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GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Combining losing games into a winning game

Bruno Rémillard [a]
Jean Vaillancourt [b]

[a] *GERAD, CRM & Department of Decision Sciences, HEC Montréal, 3000, chemin de la Côte-Sainte-Catherine, Montréal (Québec), Canada H3T 2A7*

[b] *Department of Decision Sciences, HEC Montréal, 3000, chemin de la Côte-Sainte-Catherine, Montréal (Québec), Canada H3T 2A7*

bruno.remillard@hec.ca
jean.vaillancourt@hec.ca

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Abstract: Parrondo's paradox is extended to regime switching random walks in random environments. The paradoxical behavior of the resulting random walk is explained by the effect of the random environment. Full characterization of the asymptotic behavior is achieved in terms of the dimensions of some random subspaces occurring in Oseledec's theorem. The regime switching mechanism gives our models a richer and more complex asymptotic behavior than the simple random walks in random environments appearing in the literature, in terms of transience and recurrence.

Keywords: Parrondo's paradox, random walks, random environments, Oseledec's theorem

Résumé : Le paradoxe de Parrondo est étendu à des marches aléatoires en environnements aléatoires avec changement de régimes. Le comportement paradoxal de la marche aléatoire qui en résulte est expliqué par l'effet de l'environnement aléatoire. La caractérisation complète du comportement asymptotique est obtenue en termes des dimensions de certains sous-espaces aléatoires apparaissant dans le théorème d'Oseledec. Le mécanisme de changement de régime donne à nos modèles un comportement asymptotique plus riche et plus complexe que les randonnées aléatoires simples dans des environnements aléatoires apparaissant dans la littérature, en terme de transitivité et de récurrence.

Mots clés : Paradoxe de Parrondo, marche aléatoire, environment aléatoire, théorème d'Oseledec

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1 Introduction

To illustrate what is now known as Parrondo's paradox (Harmer and Abbott, 1999), consider the following games:

- Game A: The fortune X_n of the player after n independent games is

$$X_n = \begin{cases} X_{n-1}+1 & \text{w. pr. } p, \\ X_{n-1}-1 & \text{w. pr. } 1-p, \end{cases} \quad , \quad n \geq 1.$$

- Game B: The fortune Y_n of the player after n games is given by

$$Y_n = \begin{cases} Y_{n-1}+1 & \text{w. pr. } g(Y_{n-1}), \\ Y_{n-1}-1 & \text{w. pr. } 1-g(Y_{n-1}), \end{cases} \quad , \quad n \geq 1,$$

  where g is a 3-periodic function on $\mathbb{Z}$ such that $g(0) = p_1$ and $g(1) = g(2) = p_2$.

It is well-known that Game A is fair if and only if $p = 1/2$. In fact, if $p \neq 1/2$, the Markov chain X is transient and $\lim_{n\to\infty} X_n = -\infty$ if $p < 1/2$, while $\lim_{n\to\infty} X_n = +\infty$ if $p > 1/2$. When $p = 1/2$, X is recurrent and

$$P(\liminf X_n = -\infty \text{ and } \limsup X_n = +\infty) = 1.$$

For Game B, the process Y can be seen as a particular case of a random walk in a random environment; in fact, the space E of environments has 3 elements, i.e., $E = \{T^k g; k = 0,1,2\}$ with $T^k g(x) = g(x+k)$. For more details on periodic and almost periodic environments, see, e.g., (Remillard and Dawson, 1989, Examples 1–2).

Solomon (1975) studied the special case behavior of random walks in a random environment when the latter are independent and identically distributed (i.i.d.) which does not cover the periodic environment. Instead, one can rely on (Alili, 1999, Theorem 2.1), to conclude that the process Y is recurrent if and only if $\mu = 1$, where

$$\mu = \frac{(1-p_1)(1-p_2)^2}{p_1 p_2^2}. \tag{1}$$

As a result, $P(\liminf Y_n = -\infty \text{ and } \limsup Y_n = +\infty) = 1$. Otherwise, when $\mu \neq 1$, Y_n is transient and $\lim_{n\to\infty} Y_n = -\infty$ if $\mu > 1$, while $\lim_{n\to\infty} Y_n = +\infty$ if $\mu < 1$. For example, if $p_1 = 1/10$ and $p_2 = 3/4$, then $\mu = 1$. If $p_1 < 1/10$ and $p_2 < 3/4$, then $\mu > 1$.

Following Harmer and Abbott (1999), suppose that $p = 0.499$, $p_1 = 0.099$ and $p_2 = 0.749$. Then, according to the previous observations, if a player always plays Game A or Game B, her fortune will tend to $-\infty$ with probability one. However, if she plays Game A twice, then Game B twice and so on (Game C), or if she chooses the game at random with probability 1/2 (Game D), her fortune will tend to $+\infty$. This is Parrondo's paradox and it is illustrated in Figure 1.

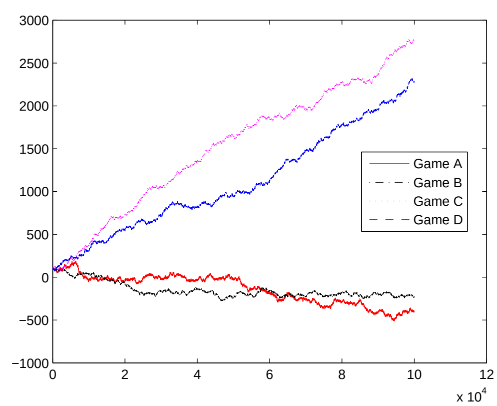

Figure 1: Evolution of the fortune of a player starting at $100\$$ over 10,000 games, where $p = 0.499$, $p_1 = 0.099$ and $p_2 = 0.749$.

Now consider Game C' , where the player alternates between Game A and Game B, i.e., she plays Game A once, then Game B once, and so on. What happens in this case? The answer will be given at the end of Example 3.2.

Games A and B are particular cases of random walks in a random environment, while Games C and D are examples of regime switching Markov chains in random environment. The aim of this paper is to study the asymptotic behavior of the latter. One of the first rigorous work on this problem is Pyke (2003), who studied some particular cases of random environment, namely the so-called periodic case, where each random walk is like in Game B, while the player chooses at random between two games. The author also consider some "deterministic" mixtures, namely the cases studied in Harmer and Abbott (1999). The earliest examples of games exhibiting this paradoxical behavior when combined can be found in Durrett et al. (1991). Those are not covered in our setting.

In what follows, the player chooses the next game to play according to a finite Markov chain, and each game is a random walk in a random environment, extending the work of Harmer and Abbott (1999) and Pyke (2003).

In Section 2, the model, which is basically a regime switching random walk in random environments, is described and one of the main characterization results is stated in Theorem 2.2, namely that the asymptotic behavior of these models does not always reflect that of independent simple random walks in random environments. Without regime switching, as shown in Alili (1999), they are transient, meaning that they converge to either of $\pm\infty$ with probability 1, for almost every environment, or they are recurrent, meaning that the limsup and liminf converge respectively to $\pm\infty$ with probability 1, for almost every environment.

With regime switching, this is not always the case and the full mathematical analysis of this generalization is the main contribution of this paper. Understanding the asymptotic behavior under regime switching is key to the development of improved tools and methods relevant to gaming strategies and control issues, when dealing with potential applications in various contexts such as financial market uncertainty Fink et al. (2017), investment portfolio solvency Abourashchi et al. (2016) and hypothesis testing under richer, more dynamic experimental designs Carter et al. (2016).

In Section 3, the possible cases are fully characterized in terms of the rank of the transition matrix that governs regime switching and the dimensions of some random subspaces occurring in Oseledec's ergodic theorem. These results generalize the best known particular cases : (i) when the games are chosen independently, the transition matrix has rank 1 and the results can be recovered as well from standard arguments applied to random walks in random environments (Alili, 1999); (ii) the periodic choices studied by many authors, starting with Pyke (2003), where the transition matrices have full rank; and continuing in a series of papers beginning with Ethier and Lee (2009), where the Markov chains associated with the matrices are irreducible. Note that Ethier and Lee (2009) consider several other structural choices not covered here. The main contribution in the current paper is in the coverage of new cases including instances of reducible Markovian switching regimes, avoided so far in the literature because of their technical intractability, as exemplified in Remark 2.1.

Finally, the proofs of the results can be found in the Appendices. They are inspired by the results of Key (1984) who studied random walks in random environments with bounded increments but no regime switching. The results of Key (1984) were later refined by several authors, notably Bolthausen and Goldsheid (2000), Keane and Rolles (2002), and Brémont (2002, 2009).

2 Regime switching random walks in random environments

We first describe the model and then study its asymptotic behavior.

2.1 Model

First, let $(E, \mathcal{E}, P)$ be a complete probability space with a measure preserving transformation T, assumed to be $\mathcal{E}$-measurable and ergodic, i.e., the T-invariant sigma-field is trivial. Next, for any $\alpha \in \{1, \ldots, m\}$, and any $k \in \mathbb{Z}$, $p_k^{(\alpha)}$ are $(0,1)$-valued variables, where $p_k^{(\alpha)}(Te) = p_{k+1}^{(\alpha)}(e)$, $e \in E$. Hence, the processes $p^{(\alpha)}$ are stationary and ergodic.

Next, for a given $\alpha \in \{1,\ldots,m\}$, let $X_n^{(\alpha)}$ be the nearest neighbor random walk in a random environment $e \in E$ defined by the process $p^{(\alpha)}$, i.e., its so-called quenched law is given by

$$P\left(X_n^{(\alpha)} = k+1 |\; X_{n-1}^{(\alpha)} = k, \mathcal{E}\right)(e) = p_k^{(\alpha)}(e),\ k \in \mathbb{Z},\ n \geq 1.$$

These random walks will be the fortunes of the player as she chooses each game. Her decision process is based on the Markov chain G on $\{1,\ldots,m\}$, with transition matrix Q. For example, in Game C, one can choose $m = 4$, $p^{(1)} = p^{(2)}$, $p^{(3)} = p^{(4)}$, where $p^{(1)}$ is the (deterministic) process determined by Game A, meaning that $p_k^{(1)} \equiv p$, $p^{(3)}$ is the stationary ergodic process determined by Game B, i.e., $p_k^{(3)}(e_j) = g(k+j)$, $j \in \{0,1,2\}$, with $e_0 = g$, $e_1 = g(1+\cdot)$, $e_2 = g(2+\cdot)$, and $Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. For Game D, $m = 2$ and $Q = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$. Further note that for Game C', then $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $p^{(1)}$ and $p^{(2)}$ are defined as before.

Finally, for any $n \geq 1$, the quenched law of the regime-switching (nearest-neighbor) random walk X_n is defined by

$$P(X_n = k+1 | X_{n-1} = k, G_n = \alpha, \mathcal{E})(e) = p_k^{(\alpha)}(e), \quad e \in E,\ k \in \mathbb{Z},\ \alpha \in \{1,\ldots,m\}.$$

Under these assumptions, given $e \in E$, (G_n, X_n) is an homogeneous Markov chain on $\{1,\ldots,m\} \times \mathbb{Z}$ with transition matrix

$$P_{\alpha i,\beta j}(e) = Q_{\alpha\beta} P_{ij}^{(\beta)}(e), \quad \alpha,\beta \in \{1,\ldots,m\},\ i,j \in \mathbb{Z},$$

where $P_{i,i+1}^{(\beta)} = p_i^{(\beta)}$ and $P_{i,i-1}^{(\beta)} = 1 - p_i^{(\beta)} = q_i^{(\beta)}$.

For a given environment e, the oft-cited Cogburn (1980) called X a Markov chain in a random environment, the sequence G_n playing, in his case, the role of the "environment". To avoid confusion, we will not make use of this terminology here, the choice for the next game G_n - the regime switching mechanism - depending solely on the previous game selection G_{n-1}, independently of the environment.

