  $W$ . Then,

$$\mathcal{Q}(\hat{\tau}) = \left\{ x \in \mathbb{R}^\ell \mid \|W_{3:n}^\top x\|^2 \leq -\rho(\hat{\tau}) + (W_1^\top x)^2 \right\}.$$

Let us define the following two sets

$$\begin{aligned} \mathcal{Q}^+(\hat{\tau}) &= \left\{ x \in \mathbb{R}^n \mid \|W_{3:n}^\top x\| \leq \xi, \left\| \begin{bmatrix} \xi & \sqrt{\rho(\hat{\tau})} \end{bmatrix}^\top \right\| \leq W_1^\top x \right\}, \\ \mathcal{Q}^-(\hat{\tau}) &= \left\{ x \in \mathbb{R}^n \mid \|W_{3:n}^\top x\| \leq \xi, \left\| \begin{bmatrix} \xi & \sqrt{\rho(\hat{\tau})} \end{bmatrix}^\top \right\| \leq -W_1^\top x \right\}, \end{aligned}$$

where  $\begin{bmatrix} \xi & \sqrt{\rho(\hat{\tau})} \end{bmatrix}^\top \in \mathbb{R}^2$ . Thus,  $\mathcal{Q}(\hat{\tau}) = \mathcal{Q}^+(\hat{\tau}) \cup \mathcal{Q}^-(\hat{\tau})$ , and each of these branches of  $\mathcal{Q}(\hat{\tau})$  are convex cylinders in the direction  $V_2$ , which is the second column of  $V$ . Also, note that  $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{Q}^-(\hat{\tau}) = \emptyset$ . Recall that in the cases under consideration the set  $\mathcal{Q}$  is either a cone or a hyperboloid of two sheets. Then, using the procedure described in Section 3.3.2 one can show that there are two convex sets  $\mathcal{Q}^+$  and  $\mathcal{Q}^-$  such that  $\mathcal{Q}^+ \cup \mathcal{Q}^- = \mathcal{Q}$ . Now consider the feasible set (6), and assume w.l.o.g. that  $\mathcal{F} = \{x \in \mathbb{R}^n \mid x = x_0 + Hw, w \in \mathcal{Q}^+\}$ . Then we obtain the following result.

**Lemma 5 (Góez [14])** *In the second case in Theorem 3 and the second case in Theorem 4 the set  $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$  is a subset of a single branch of  $\mathcal{Q}(\hat{\tau})$ .*

**Proof.** The proof is by contradiction. We show that if the set  $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$  is not a subset of a single branch of  $\mathcal{Q}(\hat{\tau})$ , then  $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} \neq (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$ , which is a contradiction. Let  $u, v \in (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$  be two vectors such that  $u \in \mathcal{Q}^+(\hat{\tau})$  and  $v \in \mathcal{Q}^-(\hat{\tau})$ .

First, consider the case when  $u, v \in \mathcal{A}^=$  or  $u, v \in \mathcal{B}^=$ . Hence, we either have  $a^\top u = \alpha$  and  $a^\top v = \alpha$ , or  $a^\top u = \beta$  and  $a^\top v = \beta$ , and then there must exist a  $0 \leq \bar{\lambda} \leq 1$  such that  $w = \bar{\lambda}v + (1 - \bar{\lambda})u \in (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$

but  $w \notin (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$ . This statement is true because  $\mathcal{Q}^+$ ,  $\mathcal{Q}^+(\hat{\tau})$ , and  $\mathcal{Q}^-(\hat{\tau})$  are convex sets, and  $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{Q}^-(\hat{\tau}) = \emptyset$ . This contradicts  $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} = (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$ .

Second, consider the case when  $u \in \mathcal{A}^=$  and  $v \in \mathcal{B}^=$ . Therefore,  $a^\top u = \alpha$ ,  $a^\top v = \beta$ , and let  $\tilde{a} = [-a_1 \ a_{2:n}^\top]^\top$ . From Theorem 1 we obtain that  $P(\hat{\tau}) = \tilde{J} - \frac{aa^\top}{(1-2a_1^2)}$ , and for any  $\theta \in \mathbb{R}$  we have that

$$(v + \theta\tilde{a})^\top P(\hat{\tau})(v + \theta\tilde{a}) + \rho(\hat{\tau}) = v^\top P(\hat{\tau})v + \rho(\hat{\tau}) \leq 0.$$

Similarly, we have for any  $\theta \in \mathbb{R}$  that

$$(u + \theta\tilde{a})^\top P(\hat{\tau})(u + \theta\tilde{a}) + \rho(\hat{\tau}) = u^\top P(\hat{\tau})u + \rho(\hat{\tau}) \leq 0,$$

Additionally, since  $a^\top \tilde{a} \neq 0$ , then  $\exists \hat{\theta}$  such that  $u + \hat{\theta}\tilde{a} \in \mathcal{Q}^+(\hat{\tau})$  and  $a^\top(u + \hat{\theta}\tilde{a}) = \beta$ , which shows that  $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \neq \emptyset$ . Similarly,  $\exists \hat{\theta}$  such that  $v + \hat{\theta}\tilde{a} \in \mathcal{Q}^-(\hat{\tau})$  and  $a^\top(v + \hat{\theta}\tilde{a}) = \alpha$ , which shows that  $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \neq \emptyset$ .

Now, we show that  $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^+ = \emptyset$  and  $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^+ = \emptyset$ . Assume to the contrary that  $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^+ \neq \emptyset$ . Then, for any  $s \in \mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^+$  there must exist a  $0 \leq \tilde{\lambda} \leq 1$  such that  $w = \tilde{\lambda}s + (1 - \tilde{\lambda})v \in \mathcal{B}^= \cap \mathcal{Q}^+$  but  $w \notin \mathcal{B}^= \cap \mathcal{Q}(\hat{\tau})$ . This is true because  $\mathcal{Q}^+$  is convex,  $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{Q}^-(\hat{\tau}) = \emptyset$ , and  $v \in \mathcal{Q}^-(\hat{\tau}) \cap \mathcal{B}^=$ . A similar contradiction would be obtained if  $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^+ \neq \emptyset$ .

Now, since  $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \neq \emptyset$  and  $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \neq \emptyset$ , and also  $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^+ = \emptyset$  and  $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^+ = \emptyset$ , we have that  $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^- \neq \emptyset$  and  $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^- \neq \emptyset$ , because  $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} = (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$ .

Let  $w \in \mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^-$ . Then,  $\lambda w + (1 - \lambda)u \in \mathcal{Q}^+(\hat{\tau})$  for  $0 \leq \lambda \leq 1$ , since  $\mathcal{Q}^+(\hat{\tau})$  is convex. Now, if  $\mathcal{Q}$  is a hyperboloid, then there exists a  $0 \leq \tilde{\lambda} \leq 1$  such that  $\tilde{\lambda}w + (1 - \tilde{\lambda})u \notin \mathcal{Q}$ , because  $u \in \mathcal{Q}^+$  and  $w \in \mathcal{Q}^-$ . Hence, we obtain that  $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} \neq (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$ , which is a contradiction. On the other hand, if  $\mathcal{Q}$  is a cone, there must exist a  $\tilde{\lambda}$  such that either  $\tilde{\lambda}w + (1 - \tilde{\lambda})u \notin \mathcal{Q}$  or  $\tilde{\lambda}w + (1 - \tilde{\lambda})u$  is the zero vector. In the first case, we find a contradiction to  $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} = (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$  again. In the second case, let us consider a vector  $s \in \mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^-$ . Then,  $\lambda s + (1 - \lambda)v \in \mathcal{Q}^-(\hat{\tau})$  for  $0 \leq \lambda \leq 1$ , since  $\mathcal{Q}^-(\hat{\tau})$  is convex. In this case, there must exist a  $\tilde{\lambda}$  such that  $\tilde{\lambda}s + (1 - \tilde{\lambda})v \notin \mathcal{Q}$ . The last statement is true because  $v \in \mathcal{Q}^+$  and  $s \in \mathcal{Q}^-$ , the zero vector is in  $\mathcal{Q}^+(\hat{\tau})$  and  $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{Q}^+(\hat{\tau}) = \emptyset$ . Hence, we obtain that  $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} \neq (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$ , which is again a contradiction. This completes the proof.  $\square$

We can now complete the derivation of the DCyCs of this section. First, note that from Proposition 2 and Lemma 5, we know that the branch of  $\mathcal{Q}(\hat{\tau})$  containing the set  $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$  is a DCyC for Problem (MISOCO). Second, we need to define a criteria to identify the branch of  $\mathcal{Q}(\hat{\tau})$  that defines the cylindrical cut. First, consider the case when  $\mathcal{Q}^+ = \{x \in \mathbb{R}^\ell \mid x \in \mathcal{Q}, x_1 \geq 0\}$ . Then, if  $W(\hat{\tau})_1^\top e_1 \geq 0$ , then the cylindrical cut is given by  $\mathcal{Q}^+(\hat{\tau})$ . On the other hand, if  $W(\hat{\tau})_1^\top e_1 \leq 0$ , then the cylindrical cut is given by  $\mathcal{Q}^-(\hat{\tau})$ . Now, consider the case when  $\mathcal{Q}^+ = \{x \in \mathbb{R}^\ell \mid x \in \mathcal{Q}, x_1 \leq 0\}$ . Then, if  $-W(\hat{\tau})_1^\top e_k \geq 0$ , then the cylindrical cut is given by  $\mathcal{Q}^+(\hat{\tau})$ . On the other hand, if  $-W(\hat{\tau})_1^\top e_1 \leq 0$ , then the cylindrical cut is given by  $\mathcal{Q}^-(\hat{\tau})$ .

