Bounds on differences between some graph theoretic invariants

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Abstract: In the present paper, we are interested in bounding differences between graph invariants as well as in characterizing the corresponding extremal graphs. This kind of results belongs to the more general form known as AGX Form 1 which is extensively studied using the AutoGraphiX system at GERAD, Montreal. The graph invariants involved in the present work are the proximity, the remoteness, the eccentricity, the average distance, the frequencies of the maximum and minimum degrees, the domination number, the stability number and the chromatic number.

Key Words: Proximity, remoteness, graph invariants, extremal graphs, AutoGraphiX.
1 Introduction

Since GRAPH [14,15], a system defined about two decades ago, several other systems for automated, or computer assisted, discovery of conjectures in graph theory have been developed. They include newGRAPH, due to Stevanović [33], Graffiti, due to Fajtlowicz [19], Graffiti.pc, due to Delavina [20], GraPHedron due to Mélot [28] and AutoGraphix (AGX for short), due to Caporossi and Hansen [4,11].

The AGX system was systematically tested in the thesis of Aouchiche [1]. Pairwise comparisons between 20 invariants were studied, and all results and/or conjectures were of the following form (called AGX Form 1):

\[ \ell_n \leq i_1(G) \oplus i_2(G) \leq u_n \]

where \(i_1(G)\) and \(i_2(G)\) are invariants, \(\oplus\) is one of the four operations \(+,−,×,\div\), and \(\ell_n\) and \(u_n\) are lower and upper bounding functions of the order \(n\) of \(G\) which are best possible, i.e., such that for each value of \(n\) (except possibly very small ones where border effects appear) there is a graph \(G\) for which the bound is tight. These pairwise comparisons gave rise to 1520 cases, more than half of which were easily proved automatically by AGX, and about 360 were proved by hand, either by the GERAD Montreal group, or by graph theorists of various countries (mainly Serbia and China).

Several papers were devoted to prove results of the AGX Form 1, a list of which can be found in [3,5]. In the present paper, we explore forward the AGX Form 1, and focus our attention on the difference between pairs of invariants in terms of their order. Our results involve the following invariants which are defined in the present paper, we explore forward the AGX Form 1, and focus our attention on the difference between pairs of invariants in terms of their order.

While the average distance is widely studied, see e.g. [9,16,21–23,29,32], the eccentricity, introduced in [16], seems to be less studied then the average distance (see [18,26,27,31]). Also, despite their recent introduction, the proximity and the remoteness attracted the attention of several graph theorists [6,25,27,30,34]. Five conjectures were listed in [7], four of them were settled in [27,30,34], and the last one (that bounds the difference between the average eccentricity and the remoteness) is still open. Particular cases of this open conjecture are solved by Sedlar [30]. Further results of the AGX Form 1 involving the proximity and the remoteness can be found in [6,8,27,34].

2 Definitions and notations

Let \(G = (V,E)\) be a finite undirected graph, where \(V\) is the vertex set, and \(E\) the edge set of \(G\). The cardinality of \(V\) is also called the order of \(G\). Two vertices \(u\) and \(v\) are said to be adjacent if \(\{u,v\}\) (also denoted by \(uv\) or \(vu\)) belongs to \(E\). For any subset \(U\) of \(V\), the subgraph of \(G\) induced by \(U\) is the graph \(H = (U,E(U))\), where \(E(U)\) consists of those edges of \(G\) with both ends in \(U\). We denote by \(n\Delta(G)\) the number of vertices of maximum degree in \(G\), while \(n_d(G)\) is the number of vertices of minimum degree in \(G\).

Given two vertices \(u\) and \(v\) of \(G\), a path of length \(\ell\) between \(u\) and \(v\) is a sequence \((u_0 = u, u_1, \ldots, u_\ell = v)\) of distinct vertices such that \(u_iu_{i+1}\) is an edge of \(G\) for all \(i \in \{0,1,\ldots,\ell - 1\}\). A cycle \(C\) is a sequence \((u_0, u_1, \ldots, u_{\ell-1})\) of distinct vertices such that \(u_0u_{\ell+1}\) is an edge of \(G\) for all \(i \in \{0,1,\ldots,\ell - 1\}\) (where the addition is modulo \(\ell\)). We denote by \(P_n\) the path of order \(n\), and by \(C_n\) the cycle of order \(n\). A graph \(G\) is connected if for any pair of vertices \(u\) and \(v\) of \(G\), there is a path between \(u\) and \(v\). If \(G\) is not connected, its vertex set can be partitioned into connected components, i.e., maximal induced subgraphs that are connected. A graph \(G\) is a tree if it is connected and has no cycle.