Remark 2.1 *Note that our setting is a particular case of a random walk in a random environment on a strip introduced in Bolthausen and Goldsheid (2000). This is also the case for the edge-reinforced random walks in Keane and Rolles (2002) where their method of proof is similar to ours. However, in both cases, their results cannot be applied in general here since they assumed that the resulting Markov chain (G_n, X_n) has only one communication class (Bolthausen and Goldsheid, 2000, Condition C), i.e., the Markov chain is almost surely irreducible (Bolthausen and Goldsheid, 2000, Remark 2). One cannot simply separate the non-communicating classes and use their results since we show that for the reducible case presented in Example 3.3, the asymptotic behavior depends on the random environment, a prohibited phenomenon in the results of Bolthausen and Goldsheid (2000) and Keane and Rolles (2002). Another example of a reducible case is Game C', where there are two closed classes: $C_0 = \{(\alpha,i);\alpha + i \text{ is even}\}$ and $C_1 = \{(\alpha,i);\alpha + i \text{ is odd}\}$. For this game, we also show in Appendix C that the main result of Bolthausen and Goldsheid (2000) does not apply.*

One is interested in the asymptotic behavior of process X describing the evolution of the player's fortune. More precisely, one would like to find conditions under which the so-called Parrondo's paradox holds, i.e., for any starting point $(\alpha,i) \in \{1,\ldots,m\} \times \mathbb{Z}$, and almost every environment e,

$$P_{\alpha i}^e\left(\lim_{n\to\infty} X_n = +\infty\right) = P\left(\lim_{n\to\infty} X_n = +\infty \middle| \mathcal{E}, G_0 = \alpha, X_0 = i\right)(e) = 1,$$

while

$$P_i^e\left(\lim_{n\to\infty} X_n^{(\alpha)} = -\infty\right) = P\left(\lim_{n\to\infty} X_n^{(\alpha)} = -\infty \middle| \mathcal{E}, X_0 = i\right)(e) = 1.$$

For $\alpha \in \{1,\ldots,m\}$, define the (stationary ergodic) process $\sigma^{(\alpha)}$ by $\sigma_i^{(\alpha)} = \frac{q_i^{(\alpha)}}{p_i^{(\alpha)}}$, $i \in \mathbb{Z}$. Then the asymptotic behavior of $X^{(\alpha)}$ is completely determined by the expectation of $\log \sigma_0^{(\alpha)}$, as proven in (Alili, 1999, Theorem 2.1).

Theorem 2.1 *Let $\alpha \in \{1,\ldots,m\}$ be given and suppose that $u = E\left(\log \sigma_0^{(\alpha)}\right)$ is well defined, with values in $[-\infty,+\infty]$. If $u > 0$, then for any $i \in \mathbb{Z}$,*

$$P_i^e\left(\lim_{n\to\infty} X_n^{(\alpha)} = -\infty\right) = 1, \; e \text{ a.s.}$$

If $u < 0$, then for any $i \in \mathbb{Z}$,

$$P_i^e\left(\lim_{n\to\infty} X_n^{(\alpha)} = +\infty\right) = 1, \; e \text{ a.s.}$$

Finally, if $u = 0$, then for any $i \in \mathbb{Z}$,

$$P_i^e\left(\liminf_{n\to\infty} X_n^{(\alpha)} = -\infty, \limsup_{n\to\infty} X_n^{(\alpha)} = +\infty\right) = 1, \; e \text{ a.s.}$$

In addition to the trivial case where $p^{(\alpha)}$ is constant, this result also covers the i.i.d. case first treated by Solomon (1975), i.e., where for a given α, $p_i^{(\alpha)}$ are i.i.d. Here are other examples of stationary ergodic sequences.

Example 2.1 (Periodic case) *Pyke (2003) studied Parrondo's paradox when for each $\alpha \in \{1,\ldots,m\}$ $p^{(\alpha)}$ is a deterministic sequence of period d_α, i.e., $p_{k+d_\alpha}^{(\alpha)} = p_k^{(\alpha)}$. These are particular cases of stationary ergodic sequences for which each environment $p^{(\alpha)}(\cdot + j)$, $j \in \{1,\ldots,d_\alpha\}$ has equal probability $1/d_\alpha$, so*

$$u = E\left(\log \sigma_0^{(\alpha)}\right) = \frac{1}{d_\alpha}\sum_{j=1}^{d_\alpha} \log \sigma_j^{(\alpha)}, \qquad \alpha \in \{1,\ldots,m\}.$$

This example contains Games A and B as particular cases.

Example 2.2 *Set $E = (0,1)$ and define $T(e) = \frac{1}{e} - \lfloor \frac{1}{e} \rfloor$, $e \in (0,1)$. Then T is a measure-preserving map for the law with density $f(x) = \frac{1}{\log 2}\frac{1}{(1+x)}$, $x \in (0,1)$. Set $p_i(e) = T^i e$ and use the two-sided extension theorem, e.g., (Durrett, 2010, Theorem 7.1.2), to obtain a stationary sequence defined for any $i \in \mathbb{Z}$. It is easy to check that $E(\log \sigma_i) = \frac{\log 2}{2} > 0$. Therefore, according to Theorem 2.1, the associated random walk converges to $-\infty$ with probability one, for almost every environment. Note that T is not invertible.*

In the next section, we study the asymptotic behavior of the process X. To reduce the notations, the random environment is fixed, unless otherwise specified.

2.2 Asymptotic behavior

For any $\ell \in \mathbb{Z}$, set $\tau_\ell = \inf\{n \ge 1; X_n = \ell\}$. The proof of the following lemma in given in Appendix B.

Lemma 2.1 *Let $\ell \in \mathbb{Z}$ be given. Then $P_{\alpha i}^e\{X_n \le \ell \text{ i.o.}\} = 0$ for every $(\alpha, i) \in \{1,\ldots,m\} \times \mathbb{Z}$ if and only if for every $(\alpha, i) \in \{1,\ldots,m\} \times \mathbb{Z}$,*

$$P_{\alpha i}^e(\tau_\ell < \infty) = 1, \text{ if } i < \ell, \text{ and } P_{\alpha i}^e(\tau_\ell < \infty) < 1, \text{ if } i > \ell. \tag{2}$$

The next result is proven in Appendix B.

Proposition 2.1 *Let $\ell > 0$ be given. Then for any $\alpha \in \{1,\ldots,m\}$,*

$$P_{\alpha i}^e(\tau_\ell < \infty \cup \tau_{-\ell} < \infty) = 1, \quad |i| < \ell. \tag{3}$$

Conditioning on the first play of the game yields the following.

Proposition 2.2 *Let ℓ be given and set $(f_{i\ell}(e))_\alpha = P^e_{\alpha i}(\tau_\ell < \infty)$, $(\alpha, i) \in \{1,\ldots,m\}\times\mathbb{Z}$. Then for any $\alpha \in \{1,\ldots,m\}$,*

$$\begin{aligned}(f_{i\ell})_\alpha &= \sum_{\beta=1}^m Q_{\alpha\beta}\left\{p_i^{(\beta)}(f_{i+1,\ell})_\beta + q_i^{(\beta)}(f_{i-1,\ell})_\beta\right\}, \quad i \notin \{\ell-1,\ell+1\},\\ (f_{i\ell})_\alpha &= \sum_{\beta=1}^m Q_{\alpha\beta}\left\{p_i^{(\beta)} + q_i^{(\beta)}(f_{i-1,\ell})_\beta\right\}, \quad i = \ell-1,\\ (f_{i\ell})_\alpha &= \sum_{\beta=1}^m Q_{\alpha\beta}\left\{p_i^{(\beta)}(f_{i+1,\ell})_\beta + q_i^{(\beta)}\right\}, \quad i = \ell+1.\end{aligned}$$

Set $\|f_{i\ell}(e)\| = \max_{\alpha\in\{1,\ldots,m\}}(f_{i\ell}(e))_\alpha$. The following proposition is an interesting consequence of the previous result. Its proof is given in Appendix B.

Proposition 2.3 *Let ℓ and e be given. If $\|f_{i\ell}(e)\| = 1$ for some $i < \ell$ then $\|f_{i\ell}(e)\| = 1$ for any $i < \ell$. Similarly, if $\|f_{i\ell}(e)\| = 1$ for some $i > \ell$ then $\|f_{i\ell}(e)\| = 1$ for any $i > \ell$.*

Next, suppose further that the Markov chain G_n is irreducible. If $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s. for some $\alpha \in \{1,\ldots,m\}$, and some $i < \ell$, then $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s. for every $\alpha \in \{1,\ldots,m\}$ and any $i < \ell$. Moreover, if $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s. for some $\alpha \in \{1,\ldots,m\}$, and some $i > \ell$, then $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s. for every $\alpha \in \{1,\ldots,m\}$ and any $i > \ell$.

The proof of the following subadditive ergodic theorem for the sequence $\|f_{\ell+k,\ell+k-1}\|$, $k \ge 1$ is given in Appendix B.

Proposition 2.4 *For any $n \ge 1$ and any $\ell \in \mathbb{Z}$,*

$$\|f_{\ell-n,\ell}\| \le \prod_{k=1}^n \|f_{\ell-k,\ell-k+1}\|. \tag{4}$$

Moreover $\|f_{\ell-k,\ell-k+1}\|$, $k \ge 1$, is a stationary ergodic sequence and

$$\gamma_- = \lim_{n\to\infty}\frac{1}{n}\log\|f_{\ell-n,\ell}(e)\| \le E\{\log\|f_{0,1}\|\} \le 0,\ e\ a.s.$$

Similarly,

$$\|f_{\ell+n,\ell}\| \le \prod_{k=1}^n \|f_{\ell+k,\ell+k-1}\|. \tag{5}$$

Moreover $\|f_{\ell+k,\ell+k-1}\|$, $k \ge 1$, is a stationary ergodic sequence and

$$\gamma_+ = \lim_{n\to\infty}\frac{1}{n}\log\|f_{\ell+n,\ell}(e)\| \le E\{\log\|f_{1,0}\|\} \le 0,\ e\ a.s.$$

We now state a useful technical result, whose proof is given in Appendix B.

Lemma 2.2 *The following statements hold:*

(i) $P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = +\infty\right) = 1\ \forall(\alpha,i)\in\{1,\ldots,m\}\times\mathbb{Z}$ *if and only if* $\forall(\alpha,i,\ell)\in\{1,\ldots,m\}\times\mathbb{Z}^2$,

$$P^e_{\alpha i}(\tau_\ell < \infty) = 1, \quad i < \ell, \tag{6}$$

$$P^e_{\alpha i}(\tau_\ell < \infty) < 1, \quad i > \ell. \tag{7}$$

(ii) $P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = -\infty\right) = 1\ \forall(\alpha, i) \in \{1,\ldots,m\}\times\mathbb{Z}$ *if and only if* $\forall(\alpha, i, \ell) \in \{1,\ldots,m\}\times\mathbb{Z}^2$,

$$P^e_{\alpha i}(\tau_\ell < \infty) = 1, \quad i > \ell, \tag{8}$$

$$P^e_{\alpha i}(\tau_\ell < \infty) < 1, \quad i < \ell. \tag{9}$$

(iii) Suppose further that the Markov chain G_n is irreducible. $P^e_{\alpha i}\left(\liminf_{n\to\infty} X_n = -\infty, \limsup_{n\to\infty} X_n = +\infty\right) = 1\ \forall(\alpha, i) \in \{1,\ldots,m\}\times\mathbb{Z}$, *if and only if* $\forall(\alpha,\ell) \in \{1,\ldots,m\}\times\mathbb{Z}$,

$$P^e_{\alpha\ell}(\tau_\ell < \infty) = 1. \tag{10}$$

We are now in a position to state the first main result, proven in Appendix B.

Theorem 2.2 *Set* $\gamma_\pm = \lim_{n\to\infty}\frac{1}{n}\log\|f_{\pm n,0}\|$. *Then under no additionnal condition, there holds* $\max(\gamma_+,\gamma_-) = 0$. *If in addition* (G_n, X_n) *is irreducible, then one of the following three mutually exclusive cases occur:*

(1) If $\gamma_+ < 0$ and $\gamma_- = 0$, then $\forall(\alpha, i) \in \{1,\ldots,m\}\times\mathbb{Z}$,

$$P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = +\infty\right) = 1,\ e\ a.s.$$

(2) If $\gamma_+ = 0$ and $\gamma_- < 0$, then $\forall(\alpha, i) \in \{1,\ldots,m\}\times\mathbb{Z}$,

$$P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = -\infty\right) = 1,\ e\ a.s.$$

(3) If $\gamma_+ = 0 = \gamma_-$, then $\forall(\alpha, i) \in \{1,\ldots,m\}\times\mathbb{Z}$,

$$P^e_{\alpha i}\left(\liminf_{n\to\infty} X_n = -\infty, \limsup_{n\to\infty} X_n = +\infty\right) = 1,\ e\ a.s.$$

This result shows that, under the hypothesis of irreducibility of the chain (G_n, X_n) - the assumption common to all previous instances in the literature quoted thus far in the present paper - the behavior of regime switching random walks in random environments mimics the (recurrence, transience-to-the-left, transience-to-the-right) trichotomy exhibited by random walks not subjected to regime switching.