### 3.4.2 Cones

We focus here on the DCCs that are derived from the family  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  associated with the first cases of Theorems 3 and 4, and the case when  $\mathcal{Q}$  is a hyperboloid of two sheets and  $a_1^2 = \frac{1}{2}$ .

Let us first analyze the case when  $\mathcal{Q}$  is a hyperboloid of two sheets and  $a_1^2 = \frac{1}{2}$ . In this case have that the numerator of (18) simplifies to

$$-\alpha\beta\tau - \rho. \tag{22}$$

Then, we have the following result [14].

**Lemma 6 (Góez [14])** *If  $\mathcal{Q}$  is a hyperboloid, and  $a_1^2 = \frac{1}{2}$ , then for  $\bar{\tau} = -\frac{\rho}{\alpha\beta}$  the quadric  $\mathcal{Q}(\bar{\tau})$  in the family  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  is a cone.*

**Proof.** In this case we have from (17) that  $P(\tau)$  is always an invertible matrix with one negative eigenvalue. On the other hand, we have from (22) that  $p(\bar{\tau})P(\bar{\tau})^{-1}p(\bar{\tau}) - \rho(\bar{\tau}) = 0$  for  $\bar{\tau} = -\rho/\alpha\beta$ . Hence, we have that the quadric  $\mathcal{Q}(\bar{\tau})$  in the family  $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  is a cone.  $\square$

Now, before the derivation of the cuts, recall that when  $\mathcal{Q}$  is a two-sided cone, we have from (2) that  $\mathcal{Q} = \{y \in \mathbb{R}^n \mid \|y_{2:n}\|^2 \leq y_1^2\}$ . In this case we define  $\mathcal{Q}^+ = \{y \in \mathbb{R}^n \mid \|y_{2:n}\| \leq y_1\}$  and  $\mathcal{Q}^- = \{y \in \mathbb{R}^n \mid \|y_{2:n}\| \leq -y_1\}$ , then  $\mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^-$  and  $\mathcal{Q}^+ \cap \mathcal{Q}^- = \emptyset$ . Also, note that  $\mathcal{Q}^+$  and  $\mathcal{Q}^-$  are two second order cones.

Additionally, recall that when  $\mathcal{Q}$  is a hyperboloid, we have from (2) that  $\mathcal{Q} = \{y \in \mathbb{R}^n \mid \|y_{2:n}\|^2 \leq y_1^2 - 1\}$ . In this case, we define  $\mathcal{Q}^+ = \{y \in \mathbb{R}^n \mid y^\top y \leq w, \|(w, 1)\| \leq y_1\}$  and  $\mathcal{Q}^- = \{y \in \mathbb{R}^n \mid y^\top y \leq w, \|(w, 1)\| \leq -y_1\}$ , then  $\mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^-$  and  $\mathcal{Q}^+ \cap \mathcal{Q}^- = \emptyset$ . Also, note that  $\mathcal{Q}^+$  and  $\mathcal{Q}^-$  are two convex sets.

Given that the result for cones and hyperboloids of two sheets are similar, we will use  $\mathcal{Q}^+$  and  $\mathcal{Q}^-$  indistinctively for cones and hyperboloids. We will specify whether we are referring to a cone or a hyperboloid of two sheets when needed. Now, for the derivation of the DCCs we need to prove that the cones in the first cases of Theorems 3 and 4, and of Lemma 6 satisfy Definition 1. To prove this, we use some intermediate results that are omitted here for the ease of understanding. The interested reader can find the details of these steps in Appendix D.3, in Lemmas 7, and 10.

**Theorem 5 (Góez [14])** *Let  $\bar{\tau}$  be the smaller root of the numerator of (18). The quadric  $\mathcal{Q}(\bar{\tau}) \in \{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$  of the first case of Theorems 3 and 4, and Lemma 6 contains a cone that satisfies Definition 1.*

**Proof.** We divide the proof into two parts. First, we show that the theorem is true for the first case of Theorem 3 when  $0 \in \mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ . Second, we show that the theorem is true when  $\mathcal{Q}$  is a hyperboloid of two sheets, or  $\mathcal{Q}$  is a cone and  $0 \notin \mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ .

**DCC when  $\mathcal{Q}$  is a cone and the vector zero is an element of  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ :** This occurs when  $\alpha$  and  $\beta$  have the same sign. Then, the smallest root of  $f(\tau)$  in this case is  $\bar{\tau}_1 = 0$ . Hence, it is enough to show that  $\mathcal{Q}^+ = \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$  in this case. First of all, since  $\mathcal{Q}^+$  is a convex set, we have that  $\text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B})) \subseteq \mathcal{Q}^+$ . Thus, to complete the proof of the first part we need to show that  $\mathcal{Q}^+ \subseteq \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$ . From the definition of convex hull it is clear that  $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \in \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$ . Now, let  $\hat{x} \in \mathcal{Q}^+$  be such that  $\hat{x} \notin \mathcal{A} \cup \mathcal{B}$ . Then, we have that  $\beta \leq a^\top \hat{x} = \sigma \leq \alpha$ . Assume first that  $0 \leq \beta \leq \alpha$ , and consequently the vector zero is contained in  $\mathcal{B}$ . Since  $\mathcal{Q}^+$  is a cone, then  $\gamma \hat{x} \in \mathcal{Q}^+$  for  $\gamma \geq 0$ . Now, we have that  $a^\top(\gamma \hat{x}) = \gamma \sigma$ . Then, for  $\gamma^1 = \frac{\alpha}{\sigma}$  we obtain  $a^\top(\gamma^1 \hat{x}) = \alpha$ , and for  $\gamma^2 = \frac{\beta}{\sigma}$  we obtain  $a^\top(\gamma^2 \hat{x}) = \beta$ . Now, consider the convex combination  $\lambda(\gamma^1 \hat{x}) + (1 - \lambda)(\gamma^2 \hat{x})$ ,  $0 \leq \lambda \leq 1$ . For  $\hat{\lambda} = \frac{1 - \gamma^2}{\gamma^1 - \gamma^2}$  we obtain that  $0 \leq \hat{\lambda} \leq 1$ , and  $\lambda(\gamma^1 \hat{x}) + (1 - \lambda)(\gamma^2 \hat{x}) = \hat{x}$ . Since  $\gamma^2 \hat{x} \in \mathcal{Q}^+ \cap \mathcal{B}$  and  $\gamma^1 \hat{x} \in \mathcal{Q}^+ \cap \mathcal{A}$ , then  $\hat{x} \in \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$ . Now, if  $\beta \leq \alpha \leq 0$ , it can be shown with a similar argument that  $\hat{x} \in \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$ . Hence  $\mathcal{Q}^+ \subseteq \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$ , and it satisfies Definition 1, i.e., it is a DCC for  $\mathcal{Q}^+$  and the disjunctive set  $\mathcal{A} \cup \mathcal{B}$ .

**DCC when  $\mathcal{Q}$  is a hyperboloid of two sheets, or  $\mathcal{Q}$  is a cone and the vector zero is not an element of  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ :** In this case we have from Lemma 10 that  $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \in \mathcal{Q}^+(\bar{\tau}_1)$  or  $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \in \mathcal{Q}^-(\bar{\tau}_1)$ . We may assume w.l.o.g. that  $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}^+(\bar{\tau}_1)$ . Since  $\mathcal{Q}^+(\bar{\tau}_1)$  is a convex set we have that  $\text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B})) \subseteq \mathcal{Q}^+(\bar{\tau}_1)$ .

