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Stabilization of Linear Systems via Delayed State Feedback Controller

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Abstract

This paper deals with the stabilization of the class of continuous-time systems. A state feedback controller with delayed states is used to stabilize this class of systems. The time delay is assumed to be time-varying and differentiable with respect to time with finite bound, not necessary less one, and appear in the state. Delay-dependent sufficient conditions on stabilizability are developed. These conditions use some weighting matrices to reduce the conservatism. A design algorithm for a state feedback controller which guarantees that the closed-loop dynamics will be stable is proposed in terms of the solutions to linear matrix inequalities.

Key Words: Delayed systems, Linear matrix inequality (LMI), Delayed input, Stabilizability, State feedback.

Résumé

Cet article traite de la stabilisation de la classe des systèmes continus. Un contrôleur de type retour d'état avec retard est utilisé pour stabiliser cette classe de système. Le retard est supposé être variant dans le temps et différentiable par rapport au temps, borné par une valeur non nécessairement égal à 1. Des conditions suffisantes dépendantes du retard pour la stabilisation sont développées. Ces conditions utilisent certaines matrices de poids pour réduire le conservatisme. Un algorithme de design du contrôleur de type retour d'état est proposé sous le formalisme des inégalités matricielles linéaires.

1 Introduction

Time delays are often encountered in practical systems and it is well known that their existence in the dynamics is one of the causes of instability and poor performance degradation. Therefore, analysis and synthesis of systems with time-delay have been and continue to be a hot subject of research. Systems with time-delay have attracted researchers from mathematics and control communities. In the literature, we can find different results on deterministic and stochastic systems with time-delay. For stochastic systems with time-delay, we refer the reader to Mahmoud et al. [8], Boukas and Liu [1, 3, 2] Boukas et al. [6], Shi and Boukas [10], Cao and Lam [5] and the references therein. For deterministic systems, we refer reader to He et al. [7], Chen and Zheng [4] and the references therein.

More recently, we witnessed the development of a new approach for the study of delay-dependent stability conditions by introducing some free weighting matrices to express the links between the terms in the Leibnitz-Newton formula (see Chen and Zheng [4], He et al. [7] and the references therein). This approach has shown less conservatism compared to the other ones that have been proposed in the past. All the results reported in the literature dealt with the stability problem and the one of stabilization (using free weighting matrices) remains an open problem.

More often when controlling linear time-invariant systems delay may occurs which may cause some problems either in stability or performance degradation if the design phase doesn't take care of it. Such problems arise in network control systems which are becoming more used in industry due to their advantages. They also pose new challenges since the time-delay is always time-varying and may in some circumstances be random.

This paper deals with the stabilization of the class of continuous-time systems via delayed states. The time delay is assumed to be time-varying and differentiable with respect to time with finite bound, not necessary less one, and appear in the state. To the best of our knowledge this class of systems has not been fully studied. In terms of a set of linear matrix inequalities (LMIs), we present first a delay-dependent sufficient condition, which guarantees stability of the closed-loop systems with a fixed controller gain. Then, based on this, a delay-dependent sufficient condition for the existence of a state feedback controller ensuring stability of the closed-loop dynamics is proposed. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed methods. Some appropriate weighting matrices are introduced in this paper to reduce the conservatism as it will be shown by the proposed example.

The rest of this paper is organized as follows. In Section 2, the problem is stated and the goal of the paper is clarified. In Section 3, the main results are given and they include results on stability for a fixed controller gain and stabilizability. A delayed state feedback controller is used in this paper and a design algorithm in terms of the solutions to linear matrix inequalities is proposed to synthesize the controller gain we are using.

Notation. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "$^\top$" denotes matrix transposition and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\mathbb{I}$ is the identity matrices with compatible dimensions. L_2 is the space of integral vector over $[0, \infty)$

$(L_2[-h,0]\stackrel{\Delta}{=}\{f(\cdot)|\int_0^\infty f^\top(t)f(t)dt<\infty\})$. $\|\cdot\|$ will refer to the Euclidean vector norm whereas $\|\cdot\|$ denotes the L_2-norm over $[0,\infty)$ defined as $\|f\|^2=\int_0^\infty f^T(t)f(t)\ dt$. We will use $\star$ as an ellipsis for terms that are introduced by symmetric in the LMIs.