A set of vertices in a graph \(G\) is stable if it induces a subgraph with no edges. The stability number of \(G\), also called independence number, and denoted by \(\alpha(G)\), is the maximum cardinality of a stable set in \(G\). A \(k\)-coloring of \(G\) is a partition of its vertex set into \(k\) stable sets. The chromatic number of \(G\), denoted by \(\chi(G)\), is the smallest integer \(k\) such that \(G\) admits a \(k\)-coloring. A set of vertices in \(G\) is dominating if all vertices of \(G\) are adjacent to at least one vertex of this set, or belongs to it. The domination number of \(G\), denoted by \(\beta(G)\) is the minimum cardinality of a dominating set in \(G\).
The distance between two vertices \( u \) and \( v \) in \( G \), denoted \( d_G(u, v) \), is the number of edges on a shortest path between \( u \) and \( v \). For a vertex \( v \) of a graph \( G \), \( \sigma_G(v) \) denotes the transmission of \( v \) in \( G \), i.e.,
\[
\sigma_G(v) = \sum_{w \in V} d_G(v, w)
\]
and \( \epsilon_G(v) \) denotes the eccentricity of \( v \) in \( G \), i.e.,
\[
\epsilon_G(v) = \max_{w \in V} d_G(v, w).
\]

The diameter of \( G \), denoted by \( D(G) \), is the maximum eccentricity of any vertex in \( G \), i.e., \( D(G) = \max_{v \in V} \epsilon_G(v) \).

We denote by \( K_{a,b} \) the complete bipartite graph having \( a \) vertices in one part, and \( b \) in the other one, and by \( K_{a,b} - e \) the graph obtained from \( K_{a,b} \) by removing an edge. The comet, denoted by \( CO_{n,k} \), is the graph obtained by linking a vertex of degree 1 of a star on \( k+1 \) vertices to one of the endpoints of the path \( P_{n-k-1} \) on \( n-k-1 \) vertices. Also, we denote by \( KI_{n,k} \) the kite which is obtained by linking a vertex of a clique on \( k \) vertices to one of the endpoints of a path \( P_{n-k} \) on \( n-k \) vertices. A double comet (double kite) is the graph obtained by considering two disjoint comets (kites) \( G_1 \) and \( G_2 \) and linking a vertex of maximum eccentricity in \( G_1 \) with a vertex of maximum eccentricity in \( G_2 \). A double comet (double kite) with \( n \) vertices and diameter not larger than \( n-3 \) is balanced if the two vertices of maximum degree have a degree that differs by at most one unit. We denote by \( DC_{n,\ell} \) (\( DK_{n,\ell} \)) the balanced double comet (balanced double kite) with \( n \) vertices and diameter \( \ell \). For illustration, the graphs \( CO_{7,4} \), \( KI_{7,4} \), \( DC_{9,5} \) and \( DK_{9,5} \) are shown in Figure 1.

Given a graph \( G = (V, E) \), we are mainly interested in the four following invariants, respectively called the eccentricity, the proximity, the remoteness, and the average distance of a connected graph \( G \):
\[
\epsilon(G) = \frac{\sum_{v \in V} \epsilon_G(v)}{n}, \quad \pi(G) = \frac{\min_{v \in V} \sigma_G(v)}{n-1}, \quad \rho(G) = \frac{\max_{v \in V} \sigma_G(v)}{n-1}, \quad \mu(G) = \frac{\sum_{v \in V} \sigma_G(v)}{n(n-1)}.
\]

In the next section, we give bounds on the differences of the form \( \pi(G) - i(G) \), where \( i(G) \) is an invariant among \( n_G(G) \), \( n_A(G) \) and \( \beta(G) \). In Section 4, we prove an upper bound on \( \rho(G) - n_{\Delta}(G) \), while an upper bound on \( \epsilon(G) - \mu(G) \) is given in Section 5 for the class of trees. Finally, in Sections 6 and 7, we prove upper and lower bounds on \( \chi(G) - n_{\Delta}(G) \) and \( \alpha(G) - n_{\beta}(G) \), respectively.

### 3 Comparing proximity to other invariants

Let \( G \) be a connected graph of order \( n \). The following result was proved in [7].

**Proposition 3.1**

\[
1 \leq \pi(G) \leq \begin{cases} \frac{n^2}{4(n-1)} & \text{if } n \text{ is even} \\ \frac{n^2}{4(n+1)} & \text{if } n \text{ is odd}. \end{cases}
\]

Moreover, the lower bound is reached if and only if \( G \) contains a dominating vertex, while the upper bound is reached if and only if \( G \) is the path \( P_n \) or the cycle \( C_n \) on \( n \) vertices.

![Figure 1: A comet, a kite, a double comet and a double kite](image.png)
We now give the second possible largest value of the proximity of a graph \( G \) of order \( n \).

**Lemma 3.1** Let \( G \) be a connected graph of order \( n \). If \( G \) is not isomorphic to the cycle \( C_n \) or to the path \( P_n \) on \( n \) vertices, then

\[
\pi(G) \leq \begin{cases} 
\frac{n^2 - 4}{4(n-1)} & \text{if } n \text{ is even} \\
\frac{n^2 - 5}{4(n-1)} & \text{if } n \text{ is odd.}
\end{cases}
\]

Moreover, the bound is reached if \( G \) is isomorphic to \( CO_{n,3} \) or \( KI_{n,3} \) (and possibly also by other graphs).