To complete the proof we need to show that  $\mathcal{Q}^+ \cap \mathcal{Q}^+(\bar{\tau}) \subseteq \text{conv}((\mathcal{A} \cup \mathcal{B}) \cap \mathcal{Q}^+)$ . For this purpose, we prove first that  $\mathcal{Q}^+ \cap \mathcal{A}^\ominus = \mathcal{Q}^+(\bar{\tau}_1) \cap \mathcal{A}^\ominus$  and  $\mathcal{Q}^+ \cap \mathcal{B}^\ominus = \mathcal{Q}^+(\bar{\tau}_1) \cap \mathcal{B}^\ominus$ . Observe that  $\mathcal{Q}^+ \cap \mathcal{A}^\ominus \subseteq \mathcal{Q}^+(\bar{\tau}_1)$ , then  $\mathcal{Q}^+ \cap \mathcal{A}^\ominus \subseteq \mathcal{Q}^+(\bar{\tau}_1) \cap \mathcal{A}^\ominus$ . Thus, it is enough to show that  $\mathcal{Q}^+(\bar{\tau}_1) \cap \mathcal{A}^\ominus \subseteq \mathcal{Q}^+ \cap \mathcal{A}^\ominus$ . Let  $u \in \mathcal{Q}^+ \cap \mathcal{A}^\ominus$ . Recall that  $\mathcal{Q} \cap (\mathcal{A}^\ominus \cup \mathcal{B}^\ominus) = \mathcal{Q}(\bar{\tau}_1) \cap (\mathcal{A}^\ominus \cup \mathcal{B}^\ominus)$ , hence if  $\mathcal{Q}^+(\bar{\tau}_1) \cap \mathcal{A}^\ominus \not\subseteq \mathcal{Q}^+ \cap \mathcal{A}^\ominus$ , then there exists a vector  $v \in \mathcal{Q}^+(\bar{\tau}_1) \cap \mathcal{A}^\ominus \setminus \mathcal{Q}^+ \cap \mathcal{A}^\ominus$ . We know that  $\mathcal{Q}^+ \cap \mathcal{Q}^- = \emptyset$  if  $\mathcal{Q}$  is a cone, and  $\mathcal{Q}^+ \cap \mathcal{Q}^- = \emptyset$  if  $\mathcal{Q}$  is a hyperboloid of two sheets. Even more, in this case if  $\mathcal{Q}$  is a cone, we know that  $0 \notin \mathcal{Q} \cap \mathcal{A}^\ominus$ . Hence, using the separation theorem we have that there exists a hyperplane  $\mathcal{H} = \{x \in \mathbb{R}^\ell \mid h^\top x = \eta\}$  separating  $\mathcal{Q}^+$  and  $\mathcal{Q}^-$ , such that  $0 \in \mathcal{H}$ . Then, there exists a  $0 \leq \lambda \leq 1$  such that  $\lambda u + (1 - \lambda)v \in \mathcal{Q}^+(\bar{\tau}_1) \cap \mathcal{A}^\ominus$  and  $h^\top(\lambda u + (1 - \lambda)v) = \eta$ , i.e.,  $(\lambda u + (1 - \lambda)v) \notin \mathcal{Q}$ . This contradicts to  $\mathcal{Q} \cap (\mathcal{A}^\ominus \cup \mathcal{B}^\ominus) = \mathcal{Q}(\bar{\tau}_1) \cap (\mathcal{A}^\ominus \cup \mathcal{B}^\ominus)$ . Hence,  $\mathcal{Q}^+(\bar{\tau}_1) \cap \mathcal{A}^\ominus \subseteq \mathcal{Q}^+ \cap \mathcal{A}^\ominus$ . Similarly, we can show that  $\mathcal{Q}^+ \cap \mathcal{B}^\ominus = \mathcal{Q}^-(\bar{\tau}_1) \cap \mathcal{B}^\ominus$ .

Now, for any  $x \in \mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B})$ , we have that  $x \in \mathcal{Q}^+ \cap \mathcal{Q}^+(\bar{\tau})$  and  $x \in \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$ . Next, we need to consider a vector  $\tilde{x} \in \mathbb{R}^n$  such that  $\tilde{x} \in \mathcal{Q}^+(\bar{\tau}) \cap \mathcal{Q}^+ \cap \bar{\mathcal{A}} \cap \bar{\mathcal{B}}$ , where  $\bar{\mathcal{A}}$  and  $\bar{\mathcal{B}}$  are the complements of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. From Lemma 7 we have that  $x(\bar{\tau}_1) \in \mathcal{A}$  or  $x(\bar{\tau}_1) \in \mathcal{B}$ . We may assume w.l.o.g. that  $x(\bar{\tau}_1) \in \mathcal{B}$ . Since  $\mathcal{Q}^+(\bar{\tau})$  is a translated cone, then  $\{x \in \mathbb{R}^n \mid x = x(\bar{\tau}_1) + \theta(\tilde{x} - x(\bar{\tau}_1)), \theta \geq 0\} \subseteq \mathcal{Q}^+(\bar{\tau})$ . Thus, there exists a scalar  $0 < \theta_1 < 1$  such that  $a^\top(x(\bar{\tau}_1) + \theta_1(x - x(\bar{\tau}_1))) = \beta$  and a scalar  $1 < \theta_2$  such that  $a^\top(x(\bar{\tau}_1) + \theta_2(x - x(\bar{\tau}_1))) = \alpha$ . Let  $\lambda = (1 - \theta_1)/(\theta_2 - \theta_1)$ , then  $\tilde{x} = (1 - \lambda)(x(\bar{\tau}_1) + \theta_1(x - x(\bar{\tau}_1))) + \lambda(x(\bar{\tau}_1) + \theta_2(x - x(\bar{\tau}_1)))$ . Therefore,  $\tilde{x} \in \text{conv}((\mathcal{A} \cup \mathcal{B}) \cap \mathcal{Q}^+)$ . The same conclusion is found if we assume that  $x(\bar{\tau}_1) \in \mathcal{A}$ . This proves that  $\mathcal{Q}^+ \cap \mathcal{Q}^+(\bar{\tau}_1) \subseteq \text{conv}((\mathcal{A} \cup \mathcal{B}) \cap \mathcal{Q}^+)$ . Thus, the cone  $\mathcal{Q}^+(\bar{\tau}_1)$  is a DCC for  $\mathcal{Q}^+$  and the disjunctive set  $\mathcal{A} \cup \mathcal{B}$ . Finally, if  $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}^-(\bar{\tau}_1)$ , then we can use a similar argument to prove that  $\mathcal{Q}^-(\bar{\tau}_1)$  is a DCC for  $\mathcal{Q}^+$  and the disjunctive set  $\mathcal{A} \cup \mathcal{B}$ .  $\square$

Now we define a criteria to identify which branch of  $\mathcal{Q}(\bar{\tau}_1)$  in Theorem 5 defines a DCC. First we consider the case when the feasible set  $\mathcal{F}$  is contained in  $\mathcal{Q}^+$ . Then, if  $W(\hat{\tau})_1^\top e_1 \geq 0$ , then the conic cut is given by  $\mathcal{Q}^+(\hat{\tau}_1)$ . On the other hand, if  $W(\hat{\tau})_1^\top e_1 \leq 0$ , then the conic cut is given by  $\mathcal{Q}^-(\hat{\tau})$ . Second, consider the case when the feasible set  $\mathcal{F}$  is contained in  $\mathcal{Q}^-$ . Then, if  $-W(\hat{\tau})_1^\top e_1 \geq 0$ , then the conic cut is given by  $\mathcal{Q}^+(\hat{\tau})$ . On the other hand, if  $-W(\hat{\tau})_1^\top e_1 \leq 0$ , then the conic cut is given by  $\mathcal{Q}^-(\hat{\tau})$ .

## 4 Conclusions

In this paper, we investigated the derivation of disjunctive conic cuts (DCCs) and Disjunctive Cylindrical Cuts (DCyC) for MISOCP problems. This was achieved by extending the ideas of disjunctive programming that have been applied successfully for obtaining linear cuts for MIP problems. We introduced first the concept of DCCs and DCyCs, which are an extension of the disjunctive cuts that have been well studied for MIP problems. In this analysis we considered disjunctions that are defined by parallel hyperplanes. Under some mild assumptions we were able to show that the intersection of these cuts with a closed convex set, given as the intersection of a SOC and an affine set, is the convex hull of the intersection of the same set with a linear disjunction. Additionally, we provided a full characterization of DCCs and DCyCs for MISOCP problems when the disjunctions are defined by parallel hyperplanes. This analysis provides a procedure for the derivation of DCCs and DCyCs separating a given point from the feasible set of a MISOCP problem.

## A Shapes of quadrics

Here we show that the shapes of the quadric  $\mathcal{Q}$  are limited to those described in Section 2. We may assume that  $\mathcal{Q}$  is not an empty set, otherwise there is no need for classification. Now, for the analysis of the shapes of  $\mathcal{Q}$  we need the following. First, recall that  $Ax^0 = b$ , then the system  $Hw = -x^0$  will have a solution if and only if  $b = 0$ . Second, recall that  $P = H^\top JH$ , and let  $H_{1\cdot}$  be the first row of  $H$ . Then, we have that

$$PH_{1\cdot} = (H^\top JH)H_{1\cdot} = (I - 2H_{1\cdot}H_{1\cdot}^\top)H_{1\cdot} = (1 - 2H_{1\cdot}^\top H_{1\cdot})H_{1\cdot}.$$

As a result,  $H_{1\cdot}$  is an eigenvector of  $P$  associated with the eigenvalue  $(1 - 2H_{1\cdot}^\top H_{1\cdot})$ . Third, let us define the set

$$\mathcal{F}^r = \{x \in \mathbb{R}^n \mid Ax = b, x^\top Jx \leq 0\} = \{x \in \mathbb{R}^n \mid x = x^0 + Hw, \text{ with } w \in \mathcal{Q}\}, \quad (23)$$

which is a relaxation of  $\mathcal{F}$ . Note that due to the constraint  $x^\top Jx \leq 0$ , if the set  $\mathcal{F}^r$  contains a wholeline, then the zero vector is an element of  $\mathcal{F}^r$ , i.e.  $b = 0$ . Finally, for the sake of clarity, we present the definition of a cylinder that is used here.