2 Problem statement

Consider a continuous-time system with the following dynamics:

$$\begin{cases}\dot{x}(t)=Ax(t)+Bu(t),\\ x(0)=x_0\end{cases}\tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the control input system, A and B are known real matrices with appropriate dimensions,

In this paper we are interested in the design of a stabilizing controller of the following form:

$$u(t)=Kx(t-d(t))\tag{2}$$

where K is a design parameter that has to be determined and $d(t)>0$ represents the system delay that will be assumed to satisfy $0\le d(t)\le h$, $\dot{d}(t)\le\mu<\infty$.

Remark 2.1 *As it was said in the introduction, this class of systems is of great importance mainly in network control systems where a linear system is controlled via a network that will introduce a time-varying delay and therefore, this case can be brought to the case of the class of systems we are dealing with in this paper.*

Plugging the controller expression (2) in (1) we get the following closed-loop dynamics:

$$\begin{cases}\dot{x}(t)=Ax(t)+A_dx(t-d(t))\\ x(s)=\phi(s),-h\le s\le 0\end{cases}\tag{3}$$

where $A_d=BK$ and $\phi(.)$ is the initial conditions such that $x(s)=\phi(s)\in L_2[-h,0]$.

This paper studies the stabilizability of the class of systems (1). Our goal is to design a state feedback controller guaranteeing that the closed-loop is stable using some appropriate weighting matrices to reduce the conservatism. In the rest of this paper, we will assume that all the required assumptions are satisfied, i.e. the complete access to the system state. The conditions we will develop here are in terms of the solutions to linear matrix inequalities that can be easily obtained using LMI control toolbox. These conditions are delay-dependent, which makes them less conservative. And the fact to use the weighting matrices will reduce more the conservatism as it was shown in many studies (see He et al. [7, 4] and the references therein).

Lemma 2.1 *For any symmetric and positive-definite matrix P and a time-varying delay $h>d(t)>0$, if there exists a differentiable vector function $x(t)$ with appropriate dimensions* *such that the integrals* $\int_{t-h}^{t} \dot{x}^\top(s)P\dot{x}(s)ds$ *and* $\int_{t-d(t)}^{t} \dot{x}(s)ds$ *are well defined, then we have:*

$$\left[\int_{t-d(t)}^{t} \dot{x}(s)ds\right]^\top P \left[\int_{t-d(t)}^{t} \dot{x}(s)ds\right] \leq h\int_{t-d(t)}^{t} \dot{x}^\top(s)P\dot{x}(s)ds \leq h\int_{t-h}^{t} \dot{x}^\top(s)P\dot{x}(s)ds$$

3 Main results

The aim of this chapter as it was presented earlier is to design a stabilizing state feedback of the form (2) for the class of systems (1). To reach this goal, we need firstly to establish the results that assure that the system (3) is stable for a given gain K. Then, using these results we will be able to design a controller of the form (2) that guarantees that the closed-loop will be stable. The following theorem gives the results on the stability of the unforced system (1).

Theorem 3.1 *System (3) is stable if there exist a symmetric and positive-definite matrix* P*, matrices* W_1, W_2, W_3 *and symmetric and positive-definite matrices* Q, R *and* S *such that the following LMI holds:*

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ \star & M_{22} & M_{23} & M_{24} \\ \star & \star & M_{33} & M_{34} \\ \star & \star & \star & M_{44} \end{bmatrix} < 0. \tag{4}$$

where

$$M_{11} = A^\top P + P^\top A + Q + R - W_1 - W_1^\top + hA^\top SA,$$

$$\begin{aligned} M_{12} &= W_1^\top - W_2 + PA_d + hA^\top SA_d, & M_{13} &= -W_3, & M_{14} &= W_1^\top, \\ M_{22} &= -(1-\mu)Q + W_2 + W_2^\top + hA_d^\top SA_d, & M_{23} &= W_3, & M_{24} &= W_2^\top, \\ M_{33} &= -R, & M_{34} &= W_3^\top, & M_{44} &= -\frac{1}{h}S. \end{aligned}$$

Proof: To prove this theorem let us consider the following Lyapunov functional:

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) + V_4(x(t))$$

where

$$\begin{aligned} V_1(x(t)) &= x^\top(t)Px(t), \\ V_2(x(t)) &= \int_{t-d(t)}^{t} x^\top(s)Qx(s)ds \\ V_3(x(t)) &= \int_{t-h}^{t} x^\top(s)Rx(s)ds, \end{aligned}$$

$$V_4(x(t)) = \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^\top(s)S\dot{x}(s)dsd\theta$$

with $P > 0$, $Q > 0$, $R > 0$ and $S > 0$.