In the next section we remove the hypothesis of irreducibility of the chain (G_n, X_n) and obtain a new, mutually exclusive breakdown of the asymptotic behavior, especially of interest in the more difficult third case of Theorem 2.2.

3 Other criteria for transience and recurrence

First, we express the relationship between hitting probabilities. These will be needed for computation purposes. Then it will be shown that γ_+ and γ_- are related to dimensions of some random spaces through the famous Oseledec's Theorem stated in Appendix A.

3.1 Recursive formulas for hitting probabilities

For any given $\ell \in \mathbb{Z}$ recall that $\tau_\ell = \inf\{n \geq 1; X_n = \ell\}$ and set $\tilde{\tau}_\ell = \inf\{n \geq 0; X_n = \ell\}$. For any choice of $\alpha, \beta \in \{1,\ldots,m\}$ set $\left(\tilde{f}^{(\beta)}_{i\ell}\right)_\alpha(e) = P^e_{\alpha i}(\tilde{\tau}_\ell < \infty, G_{\tilde{\tau}_\ell} = \beta)$ and similarly $\left(f^{(\beta)}_{i\ell}\right)_\alpha(e) = P^e_{\alpha i}(\tau_\ell < \infty, G_{\tau_\ell} = \beta)$. It is easy to check that $\left(\tilde{f}^{(\beta)}_{\ell\ell}\right)_\alpha = \delta_{\alpha\beta}$, for any $\alpha, \beta \in \{1,\ldots,m\}$. Also, when $i \neq \ell$, then $\left(f^{(\beta)}_{i\ell}\right)_\alpha = \left(\tilde{f}^{(\beta)}_{i\ell}\right)_\alpha$. It then follows that

$$\left(\tilde{f}^{(\beta)}_{i\ell}\right)_\alpha = \sum_{k=1}^m Q_{\alpha k}\left\{p^{(k)}_i\left(\tilde{f}^{(\beta)}_{i+1,\ell}\right)_k + q^{(k)}_i\left(\tilde{f}^{(\beta)}_{i-1,\ell}\right)_k\right\}. \tag{11}$$

First, let Δ_i be the random diagonal matrix with entries $\left\{p_i^{(1)}, \ldots, p_i^{(m)}\right\}$.

Then, using (11), $\tilde{f}_{\ell\ell}^{(\beta)} = e^{(\beta)}$, where $e_\alpha^{(\beta)} = \delta_{\alpha\beta}$, $\alpha \in \{1, \ldots, m\}$, and

$$\tilde{f}_{i\ell}^{(\beta)} = M_i \tilde{f}_{i+1,\ell}^{(\beta)} + N_i \tilde{f}_{i-1,\ell}^{(\beta)}, \quad i \neq \ell, \beta \in \{1, \ldots, m\}, \tag{12}$$

where

$$M_i = Q\Delta_i, \qquad N_i = Q(I - \Delta_i), \tag{13}$$

so that $M_i + N_i = Q$ for any $i \in \mathbb{Z}$. Note that for $i \neq \ell$, $P_{\alpha i}^e(\tau_\ell < \infty) = 1$ for all $\alpha \in \{1, \ldots, m\}$ if and only if $f_{i\ell}(e) = \tilde{f}_{i\ell}(e) = \sum_{\beta=1}^m \tilde{f}_{i\ell}^{(\beta)}(e) = \mathbf{1}$, with $\mathbf{1}_\alpha = 1$ for all $\alpha \in \{1, \ldots, m\}$. Further note that $\tilde{f}_{\ell\ell} = \mathbf{1}$.

3.2 First case: Q has rank 1

It then follows that for some positive vector π, $Q_{\alpha\beta} = \pi_\beta > 0$ for all $\alpha, \beta \in \{1, \ldots, m\}$. Note that in this case, the chain (G_n, X_n) is obviously irreducible. This corresponds to choosing the regimes independently, so X is a Markov chain with transition matrix $P = P(e)$ given by $P_{ij} = p_i$ if $j = i + 1$, $P_{ij} = q_i$ if $j = i - 1$, and $P_{ij} = 0$, whenever $|j - i| > 1$, where $p_i = \sum_{\beta=1}^m \pi_\beta p_i^{(\beta)}$, $q_i = 1 - p_i$, $i \in \mathbb{Z}$. Then it follows from (12) that $f_{i\ell}^{(\beta)} = g_{i\ell}^{(\beta)} \mathbf{1}$, for every $i \neq \ell$. Set $\sigma_i = \frac{q_i}{p_i}$, and $g_{i\ell} = \sum_{\beta=1}^m g_{i\ell}^{(\beta)}$, $i \in \mathbb{Z}$. Next, it is easy to check that for every $\beta \in \{1, \ldots, m\}$,

$$P_{\alpha i}^e \left(G_{\tau_\ell} = \beta | \tau_\ell < \infty\right) = \frac{g_{i\ell}^{(\beta)}(e)}{g_{i\ell}(e)} = \begin{cases} \pi_\beta \frac{q_{\ell+1}^{(\beta)}(e)}{q_{\ell+1}(e)} & \text{for } i > \ell, \\ \\ \pi_\beta \frac{p_{\ell-1}^{(\beta)}(e)}{p_{\ell-1}(e)} & \text{for } i < \ell. \end{cases} \tag{14}$$

Finally, since X is itself a random walk in a random environment, one can apply Theorem 2.1, to obtain the following corollary.

Corollary 3.1 *If $E(\log \sigma_0) > 0$, then for any $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$,*

$$P_{\alpha i}^e \left(\lim_{n\to\infty} X_n = -\infty\right) = 1, \ e \ a.s.$$

If $E(\log \sigma_0) < 0$, then for any $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$,

$$P_{\alpha i}^e \left(\lim_{n\to\infty} X_n = +\infty\right) = 1, \ e \ a.s.$$

If $E(\log \sigma_0) = 0$, then for any $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$,

$$P_{\alpha i}^e \left(\liminf_{n\to\infty} X_n = -\infty, \limsup_{n\to\infty} X_n = +\infty\right) = 1, \ e \ a.s.$$

Example 3.1 *For Game D, one finds that p is 3-periodic with values $\{.299, .624, .624\}$. Using Corollary 3.1, one obtains that $E(\log \sigma_0) = \frac{1}{3} \log \mu$, where μ given by formula (1). In this specific example, $\mu = 0.8512 < 1$, so that for any $(\alpha, i) \in \{1, 2\} \times \mathbb{Z}$, $P_{\alpha i}^e \left(\lim_{n\to\infty} X_n = +\infty\right) = 1$ a.s. More generally, the value of μ depends on π_1 which is the probability of choosing Game A. It then follows that*

$$\mu(\pi_1) = \left(\frac{1}{.099 + .4\pi_1} - 1\right) \left(\frac{1}{.749 - .25\pi_1} - 1\right)^2.$$

Figure 2 illustrates that $\mu(\pi_1) < 1$ quite often.

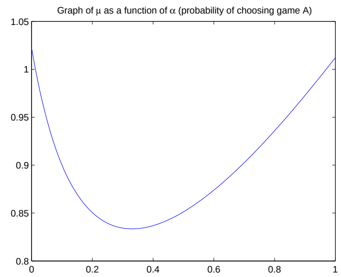

Figure 2: Graph of μ for Game D as a function of the probability π_1 of choosing at Game A.

3.3 Second case: Q has full rank

Since Q is invertible, it follows that M_i and N_i, as defined in (13), are invertible. For $i \in \mathbb{Z}$, set $A_i = \begin{pmatrix} M_i^{-1} & -\sigma_i \\ I & 0 \end{pmatrix}$, where $\sigma_i = M_i^{-1} N_i = \Delta_i^{-1} - I$. Since each A_i is invertible and the sequence of $2m \times 2m$ matrices A_i is stationary and ergodic, it follows from Oseledec's Theorem (Appendix A, Theorem A.1) that with probability 1, the random sets

$$\bar{V}_0 = \left\{ v \in \mathbb{R}^{2m} : \lim_{n\to\infty} n^{-1} \log \|A_n \cdots A_1 v\| \le 0 \right\}$$

and

$$\bar{V}_{0-} = \left\{ v \in \mathbb{R}^{2m} : \lim_{n\to\infty} n^{-1} \log \|A_n \cdots A_1 v\| < 0 \right\}$$

are subspaces with deterministic dimensions $\bar{d}_0$ and $\bar{d}_{0-}$ respectively, provided $\log^+ \|A_1\|$ is integrable. Note that for any $i \in \mathbb{Z}$, $A_i \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$. The norm $\|\|$ is arbitrary but fixed throughout since they are all equivalent.

Also, with probability 1, the random sets

$$\tilde{V}_0 = \left\{ v \in \mathbb{R}^{2m} : \lim_{n\to\infty} n^{-1} \log \left\| (A_n \cdots A_1)^{-1} v \right\| \le 0 \right\}$$

and

$$\tilde{V}_{0-} = \left\{ v \in \mathbb{R}^{2m} : \lim_{n\to\infty} n^{-1} \log \left\| (A_n \cdots A_1)^{-1} v \right\| < 0 \right\}$$

are subspaces with deterministic dimensions $\tilde{d}_0$ and $\tilde{d}_{0-}$ respectively, provided $\log^+ \left\|A_1^{-1}\right\|$ is integrable.

The next result is proven in Appendix B.

Theorem 3.1 *Suppose that Q is invertible and assume that $\log^+ \|A_1\|$ and $\log^+ \left\|A_1^{-1}\right\|$ are integrable. Then $\bar{d}_0 \ge m$, $\tilde{d}_0 \ge m$, $\bar{d}_0 > \bar{d}_{0-}$, and $\tilde{d}_0 > \tilde{d}_{0-}$. Furthermore,*

(1) If $\bar{d}_0 = m$, then $\gamma_+ = 0$ and $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s., for any $\alpha \in \{1, \ldots, m\}$ and any $i > \ell$.
(2) $\bar{d}_{0-} \ge m$ if and only if $\gamma_+ < 0$. In this case, there holds $\lim_{i\to\infty} P^e_{\alpha i}(\tau_\ell < \infty) = 0$ for any $\alpha \in \{1, \ldots, m\}$.
(3) If $\tilde{d}_0 = m$, then $\gamma_- = 0$, and $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s., for any $\alpha \in \{1, \ldots, m\}$ and any $i < \ell$.
(4) $\tilde{d}_{0-} \ge m$ if and only if $\gamma_- < 0$. In this case, there holds $\lim_{i\to-\infty} P^e_{\alpha i}(\tau_\ell < \infty) = 0$ for any $\alpha \in \{1, \ldots, m\}$.

Remark 3.1 *It follows from Theorem 2.2 that $\max(\gamma_+, \gamma_-) = 0$, so Theorem 3.1 implies that one cannot have at the same time $\bar{d}_{0-} \ge m$ and $\tilde{d}_{0-} \ge m$. Hence, if $\bar{d}_{0-} \ge m$, there holds $\gamma_+ < 0$ and $\gamma_- = 0$. Combining Theorem 3.1 with Proposition 2.3 first yields $\|f_{i\ell}(e)\| < 1$ for any $i > \ell$. Proposition 2.4 together with*

Proposition 2.3 yield that $\|f_{i\ell}(e)\| = 1$ *holds for any* $i < \ell$. *If in addition,* $\tilde{d}_0 = m$, *then*

$$P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = +\infty\right) = 1, \; e \text{ a.s.}$$

for every $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$, *using Lemma 2.2.*

On the other hand, if $\tilde{d}_{0-} \geq m$, *there holds* $\gamma_- < 0$, $\gamma_+ = 0$. *If in addition,* $\bar{d}_0 = m$, *then it follows in a similar fashion that*

$$P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = -\infty\right) = 1, \; e \text{ a.s.}$$

for every $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$.