**Definition 3 (Convex Cylinder [9])** *Let  $\mathcal{D} \subseteq \mathbb{R}^n$  be a convex set and  $d_0 \in \mathbb{R}^n$  a vector. Then, the set  $\mathcal{C} = \{x \in \mathbb{R}^n \mid x = d + \sigma d_0, d \in \mathcal{D}, \sigma \in \mathbb{R}\}$  is a convex cylinder in  $\mathbb{R}^n$ .*

We divide the classification of the shapes of  $\mathcal{Q}$  in two cases:  $P$  is singular, and  $P$  is non-singular.

Let us begin classifying the shapes of  $\mathcal{Q}$  when  $P$  is singular. First of all, from Lemma 2 we know that if  $P$  is singular, then  $P \succeq 0$  and  $(1 - 2H_{1\cdot}^\top H_{1\cdot}) = 0$ . Consequently,  $H_{1\cdot}$  is an eigenvector of  $P$  associated with



its zero eigenvalue. Now, from Section 2 we know that  $\mathcal{Q}$  may be a paraboloid or a cylinder. To decide which is the case, one has to verify if the system  $Pw = -p$  is solvable. On one hand, if  $Pw = -p$  has no solution, then we obtain that  $\mathcal{Q}$  is a paraboloid. On the other hand, if the system  $Pw = -p$  is solvable, then  $\mathcal{Q}$  is a cylinder. We show now that if  $Pw = -p$ , then given the setup of Section 2,  $\mathcal{Q}$  is always a line, i.e., a cylinder whose base is a point.

Let  $w^c \in \mathbb{R}^\ell$  be such that  $Pw^c = -p$ . Hence,  $\mathcal{Q}$  is a cylinder, and contains a whole line. Now, consider the set  $\mathcal{L} = \{w \in \mathbb{R}^\ell \mid w = w^c + \sigma H_{1\cdot}, \sigma \in \mathbb{R}\}$ . Note that  $\mathcal{L} \subseteq \mathcal{Q}$ , which follows from the following inequality

$$(w^c + \sigma H_{1\cdot})^\top P(w^c + \sigma H_{1\cdot}) + 2p^\top (w^c + \sigma H_{1\cdot}) + \rho = (w^c)^\top Pw^c + 2p^\top w^c + \rho \leq 0.$$

The first equality is true because  $p^\top H_{1\cdot} = 0$ . The last inequality follows from the assumption that  $\mathcal{Q}$  is not an empty set. Thus,  $H_{1\cdot}$  is the vector defining the direction of the cylinder  $\mathcal{Q}$ . Let us define the set

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid x = x^0 + H(w^c + \sigma H_{1\cdot}), \sigma \in \mathbb{R}\},$$

which is a line in  $\mathbb{R}^n$ . Hence, since  $\mathcal{L} \subseteq \mathcal{Q}$ , we obtain from (23) that  $\mathcal{S} \subseteq \mathcal{F}^r$ . Additionally, recall that  $\mathcal{F}^r$  may contain a whole line if and only if  $b = 0$ . Hence, it follows from  $\mathcal{S} \subseteq \mathcal{F}^r$  that  $b = 0$ , and the system  $Hw = -x^0$  is solvable because  $-x^0$  is in the null space of  $A$ . Let  $w^c = (\hat{w} + \sigma H_{1\cdot})$ , where  $\sigma \in \mathbb{R}$  and  $H\hat{w} = -x^0$ . Note that  $\hat{w}$  is unique since the columns of  $H$  are linearly independent. Then, for  $\sigma \in \mathbb{R}$  we have that  $Pw^c = H^\top JH\hat{w} = -H^\top Jx^0 = -p$ , and we obtain that

$$(w^c)^\top Pw^c + 2p^\top w^c + \rho = -(\hat{w})^\top P\hat{w} + \rho = -(\hat{w})^\top H^\top JH\hat{w} + (x^0)^\top Jx^0 = 0.$$

This shows that if the system  $Pw = -p$  is solvable, then we have that  $\mathcal{Q}$  is a line.

We now classify the shapes of  $\mathcal{Q}$  when  $P$  is non-singular. In this case we have that (5) is equivalent to

$$\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid (w + P^{-1}p)^\top P(w + P^{-1}p) \leq p^\top P^{-1}p - \rho\}. \quad (24)$$

The shape of  $\mathcal{Q}$  in this case is determined by the inertia of  $P$  and the value of the right hand side of (24) [8]. The first case to consider is when  $P \succ 0$ , in which case we have that  $\mathcal{Q}$  is an ellipsoid. Now, to complete the classification of  $\mathcal{Q}$  we need to consider the case when  $P$  is an ID1 matrix. We have the following possibilities [8]:

- if  $p^\top P^{-1}p - \rho \leq 0$ , then  $\mathcal{Q}$  is a hyperboloid of two sheets;
- if  $p^\top P^{-1}p - \rho = 0$ , then  $\mathcal{Q}$  is a scaled and translated second order cone;
- if  $p^\top P^{-1}p - \rho \geq 0$ , then  $\mathcal{Q}$  is a hyperboloid of one sheet.

We show here that the setup of Section 2 only allows  $p^\top P^{-1}p - \rho \leq 0$ . In other words, we need to show that  $\mathcal{Q}$  is never a hyperboloid of one sheet.

We know that the vector  $-P^{-1}p$  is either the vertex of a scaled second order cone or the intersection point of the asymptotes of a hyperboloid [8]. Now, note that if  $\mathcal{Q}$  is a cone or a hyperboloid of one sheet, then  $-P^{-1}p \in \mathcal{Q}$ . In this case, we need to show that  $p^\top P^{-1}p - \rho = 0$  is always true to exclude the possibility of hyperboloid of one sheet. From Lemma 2 we know that if  $P$  is ID1, then  $(1 - 2H_{1\cdot}^\top H_{1\cdot}) < 0$ , and  $H_{1\cdot}$  is an eigenvector of  $P$  associated with its negative eigenvalue. Recall the set  $\mathcal{L}$ , then we have the following inequality

$$(-P^{-1}p + \sigma H_{1\cdot} + P^{-1}p)^\top P(-P^{-1}p + \sigma H_{1\cdot} + P^{-1}p) = \sigma^2 H_{1\cdot}^\top P H_{1\cdot} \leq 0,$$

which shows that  $\mathcal{L} \subseteq \mathcal{Q}$  when  $\mathcal{Q}$  is either a cone or a hyperboloid of one sheet. Define the set

$$\mathcal{T} = \{x \in \mathbb{R}^n \mid x = x^0 + H(-P^{-1}p + \sigma H_{1\cdot}), \sigma \in \mathbb{R}\}.$$

Then,  $\mathcal{T} \subseteq \mathcal{F}^r$  when  $\mathcal{Q}$  is either a cone or a hyperboloid of one sheet. Now, since  $\mathcal{T} \subset \mathbb{R}^n$  is a line, from  $\mathcal{T} \subseteq \mathcal{F}^r$  follows that  $b = 0$ , which implies the existence of a unique vector  $w^c \in \mathbb{R}^\ell$  such that  $Hw^c = -x^0$ . Further, we have that

$$\begin{aligned}
(w^c + P^{-1}p)^\top P(w^c + P^{-1}p) &= (w^c)^\top Pw^c + 2p^\top w^c + p^\top P^{-1}p \\
&= (w^c)^\top H^\top JHw^c + 2(x^0)^\top JHw^c + p^\top P^{-1}p \\
&= p^\top P^{-1}p + (x^0)^\top Jx^0 - 2(x^0)^\top Jx^0 \\
&= p^\top P^{-1}p - \rho.
\end{aligned}$$

On the other hand, we have that

$$P(w^c + P^{-1}p) = Pw^c + p = H^\top JHw^c + H^\top Jx^0 = -H^\top Jx^0 + H^\top Jx^0 = 0.$$

Henceforth, we have that  $p^\top P^{-1}p - \rho = 0$ , and the quadric  $\mathcal{Q}$  cannot be a hyperboloid of one sheet.

## B Normalized quadrics

To facilitate the algebra in Sections 3.3 and 3.4 we discuss here the normalization of the quadrics. It is always possible to find a linear transformation  $\mathbf{L}$  in order to normalize the quadric  $\mathcal{Q}$ . Here, normalizing implies three things. First that the matrix  $Q$  is a diagonal matrix with all its diagonal entries taking values in  $\{-1, 0, 1\}$ . Second, the scalar  $\rho$  takes a value in  $\{-1, 0, 1\}$ . Third, for the case of ellipsoids, hyperboloids, or cones, we have that  $p = 0$ .