**Proof.** Since \( G \) is not isomorphic to \( C_n \) or \( P_n \), it follows from Property 3.1 that \( \pi(G) \) is strictly smaller than \( \pi(C_n) = \pi(P_n) \). It then follows from the definition of the proximity that \( \pi(G) \) is at most equal to \( \pi(C_n) - \frac{1}{n-1} \). Hence,

\[
\pi(G) \leq \begin{cases} 
\frac{n^2 - 4}{4(n-1)} - \frac{1}{n-1} = \frac{n^2 - 5}{4(n-1)} & \text{if } n \text{ is even} \\
\frac{n+1}{4} + \frac{1}{n-1} = \frac{n^2 - 5}{4(n-1)} & \text{if } n \text{ is odd.}
\end{cases}
\]

It is easy to verify that this bound is attained for \( G \) equal to \( CO_{n,3} \) or \( KI_{n,3} \).

**Theorem 3.2** Let \( G \) be a connected graph of order \( n \geq 4 \). Then

\[
1 - n \leq \min\{\pi(G) - n_{\delta}(G), \pi(G) - n_{\Delta}(G)\} \leq \max\{\pi(G) - n_{\delta}(G), \pi(G) - n_{\Delta}(G)\} \leq \begin{cases} 
\frac{n^2 - 4n}{4(n-1)} & \text{if } n \text{ is even} \\
\frac{n^2 - 4n - 1}{4(n-1)} & \text{if } n \text{ is odd.}
\end{cases}
\]

Moreover, the lower bound is reached if and only if \( G \) is a clique, while the upper bound is reached if \( G \) is isomorphic to \( CO_{n,3} \) or \( KI_{n,3} \) (and possibly also by other graphs).

**Proof.** Notice first that \( \pi(G) \) reaches its minimum value 1 when \( G \) has a dominating vertex, while \( n_{\Delta}(G) \) and \( n_{\delta}(G) \) reach their maximum value \( n \) only for regular graphs. Hence, both \( \pi(G) - n_{\Delta}(G) \) and \( \pi(G) - n_{\delta}(G) \) are at least equal to \( 1 - n \) and this lower bound is attained only for cliques.

For the upper bound, let us first show that it is not reached when \( G \) is isomorphic to \( C_n \) or \( P_n \). Since \( n_{\Delta}(C_n) = n_{\delta}(C_n) = n > n_{\Delta}(P_n) = n - 2 \geq n_{\delta}(P_n) = 2 \), it is sufficient to show that \( \pi(P_n) - n_{\delta}(P_n) \) is strictly smaller than the proposed upper bound.

\[
\pi(P_n) - n_{\delta}(P_n) = \begin{cases} 
\frac{n^2}{4(n-1)} - 2 & \text{if } n \text{ is even} \\
\frac{n+1}{4} - 2 & \text{if } n \text{ is odd}
\end{cases}
\]

\[
= \begin{cases} 
\frac{n^2 - 8n + 8}{4(n-1)} & \text{if } n \text{ is even} \\
\frac{n^2 - 8n + 7}{4(n-1)} & \text{if } n \text{ is odd}
\end{cases}
\]

\[
< \begin{cases} 
\frac{n^2 - 4n}{4(n-1)} & \text{if } n \text{ is even} \\
\frac{n^2 - 4n - 1}{4(n-1)} & \text{if } n \text{ is odd.}
\end{cases}
\]

If \( G \) is not isomorphic to \( C_n \) or \( P_n \), then it follows from Lemma 3.1 and the inequalities \( n_{\Delta}(G) \geq 1 \) and \( n_{\delta}(G) \geq 1 \) that

\[
\max\{\pi(G) - n_{\delta}(G), \pi(G) - n_{\Delta}(G)\} \leq \begin{cases} 
\frac{n^2 - 4}{4(n-1)} - 1 & \text{if } n \text{ is even} \\
\frac{n^2 - 5}{4(n-1)} - 1 & \text{if } n \text{ is odd.}
\end{cases}
\]

It is easy to verify that this bound is attained for \( G \) equal to \( CO_{n,3} \) or \( KI_{n,3} \).
Theorem 3.3 Let $G$ be a connected graph of order $n \geq 2$ and with a given diameter $D(G) = \ell$. Then

$$\pi(G) \leq \begin{cases} 
\frac{2n-\ell^2}{4(n-1)} & \text{if } \ell \text{ is even} \\
\frac{2n-\ell^2+1}{4(n-1)} & \text{if } \ell \text{ is odd and } n \text{ is even} \\
\frac{2n-\ell^2-1}{4(n-1)} & \text{if both } \ell \text{ and } n \text{ are odd.}
\end{cases}$$

Moreover, the upper bound is reached by balanced double comets $DC_{n,\ell}$ and balanced double kites $DK_{n,\ell}$.