Note that these two transient behavior occur without the assumption that (G_n, X_n) *is irreducible. Finally, if* $\bar{d}_{0-} < m$ *and* $\tilde{d}_{0-} < m$, *then* $\gamma_+ = \gamma_- = 0$; *nevertheless, one cannot show in general that a form of recurrence occurs, i.e.,*

$$P^e_{\alpha i}\left(\liminf_{n\to\infty} X_n = -\infty, \limsup_{n\to\infty} X_n = +\infty\right) = 1, \; e \text{ a.s.}$$

for every $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$. *In fact, a counterexample is given in Example 3.3, where* (G_n, X_n) *is not irreducible.*

Note that to obtain a "Parrondo's paradox" which is basically a transient phenomenon, we do not need the hypothesis of irreducibility. In fact, combining Theorem 2.1 and Remark 3.1, we end up with the following sufficient condition.

Corollary 3.2 *If for every* $\alpha \in \{1, \ldots, m\}$, *there holds* $E\left(\log \sigma_0^{(\alpha)}\right) > 0$, $\tilde{d}_0 = m$, *and* $\bar{d}_{0-} \geq m$, *then for any* $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$ *and any e a.s.,*

$$P^e_i\left(\lim_{n\to\infty} X^{(\alpha)}_n = -\infty\right) = 1 \quad \text{and} \quad P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = +\infty\right) = 1.$$

Note also that for an invertible measure preserving mapping T, it follows from (Ruelle, 1979, Theorem 3.1) that the Lyapunov exponents $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{2m}$ associated with the time reversed sequence $(A_n \cdots A_1)^{-1}$ in Oseledec's Theorem (Appendix A) are $-\bar{\lambda}_{2m}, \ldots, -\bar{\lambda}_1$, that is, minus those associated with $(A_n \cdots A_1)$. Hence, $\tilde{d}_{0-} = \dim \tilde{V}_{0-} = 2m - \bar{d}_0$ and $\tilde{d}_0 = \dim \tilde{V}_0 = 2m - \bar{d}_{0-}$. In particular, if $\bar{d}_0 = m$, then $\tilde{d}_{0-} = m$ and if $\bar{d}_{0-} = m$, then $\tilde{d}_0 = m$.

The proof of the following corollary follows directly from Theorem 3.1 and Remark 3.1.

Corollary 3.3 *Suppose that the measure preserving mapping* T *defined in Section 2 is invertible. If* $\bar{d}_0 = m$, *then for every* $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$, $P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = -\infty\right) = 1$, *e a.s. Similarly, if* $\tilde{d}_0 = m$, *then for any* $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$, $P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = +\infty\right) = 1$, *e a.s..*

Example 3.2 *[Periodic probabilities] Suppose that the processes* $p^{(\alpha)}$ *are periodic for all* $\alpha \in \{1, \ldots, m\}$, *and denote by* $\mathfrak{p}$ *the least common multiple of their respective periods. It then follows that the measure preserving mapping is invertible. In fact, if e is such that* $e^{(\alpha)}(i) = p^{(\alpha)}_i$, *then* $E = \{e, Te, \ldots, T^{\mathfrak{p}-1}e\}$, *where* $Te(i) = e(i+1)$, $i \in \mathbb{Z}$. *Further set* $\mathcal{A} = A_{\mathfrak{p}} \cdots A_1$. *Then* $\bar{d}_0$ *is the number of eigenvalues of* $\mathcal{A}$ *less than or equal to 1 in absolute value, while* $\tilde{d}_0$ *is the number of eigenvalues of* $\mathcal{A}$ *greater of equal to 1 in absolute value. Game C is an example of a Q with full rank and periodic probabilities with period* $\mathfrak{p} = 3$. *In this case* $m = 4$, $\bar{d}_0 = 6$, *and* $\tilde{d}_0 = 4$. *As a result, from Corollary 3.3,* $P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = +\infty\right) = 1$, *e a.s., for every* $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$. *This explains why Game C has a paradoxical behavior.*

Suppose now that one alternates between Games A and B, i.e., one plays Game C'. Then $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $m = 2$, $\bar{d}_0 = 2$, *and* $\tilde{d}_0 = 4$.

From Corollary 3.3, $P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = -\infty\right) = 1$, e a.s., for every $(\alpha, i) \in \{1,\ldots,m\}\times\mathbb{Z}$. Hence, in this case, the game does not have a paradoxical behavior.

Example 3.3 *Suppose that $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and consider the case of periodic probabilities of period $\mathfrak{p} = 2$. Then, starting from $(G_0, X_0) = (1, 0)$ and environment e, (X_n) is a random walk, and if $\rho_k(e) = \frac{P^e(X_{n+1}=k-1|X_n=k)}{P^e(X_{n+1}=k+1|X_n=k)}$, then $\rho_k(e) = \sigma_1^{(2)}(e) = \frac{q_1^{(2)}(e)}{p_1^{(2)}(e)}$ if k is odd, and $\rho_k(e) = \sigma_0^{(1)}(e) = \frac{q_0^{(1)}(e)}{p_0^{(1)}(e)}$ if k is even. Therefore, for any $k \in \mathbb{Z}$, $\rho_k(e)\rho_{k+1}(e) = \lambda_1(e) = \sigma_2^{(1)}(e)\sigma_1^{(2)}(e)$, and $\lambda_2(e) = \lambda_1(Te) = \sigma_1^{(1)}(e)\sigma_2^{(2)}(e)$. Note that the eigenvalues of $A_2(e)A_1(e)$ are $1, 1, \lambda_1(e), \lambda_2(e)$, which are the same as the eigenvalues of $A_2(Te)A_1(Te)$.*

As in Alili (1999), set

$$S(e) = \sum_{n=0}^{\infty} \rho_1(e)\cdots\rho_n(e), \quad F(e) = \sum_{n=0}^{\infty} \frac{1}{\rho_{-1}(e)}\cdots\frac{1}{\rho_{-n}(e)}.$$

Then S is finite iff $\lambda_1 < 1$ and F is finite iff $\lambda_1 > 1$.

As a numerical example, for any $i \in \mathbb{Z}$, define $p_i^{(1)} = 0.49$, $p_{2i}^{(2)} = 0.48$, and $p_{2i-1}^{(2)} = 1/1.95$. Here $E = \{e, Te\}$, where $e(i) = p_i^{(2)}$, so $Te(i) = p_{i+1}^{(2)}$. Then, if k is even, $\rho_k(e) = 1.0408$, while if k is odd, then $\rho_k(e) = 0.95$ and $\rho_k(Te) = 1.0833$. Hence, $\lambda_1(e) = \rho_0(e)\rho_1(e) = 0.9888 < 1$ and $\lambda_2(e) = \rho_0(Te)\rho_1(Te) = 1.1276 > 1$. As a result, $\bar{d}_0 = 3 = \tilde{d}_0$. Also, $S(e) < \infty$, $S(Te) = \infty$, $F(e) = \infty$, and $F(Te) < \infty$. Thus, starting from $P^e_{(1,0)}(X_n \to +\infty) = 1$ and $P^{Te}_{(1,0)}(X_n \to -\infty) = 1$.

On the other hand, starting from $(G_0, X_0) = (2, 0)$ and environment e, (X_n) is a random walk with $\rho_k(e) = \sigma_1^{(1)}(e) = \frac{q_1^{(1)}(e)}{p_1^{(1)}(e)}$ if k is odd, and $\rho_k(e) = \sigma_0^{(2)}(e) = \frac{q_0^{(2)}(e)}{p_0^{(2)}(e)}$ if k is even. As a result, if k is even, $\rho_k(e) = 1.0833$ and $\rho_k(Te) = 0.95$, while if k is odd, then $\rho_k(e) = 1.0408$. Hence $\rho_0(e)\rho_1(e) = 1.1276 > 1$ and $\rho_0(Te)\rho_1(Te) = 0.9888 < 1$. Therefore $S(e) = \infty$, $S(Te) < \infty$, $F(e) < \infty$, and $F(Te) = \infty$. Thus, $P^e_{(2,0)}(X_n \to -\infty) = 1$ and $P^{Te}_{(2,0)}(X_n \to +\infty) = 1$. Summarizing, for the same environment, the asymptotic behavior of the random walk depends on the starting point, and for the same starting point, the asymptotic behavior of the random walk depends on the environment. This shows that the case $\bar{d}_0 > m$ and $\tilde{d}_0 > m$ can lead to chaotic behavior.

3.4 General case: Q has rank r

Suppose that Q has rank $1 < r < m$. One can assume, without loss of generality that $Q = \begin{pmatrix} \pi \\ \Theta\pi \end{pmatrix}$, where $\pi \in \mathbb{R}^{r\times m}$ has rank r, and $\Theta \in \mathbb{R}^{(m-r)\times r}$, so that $\Theta\mathbf{1}_r = \mathbf{1}_{(m-r)}$. Further let $\Delta_i^{(1)}$ be the $r\times r$ diagonal matrix formed with the first r rows and columns of Δ_i, and let $\Delta_i^{(2)}$ stand for the diagonal matrix formed with the last $m - r$ rows and columns of Δ_i. Finally, let $\pi^{(1)}$ be the matrix composed for the first r columns of π and set $\pi^{(2)}$ for the remaining $m - r$ columns of π.

For $i \in \mathbb{Z}$, set $\check{M}_i = \pi^{(1)}\Delta_i^{(1)} + \pi^{(2)}\Delta_i^{(2)}\Theta$ and $\check{N}_i = \pi^{(1)}\left(I - \Delta_i^{(1)}\right) + \pi^{(2)}\left(I - \Delta_i^{(2)}\right)\Theta$.

Hypothesis 3.1 *With probability 1, $\check{M}_i$ and $\check{N}_i$ are invertible.*

Under this assumption, for any $i \in \mathbb{Z}$, define the matrices $\check{A}_i = \begin{pmatrix} \check{M}_i^{-1} & -\check{\sigma}_i \\ \mathbf{1}_r & 0 \end{pmatrix}$ and $\check{B}_i = \begin{pmatrix} \check{N}_i^{-1} & -\check{\sigma}_i^{-1} \\ I & 0 \end{pmatrix}$, where $\check{\sigma}_i = \check{M}_i^{-1}\check{N}_i$. Note that both $\check{M}_i$ and $\check{N}_i$ are stationary ergodic sequences since Θ is not random. As before, it follows from Oseledec's Theorem that with probability 1, the random sets

$$\check{V}_0 = \left\{v \in \mathbb{R}^{2r} : \lim_{n\to\infty} n^{-1}\log\|\check{A}_n\cdots\check{A}_1 v\| \le 0\right\}$$

and

$$\check{V}_{0-} = \left\{v \in \mathbb{R}^{2r} : \lim_{n\to\infty} n^{-1}\log\|\check{A}_n\cdots\check{A}_1 v\| < 0\right\}$$

are subspaces with deterministic dimensions $\check{d}_0$ and $\check{d}_{0-}$ respectively, provided $\log^+\|\check{A}_1\|$ is integrable. Also, with probability 1, the random sets

$$\overset{\circ}{V}_0 = \left\{ v \in \mathbb{R}^{2r} : \lim_{n\to\infty} n^{-1} \log \left\| \left(\check{A}_n \cdots \check{A}_1\right)^{-1} v \right| \le 0 \right\}$$

and

$$\overset{\circ}{V}_{0-} = \left\{ v \in \mathbb{R}^{2r} : \lim_{n\to\infty} n^{-1} \log \left\| \left(\check{A}_n \cdots \check{A}_1\right)^{-1} v \right\| < 0 \right\}$$

are subspaces with deterministic dimensions $\overset{\circ}{d}_0$ and $\overset{\circ}{d}_{0-}$ respectively, provided $\log^+\left\|\check{A}_1^{-1}\right\|$ is integrable.

The proof of the following result is given in Appendix B.

Theorem 3.2 *Suppose that Hypothesis 3.1 holds and that $\log^+\|\check{A}_1\|$ and $\log^+\left\|\check{A}_1^{-1}\right|$ are integrable. Then $\check{d}_0 \ge r$, $\overset{\circ}{d}_0 \ge r$, $\check{d}_0 > \check{d}_{0-}$, and $\overset{\circ}{d}_0 > \overset{\circ}{d}_{0-}$. Furthermore,*

(1) If $\check{d}_0 = r$, then $\gamma_+ = 0$ and $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s., for any $\alpha \in \{1,\ldots,m\}$ and any $i > \ell$.

(2) $\check{d}_{0-} \ge r$ if and only if $\gamma_+ < 0$. In this case, $\lim_{i\to\infty} P^e_{\alpha i}(\tau_\ell < \infty) = 0$ for any $\alpha \in \{1,\ldots,m\}$.