Before discussing the normalization we need the following elements. First, let us define

$$\tilde{J} = \begin{bmatrix} \tilde{J}_{1,1} & 0 \\ 0 & I \end{bmatrix}.$$

Also, since  $P$  is a real symmetric matrix, recall that  $P$  can be factorized as  $P = VDV^\top$ , where  $V \in \mathbb{R}^{\ell \times \ell}$  is an orthonormal matrix and  $D \in \mathbb{R}^{\ell \times \ell}$  is a diagonal matrix. Finally, we may assume w.l.o.g. that the diagonal elements of  $D$  are arranged from smaller to bigger, where  $D_{1,1}$  is the smallest value [15].

Using this framework, we consider two possible normalized descriptions for the quadric  $\mathcal{Q}$ .

### B.1 $P$ is non-singular

When  $\mathcal{Q}$  is a cone, a hyperboloid of two sheets, or an ellipsoid, then  $P$  is non-singular and the quadric  $\mathcal{Q}$  can be written as

$$\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid (w + P^{-1}q)^\top P(w + P^{-1}p) \leq p^\top P^{-1}p - \rho\}. \quad (25)$$

Equation (25) can be expressed in terms of  $V$  and  $D$  using the eigenvalue and eigenvector decomposition  $P = VDV^\top$ . First, for  $\tilde{J}$  let

$$\tilde{J}_{i,i} = \frac{D_{i,i}}{|D_{i,i}|}, i = 1, \dots, \ell. \quad (26)$$

Hence,  $\tilde{J}$  is the identity matrix if  $D_{1,1} > 0$ , and if  $D_{1,1} < 0$ , then we have that  $\tilde{J}_{1,1} = -1$ . Now, let  $\tilde{D} \in \mathbb{R}^{\ell \times \ell}$  be a diagonal matrix defined as  $\tilde{D}_{i,i} = |D_{i,i}|$ ,  $i = 1, \dots, \ell$ . We obtain that

$$\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid (w + P^{-1}p)^\top (V\tilde{D}^{\frac{1}{2}})\tilde{J}(\tilde{D}^{\frac{1}{2}}V^\top)(w + P^{-1}p) \leq p^\top P^{-1}p - \rho\}. \quad (27)$$

For this first normalized description we define an affine transformation  $\mathbf{L} : \mathbb{R}^\ell \mapsto \mathbb{R}^\ell$  as follows

$$\mathbf{L}(w) = \tilde{D}^{\frac{1}{2}}V^\top(w + P^{-1}p). \quad (28)$$

Recall that  $V$  is an orthonormal matrix, and that  $\tilde{D}$  is non-singular by definition. Hence, the matrix  $\tilde{D}^{\frac{1}{2}}V^\top$  is non-singular.

To complete the description of the first normalization we need now to examine the term  $p^\top P^{-1}p - \rho$ . Consider the case  $p^\top P^{-1}p - \rho \neq 0$ , and define

$$u = \frac{1}{\sqrt{|p^\top P^{-1}p - \rho|}}\mathbf{L}(w) \quad \text{and} \quad \delta = -\frac{p^\top P^{-1}p - \rho}{|p^\top P^{-1}p - \rho|}. \quad (29)$$

Then, since  $\tilde{D}^{\frac{1}{2}}V^{\top}$  is non-singular, using (29) we obtain a one-to-one mapping between every element of  $\mathcal{Q}$  and the set

$$\tilde{\mathcal{Q}} = \left\{ u \in \mathbb{R}^n \mid u^{\top} \tilde{J}u + \delta \leq 0 \right\}. \quad (30)$$

Now, for the case  $p^{\top}P^{-1}p - \rho = 0$  let

$$u = L(w) \quad \text{and} \quad \delta = 0. \quad (31)$$

In this case, using (31) we obtain a one-to-one mapping between  $\mathcal{Q}$  and  $\tilde{\mathcal{Q}}$ . Hence, the set  $\tilde{\mathcal{Q}}$  in (30) is a normalization of the quadric when  $\mathcal{Q}$  is a cone, a hyperboloid of two sheets or an ellipsoid.

## B.2 $P$ is singular

When  $\mathcal{Q}$  is a paraboloid  $P$  has at most one non-positive eigenvalue, thus its non-positive eigenvalue in this case is 0. Hence, we have that  $D_{1,1} = 0$  for the matrix  $D$  of the diagonalization of  $P$ . Define a diagonal matrix  $\tilde{D} \in \mathbb{R}^{\ell \times \ell}$  as  $\tilde{D}_{i,i} = D_{i,i}$  for  $i \in \{2, \dots, \ell\}$  and  $\tilde{D}_{1,1} = 1$ . Additionally, let

$$\tilde{J}_{i,i} = 1, i \in \{2, \dots, \ell\}, \quad \text{and} \quad \tilde{J}_{1,1} = 0. \quad (32)$$

Thus, we have the following equivalent description

$$\mathcal{Q} = \left\{ w \in \mathbb{R}^{\ell} \mid w^{\top} V \tilde{D}^{\frac{1}{2}} \tilde{J} \tilde{D}^{\frac{1}{2}} V^{\top} w + 2(p^{\top} V \tilde{D}^{-\frac{1}{2}})(\tilde{D}^{\frac{1}{2}} V^{\top} w) + \rho \leq 0 \right\}. \quad (33)$$

For this normalization we define the affine transformation  $L : \mathbb{R}^{\ell} \mapsto \mathbb{R}^{\ell}$  as follows

$$L(w) = \tilde{D}^{\frac{1}{2}} V^{\top} w. \quad (34)$$

Recall that  $V$  is an orthonormal matrix, and that by construction  $\tilde{D}$  is non-singular. Hence, the matrix  $\tilde{D}^{\frac{1}{2}} V^{\top}$  is non-singular.

To complete the description of the second normalization we need to examine the value of  $\rho$ . Consider the case  $\rho \neq 0$ , and define

$$u = \frac{1}{\sqrt{|\rho|}} L(w), \quad \tilde{p} = \frac{1}{\sqrt{|\rho|}} \tilde{D}^{-\frac{1}{2}} V^{\top} p, \quad \omega = \frac{\rho}{|\rho|}. \quad (35)$$

Then, since  $\tilde{D}^{\frac{1}{2}} V^{\top}$  is non-singular, using (35) we obtain a one-to-one mapping between  $\mathcal{Q}$  and the set

$$\tilde{\mathcal{Q}} = \left\{ u \in \mathbb{R}^{\ell} \mid u^{\top} \tilde{J}u + 2\tilde{p}^{\top}u + \omega \leq 0 \right\}. \quad (36)$$

Now, for the case  $\rho = 0$  define

$$u = L(w), \quad \tilde{p} = \tilde{D}^{-\frac{1}{2}} V^{\top} p, \quad \omega = 0. \quad (37)$$

In this case, using (37) we obtain a one-to-one mapping between  $\mathcal{Q}$  and  $\tilde{\mathcal{Q}}$ . Hence, the set  $\tilde{\mathcal{Q}}$  in (36) is a normalization of the quadric when  $\mathcal{Q}$  is a paraboloid.

The sets (30) and (36) define our two possible normalizations for  $\mathcal{Q}$ . Note that  $P$  and  $\tilde{J}$  have always the same inertia. Hence, the classification of the quadrics  $\mathcal{Q}$  and  $\tilde{\mathcal{Q}}$  is the same. Additionally, if we apply the affine transformations  $L$  as given in (35) and (37) to two parallel hyperplanes, then the resulting hyperplanes are still parallel. Finally, by construction the transformation  $L$  has an inverse in both normalizations.

## C Definitions and known results

**Definition 4 (Base of a Convex Cylinder)** Let  $\mathcal{C} \subset \mathbb{R}^n$  be a convex cylinder with the direction  $d^0 \in \mathbb{R}^n$ . A set  $\mathcal{D} \subset \mathcal{C}$  is called a base of  $\mathcal{C}$  if for every vector  $x \in \mathcal{C}$ , there is a unique  $d \in \mathcal{D}$  and  $\sigma \in \mathbb{R}$  such that  $x = d + \sigma d_0$ .

## C.1 Eigenvalues of a rank one update

Recall that the eigenvalues of  $P + \tau aa^\top$  can be computed by finding the roots of the equation

$$\det(P + \tau aa^\top - \lambda I) = 0.$$

This equation is shown [16] to be equivalent to the characteristic equation

$$\prod_{i=1}^n (P_{i,i} - \lambda) + \tau \sum_{i=1}^n a_i^2 \prod_{\substack{j=1 \\ j \neq i}}^n (P_{i,i} - \lambda) = 0. \quad (38)$$

We use this result in the proofs of this paper.