Proof. Consider an induced path $P = (v_0, v_1, \ldots, v_\ell)$ of length $\ell$ in $G$. Let $a = \lceil \frac{\ell+1}{2} \rceil$ and $b = \lceil \frac{\ell}{2} \rceil$. We obviously have $\pi(G) \leq \min \{ \frac{\sigma_G(v_a)}{n-1}, \frac{\sigma_G(v_b)}{n-1} \}$. The sum of the distances from $v_a$ or from $v_b$ to the other vertices of $P$ is $\frac{1}{2} \lceil \frac{\ell}{2} \rceil (\lceil \frac{\ell}{2} \rceil + 1) + \frac{1}{2} \lceil \frac{\ell}{2} \rceil (\lceil \frac{\ell}{2} \rceil + 1) + \lceil \frac{n-\ell-1}{2} \rceil \lfloor \frac{\ell}{2} \rfloor + \lceil \frac{n-\ell-1}{2} \rceil \lfloor \frac{\ell}{2} \rfloor$. Also, $\min \{ \frac{\sigma_G(v_a)}{n-1}, \frac{\sigma_G(v_b)}{n-1} \}$ is maximised by linking half of the $n-\ell-1$ other vertices of $G$ to $v_1$ and the other half to $v_{\ell-1}$. More precisely, without loss of generality, we may assume that $\lceil \frac{n-\ell-1}{2} \rceil$ vertices of $G$ not in $P$ are linked to $v_1$ and are at distance $\lceil \frac{\ell+1}{2} \rceil$ of $v_a$, while $\lfloor \frac{n-\ell-1}{2} \rfloor$ vertices of $G$ not in $P$ are linked to $v_{\ell-1}$ and are at distance $\lfloor \frac{\ell}{2} \rfloor$ of $v_a$. In summary,

$$(n-1)\pi(G) \leq \sigma_G(v_a) \leq \frac{1}{2} \lfloor \frac{\ell}{2} \rfloor (\lfloor \frac{\ell}{2} \rfloor + 1) + \frac{1}{2} \lfloor \frac{\ell}{2} \rfloor (\lfloor \frac{\ell}{2} \rfloor + 1) + \lceil \frac{n-\ell-1}{2} \rceil \lfloor \frac{\ell}{2} \rfloor + \lceil \frac{n-\ell-1}{2} \rceil \lfloor \frac{\ell}{2} \rfloor.$$  

If $\ell$ is even, we therefore get

$$(n-1)\pi(G) \leq \frac{\ell}{2} (\frac{\ell}{2} + 1) + (n-\ell-1)\frac{\ell}{2} = \frac{2n-\ell^2}{4}.$$  

If $\ell$ is odd, while $n$ is even, then

$$(n-1)\pi(G) \leq \frac{\ell+1}{2} (\frac{\ell+1}{2} + 1) + \frac{\ell}{2} (\frac{\ell}{2} + 1) + \frac{n-\ell-1}{2} \frac{\ell+1}{2} + \frac{n-\ell-1}{2} \frac{\ell}{2} = \frac{2n-\ell^2+1}{4}.$$  

Finally, if both $\ell$ and $n$ are odd, then

$$(n-1)\pi(G) \leq \frac{\ell+1}{2} (\frac{\ell+1}{2} + 1) + \frac{\ell}{2} (\frac{\ell}{2} + 1) + \frac{n-\ell}{2} \frac{\ell+1}{2} + \frac{n-\ell}{2} \frac{\ell}{2} = \frac{2n-\ell^2-1}{4}.$$  

\[\square\]

Theorem 3.4 Let $G$ be a connected graph of order $n \geq 2$. Then

$$\pi(G) - \beta(G) \leq \frac{n^2 - 8n + z}{36(n-1)}$$  

with $z = \begin{cases} 
-9 & \text{if } n = 9 \text{ or } 17 \pmod{18} \\
0 & \text{if } n = 0 \text{ or } 8 \pmod{18} \\
3 & \text{if } n = 11 \text{ or } 15 \pmod{18} \\
7 & \text{if } n = 1, 7 \text{ or } 13 \pmod{18} \\
12 & \text{if } n = 2 \text{ or } 6 \pmod{18} \\
15 & \text{if } n = 3 \text{ or } 5 \pmod{18} \\
16 & \text{if } n = 4, 10 \text{ or } 16 \pmod{18} \\
24 & \text{if } n = 12 \text{ or } 14 \pmod{18}.
\end{cases}$$

Moreover, if $2 \leq n \leq 9$, then the bound is attained by all graphs $G$ with a dominating vertex, while if $n \geq 10$, then the bound is reached by balanced double comets $DC_{n,\ell}$ and balanced double kites $DK_{n,\ell}$ with diameter $\ell = 2 + 3\lceil \frac{n}{9} \rceil$.

Proof. Since the statement of the theorem is clearly true for $n = 2$, we can assume $n \geq 3$. Also, notice that if $D(G) = 1$ (i.e., $G$ is a clique), then $\pi(G) - \beta(G) = 0$, while if $G$ is a star (i.e., $G = K_{1,n-1})$, then $D(G) = 2$ and $\pi(G) - \beta(G)$ also equals 0. Hence, the upper bound on the difference $\pi(G) - \beta(G)$ is always attained for at least one graph which is not a clique. From now on, we therefore assume $D(G) \geq 2$. 