(3) If $\overset{\circ}{d}_0 = r$, then $\gamma_- = 0$, and $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s., for any $\alpha \in \{1,\ldots,m\}$ and any $i < \ell$.

(4) $\overset{\circ}{d}_{0-} \ge r$ if and only if $\gamma_- < 0$. In this case, $\lim_{i\to-\infty} P^e_{\alpha i}(\tau_\ell < \infty) = 0$ for any $\alpha \in \{1,\ldots,m\}$.

Remark 3.2 *Theorem 3.2 implies that one cannot have at the same time $\check{d}_{0-} \ge r$ and $\overset{\circ}{d}_{0-} \ge r$. Hence, if $\overset{\circ}{d}_0 = r$ and $\check{d}_{0-} \ge r$, $\gamma_- = 0$, $\gamma_+ < 0$, and it follows from Lemma 2.2 that*

$$P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = +\infty\right) = 1,\ e \text{ a.s.}$$

for every $(\alpha, i) \in \{1,\ldots,m\} \times \mathbb{Z}$. On the other hand, if $\check{d}_0 = r$ and $\overset{\circ}{d}_{0-} \ge r$, then $\gamma_+ = 0$, $\gamma_- < 0$, and

$$P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = -\infty\right) = 1,\ e \text{ a.s.}$$

for every $(\alpha, i) \in \{1,\ldots,m\} \times \mathbb{Z}$.

These results are summarized in the following corollary if the mapping T is invertible.

Corollary 3.4 *Suppose that the mapping $e \mapsto Te$ is invertible. If $\check{d}_0 = r$, then $P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = -\infty\right) = 1$, e a.s., for every $(\alpha, i) \in \{1,\ldots,m\} \times \mathbb{Z}$.*

Also, if $\check{d}_{0-} = r$, then $P^e_{\alpha i}\left(\lim_{n\to\infty} X_n = +\infty\right) = 1$, e a.s., for every $(\alpha, i) \in \{1,\ldots,m\} \times \mathbb{Z}$.

Using our methodology when $r = 1$, one recovers Theorem 2.1 due to Alili (1999).

Corollary 3.5 *Suppose $r = 1$ and set $u = E(\log \sigma_0)$. Then $u < 0$ if and only if $\check{d}_{0-} = 1$; $u > 0$ if and only if $\overset{\circ}{d}_{0-} = 1$; $u = 0$ if and only if $\check{d}_{0-} = \overset{\circ}{d}_{0-} = 0$.*

Proof. Note that is this case, $\check{A}_n \cdots \check{A}_1 = \begin{pmatrix} 1+U_n & -U_n \\ 1+U_{n-1} & -U_{n-1} \end{pmatrix}$, where $U_n = \sigma_1 + \sigma_1\sigma_2 + \ldots + \sigma_1 \cdots \sigma_n$.

Since $A_n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\check{A}_n \cdots \check{A}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} U_n \\ U_{n-1} \end{pmatrix}$, and $\left(\check{A}_n \cdots \check{A}_1\right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} U_n/s_n \\ (1+U_n)/s_n \end{pmatrix}$, where $s_n = \sigma_1 \cdots \sigma_n$. Next, it is easy to check that $\frac{1}{n}\log U_n \to \max(0,u)$ and $\frac{1}{n}\log\{(1+U_n)/s_n\} \to \max(0,-u)$, as $n \to \infty$. It then follows from Oseledec's Theorem (Theorem A.1 in Appendix A) that $u < 0$ if and only if $\check{d}_{0-} = 1$, $\check{d}_0 = 2$, $\overset{\circ}{d}_{0-} = 0$ and $\overset{\circ}{d}_0 = 1$. Just take $v = (U_\infty, 1+U_\infty)^\top$ to get $\check{\lambda}_1 = u < 0 = \lambda_2$. Also take $v = (0,1)^\top$ to obtain $\overset{\circ}{\lambda}_2 = -u > 0 = \overset{\circ}{\lambda}_1$. Similarly, $u > 0$ if and only if $\check{d}_{0-} = 0$, $\check{d}_0 = 1$, $\overset{\circ}{d}_{0-} = 1$ and $\overset{\circ}{d}_0 = 2$. Finally, $u = 0$ if and only if $\check{d}_{0-} = 0 = \overset{\circ}{d}_{0-}$, and $\check{d}_0 = 2 = \overset{\circ}{d}_0$, since in this case, taking $v = (0,1)^\top$, one gets $\check{\lambda}_1 = \check{\lambda}_2 = \overset{\circ}{\lambda}_1 = \overset{\circ}{\lambda}_2 = 0$. □

Example 3.4 *Suppose* $Q = \frac{1}{24}\begin{pmatrix} 8 & 8 & 8 \\ 6 & 6 & 12 \\ 7 & 7 & 10 \end{pmatrix}$. *Then* $\Theta = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

In this example, $\check{M}_i = \frac{1}{24}\begin{pmatrix} 8p_i^{(1)} + 4p_i^{(3)} & 8p_i^{(2)} + 4p_i^{(3)} \\ 6p_i^{(1)} + 6p_i^{(3)} & 6p_i^{(2)} + 6p_i^{(3)} \end{pmatrix}$, *and* $\check{N}_i = \frac{1}{24}\begin{pmatrix} 8q_i^{(1)} + 4q_i^{(3)} & 8q_i^{(2)} + 4q_i^{(3)} \\ 6q_i^{(1)} + 6q_i^{(3)} & 6q_i^{(2)} + 6q_i^{(3)} \end{pmatrix}$. *It follows that* $\det\left(\check{M}_i\right) = \frac{p_i^{(3)}}{24}\left(p_i^{(1)} - p_i^{(2)}\right)$. *Hence* $\check{M}_i$ *is invertible if and only if* $p_i^{(1)} \neq p_i^{(2)}$ *a.s.*

Remark 3.3 *What happens if Hypothesis 3.1 is not met? In this case, one proceeds almost as before, reducing the dimension of* $\tilde{M}_i$ *instead. For example, taking* $p_i^{(1)} = p_i^{(2)}$ *in Example 3.4, i.e., Games 1 and 2 are the same,* $\check{M}_i = \begin{pmatrix} \alpha_i & \alpha_i \\ \beta_i & \beta_i \end{pmatrix}$, $\check{N}_i = \begin{pmatrix} \frac{1}{2} - \alpha_i & \frac{1}{2} - \alpha_i \\ \frac{1}{2} - \beta_i & \frac{1}{2} - \beta_i \end{pmatrix}$, *where* $\alpha_i = \frac{1}{24}\left(8p_i^{(1)} + 4p_i^{(3)}\right)$, *and* $\beta_i = \frac{1}{24}\left(6p_i^{(1)} + 6p_i^{(3)}\right)$. *Hence,* $p_i = \alpha_i + \beta_i = \frac{1}{24}\left(14p_i^{(1)} + 10p_i^{(3)}\right) < 1$. *Thus Hypothesis 3.1 does not hold. However, setting* $\check{f}_{i\ell} = \begin{pmatrix} P_{1i}(\tilde{\tau}_\ell < \infty) \\ P_{2i}(\tilde{\tau}_\ell < \infty) \end{pmatrix}$, *one has* $\check{f}_{\ell\ell} = \mathbf{1}_2$ *and*

$$\check{f}_{i\ell} = \check{M}_i \check{f}_{i+1,\ell} + \check{N}_i \check{f}_{i-1,\ell}, \qquad i \neq \ell. \tag{15}$$

Also, $g_{i\ell} = P_{3i}(\tilde{\tau}_\ell < \infty) = \frac{1}{2}\mathbf{1}_2^\top \check{f}_{i\ell}$. *It then follows from (15) that*

$$g_{i\ell} = p_i g_{i+1,\ell} + (1 - p_i) g_{i-1,\ell}, \qquad i \neq \ell, \quad g_\ell = 1, \tag{16}$$

and $\check{f}_{i\ell} = \begin{pmatrix} 2\alpha_i g_{i+1,\ell} + (1 - 2\alpha_i) g_{i-1,\ell} \\ 2\beta_i g_{i+1,\ell} + (1 - 2\beta_i) g_{i-1,\ell} \end{pmatrix}$. *Hence we are back to the first case considered, i.e., the rank* 1 *case.*

Appendix A Oseledec's multiplicative ergodic theorem

The following statement of the celebrated Oseledec's Theorem is taken from Walters (1982).

Theorem A.1 *Let* $A_1, A_2, \ldots$ *be a stationary ergodic sequence of* $d \times d$ *matrices such that* $E\left\{\log^+(\|A_1\|)\right\} < \infty$. *Then there exists constants* $-\infty \le \lambda_d \le \lambda_{d-1} \le \cdots \le \lambda_1 < \infty$ *with the following properties:*

(a) With probability 1, the random sets

$$\mathcal{V}_q = \mathcal{V}_q(e) = \left\{ v \in \mathbb{R}^d : \lim_{n\to\infty} n^{-1} \log\left(\|A_n(e) \cdots A_1(e) v\| \le \lambda_q\right) \right\}$$

are subspaces. The map $e \mapsto \mathcal{V}_q(e)$ *is measurable and if* T *is the measure preserving map for which* $A_i(Te) = A_{i+1}(e)$, *then* $\mathcal{V}_q(Te) = A_1(e)\mathcal{V}_q(e)$.

(b) $\mathrm{Dim}(\mathcal{V}_q) = \mathrm{card}\{i : \lambda_i \le \lambda_q\}$.

(c) Set $\mathcal{V}_{d+1} = \{0\}$ *and let* $i_1 = 1 < i_2 \cdots < i_{p+1} = d + 1$ *be the unique indices at which* λ_i *jumps, i.e.,* $\lambda_1 = \lambda_2 = \cdots = \lambda_{i_2 - 1} > \lambda_{i_2} \cdots$. *Then for* $v \in \mathcal{V}_{i_{s-1}} \setminus \mathcal{V}_{i_s}$, *one has*

$$\lim_{n\to\infty} n^{-1} \log\left(\|A_n \cdots A_1 v\|\right) = \lambda_{i_{s-1}}, \quad 2 \le s \le p + 1.$$

(d) The sequence of matrices $\left(A_1^\top \cdots A_n^\top A_n \cdots A_1\right)^{1/(2n)}$ *converges almost surely to a limit matrix* B *with eigenvalues* $\mu_1 = e^{\lambda_1}, \ldots, \mu_d = e^{\lambda_d}$. *The orthogonal complement of* $\mathcal{V}_{i_s}$ *in* $\mathcal{V}_{i_{s-1}}$ *is the eigenspace of* B *corresponding to* $\mu_{i_{s-1}}$.

(e) If $\limsup_{n\to\infty} n^{-1} E\left\{\log\left(\|A_n \cdots A_1\|\right)\right\} > 0$ *and* $\det(A_1) = 1$ *with probability 1, then* $\lambda_d < 0 < \lambda_1$ *and* $\mathcal{V}_d$, *the subspace corresponding to* λ_d *is a proper nonempty subspace of* $\mathbb{R}^d$.

Appendix B Proofs of the main results

Proof of Lemma 2.1

Proof. Set $\mathcal{A}_\ell = \{X_n \le \ell \; i.o\}$. First,

$$\begin{aligned} P^e_{\alpha i}(\mathcal{A}_\ell) &= \sum_{\beta=1}^m P^e_{\alpha i}(\{\tau_\ell < \infty, G_{\tau_\ell} = \beta\} \cap \mathcal{A}_\ell) + P^e_{\alpha i}(\{\tau_\ell = \infty\} \cap \mathcal{A}_\ell) \\ &= \sum_{\beta=1}^m P^e_{\alpha i}(\tau_\ell < \infty, G_{\tau_\ell} = \beta) P^e_{\beta\ell}(\mathcal{A}_\ell) + P^e_{\alpha i}(\{\tau_\ell = \infty\} \cap \mathcal{A}_\ell). \end{aligned}$$

As a result, for $i < \ell$, one obtains

$$P^e_{\alpha i}(\mathcal{A}_\ell) = \sum_{\beta=1}^m P^e_{\alpha i}(\tau_\ell < \infty, G_{\tau_\ell} = \beta) P^e_{\beta\ell}(\mathcal{A}_\ell) + P^e_{\alpha i}(\tau_\ell = \infty), \tag{B1}$$

while if $i > \ell$, then

$$P^e_{\alpha i}(\mathcal{A}_\ell) = \sum_{\beta=1}^m P^e_{\alpha i}(\tau_\ell < \infty, G_{\tau_\ell} = \beta) P^e_{\beta\ell}(\mathcal{A}_\ell). \tag{B2}$$

First, suppose that $P^e_{\alpha i}(\mathcal{A}_\ell) = 0$ for any $(\alpha, i) \in \{1, \ldots, m\} \times \mathbb{Z}$. Then, according to (B1), $P^e_{\alpha i}(\tau_\ell < \infty) = 1$ for any $\alpha \in \{1, \ldots, m\}$ and any $i < \ell$. Next, if $i > \ell$, then $P^e_{\alpha i}(\mathcal{A}_\ell) = 0$ implies that $P^e_{\alpha i}(\tau_\ell < \infty) < 1$. Hence (2.1) holds true.