## D Results for proofs with the unbounded intersections in Section 3.4

### D.1 Proof of Theorem 3

We first show that

$$\bar{\tau} \geq -\frac{1}{(1 - 2a_1^2)}.$$

From (20) we have that the most negative value  $\bar{\tau}$  can take is achieved when  $\alpha\beta < 0$ . We have

$$\frac{-4|\alpha\beta|}{(1 - 2a_1^2)(\alpha - \beta)^2} = \left( \frac{-1}{(1 - 2a_1^2)} \right) \left( \frac{4|\alpha\beta|}{(\alpha - \beta)^2} \right) \geq \frac{-1}{(1 - 2a_1^2)}. \quad (39)$$

The last inequality follows because if  $\alpha\beta < 0$ , then  $\alpha^2 - 2\alpha\beta + \beta^2 \geq 4|\alpha\beta|$ , since  $\alpha^2 + \beta^2 \geq 2|\alpha\beta|$ . From (17) we know that  $P(\bar{\tau})$  has one negative eigenvalue and  $n - 1$  positive eigenvalues if the inequality (39) is strict. If (39) is satisfied with equality, then  $P(\bar{\tau})$  has one negative eigenvalue, one zero eigenvalue, and  $n - 2$  positive eigenvalues.

If  $\bar{\tau} > -\frac{1}{(1 - 2a_1^2)}$ , then we obtain that  $\mathcal{Q}(\bar{\tau})$  is a cone. Now, we analyze the case when  $\bar{\tau} = -\frac{1}{(1 - 2a_1^2)}$ , which by (39) can happen only when  $\beta = -\alpha$ . In this case  $P(\bar{\tau})$  is singular,  $p(\bar{\tau}) = 0$ , and  $\rho(\bar{\tau}) > 0$ . Recall that since  $P(\bar{\tau})$  is symmetric, then there exist  $D(\bar{\tau}) \in \mathbb{R}^{\ell \times \ell}$  and  $V(\bar{\tau}) \in \mathbb{R}^{\ell \times \ell}$  such that  $P(\bar{\tau}) = V(\bar{\tau})^\top D(\bar{\tau}) V(\bar{\tau})$ .

Let us now characterize the shape of the quadric  $\mathcal{Q}(\bar{\tau})$ . First, recall that when  $\bar{\tau} = -\frac{1}{(1 - 2a_1^2)}$  then  $P(\bar{\tau})$  has one negative eigenvalue, one zero eigenvalue, and  $\ell - 2$  positive eigenvalues. We may assume w.l.o.g. that  $D_{1,1}(\bar{\tau}) < 0$ ,  $D_{2,2}(\bar{\tau}) = 0$ , and  $D_{i,i}(\bar{\tau}) > 0$ ,  $i \in \{3, \dots, n\}$ . Then

$$P(\bar{\tau}) = V(\bar{\tau}) \hat{D}(\bar{\tau})^{\frac{1}{2}} \hat{J} \hat{D}(\bar{\tau})^{\frac{1}{2}} V(\bar{\tau})^\top,$$

where  $\hat{D}(\bar{\tau})$  is a diagonal matrix with  $\hat{D}_{i,i}(\bar{\tau}) = |D_{i,i}(\bar{\tau})|$ ,  $i \in \{1, \dots, n\} \setminus \{2\}$ , and  $\hat{D}_{2,2}(\bar{\tau}) = 1$ . Additionally,  $\hat{J}$  is a diagonal matrix defined as  $\hat{J}_{1,1} = -1$ ,  $\hat{J}_{2,2} = 0$ , and  $\hat{J}_{i,i} = 1$ ,  $i \in \{3, \dots, n\}$ . Thus, using the transformation

$$u = \frac{\hat{D}(\bar{\tau})^{\frac{1}{2}} V(\bar{\tau})^\top w}{\sqrt{\rho(\bar{\tau})}}, \quad \forall w \in \mathcal{Q}(\bar{\tau}),$$

we obtain that  $\mathcal{Q}(\bar{\tau})$  is an affine transformation of the set

$$\{u \in \mathbb{R}^\ell \mid u^\top \hat{J} u \leq -1\}, \quad (40)$$

which is a hyperbolic cylinder of two sheets. The right hand side of the quadratic equation in (40) is  $-1$  because

$$\rho(\bar{\tau}) = -\frac{\beta\alpha}{(1 - 2a_1^2)} = \frac{\alpha^2}{(1 - 2a_1^2)} > 0.$$

Finally, given that  $\hat{J}$  and  $P(\bar{\tau})$  have the same inertia, we have shown that  $\mathcal{Q}(\bar{\tau})$  is a hyperbolic cylinder of two sheets. This proves the result.

## D.2 Proof of Theorem 4

First, to facilitate the discussion let  $f : \mathbb{R} \mapsto \mathbb{R}$  be such that  $f(\tau) = \tau^2(1 - 2a_1^2)\frac{(\alpha - \beta)^2}{4} - \tau((1 - 2a_1^2) + \alpha\beta) - 1$ , which is the numerator of (21). We need to compare the roots of  $f$  with the critical value  $\hat{\tau} = -\frac{1}{(1 - 2a_1^2)}$ . Based on this comparison, we then classify the shapes of the quadrics  $\mathcal{Q}(\tau)$  at these two roots.

Recall that  $\alpha \neq \beta$ , and then the roots  $\bar{\tau}_1$  and  $\bar{\tau}_2$  of  $f$  are

$$\frac{2(1 - 2a_1^2 + \alpha\beta \pm \sqrt{(1 - 2a_1^2 + \alpha\beta)^2 + (1 - 2a_1^2)(\alpha - \beta)^2})}{(1 - 2a_1^2)(\alpha - \beta)^2}. \quad (41)$$

Hence, since  $(1 - 2a_1^2)(\alpha - \beta)^2 > 0$ , we have that one root is positive and the other is negative. We may assume w.l.o.g. that  $\bar{\tau}_1 \leq \bar{\tau}_2$ . Also, observe that the roots are always different, since the discriminant of (41) is never zero for  $a_1^2 < 1/2$ .

Let us compare these two roots with the critical value  $\hat{\tau} = -\frac{1}{(1 - 2a_1^2)}$ . First of all, note that  $f(\hat{\tau}) > 0$ , and that the coefficient of  $\tau^2$  in  $f(\tau)$  is positive since  $a_1^2 < \frac{1}{2}$ . Hence,  $\hat{\tau} \in (\bar{\tau}_1, \bar{\tau}_2)$ . Additionally, if  $\alpha \neq -\beta$ , then  $f(\tau) > 0$ . To complete the comparison we need to check the value of the derivative  $f'(\hat{\tau})$  to verify in which branch of  $f$  the value  $\hat{\tau}$  lies. We have that

$$f'(\hat{\tau}) = -\frac{(\alpha - \beta)^2}{2} - (1 - 2a_1^2 + \alpha\beta) = -\frac{(\alpha^2 + \beta^2)}{2} - (1 - 2a_1^2) \leq 0.$$

Hence, the inequality  $\hat{\tau} \leq \bar{\tau}_1$  is always satisfied, and it is strict if  $\alpha \neq -\beta$ .

From (17), we know that if  $\hat{\tau} < \bar{\tau}_1$ , then  $P(\bar{\tau}_1)$  and  $P(\bar{\tau}_2)$  have  $\ell - 1$  positive eigenvalues and one negative eigenvalue. As a result,  $\mathcal{Q}(\bar{\tau}_1)$  and  $\mathcal{Q}(\bar{\tau}_2)$  are two different scaled second order cones. On the other hand, if  $\alpha = -\beta$ , then the roots of  $f$  are given by

$$\frac{1 - 2a_1^2 - \alpha^2 \pm \sqrt{(1 - 2a_1^2 + \alpha^2)^2}}{2(1 - 2a_1^2)\alpha^2}.$$

Thus,  $\hat{\tau} = \bar{\tau}_1$  when the hyperplanes are symmetric with respect to the origin. From (17) we know that  $P(\bar{\tau}_1)$  has one negative eigenvalue, one zero eigenvalue, and  $\ell - 2$  positive eigenvalues. Additionally, note that

$$\rho(\bar{\tau}_1) = 1 + \frac{\alpha^2}{(1 - 2a_1^2)} > 0.$$

Thus, similarly to the proof of Theorem 3, one can use the eigenvalue decomposition of  $P(\bar{\tau}_1)$  to show that  $\mathcal{Q}(\bar{\tau}_1)$  is an affine transformation of the set (40). Thus,  $\mathcal{Q}(\bar{\tau}_1)$  is a cylindrical hyperboloid of two sheets. Finally, since  $\bar{\tau}_1 < \bar{\tau}_2$ , we have that  $P(\bar{\tau}_2)$  has one negative eigenvalue and  $\ell - 1$  positive eigenvalues, and we obtain that  $\mathcal{Q}(\bar{\tau}_2)$  is a cone. This proves the result.