It was proved in [24] that \( \beta(G) \geq \left\lfloor \frac{D(G)+1}{3} \right\rfloor \) and we know from Theorem 3.3 that \( \pi(G) \leq \frac{2D(G)n-D(G)^2+x}{4(n-1)} \), with \( x = 0, 1 \) or \(-1\), depending on the parity of \( n \) and \( D(G) \). Hence,

\[
\pi(G) - \beta(G) \leq \frac{2D(G)n-D(G)^2+x}{4(n-1)} - \left[ \frac{D(G)+1}{3} \right].
\]

Let \( F(\ell) \) and \( f(\ell, n) \) be defined as follows:

\[
F(\ell) = \frac{2\ell n - \ell^2}{4(n-1)} - \left[ \frac{\ell + 1}{3} \right]
\]

\[
f(\ell, n) = \frac{x}{4(n-1)} \quad \text{with} \quad x = \begin{cases} 0 & \text{if } \ell \text{ is even} \\ 1 & \text{if } \ell \text{ is odd and } n \text{ is even} \\ -1 & \text{if both } \ell \text{ and } n \text{ are odd.}
\end{cases}
\]

We then have \( \pi(G) - \beta(G) \leq F(D(G)) + f(D(G), n) \). Moreover,

- if \( \ell = 0(\mod 3) \), then \( F(\ell + 2) + f(\ell + 2, n) = F(\ell) + f(\ell, n) + \frac{n-\ell+1}{n-1} \);
- if \( \ell = 1(\mod 3) \) then \( F(\ell - 2) + f(\ell - 2, n) = F(\ell) + f(\ell, n) + \frac{\ell-2}{n-1} \).

This implies that given any \( n \), the integer \( \ell \) for which \( F(\ell) + f(\ell, n) \) reaches its maximal value is equal to \( 2 + 3a \) for some integer \( a \geq 0 \), which implies \( \left\lfloor \frac{\ell+1}{3} \right\rfloor = \frac{\ell+1}{3} \). We now show that this optimum value is reached for \( a = \left\lfloor \frac{n}{3} \right\rfloor \).

So assume \( \ell = 2 + 3\left\lfloor \frac{n}{3} \right\rfloor \). Then

\[
F(\ell + 3) = F(\ell) + \frac{2n - 6\ell - 5}{4(n-1)} = F(\ell) + \frac{2n - 18\left\lfloor \frac{n}{3} \right\rfloor - 17}{4(n-1)} \leq F(\ell) - \frac{1}{4(n-1)}
\]

and

\[
F(\ell - 3) = F(\ell) - \frac{2n - 6\ell + 13}{4(n-1)} = F(\ell) - \frac{2n - 18\left\lfloor \frac{n}{3} \right\rfloor + 1}{4(n-1)} \leq F(\ell) - \frac{1}{4(n-1)}.
\]

Since considering \( F(\ell) \) as a continuous function, it is quadratic concave, this proves that the maximum value of \( F(\ell) \) with \( \ell \) integer is reached when \( \ell = 2 + 3\left\lfloor \frac{n}{3} \right\rfloor \).

Consider now any integer \( \ell' = 2 + 3a' \) with \( a' \neq \left\lfloor \frac{n}{3} \right\rfloor \). It follows from the definition of \( f \) that \( f(\ell', n) \leq f(\ell, n) + \frac{1}{4(n-1)} \). Also, we have shown that \( F(\ell') \leq \max\{F(\ell - 3), F(\ell + 3)\} \). Hence,

\[
F(\ell') + f(\ell', n) \leq \left( F(\ell) - \frac{1}{4(n-1)} \right) + \left( f(\ell, n) + \frac{1}{4(n-1)} \right) = F(\ell) + f(\ell, n).
\]

This means that the upper bound on \( \pi(G) - \beta(G) \) is reached for graphs \( G \) of order \( n \) with diameter \( D(G) = 2 + 3\left\lfloor \frac{n}{3} \right\rfloor \). We now analyze the value of \( F(D(G)) + f(D(G), n) \) according to the value of \( n \mod 18 \).

If \( n = 0(\mod 18) \), then \( D(G) = 2 + 3\left\lfloor \frac{n}{3} \right\rfloor = \frac{n+6}{3} \). Hence, both \( n \) and \( D(G) \) are even, which means that \( f(D(G), n) = 0 \) and

\[
F(D(G)) + f(D(G), n) = \frac{2\frac{n+6}{3}n - \frac{(n+6)^2}{9}}{4(n-1)} - \frac{n+9}{9} = \frac{n^2 - 8n}{36(n-1)}.
\]

If \( n = 1(\mod 18) \), then \( D(G) = 2 + 3\frac{n-1}{3} = \frac{n+5}{3} \). Hence \( n \) is odd while \( D(G) \) is even, which means that \( f(D(G), n) \) is again equal to 0. We therefore have

\[
F(D(G)) + f(D(G), n) = \frac{2\frac{n+5}{3}n - \frac{(n+5)^2}{9}}{4(n-1)} - \frac{n+8}{9} = \frac{n^2 - 8n + 7}{36(n-1)}.
\]

Similar computations can be done for the other values of \( n \mod 18 \), and we get the result in the statement of the theorem. It is easy to check that the upper bound equals 0 for \( 2 \leq n \leq 9 \), which means that it is reached by all graphs having a dominating vertex. For \( n \geq 10 \), it follows from Theorem 3.3 and the above proof that the upper bound is reached by balanced double comets \( DC_{n, \ell} \) and balanced double kites \( DK_{n, \ell} \) with diameter \( \ell = 2 + 3\left\lfloor \frac{n}{3} \right\rfloor \).
4 The remoteness and the maximum degree frequency

Let $G$ be a connected graph of order $n$. The following result was proved in [7].