Suppose now that (2.1) holds true. Then combining (B1) and (B2), one obtains

$$P^e_{\alpha i}(\mathcal{A}_\ell) = \sum_{\beta=1}^m P^e_{\alpha i}(\tau_\ell < \infty, G_{\tau_\ell} = \beta) P^e_{\beta\ell}(\mathcal{A}_\ell), \quad \alpha \in \{1, \ldots, m\}, i \ne \ell. \tag{B3}$$

Therefore, to complete the proof, it suffices to show that $P^e_{\alpha\ell}(\mathcal{A}_\ell) = 0$ for any $\alpha \in \{1, \ldots, m\}$. To this end, note that using (B3), one gets

$$\begin{aligned} P^e_{\alpha\ell}(\mathcal{A}_\ell) &= \sum_{\beta=1}^m P^e_{\alpha\ell}(\{G_1 = \beta\} \cap \mathcal{A}_\ell) \\ &= \sum_{\beta=1}^m Q_{\alpha\beta} \left[p_\ell^{(\beta)} P^e_{\beta,\ell+1}(\mathcal{A}_\ell) + q_\ell^{(\beta)} P^e_{\beta,\ell-1}(\mathcal{A}_\ell) \right] \\ &= \sum_{\beta=1}^m \sum_{\gamma=1}^m Q_{\alpha\beta} p_\ell^{(\beta)} P^e_{\beta,\ell+1}(\tau_\ell < \infty, G_{\tau_\ell} = \gamma) P^e_{\gamma\ell}(\mathcal{A}_\ell) \\ &\qquad + \sum_{\beta=1}^m \sum_{\gamma=1}^m Q_{\alpha\beta} q_\ell^{(\beta)} P^e_{\beta,\ell-1}(\tau_\ell < \infty, G_{\tau_\ell} = \gamma) P^e_{\gamma\ell}(\mathcal{A}_\ell) \\ &= \sum_{\gamma=1}^m P^e_{\alpha\ell}(\tau_\ell < \infty, G_{\tau_\ell} = \gamma) P^e_{\gamma\ell}(\mathcal{A}_\ell), \end{aligned} \tag{B4}$$

since

$$\begin{aligned} P^e_{\alpha\ell}(\tau_\ell < \infty, G_{\tau_\ell} = \gamma) &= \sum_{\beta=1}^m Q_{\alpha\beta} p_\ell^{(\beta)} P^e_{\beta,\ell+1}(\tau_\ell < \infty, G_{\tau_\ell} = \gamma) \\ &\quad + \sum_{\beta=1}^m Q_{\alpha\beta} q_\ell^{(\beta)} P^e_{\beta,\ell-1}(\tau_\ell < \infty, G_{\tau_\ell} = \gamma). \end{aligned} \tag{B5}$$

By hypothesis, $P^e_{\alpha,\ell+1}(\tau_\ell < \infty) < 1$ for any $\alpha \in \{1, \ldots, m\}$, so by (B5) it follows that $f_{ll}^{e,*} = \max_{1 \le \alpha \le m} P^e_{\alpha\ell}$ $(\tau_\ell < \infty) < 1$. Further set $\mathcal{P}_\ell = \max_{1 \le \alpha \le m} P^e_{\alpha\ell}(\mathcal{A}_\ell)$.

Then, from (B4), $\mathcal{P}_\ell = \max_{1 \le \alpha \le m} P^e_{\alpha\ell}(\mathcal{A}_\ell) \le f_{ll}^{e,*} \mathcal{P}_\ell$, so $\mathcal{P}_\ell \left(1 - f_{ll}^{e,*}\right) \le 0$. Since $f_{ll}^{e,*} < 1$, it follows that $\mathcal{P}_\ell = 0$. Hence the result. □

Proof of Proposition 2.1

Proof. Let e be given. First, (2.2) is obviously true for $\ell = 1$. If (2.2) is not true for some $\ell > 1$, then for some $\alpha \in \{1, \ldots, m\}$, and some $|i| < \ell$,

$$P^e_{\alpha i}(|X_n| < \ell \text{ for all } n \geq 1) > 0.$$

Next, for $|i| < \ell$ and $\alpha \in \{1, \ldots, m\}$, set $g_{\alpha i}(e) = P^e_{\alpha i}(|X_n| < \ell \text{ for all } n \geq 1)$. It then follows that for any $\alpha \in \{1, \ldots, m\}$, and $|i| < \ell - 1$,

$$g_{\alpha i} = \sum_{\beta=1}^{m} Q_{\alpha\beta} \left(p_i^{(\beta)} g_{\beta,i+1} + q_i^{(\beta)} g_{\beta,i-1} \right),$$

while $g_{\alpha,\ell-1} = \sum_{\beta=1}^m Q_{\alpha\beta} q_{\ell-1}^{(\beta)} g_{\beta\ell-2}$, and $g_{\alpha,-\ell+1} = \sum_{\beta=1}^m Q_{\alpha\beta} p_{-\ell+1}^{(\beta)} g_{\beta,-\ell+2}$. It follows that for every e, there is a sub-stochastic matrix $\tilde{P}_{\alpha i,\beta j}(e)$ on $S = \{1, \ldots, m\} \times \{-\ell+1, \ldots, \ell-1\}$, so that $g_{\alpha i} = \sum_{(\beta,j)\in S} \tilde{P}_{\alpha i,\beta j} g_{\beta j}$, for any $(\alpha, i) \in S$. Note that $\left(\tilde{P}\mathbf{1}\right)_{\alpha i} = 1$ if $|i| < \ell - 1$, while $\left(\tilde{P}\mathbf{1}\right)_{\alpha,\ell-1} = \sum_{\beta=1}^m Q_{\alpha\beta} q_{\ell-1}^{(\beta)} < 1$, and $\left(\tilde{P}\mathbf{1}\right)_{\alpha,-\ell+1} = \sum_{\beta=1}^m Q_{\alpha\beta} p_{\ell+1}^{(\beta)} < 1$. Next, $\left(\tilde{P}^2\mathbf{1}\right)_{\alpha i} < 1$ if $i \in \{-\ell+1, -\ell+2, \ell-2, \ell-1\}$, and by induction, $\left(\tilde{P}^k\mathbf{1}\right)_{\alpha i} < 1$ if $i \in \{-\ell+1, \ldots, \ell+k, \ell-k, \ldots, \ell-1\}$. Therefore, for any e, there exists $c = c(e) \in (0, 1)$ so that $\tilde{P}^\ell \mathbf{1} \leq c\mathbf{1}$. As a result, using (Billingsley, 1995, Theorem 8.4), one obtains that $g = \lim_{n\to\infty} \tilde{P}^n \mathbf{1} = 0$, contradicting the hypothesis that $g_{\alpha i} > 0$ for some $(\alpha, i) \in S$. Hence the result. □

Proof of Proposition 2.3

Proof. One only proves the proposition for $i < \ell$, the proof of the case $i > \ell$ being similar. If $\|f_{i\ell}(e)\| = 1$ for $i = \ell - 1$, then for some $\alpha \in \{1, \ldots, m\}$, $P^e_{\alpha,\ell-1}(\tau_\ell < \infty) = 1$. Hence, for any β so that $Q_{\alpha\beta} > 0$, one has $P^e_{\beta,\ell-2}(\tau_\ell < \infty) = 1$, according to Proposition 2.2. Therefore $\|f_{\ell-2,\ell}(e)\| = 1$. Next, if $\|f_{i\ell}(e)\| = 1$ for some $i < \ell - 1$, then Proposition 2.2 implies that $\|f_{i\pm1,\ell}(e)\| = 1$. This proves the first part of the proposition. Suppose now that $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s. Since G_n is irreducible, for a given β, one can find $n \geq 1$ so that $Q^n_{\alpha\beta} > 0$. As a result, it follows from Proposition 2.2 that $P^e_{\beta,i-n}(\tau_\ell < \infty) = 1$ a.s. Hence, using stationarity, $P^e_{\beta,i}(\tau_{\ell+n} < \infty) = 1$, e a.s., entailing that $P^e_{\beta,i}(\tau_\ell < \infty) = 1$, e a.s. □

Proof of Proposition 2.4

Proof. For any $i < j < \ell$,

$$(f_{i\ell}(e))_\alpha = P^e_{\alpha i}(\tau_\ell < \infty) = \sum_{\beta=1}^m P^e_{\alpha i}(\tau_j < \infty, G_{\tau_j} = \beta) P^e_{\beta j}(\tau_\ell < \infty).$$

As a result, for any e, $\|f_{i\ell}(e)\| \leq \|f_{ij}(e)\| \|f_{j\ell}(e)\|$, showing that the logarithm of the sequence is subadditive. Then (2.3) follows easily. Next, $\|f_{ij}(e)\| = \|f_{0,j-i}(T^i e)\|$. This proves that $\|f_{\ell-k,\ell-k+1}\|$, $k \geq 1$, is a stationary ergodic sequence, so by using the Subadditive Ergodic Theorem (Durrett, 2010), there exists a constant γ_- so that for almost every $e \in E$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|f_{\ell-n,\ell}(e)\| = \gamma_- \leq E \{\log \|f_{0,1}\|\},$$

proving the first part. Next, if $i > j > \ell$, then

$$(f_{i\ell}(e))_\alpha = P^e_{\alpha i}(\tau_\ell < \infty) = \sum_{\beta=1}^m P^e_{\alpha i}(\tau_j < \infty, G_{\tau_j} = \beta) P^e_{\beta j}(\tau_\ell < \infty),$$

so again $\|f_{i\ell}\| \leq \|f_{ij}\| \|f_{j\ell}\|$. The rest of the proof follows along the same lines as the previous case $i < j < \ell$. □

Proof of Lemma 2.2

Proof. Set $\mathcal{A}_\ell = \{X_n \le \ell \ i.o\}$. First, $P^e_{\alpha i}(\lim_{n\to\infty} X_n = +\infty) = 1$ for every $(\alpha, i) \in \{1,\ldots,m\} \times \mathbb{Z}$ if and only if for every $(\alpha, i, \ell) \in \{1,\ldots,m\} \times \mathbb{Z}^2$, $P^e_{\alpha i}(\mathcal{A}_\ell) = 0$. By Lemma 2.1, this is equivalent to (2.5) and (2.6). This proves (i). Next, (ii) follows from (i) applied to $-X_n$. To prove (iii), note that $P^e_{\alpha i}(\liminf_{n\to\infty} X_n = -\infty) = 1$ for every $(\alpha, i) \in \{1,\ldots,m\} \times \mathbb{Z}$ if and only if for every $(\alpha, i, \ell) \in \{1,\ldots,m\} \times \mathbb{Z}^2$, $P^e_{\alpha i}(\mathcal{A}_\ell) = 1$. The latter implies that $P^e_{\alpha i}(\tau_\ell < \infty) = 1$ whenever $i > \ell$. Using this result with $-X_n$, one obtains that $P^e_{\alpha i}(\limsup_{n\to\infty} X_n = +\infty) = 1$ for every $(\alpha, i) \in \{1,\ldots,m\} \times \mathbb{Z}$ implies that for every $(\alpha, i, \ell) \in \{1,\ldots,m\} \times \mathbb{Z}^2$, $P^e_{\alpha i}(\tau_\ell < \infty) = 1$ whenever $i < \ell$. Then (B5) yields that $P^e_{\alpha \ell}(\tau_\ell < \infty) = 1$. (Irreducibility has not been used yet.) To complete the proof, suppose now that (2.9) holds true. Proposition 2.2 with $i = \ell$ combined with Proposition 2.3 first implies that $\|f_{i\ell}(e)\| = 1$ for any i and ℓ. With the additional condition that the Markov chain G_n is irreducible, Proposition 2.3 further implies that $P^e_{\alpha i}(\tau_\ell < \infty) = 1$ for any $(\alpha, i, \ell) \in \{1,\ldots,m\} \times \mathbb{Z}^2$. By using (B3), it suffices to show that $P^e_{\alpha\ell}(\mathcal{A}_\ell) = 1$ for every $\alpha \in \{1,\ldots,m\}$. Now, by hypothesis, $\mathcal{U}_\ell$, defined by $(\mathcal{U}_\ell)_{\alpha\beta} = \left(f^{(\beta)}_{\ell\ell}\right)_\alpha$, $\alpha, \beta \in \{1,\ldots,m\}$, is a stochastic matrix, so $P^e_{\alpha\ell}(\mathcal{A}_\ell) \ge 1$. ☐