## D.3 Additional lemmas for Section 3.4.2

**Lemma 7** *Let  $\bar{\tau}$  be the smaller root of the numerator of (18). In the first cases of Theorems 3 and 4, one has that in Lemma 6 the vertex  $x(\bar{\tau})$  of the quadric  $\mathcal{Q}(\bar{\tau})$  is either in  $\mathcal{A}$  or  $\mathcal{B}$ .*

**Proof.** From Theorem 1 we have that

$$a^\top x(\bar{\tau}) = -a^\top P(\bar{\tau})^{-1} p(\bar{\tau}) = \bar{\tau} \frac{(\alpha + \beta)(1 - 2a_1^2)}{2(1 + \bar{\tau}(1 - 2a_1^2))}.$$

First of all, for Lemma 6 we obtain that  $a^\top x(\bar{\tau}) = 0$ . Note that if  $\alpha$  and  $\beta$  have opposite signs, then each hyperplane is intersecting a different branch of  $\mathcal{Q}$ . This is not possible for MISOCO problems, because the feasible set of its SOCO relaxation would be non-convex. Now note that  $\alpha \neq 0$  and  $\beta \neq 0$ , since otherwise one of the intersections  $\mathcal{Q} \cap \mathcal{A} = \emptyset$  or  $\mathcal{Q} \cap \mathcal{B} = \emptyset$ . Hence, we have that  $x(\bar{\tau}) \in \mathcal{A}$  or  $x(\bar{\tau}) \in \mathcal{B}$ .

Now, recall also from Sections D.1 and D.2 that  $-\frac{1}{(1-2a_1^2)} \leq \bar{\tau}$ . Hence,

$$\lim_{\bar{\tau} \rightarrow \infty} a^\top x(\bar{\tau}) = \frac{(\alpha + \beta)}{2}.$$

On the other hand, we have

$$\lim_{\bar{\tau} \searrow -\frac{1}{(1-2a_1^2)}} a^\top x(\bar{\tau}) = \begin{cases} -\infty & \text{if } \alpha + \beta > 0, \\ +\infty & \text{if } \alpha + \beta < 0. \end{cases}$$

Thus, if  $\alpha + \beta > 0$  then  $a^\top x(\bar{\tau}) < \alpha$ . Now, if  $a^\top x(\bar{\tau}) \leq \beta$  is true, then we obtain that

$$\bar{\tau} \frac{(\alpha + \beta)(1 - 2a_1^2)}{2(1 + \bar{\tau}(1 - 2a_1^2))} \leq \beta \quad \text{which implies} \quad \bar{\tau} \leq \frac{2\beta}{(\alpha - \beta)(1 - 2a_1^2)}.$$

On the other hand, if  $\alpha + \beta < 0$ , then  $a^\top x(\bar{\tau}) > \beta$ . Now, if  $a^\top x(\bar{\tau}) \geq \alpha$  is true, then we obtain that

$$\bar{\tau} \frac{(\alpha + \beta)(1 - 2a_1^2)}{2(1 + \bar{\tau}(1 - 2a_1^2))} \geq \alpha \quad \text{that implies} \quad \bar{\tau} \leq \frac{-2\alpha}{(\alpha - \beta)(1 - 2a_1^2)}.$$

Recall that  $\beta < \alpha$ . Then,  $\alpha + \beta > 0$  implies that  $\alpha > 0$  and  $\alpha > |\beta|$ . Additionally,  $\alpha + \beta < 0$  implies that  $\beta < 0$  and  $\beta < -|\alpha|$ .

For the first case of Theorem 3 we need to consider two cases. On one hand if  $\alpha\beta \geq 0$ , then  $\bar{\tau} = 0$ . In this case, if  $\alpha + \beta > 0$  then  $\frac{2\beta}{(\alpha - \beta)(1 - 2a_1^2)} \geq 0$ , and  $x(\bar{\tau}) \in \mathcal{B}$ . Additionally, if  $\alpha + \beta < 0$  then  $\frac{-2\alpha}{(\alpha - \beta)(1 - 2a_1^2)} \geq 0$  and  $x(\bar{\tau}) \in \mathcal{A}$ . On the other hand, if  $\alpha\beta \leq 0$  then  $\bar{\tau} = \frac{4\alpha\beta}{(1 - 2a_1^2)(\alpha - \beta)^2} \leq 0$ . Hence, if  $\alpha + \beta > 0$  then

$$\frac{4\alpha\beta}{(1 - 2a_1^2)(\alpha - \beta)^2} = \left( \frac{2\beta}{(1 - 2a_1^2)(\alpha - \beta)} \right) \left( \frac{2\alpha}{(\alpha - \beta)} \right) \leq \frac{2\beta}{(\alpha - \beta)(1 - 2a_1^2)},$$

and the vertex  $x(\bar{\tau}) \in \mathcal{B}$ . Additionally, if  $\alpha + \beta < 0$  then

$$\frac{4\alpha\beta}{(1 - 2a_1^2)(\alpha - \beta)^2} = \left( \frac{2\alpha}{(1 - 2a_1^2)(\alpha - \beta)} \right) \left( \frac{2\beta}{(\alpha - \beta)} \right) \leq \frac{-2\alpha}{(\alpha - \beta)(1 - 2a_1^2)},$$

and the vertex  $x(\bar{\tau}) \in \mathcal{A}$ .

For the first case of Theorem 4 recall that

$$\begin{aligned} \bar{\tau} &= \frac{2 \left( 1 - 2a_1^2 + \alpha\beta - \sqrt{(1 - 2a_1^2 + \alpha\beta)^2 + (1 - 2a_1^2)(\alpha - \beta)^2} \right)}{(1 - 2a_1^2)(\alpha - \beta)^2} \\ &= \frac{2 \left( 1 - 2a_1^2 + \alpha\beta - \sqrt{(1 - 2a_1^2 + \alpha^2)(1 - 2a_1^2 + \beta^2)} \right)}{(1 - 2a_1^2)(\alpha - \beta)^2}. \end{aligned}$$

Hence, if  $\alpha + \beta > 0$ , then

$$\frac{2 \left( 1 - 2a_1^2 + \alpha\beta - \sqrt{(1 - 2a_1^2 + \alpha^2)(1 - 2a_1^2 + \beta^2)} \right)}{(1 - 2a_1^2)(\alpha - \beta)^2} \leq \frac{2\beta}{(\alpha - \beta)(1 - 2a_1^2)},$$

and the vertex  $x(\bar{\tau}) \in \mathcal{B}$ . Additionally, if  $\alpha + \beta < 0$  then

$$\frac{2 \left( 1 - 2a_1^2 + \alpha\beta - \sqrt{(1 - 2a_1^2 + \alpha^2)(1 - 2a_1^2 + \beta^2)} \right)}{(1 - 2a_1^2)(\alpha - \beta)^2} \leq \frac{-2\alpha}{(\alpha - \beta)(1 - 2a_1^2)},$$

and the vertex  $x(\bar{\tau}) \in \mathcal{A}$ . This shows that  $x(\bar{\tau})$  is contained in one of the sets  $\mathcal{A}$  or  $\mathcal{B}$ .  $\square$

**Lemma 8** *Let  $\bar{\tau}$  be the smaller root of the numerator of (18). In the first cases of Theorems 3 and 4, and in Lemma 6, we have that  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}(\bar{\tau}_1)$ .*

**Proof.** Recall that  $\mathcal{Q}(\bar{\tau}) = \{x \in \mathbb{R}^\ell \mid x^\top P(\bar{\tau})x + 2p(\bar{\tau})^\top x + \rho(\bar{\tau}) \leq 0\}$ , then from Theorem 1 we have for the first case of Theorem 3 that

$$x^\top P(\bar{\tau})x + 2p(\bar{\tau})^\top x + \rho(\bar{\tau}) = x^\top Jx + \bar{\tau}_1 ((a^\top x)^2 - \alpha a^\top x - \beta a^\top x + \alpha\beta),$$

and for the first case of Theorem 4 and Lemma 6 we have that

$$x^\top P(\bar{\tau})x + 2p(\bar{\tau})^\top x + \rho(\bar{\tau}) = x^\top Jx + 1 + \bar{\tau} ((a^\top x)^2 - \alpha a^\top x - \beta a^\top x + \alpha\beta).$$

Recall that in the case of Lemma 6, we have that  $\alpha \neq 0$  and  $\beta \neq 0$  have the same sign. From (20), (41), and (22) we know that  $\bar{\tau}_1 \leq 0$  and for  $\tilde{x} \in \mathcal{Q}$  we have either  $\tilde{x}^\top J\tilde{x} \leq 0$  or  $\tilde{x}^\top J\tilde{x} + 1 \leq 0$ . Now, observe that  $(a^\top \tilde{x})^2 - \alpha a^\top \tilde{x} - \beta a^\top \tilde{x} + \alpha\beta = (a^\top \tilde{x} - \alpha)(a^\top \tilde{x} - \beta)$ . On one hand, if  $\tilde{x} \in \mathcal{B} \cap \mathcal{Q}$ , then  $(a^\top \tilde{x} - \alpha) \leq 0$  and  $(a^\top \tilde{x} - \beta) \leq 0$ . On the other hand, if  $\tilde{x} \in \mathcal{A} \cap \mathcal{Q}$ , then  $(a^\top \tilde{x} - \alpha) \geq 0$  and  $(a^\top \tilde{x} - \beta) \geq 0$ . Thus, if  $\tilde{x} \in \mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ , we have that

$$(a^\top \tilde{x})^2 - \alpha(a^\top \tilde{x}) - \beta(a^\top \tilde{x}) + \alpha\beta \geq 0,$$

and we obtain that  $\tilde{x}^\top P(\bar{\tau})\tilde{x} + 2p(\bar{\tau})^\top \tilde{x} + \rho(\bar{\tau}) \leq 0$  for  $\tilde{x} \in \mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B})$ . Thus,  $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}(\bar{\tau}_1)$ .  $\square$