**Proposition 4.1**

$$1 \leq \rho(G) \leq \frac{n}{2}$$

Moreover, the lower bound is reached if and only if $G$ is a clique, while the upper bound is reached if and only if $G$ is the path $P_n$ on $n$ vertices.

We now give the second possible largest value of the remoteness of a graph $G$ of order $n$

**Lemma 4.1** Let $G$ be a connected graph of order $n$. If $G$ is not the path $P_n$, then

$$\rho(G) \leq \frac{n^2 - n - 2}{2(n - 1)}$$

Moreover, the bound is reached if $G$ is isomorphic to $CO_{n,3}$ or $KI_{n,3}$ (and possibly also by other graphs).

**Proof.** Since $G$ is not isomorphic to $P_n$, it follows from Property 4.1 that $\rho(G)$ is strictly smaller than $\rho(P_n) = \frac{n}{2}$. It then follows from the definition of the proximity that $\rho(G)$ is at most equal to $\frac{n}{2} - \frac{1}{n - 1} = \frac{n^2 - n - 2}{2(n - 1)}$. It is easy to verify that this bound is attained for $G$ equal to $CO_{n,3}$ or $KI_{n,3}$.

**Theorem 4.2** Let $G$ be a connected graph of order $n \geq 4$. Then

$$1 - n \leq \rho(G) - n_\Delta(G) \leq \frac{n^2 - 3n}{2(n - 1)}$$

Moreover, the lower bound is reached if and only if $G$ is a clique, while the upper bound is reached if $G$ is isomorphic to $CO_{n,3}$ or $KI_{n,3}$ (and possibly also by other graphs).

**Proof.** Notice first that $\rho(G)$ reaches its minimum value 1 when $G$ is a clique, while $n_\Delta(G)$ reaches its maximum value $n$ for regular graphs. We therefore have $\rho(G) - n_\Delta(G) \geq 1 - n$, and this lower bound is attained only for cliques.

For the upper bound, let us first show that it is not reached when $G$ is isomorphic to $P_n$. This follows from the fact that $\rho(P_n) - n_\Delta(P_n) = \frac{n}{2} - (n - 2) \leq 0 < \frac{n^2 - 3n}{2(n - 1)}$.

If $G$ is not isomorphic to $P_n$, then Lemma 4.1 and the inequality $n_\Delta(G) \geq 1$ give

$$\rho(G) - n_\Delta(G) \leq \frac{n^2 - n - 2}{2(n - 1)} - 1 = \frac{n^2 - 3n}{2(n - 1)}.$$

It is easy to verify that this bound is attained for $G$ equal to $CO_{n,3}$ or $KI_{n,3}$.

5 Average eccentricity and average distance

**Lemma 5.1** Let $T$ be a tree of order $n$ and let $P = (v_0, v_1, \cdots, v_{D(T)})$ be a path of length $D(T)$ in $T$. If there is $j \leq \frac{D(T)}{2}$ such that the degree of $v_k$ is at most 2 for $k \geq j + 1$, then $\epsilon(T) - \mu(T) \leq \epsilon(P_n) - \mu(P_n)$.

**Proof.** Let $\ell = D(T)$ and suppose $T$ is not the path $P_n$. Let $w$ be a leaf in $T$ different from $v_0$ and $v_\ell$, and let $T'$ be the graph obtained from $T$ by deleting the edge incident to $w$ and adding an edge between $w$ and $v_\ell$. Note that it follows from the assumptions that $d_T(v_\ell, w) \geq \frac{\ell}{2} + 1$. It is now sufficient to prove that $\epsilon(T) - \mu(T) \leq \epsilon(T') - \mu(T')$ since we can then repeat this transformation until we obtain the path $P_n$. 


Similarly, we get increases by $\ell$ units since the distance from $w$ to the vertices outside $P$ is minimized when $w$ is adjacent to $v_i$ ($i = 1, \ldots, \ell - 1$). In such a case, we therefore have

$$
\sigma_T(w) \geq \frac{\lceil \frac{\ell}{2} \rceil + 1}{2} + \frac{\lceil \frac{\ell}{2} \rceil + 1}{2} - 1 + 2(n - \ell - 2)
$$

$$
\geq \frac{\ell + 1}{2} + \frac{\ell + 2}{4} - 1 + 2(n - \ell - 2)
$$

$$
= 2n + \frac{\ell^2}{4} - \frac{\ell}{2} - 3.
$$

This is the best possible lower bound on $\sigma_T(w)$ because if $w$ is not adjacent to a vertex in $P$, then the bound increases by $\ell$ units since the distance from $w$ to the $\ell + 1$ vertices in $P$ increases by one unit for each of them, while the distance from $w$ to its neighbour outside $P$ decreases from 2 to 1.

Putting together (2) to (5), we get

$$
\mu(T') - \mu(T) = \frac{2(\sigma_T(w) - \sigma_T(w))}{n(n-1)}
$$

$$
\leq \frac{2n - 4 - \ell + 2(\sigma_T(v_\ell) - \sigma_T(w))}{n(n-1)}
$$
Note that the diameter of $T$ for every vertex in $T$

Consider the following partition of the vertices of $T$

We follow a similar proof as the one used in [30] for determining an upper bound on $\epsilon(T) - \rho(T)$.