Proof of Theorem 2.2

Proof. One first proves that $\max(\gamma_+, \gamma_-) = 0$. In fact, it follows from Proposition 2.4 that $\max(\gamma_+, \gamma_-) \le 0$. One now shows by contradiction that it is impossible to have $\gamma_+ < 0$ and $\gamma_- < 0$. So suppose that $\max(\gamma_+, \gamma_-) < 0$. Since (2.2) holds by Proposition 2.1, it follows that

$$\|f_{0,\ell}(e)\| + \|f_{0,-\ell}(e)\| \ge P^e_{\alpha 0}(\tau_\ell < \infty) + P^e_{\alpha 0}(\tau_{-\ell} < \infty) \ge 1,$$

which is impossible since $\gamma_+ < 0$ and $\gamma_- < 0$. Hence, $\max(\gamma_+, \gamma_-) = 0$. It follows from Propositions 2.3–2.4 that if $\gamma_+ < 0$, then for any $\alpha \in \{1,\ldots,m\}$ and any $i > \ell$, $P^e_{\alpha i}(\tau_\ell < \infty) < 1$, e a.s. Similarly, if $\gamma_- < 0$, then for any $\alpha \in \{1,\ldots,m\}$ and any $i < \ell$, $P^e_{\alpha i}(\tau_\ell < \infty) < 1$, e a.s. Next, if $\gamma_- = 0$, then $\|f_{01}\| = 1$ a.s. by Proposition 2.4. It then follows from the irreducibility of (G_n, X_n) that $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s., for any $\alpha \in \{1,\ldots,m\}$ and any $i < \ell$. In fact, if for a given e, $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, then it follows that $P^e_{\beta i}(\tau_\ell < \infty) = 1$, for all $\beta \in \{1,\ldots,m\}$. Similarly, if $\gamma_+ = 0$, then $\|f_{10}\| = 1$ a.s. by Proposition 2.4, so the irreducibility of (G_n, X_n) entails that $P^e_{\alpha i}(\tau_\ell < \infty) = 1$, e a.s., for any $\alpha \in \{1,\ldots,m\}$ and any $i > \ell$. As a result, using Lemma 2.2, one gets (1), (2), and (3). ☐

Proof of Theorem 3.1

Proof. By Oseledec's Theorem (Theorem A.1), there exists constants $-\infty \le \bar\lambda_{2m} \le \bar\lambda_{2m-1} \le \cdots \le \bar\lambda_1 < \infty$ with the following properties:

(a) With probability 1, the random sets

$$\bar V_q = \bar V_q(e) = \left\{ v \in \mathbb{R}^{2m} : \lim_{n\to\infty} n^{-1} \log\left(\|A_{n-1}(e)\cdots A_0(e) v\| \le \bar\lambda_q\right) \right\}$$

are linear subspaces. The map $e \mapsto \bar V_q(e)$ is measurable and $\bar V_q(Te) = A_0(e)\bar V_q(e)$.

(b) $\mathrm{Dim}(\bar V_q) = \mathrm{card}\{i : \bar\lambda_i \le \bar\lambda_q\}$.

(c) Set $\bar V_{2m+1} = \{0\}$ and let $i_1 = 1 < i_2 \cdots < i_{p+1} = 2m+1$ be the unique indices at which $\bar\lambda_i$ jumps, i.e., $\bar\lambda_1 = \bar\lambda_2 = \cdots = \bar\lambda_{i_2-1} > \bar\lambda_{i_2} \cdots$. Then for $v \in \bar V_{i_{s-1}} \setminus \bar V_{i_s}$, one has $\lim_{n\to\infty} n^{-1}\log\left(\|A_{n-1}\cdots A_0 v\|\right) = \bar\lambda_{i_{s-1}}, \quad 2 \le s \le p+1$.

Since each A_i is invertible, it then follows from Oseledec's Theorem that for any $\ell \in \mathbb{Z}$, the product $A_{n-1}\cdots A_0$ can be replaced with the product $A_{\ell+n}\cdots A_{\ell+1}$ and one still gets the same constants $\bar\lambda_j$ and the dimensions of the associated subspaces are exactly the same.

Let ℓ be given. For $i > \ell$, set $\bar{v}_i^{(\beta)} = \begin{bmatrix} \tilde{f}_{i\ell}^{(\beta)} \\ \tilde{f}_{i-1,\ell}^{(\beta)} \end{bmatrix}$, $i > \ell$, $\beta \in \{1, \ldots, m\}$. Using (3.2) , one can write

$$\bar{v}_{i+1}^{(\beta)} = A_i \bar{v}_i^{(\beta)} = A_i \cdots A_{\ell+1} \bar{v}_{\ell+1}^{(\beta)}, \quad i > \ell. \tag{B6}$$

Since $\tilde{f}_{\ell\ell}^{(\beta)} = e^{(\beta)}$ for every $\beta \in \{1, \ldots, m\}$, it follows that the vectors $\bar{v}_{\ell+1}^{(\beta)}$ are linearly independent, and they belong to $\bar{V}_0$. As a result, $\bar{d}_0 = \dim \bar{V}_0 \geq m$. Also, $A_i \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$, so $\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \in \bar{V}_0$. Therefore $\bar{d}_0 > \bar{d}_{0-}$.

Proof of (1)

Suppose $\bar{d}_0 = m$. Then for almost every environment, there is a random vector $\theta \in \mathbb{R}^m$ such that $\sum_{\beta=1}^m \theta_\beta v_{\ell+1}^{(\beta)} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$. In particular, $\sum_{\beta=1}^m \theta_\beta \tilde{f}_{\ell+1,\ell}^{(\beta)} = \mathbf{1}$ a.s. and $\sum_{\beta=1}^m \theta_\beta \tilde{f}_{\ell,\ell}^{(\beta)} = \mathbf{1}$ a.s., which implies $\sum_{\beta=1}^m \theta_\beta \tilde{f}_{\ell,\ell}^{(\beta)} = \sum_{\beta=1}^m \theta_\beta e^{(\beta)} = \theta = \mathbf{1}$. It then follows that $\sum_{\beta=1}^m \bar{v}_{\ell+1}^{(\beta)} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$, and one can deduce from (B6) that for any $i > \ell$, $\sum_{\beta=1}^m \bar{v}_i^{(\beta)} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$. Hence, for any $i > \ell$, one has $\tilde{f}_{il}(e) = f_{i\ell}(e) = \mathbf{1}$, and consequently, for any $\alpha \in \{1, \ldots, m\}$,

$$\left(\tilde{f}_{il}(e)\right)_\alpha = P_{\alpha i}^e(\tau_\ell < \infty) = 1 \qquad e \text{ a.s..}$$

As a result, $\gamma_+ = 0$. This completes the proof of (1).

Proof of (2)

The proof proceeds somewhat along the lines of Key (1984). First note that for any (possibly random) element $h_\ell \in \bar{V}_{0-}$ with V_{0-} as in Section 3, we can define a sequence $h_i = A_i h_{i-1}$ for $i > \ell$ with the property that h_i has the form $h_i = \begin{bmatrix} z_{i+1} \\ z_i \end{bmatrix}$ for some (unique, possibly random) sequence $z_\ell, z_{\ell+1}, \ldots \in \mathbb{R}^m$, because of the structure of A_i defined in Section 3. Set $g(\alpha, i) = (z_i)_\alpha$, $\alpha \in \{1, \ldots, m\}$, $i \geq \ell$ and notice that, because of $h_\ell \in \bar{V}_{0-}$, $\lim_{n\to\infty} g(\alpha, n) = 0$ ensues.

Let (G_n, Y_n) be the Markov chain starting at (α, i) associated with (G_n, X_n) but absorbed on $\{1, \ldots, m\} \times \{\ell\}$, and set $\mathcal{M}_n = g(G_n, Y_n)$. Since $\lim_{n\to\infty} g(\alpha, n) = 0$, $\mathcal{M}$ is a bounded martingale with respect to its natural filtration (provided the initial sigma field is enlarged for z_i to be measurable with respect to it) started at $\mathcal{M}_0 = g(\alpha, i)$. Because of its random walk nature, either (G_n, Y_n) is absorbed on the boundary or $\limsup_{n\to\infty} Y_n = +\infty$. As a result, by the martingale convergence theorem, it follows that $\mathcal{M}_n \to \mathcal{M}_\infty$, so $g(\alpha, i) = E(\mathcal{M}_0) = E(\mathcal{M}_\infty)$ for any $i > \ell$ and any $\alpha \in \{1, \ldots, m\}$.

Suppose first that $\bar{d}_{0-} \geq m$. Let $\bar{v}_{-1}, \ldots, \bar{v}_{-m}$ be a set of (possibly random) linearly independent vectors in $\bar{V}_{0-}$. The last m components of $\bar{v}_{-1}, \ldots, \bar{v}_{-m}$ are linearly independent. If not, there exists a member $h_\ell \neq 0$ of $\bar{V}_{0-}$ so that its last m components are zero. In this case, it follows that $\mathcal{M}_\infty = 0$, so $g(\alpha, i) = E(\mathcal{M}_0) = E(\mathcal{M}_\infty) = 0$. Since the latter is true for any $i > \ell$ and any $\alpha \in \{1, \ldots, m\}$, one may conclude that $h_\ell \equiv 0$, contradicting the assumption. Thus, the last m components of $\bar{v}_{-1}, \ldots, \bar{v}_{-m}$ are linearly independent. Because of this, there exists $h_\ell \in \bar{V}_{0-}$ such that its last m components are 1. We set $z_\ell = \mathbf{1}$ henceforth. Recall that for any $\alpha \in \{1, \ldots, m\}$, $\left(\tilde{f}_{i\ell}(e)\right)_\alpha = P_{\alpha i}^e(\tilde{\tau}_\ell < \infty)$ and define $w_i = \begin{bmatrix} \tilde{f}_{i+1,\ell} \\ \tilde{f}_{i\ell} \end{bmatrix}$, $i \geq \ell$. By (3.2) , we get $w_i = A_i w_{i-1}$, $i > \ell$. Note also that $\tilde{f}_{\ell\ell} = \mathbf{1}$ by definition. Now set $\tilde{\mathcal{M}}_n = \left(\tilde{f}_{Y_n,\ell}\right)_{G_n}$; $\tilde{\mathcal{M}}$ also forms a bounded martingale with $\tilde{\mathcal{M}}_0 = \left(\tilde{f}_{i\ell}(e)\right)_\alpha$, and

$$\begin{aligned}
\left(\tilde{f}_{i\ell}\right)_\alpha &= E(\tilde{\mathcal{M}}_0) = E(\tilde{\mathcal{M}}_\infty) = E\{\tilde{\mathcal{M}}_\infty \mathbb{I}(\tilde{\tau}_\ell < \infty)\} + E\{\tilde{\mathcal{M}}_\infty \mathbb{I}(\tilde{\tau}_\ell = \infty)\} \\
&= \left(\tilde{f}_{i\ell}\right)_\alpha + E\{\tilde{\mathcal{M}}_\infty \mathbb{I}(\tilde{\tau}_\ell = \infty)\},
\end{aligned}$$

since $\tilde{\mathcal{M}}_\infty = 1$ on $\{\tilde\tau_\ell < \infty\}$. As a result, $E\{\tilde{\mathcal{M}}_\infty \mathbb{I}(\tilde\tau_\ell = \infty)\} = 0$. The bounded martingale $\tilde{\mathcal{D}}_n = \tilde{\mathcal{M}}_n - \mathcal{M}_n$ satisfies

$$\begin{aligned}\left(\tilde f_{i\ell} - z_i\right)_\alpha &= E(\tilde{\mathcal{D}}_0) = E(\tilde{\mathcal{D}}_\infty) = E\{\tilde{\mathcal{D}}_\infty \mathbb{I}(\tilde\tau_\ell = \infty)\} \\ &= E\{\tilde{\mathcal{M}}_\infty \mathbb{I}(\tilde\tau_\ell = \infty)\} - 0 = 0,\end{aligned}$$

since $\tilde f_{\ell\ell} - z_\ell = 0$ and $\limsup_{n\to\infty} Y_n = +\infty$ on $\{\tilde\tau_\ell = \infty\}$ implies $\lim_{n\to\infty} g(G_n, Y_n)\mathbb{I}(\tilde\tau_\ell = \infty) = 0$. Hence $\tilde f_{i\ell} = z_i$ for any $i \ge \ell$. It then follows that $\gamma_+ < 0$, and so $\lim_{i\to\infty} P^e_{\alpha i}(\tau_\ell < \infty) = 0$, e a.s., for any $\alpha \in \{1,\ldots,m\}$. To complete the proof of (2), suppose now that $\gamma_+ < 0$. Combined with equation (B6) this assumption implies $\bar v_i^{(\beta)} = \begin{bmatrix} \tilde f_{i\ell}^{(\beta)} \\ \tilde f_{i-1,\ell}^{(\beta)} \end{bmatrix} \in \bar V_{0-}$ for every $i > \ell$ and $\beta \in \{1,\ldots,m\}$. Since the m vectors $\bar v_{\ell+1}^{(\beta)}$ are linearly independent, we conclude $\bar d_{0-} \ge m$.