**Lemma 9** *Let  $\bar{\tau}$  be the smaller root of the numerator of (18). In the first case of Theorems 3 and 4, and Lemma 6, each of the subsets  $\mathcal{Q}^+ \cap \mathcal{A}$ ,  $\mathcal{Q}^+ \cap \mathcal{B}$ ,  $\mathcal{Q}^- \cap \mathcal{A}$ , and  $\mathcal{Q}^- \cap \mathcal{B}$ , is a subset of one of the branches  $\mathcal{Q}^+(\bar{\tau})$  or  $\mathcal{Q}^-(\bar{\tau})$ .*

**Proof.** First, we show that either  $\mathcal{Q}^+ \cap \mathcal{A} \subseteq \mathcal{Q}^+(\bar{\tau})$  or  $\mathcal{Q}^+ \cap \mathcal{A} \subseteq \mathcal{Q}^-(\bar{\tau})$ . We know from the definition of the sets in Section 3.4.2 that  $\mathcal{Q}^+ \cap \mathcal{A}$ ,  $\mathcal{Q}^+(\bar{\tau})$ ,  $\mathcal{Q}^-(\bar{\tau})$  are convex sets and from Lemma 8 we have that  $\mathcal{Q}^+ \cap \mathcal{A} \subseteq \mathcal{Q}(\bar{\tau}_1)$ . Recall also that  $\mathcal{Q}(\bar{\tau})$  is a cone, which vertex is denoted by  $x(\bar{\tau})$ , and that  $\mathcal{Q}^+(\bar{\tau}) \cap \mathcal{Q}^-(\bar{\tau}) = x(\bar{\tau})$ . Then, observe that if  $\mathcal{Q}^+ \cap \mathcal{A} \cap \mathcal{Q}^+(\bar{\tau}) \neq \emptyset$  and  $\mathcal{Q}^+ \cap \mathcal{A} \cap \mathcal{Q}^-(\bar{\tau}) \neq \emptyset$ , then  $x(\bar{\tau}) \in \mathcal{Q}^+ \cap \mathcal{A}$ , otherwise  $\mathcal{Q}^+ \cap \mathcal{A} \not\subseteq \mathcal{Q}(\bar{\tau})$ . We have

$$\begin{aligned} x(\bar{\tau}) &= -P(\bar{\tau})^{-1}p(\bar{\tau}) = -\left(J - \bar{\tau} \frac{Jaa^\top J}{1 + \bar{\tau}(1 - 2a_1^2)}\right) \left(-\bar{\tau}_1 \frac{\alpha + \beta}{2} a\right) \\ &= \bar{\tau} \frac{\alpha + \beta}{2} \left(1 - \bar{\tau} \frac{(1 - 2a_1^2)}{1 + \bar{\tau}(1 - 2a_1^2)}\right) Ja \\ &= \bar{\tau} \frac{\alpha + \beta}{2(1 + \bar{\tau}(1 - 2a_1^2))} Ja. \end{aligned}$$

Then, we obtain that

$$x(\bar{\tau})^\top Jx(\bar{\tau}) = \bar{\tau}^2 \frac{(\alpha + \beta)^2(1 - 2a_1^2)}{4(1 + \bar{\tau}(1 - 2a_1^2))^2} \geq 0.$$

Now, if  $\bar{\tau} = 0$ , then  $\mathcal{Q}(\bar{\tau}) = \mathcal{Q}$ , and it is clear that  $\mathcal{Q}^+$  is a subset of  $\mathcal{Q}^+(\bar{\tau})$ . On the other hand, if  $\bar{\tau} \neq 0$ , then  $x(\bar{\tau}) \notin \mathcal{Q}$ . For that reason  $x(\bar{\tau}) \notin \mathcal{Q}^+ \cap \mathcal{A}$ , and either  $\mathcal{Q}^+ \cap \mathcal{A} \cap \mathcal{Q}^+(\bar{\tau}) = \emptyset$  or  $\mathcal{Q}^+ \cap \mathcal{A} \cap \mathcal{Q}^-(\bar{\tau}) = \emptyset$ . Hence,  $\mathcal{Q}^+ \cap \mathcal{A}$  must be a subset of either  $\mathcal{Q}^+(\bar{\tau})$  or  $\mathcal{Q}^-(\bar{\tau})$ . A similar argument can be built to show that each subsets  $\mathcal{Q}^+ \cap \mathcal{B}$ ,  $\mathcal{Q}^- \cap \mathcal{A}$ , and  $\mathcal{Q}^- \cap \mathcal{B}$ , must be a subset of either  $\mathcal{Q}^+(\bar{\tau})$  or  $\mathcal{Q}^-(\bar{\tau})$ .  $\square$

**Lemma 10** *In the first case of Theorems 3 and 4 if  $\mathcal{Q}^+ \cap \mathcal{A} \neq \emptyset$  and  $\mathcal{Q}^+ \cap \mathcal{B} \neq \emptyset$ , then we have either  $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}^+(\bar{\tau}_1)$  or  $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}^-(\bar{\tau}_1)$ .*

**Proof.** From Lemma 9 we know that  $\mathcal{Q}^+ \cap \mathcal{A}$  and  $\mathcal{Q}^+ \cap \mathcal{B}$  are subsets of one of the branches  $\mathcal{Q}^+(\bar{\tau}_1)$  or  $\mathcal{Q}^-(\bar{\tau}_1)$ . Recall that  $\mathcal{Q}^+$ ,  $\mathcal{Q}^-$ ,  $\mathcal{Q}^+(\bar{\tau}_1)$ , and  $\mathcal{Q}^-(\bar{\tau}_1)$  are convex sets.

Now, assume to the contrary that  $\mathcal{Q}^+ \cap \mathcal{A} \subseteq \mathcal{Q}^+(\bar{\tau}_1)$  and  $\mathcal{Q}^+ \cap \mathcal{B} \subseteq \mathcal{Q}^-(\bar{\tau}_1)$ . We need to consider two cases. First, if  $\mathcal{Q}$  is a cone and  $0 \in \mathcal{A} \cup \mathcal{B}$ , then from (20) we obtain that  $\bar{\tau} = 0$ , i.e.,  $\mathcal{Q} = \mathcal{Q}(\bar{\tau})$ . Hence it is clear that  $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}^+(\bar{\tau}_1)$ , which contradicts the assumption.

Second, if  $\mathcal{Q}$  is a hyperboloid of two sheets, or  $\mathcal{Q}$  is a cone and  $0 \notin \mathcal{A} \cup \mathcal{B}$ , then from the proof of Lemma 9 we know that  $x(\bar{\tau}) \notin \mathcal{Q}$ . Recall that  $\mathcal{Q}^+(\bar{\tau}) \cap \mathcal{Q}^-(\bar{\tau}) = x(\bar{\tau})$ . Hence, using the separation theorem we know that there exist a hyperplane  $\mathcal{H} = \{x \in \mathbb{R}^\ell \mid h^\top x = \eta\}$  separating  $\mathcal{Q}^+(\bar{\tau}_1)$  and  $\mathcal{Q}^-(\bar{\tau}_1)$ , such that  $x(\bar{\tau}_1) \in \mathcal{H}$ . Given the assumption  $\mathcal{Q}^+ \cap \mathcal{A} \subset \mathcal{Q}^+(\bar{\tau}_1)$  and  $\mathcal{Q}^+ \cap \mathcal{B} \subset \mathcal{Q}^-(\bar{\tau}_1)$ , we have that  $\mathcal{H}$  must separate  $\mathcal{Q}^+ \cap \mathcal{A}$  and  $\mathcal{Q}^+ \cap \mathcal{B}$  as well. Hence,  $\mathcal{H}$  must be parallel to  $\mathcal{A}$  and  $\mathcal{B}$ , and  $\beta \leq \eta \leq \alpha$ . Now, if  $\beta < \eta < \alpha$ , then we obtain that  $x(\bar{\tau}) \notin \mathcal{A} \cup \mathcal{B}$ , which contradicts to Lemma 7. On the other hand, if  $\eta = \alpha$  or  $\eta = \beta$ , we obtain that  $x(\bar{\tau}_1) \in \mathcal{Q}$ , which is also a contradiction. This proves the lemma.  $\square$

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