Proof. We follow a similar proof as the one used in [30] for determining an upper bound on $\epsilon(T) - \rho(T)$.

Let $T$ be a tree which maximizes the difference $\epsilon(T) - \mu(T)$. Assume, by contradiction, that $T$ is not the path $P_n$. Let $P = (v_0, v_1, \ldots, v_{D(T)})$ be a longest path in $T$. Let $G_i$ be the connected component of $T \setminus P$ rooted in $v_i$, and let $V_i$ be the vertex set of $G_i$. We know from Lemma 5.1 that there are two vertices $v_j$ and $v_k$ on $P$ of degree at least 3 such that $j \leq \frac{D(T)}{2} = k$. Let us choose such a pair of vertices on $P$ with minimum value $k - j$. Let $w_j$ be a neighbor of $v_j$ outside $P$, and let $w_k$ be a neighbor of $v_k$ outside $P$. Let $T'$ be the tree obtained from $T$ as follows:

- for every $w \neq w_j, v_{j+1}$ adjacent to $v_j$, remove the edge $v_jw$ and add the edge $w_jw$;
- for every $w \neq w_k, v_{k-1}$ adjacent to $v_k$, remove the edge $v_kw$ and add the edge $w_kw$.

Note that the diameter of $T'$ is two units larger than the diameter of $T$. Now, let

$$V'_j = \{v \in V_j : d_T(v, w_j) < d_T(v, v_j)\},$$

$$V'_k = \{v \in V_k : d_T(v, w_k) < d(v, v_k)\}.$$

Consider the following partition of the vertices of $T$

$$X_1 = V_0 \cup \ldots \cup V_{j-1} \cup (V_j \setminus (V'_j \cup \{v_j\})},$$

$$X_2 = V'_j,$$

$$X_3 = \{v_j\} \cup V_{j+1} \cup \ldots \cup V_{k-1} \cup \{v_k\},$$

$$X_4 = V'_k,$$

$$X_5 = (V_k \setminus (V'_k \cup \{v_k\})) \cup V_{j+1} \cup \ldots \cup V_{D(T)}.$$

Let $x_i = |X_i|$. We clearly have $\epsilon_T(v) = \epsilon_T(v) + 1$ for every vertex in $X_2 \cup X_3 \cup X_4$, while $\epsilon_T(v) = \epsilon_T(v) + 2$ for every vertex in $X_1 \cup X_5$. Hence

$$\epsilon(T') = \epsilon(T) + \frac{\phi}{n}.$$
where \( \phi = 2x_1 + x_2 + x_3 + x_4 + 2x_5 \). Let us now analyze how the average distance varies when transforming \( T \) to \( T' \). If \( v \in X_1 \), then

\[
d_{T'}(v, u) - d_T(v, u) = \begin{cases} 
0 & \text{if } u \in X_1 \\
-1 & \text{if } u \in X_2 \\
1 & \text{if } u \in X_3 \cup X_4 \\
2 & \text{if } u \in X_5.
\end{cases}
\]

Hence, \( \sigma_{T'}(v) = \sigma_T(v) + \tau_1 \) for all \( v \in X_1 \), where \( \tau_1 = -x_2 + x_3 + x_4 + 2x_5 \).

If \( v \in X_2 \), then

\[
d_{T'}(v, u) - d_T(v, u) = \begin{cases} 
-1 & \text{if } u \in X_1 \\
0 & \text{if } u \in X_2 \cup X_3 \cup X_4 \\
1 & \text{if } u \in X_5.
\end{cases}
\]

Hence, \( \sigma_{T'}(v) = \sigma_T(v) + \tau_2 \) for all \( v \in X_2 \), where \( \tau_2 = -x_1 + x_5 \).

If \( v \in X_3 \), then

\[
d_{T'}(v, u) - d_T(v, u) = \begin{cases} 
1 & \text{if } u \in X_1 \cup X_5 \\
0 & \text{if } u \in X_2 \cup X_3 \cup X_4.
\end{cases}
\]

Hence, \( \sigma_{T'}(v) = \sigma_T(v) + \tau_3 \) for all \( v \in X_3 \), where \( \tau_3 = x_1 + x_5 \).

By symmetry, we also have \( \sigma_{T'}(v) = \sigma_T(v) - \tau_4 \) for all \( v \in X_4 \) and \( \sigma_{T'}(v) = \sigma_T(v) - \tau_5 \) for all \( v \in X_5 \), where \( \tau_4 = x_1 - x_5 \) and \( \tau_5 = 2x_1 + x_2 + x_3 - x_4 \).