Proofs of (3) and (4)

They are similar to those of (1) and (2). In fact, setting $\tilde v_i^{(\beta)} = \begin{bmatrix} \tilde f_{i-1}^{(\beta)} \\ \tilde f_i^{(\beta)} \end{bmatrix}$, $i < \ell$, $\beta \in \{1,\ldots,m\}$, and using (3.2), one can write

$$\tilde v_{i-1}^{(\beta)} = B_i \tilde v_i^{(\beta)}, \quad i < \ell, \tag{B7}$$

with $B_i = \begin{pmatrix} N_i^{-1} & -\sigma_i^{-1} \\ I & 0 \end{pmatrix}$, M_i, N_i as in (3.3) and $\sigma_i = M_i^{-1} N_i$. We have $B_i = G A_i^{-1} G$, where $G = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and $A_i^{-1} = \begin{pmatrix} 0 & I \\ -\sigma_i^{-1} & N_i^{-1} \end{pmatrix}$. As a result, $B_{\ell-n}\cdots B_{\ell-1} = G(A_{\ell-1}\cdots A_{\ell-n})^{-1} G$. □

Proof of Theorem 3.2

Recall that from (3.2) , $\tilde f_{i\ell}^{(\beta)} = Q\Delta_i \tilde f_{i+1,\ell}^{(\beta)} + Q(I-\Delta_i)\tilde f_{i-1,\ell}^{(\beta)}$, $i \ne \ell, \beta \in \{1,\ldots,m\}$, with $\tilde f_{\ell\ell}^{(\beta)} = e^{(\beta)}$. Also, since $\tilde f_{i\ell} = \sum_{\beta=1}^m \tilde f_{i\ell}^{(\beta)}$, it follows that $\tilde f_{i\ell} = Q\Delta_i \tilde f_{i+1,\ell} + Q(I-\Delta_i)\tilde f_{i-1,\ell}$, $i \ne \ell$, with $\tilde f_{\ell\ell} = \mathbf{1}$.

Let $\tilde f_{i\ell}^{(1,\beta)}$ and $\tilde f_{i\ell}^{(2,\beta)}$ represent respectively the first r components and the last $m-r$ components of $\tilde f_{i\ell}^{(\beta)}$. For simplicity, set $\check f_{i\ell}^{(\beta)} = \tilde f_{i\ell}^{(1,\beta)}$. It then follows from the representation of Q and (3.2) that for any $i \ne \ell$ and any $\beta \in \{1,\ldots,m\}$,

$$\begin{aligned}\check f_{i\ell}^{(\beta)} &= \pi^{(1)}\Delta_i^{(1)} \check f_{i+1,\ell}^{(\beta)} + \pi^{(2)}\Delta_i^{(2)} \tilde f_{i+1,\ell}^{(2,\beta)} \\ &\quad + \pi^{(1)}\left(I - \Delta_i^{(1)}\right)\check f_{i-1,\ell}^{(\beta)} + \pi^{(2)}\left(I - \Delta_i^{(2)}\right)\tilde f_{i-1,\ell}^{(2,\beta)},\end{aligned} \tag{B8}$$

$$\tilde f_{i\ell}^{(2,\beta)} = \Theta \check f_{i\ell}^{(\beta)}. \tag{B9}$$

As a result, one can write, for any $i \ne \ell$ and any $\beta \in \{1,\ldots,m\}$,

$$\tilde f_{i\ell}^{(\beta)} = \begin{bmatrix} \check f_{i\ell}^{(\beta)} \\ \Theta \check f_{i\ell}^{(\beta)} \end{bmatrix}. \tag{B10}$$

Furthermore, since $\tilde f_{\ell\ell} = \mathbf{1}$ and $\Theta \mathbf{1}_r = \mathbf{1}_{m-r}$, it follows from (B8)–(B10) that for any $i \in \mathbb{Z}$,

$$\tilde f_{i\ell} = \begin{bmatrix} \check f_{i\ell} \\ \Theta \check f_{i\ell} \end{bmatrix}. \tag{B11}$$

Next, using (B8)–(B11), one can write, for any $|i-\ell| > 1$ and any $\beta \in \{1,\ldots,m\}$,

$$\check f_{i\ell}^{(\beta)} = \check M_i \check f_{i+1,\ell}^{(\beta)} + \check N_i \check f_{i-1,\ell}^{(\beta)}, \tag{B12}$$

while for any $i \neq \ell$,

$$\check{f}_{i\ell} = \check{M}_i \check{f}_{i+1,\ell} + \check{N}_i \check{f}_{i-1,\ell}. \tag{B13}$$

Also, for any $\beta \in \{1, \ldots, m\}$,

$$\check{f}^{(\beta)}_{\ell+1,\ell} = \check{M}_{\ell+1}\check{f}^{(\beta)}_{\ell+2,\ell} + \pi(I-\Delta_{\ell+1})e^{(\beta)}, \tag{B14}$$

$$\check{f}^{(\beta)}_{\ell-1,\ell} = \pi\Delta_{\ell-1}e^{(\beta)} + \check{N}_{\ell-1}\check{f}^{(\beta)}_{\ell-2,\ell}. \tag{B15}$$

Now, set $h^{(\beta)}_{\ell\ell} = \begin{bmatrix} e^{(\beta)}_r \\ \Theta e^{(\beta)}_r \end{bmatrix}$, where $e^{(\beta)}_r$ is the vector composed of the first r components of $e^{(\beta)}$. Next, taking linear combinations of the functions $f^{(\beta)}_{i\ell}$, $\beta \in \{1, \ldots, m\}$, one can define, for any $i \neq \ell$ and any $\beta \in \{1, \ldots, r\}$,

$$\check{h}^{(\beta)}_{i,\ell} = \check{M}_i \check{h}^{(\beta)}_{i+1,\ell} + \check{N}_i \check{h}^{(\beta)}_{i-1,\ell}, \tag{B16}$$

where $\check{h}^{(\beta)}_{i\ell}$ is the first r components of $h^{(\beta)}_{i\ell} = \begin{bmatrix} \check{h}^{(\beta)}_{i\ell} \\ \Theta \check{h}^{(\beta)}_{i\ell} \end{bmatrix}$. Also $\sum_{\beta=1}^r \check{h}^{(\beta)}_{\ell\ell} = \mathbf{1}_r$, and $\sum_{\beta=1}^r \check{h}^{(\beta)}_{i\ell} = \check{f}_{i\ell}$, for any $i \in \mathbb{Z}$. Under Hypothesis 3.1, one can then apply Oseledec's theorem. Define $\check{d}_{0-}$ as the dimension of the set $\check{V}_{0-}$ of vectors $v \in \mathbb{R}^{2r}$ so that $\lim_{n\to\infty} \frac{1}{n}\log\left\|\check{A}_n \cdots \check{A}_\ell\right\| < 0$. Next, let ℓ be given. For $i > \ell$, set $\check{v}^{(\beta)}_i = \begin{bmatrix} \check{h}^{(\beta)}_{i\ell} \\ \check{h}^{(\beta)}_{i-1,\ell} \end{bmatrix}$, $i > \ell$, $\beta \in \{1, \ldots, r\}$. Using (B12)–(B14), one can write

$$\check{v}^{(\beta)}_{i+1} = \check{A}_i \check{v}^{(\beta)}_i = \check{A}_i \cdots \check{A}_{\ell+1}\check{v}^{(\beta)}_{\ell+1}, \quad i > \ell. \tag{B17}$$

Since the vectors $\check{h}^{(\beta)}_{\ell\ell}$ are linearly independent for any $\beta \in \{1, \ldots, r\}$, it follows that the vectors $\check{v}^{(\beta)}_{\ell+1}$ are linearly independent as well, and belong to $\check{V}_0$. As a result, $\check{d}_0 = \dim \check{V}_0 \geq r$. Also, $\check{A}_i \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$, so $\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \in \check{V}_0$. Then one proceeds as in the proof of Theorem 3.1, with a few minor changes. □

Appendix C Explicit computations in the reducible case

Set $P_n = Q\Delta_n$, $Q_n = Q(I-\Delta_n)$ and $R_n = 0$. Using Bolthausen and Goldsheid (2000) notations, for a given $a \in \mathbb{Z}$ and a given stochastic matrix ρ, for $n > a$, define $\psi_n = \psi_{n,a,\rho} = (I - R_n - Q_n\psi_{n-1})^{-1}P_n$, where $\psi_a = \psi_{a,a,\rho} = \rho$. It is not easy to compute $\psi_{n,a}$ for a small a, but if the probabilities are periodic with period 3 for example, then $\psi_{0,-6n} = \psi_{6n,0}$, $\psi_{0,-6n-1} = \psi_{6n+3,2}$, $\psi_{0,-6n-2} = \psi_{6n+3,1}$, $\psi_{0,-6n-3} = \psi_{6n+3,0}$, $\psi_{0,-6n-4} = \psi_{6n+6,2}$ and $\psi_{0,-6n-5} = \psi_{6n+6,1}$.

In Bolthausen and Goldsheid (2000, Theorem 1), the authors claim that the limit $\xi_n = \lim_{a\to-\infty}\psi_{n,a,\rho}$ exists and is independent of ρ. One crucial hypothesis for the proof is the existence of only one communication class (a.e.). As noted in Remark 2.1, this is usually not the case here, especially for Game C'.

We show next that Bolthausen and Goldsheid (2000, Theorem 1) does not hold for Game C', because either the limit does not exist, or it depends on the initial value ρ. In fact, starting from $\rho = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, the limit does not exist. Now, $\lim_{n\to\infty}\psi_{0,-2n} = \begin{pmatrix} 0 & 1 \\ 0.9987 & 0.0013 \end{pmatrix}$ and $\lim_{n\to\infty}\psi_{0,-2n-1} = \begin{pmatrix} 0.0039 & 0.9961 \\ 1 & 0 \end{pmatrix}$. Next, if one starts from $\rho = Q$, then $\xi_n \equiv Q$. The above computations ensue from the following set of equations for Game C': if $\rho = \begin{pmatrix} x_a & 1-x_a \\ 1-y_a & y_a \end{pmatrix}$, then $\psi_n = \begin{pmatrix} x_n & 1-x_n \\ 1-y_n & y_n \end{pmatrix}$, where, for any $n \geq a$,

$$\begin{aligned} z_{n+1} &= p^{(1)}_{n+1}p^{(2)}_{n+1} + p^{(2)}_{n+1}q^{(1)}_{n+1}x_n + p^{(1)}_{n+1}q^{(2)}_{n+1}y_n, \\ x_{n+1} &= p^{(1)}_{n+1}q^{(2)}_{n+1}y_n/z_{n+1}, \\ y_{n+1} &= p^{(2)}_{n+1}q^{(1)}_{n+1}x_n/z_{n+1}. \end{aligned}$$

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