We thus have

\[
n(n-1)\mu(T') = \sum_{i=1}^{5} \sum_{v \in X_i} \sigma_{T'}(v) = \sum_{i=1}^{5} \sum_{v \in X_i} (\sigma_T(v) + \tau_i) = n(n-1)\mu(T) + \sum_{i=1}^{5} x_i \tau_i.
\]

Notice that \((n-1)\phi - n\tau_1 = (2n-2)x_1 + (2n-1)x_2 - x_3 - x_4 - 2x_5 > 0 \) because \( x_3, x_4, x_5 \) are at most equal to \( n-4 \), while \( x_1 \) and \( x_2 \) are at least equal to 1. Similarly \((n-1)\phi - n\tau_i > 0 \) for \( i = 2, 3, 4, 5 \). Hence

\[
\epsilon(T') - \mu(T') = (\epsilon(T) - \mu(T)) = \frac{\phi}{n} - \frac{\sum_{i=1}^{5} x_i \tau_i}{n(n-1)} = \frac{\phi}{n} - \frac{\sum_{i=1}^{5} x_i \phi}{n^2},
\]

which contradicts that fact that \( T \) maximizes the difference \( \epsilon(T) - \mu(T) \).

Hence the path \( P_n \) maximizes \( \epsilon(T) - \mu(T) \) among all trees \( T \) and it is not difficult to check that \( \epsilon(P_n) - \mu(P_n) = \frac{5n-10}{12} - \frac{n \mod 2}{4n} \).

6 The chromatic number and the maximum degree frequency

**Theorem 6.1** Let \( G \) be a connected graph of order \( n \geq 3 \). Then.

\[
\begin{cases} 
2 - n & \text{if } n \text{ is even}, \\
3 - n & \text{if } n \text{ is odd},
\end{cases}
\]

\[
\chi(G) - n_{\Delta}(G) \leq n - 2.
\]

Moreover, the lower bound is reached, for example, by the cycle \( C_n \) as well as by the regular bipartite graphs. The upper bound is reached if and only if \( G \) is a kite \( KI_{n,n-1} \).

**Proof.** Since \( \chi(G) \geq 2 \) and \( n_{\Delta}(G) \leq n \), we clearly have \( \chi(G) - n_{\Delta}(G) \geq 2 - n \). This lower bound is clearly only reached by bipartite regular graphs, which means that \( n \) must be even. Hence, if \( n \) is odd, then \( \chi(G) - n_{\Delta}(G) \geq 3 - n \). The bound is then reached, for example by the cycle \( C_n \).

We now prove the upper bound \( n - 2 \) is valid and can only be reached if \( G = KI_{n,n-1} \). If \( \chi(G) = n \), then \( G \) is a clique and \( n_{\Delta}(G) = n \), which means that \( \chi(G) - n_{\Delta}(G) = 0 < n - 2 \). Also, if \( \chi(G) \neq n - 1 \) or \( n_{\Delta}(G) \neq 1 \), then \( \chi(G) - n_{\Delta}(G) < n - 2 \). So assume \( \chi(G) = n - 1 \) and \( n_{\Delta}(G) = 1 \). We then know that \( G \)
is not the cycle $C_n$. Since $G$ is not a clique, Brook’s theorem [10] implies $n - 1 = \chi(G) \leq \Delta(G)$. Hence, $G$ contains a dominating vertex $v$.

Let $G'$ be the induced subgraph obtained by removing $v$ from $G$. Clearly, $\chi(G') = n - 2$. Hence, $G'$ is not the cycle $C_{n-1}$ since we would have $\chi(G') = 2 < n - 2$ if $n = 5$, and $\chi(G') = 3 < n - 2$ if $n > 5$. Also, since $n_{\Delta}(G) = 1$, we necessarily have $\Delta(G') \leq n - 3$, which implies that $G'$ is not a clique. If $G'$ is connected, then we can again apply Brook’s theorem, and we get $\chi(G') \leq \Delta(G') \leq n - 3$, a contradiction. So assume $G'$ is not connected.

Let $H_1, \ldots, H_r$ denote the connected components of $G'$. We then have

$$n - 1 = \chi(G) = \max_i \chi(H_i) + 1.$$ 

Hence, there is a connected component $H_i$ such that $\chi(H_i) = n - 2$. But since every $H_i$ contains at most $n - 2$ vertices, we deduce that $G'$ contains only two connected components, one being a clique or order $n - 2$, and the other one an isolated vertex. But this implies that $G = K_{I, n-1}$.

\section{The stability number and the minimum degree frequency}

\begin{theorem}
Let $G$ be a connected graph of order $n \geq 5$. Then

$$1 - n \leq \alpha(G) - n_{\delta}(G) \leq n - 3.$$ 

Moreover, the lower bound is reached if and only if $G$ is a clique, while the upper bound is reached, for example, by $K_{2, n-2} - e$.
\end{theorem}

\begin{proof}
Since $\alpha(G) \geq 1$ and $n_{\delta}(G) \leq n$, we clearly have $\alpha(G) - n_{\delta}(G) \geq 1 - n$, and this bound is only reached if $G$ is a clique. Let us now prove that the upper bound is valid. Since $G$ is connected, we have $\alpha(G) \leq n - 1$, and this bound is attained only for $G = S_n$. In such a case, $\alpha(S_n) - n_{\delta}(S_n) = 0 < n - 3$. So assume $\alpha(G) \leq n - 2$. We then have $\alpha(G) - n_{\delta}(G) \leq n - 3$. The bound is reached for, example, by $G = K_{2,n-2} - e$.
\end{proof}

\section*{References}


[33] Stevanović, D., Brankov, V., Cvetković, D., Simić, S., newGRAPH system available at the website http://www.mi.sanu.ac.rs/newgraph/.