The derivatives of these Lyapunov functionals with respect to time along the solution of system (3) are given by:

$$\begin{aligned}
\dot{V}_1(x(t)) &= x^\top(t)\left[A^\top P + AP\right]x(t) + 2x^\top(t)PA_d x(t-d(t)) \\
\dot{V}_2(x(t)) &= x^\top(t)Qx(t) - (1-\dot{d}(t))x^\top(t-d(t))Qx(t-d(t)) \\
&\leq x^\top(t)Qx(t) - (1-\mu)x^\top(t-d(t))Qx(t-d(t)) \\
\dot{V}_3(x(t)) &= x^\top(t)Rx(t) - x^\top(t-h)Rx(t-h) \\
\dot{V}_4(x(t)) &= h\dot{x}^\top(t)S\dot{x}(t) - \int_{t-h}^{t}\dot{x}(s)S\dot{x}(s)ds \\
&= x^\top(t)hA^\top SAx(t) + x^\top(t)hA^\top SA_d x(t-d(t)) + x^\top(t-d(t))hA_d^\top SAx(t) \\
&\quad + x^\top(t-d(t))hA_d^\top SA_d x(t-d(t)) - \left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top S\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)
\end{aligned}$$

Notice that from Leibnitz-Newton formula, we have:

$$[\Psi(t,s)]^\top\left[\int_{t-d(t)}^{t}\dot{x}(s)ds - x(t) + x(t-d(t))\right] = 0$$

$$\left[\int_{t-d(t)}^{t}\dot{x}(s)ds - x(t) + x(t-d(t))\right]^\top\Psi(t,s) = 0$$

with $\Psi(t,s) = W_1x(t) + W_2x(t-d(t)) + W_3x(t-h)$.

Using all these relations, we get:

$$\begin{gathered}
\dot{V}(x(t)) \leq x^\top(t)M_{11}x(t) + x^\top(t)M_{12}x(t-d(t)) \\
+x^\top(t)M_{13}x(t-h) + x^\top(t)M_{14}\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right) \\
+x^\top(t-d(t))M_{12}^\top x(t) + x^\top(t-d(t))M_{22}x(t-d(t)) \\
+x^\top(t-d(t))M_{23}x(t-h) + x^\top(t-d(t))M_{24}\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right) \\
+x^\top(t-h)M_{13}^\top x(t) + x^\top(t-h)M_{23}^\top x(t-d(t)) \\
+x^\top(t-h)M_{33}x(t-h) + x^\top(t-h)M_{34}\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right) \\
+\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top M_{14}^\top x(t) + \left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top M_{24}^\top x(t-d(t)) \\
+\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top M_{34}^\top x(t-h) + \left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^\top M_{44}\left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)
\end{gathered}$$

which can be rewritten as follows:

$$\dot{V}(x(t)) \quad \leq \quad \eta^\top(t)M\eta(t)$$

where

$$\eta(t) = \left[x^\top(t)\ x^\top(t-d(t))\ x^\top(t-h)\ \left(\int_{t-d(t)}^{t} \dot{x}(s)ds \right)^\top \right]^\top.$$

Using (4) and following similar steps as in [3], we can deduce that system (3) is stable. This completes the proof. □

Let us now concentrate on the design of a state feedback controller of the form (2) which guarantees that the closed-loop system will be stable. For this purpose, using the results of Theorem 3.1, the dynamics (3) will be stable if there exist a symmetric and positive-definite matrix P, matrices W_1, W_2, W_3 and symmetric and positive-definite matrices Q, R and S such that the LMI (4) holds with A_d replaced by BK.

Firstly, notice that $\widetilde{M}$ can be rewritten as follows:

$$\widetilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} & \tilde{M}_{13} & \tilde{M}_{14} \\ \star & \tilde{M}_{22} & \tilde{M}_{23} & \tilde{M}_{24} \\ \star & \star & \tilde{M}_{33} & \tilde{M}_{34} \\ \star & \star & \star & \tilde{M}_{44} \end{bmatrix} + \begin{bmatrix} A^\top \\ (BK)^\top \\ 0 \\ 0 \end{bmatrix} [hS] \begin{bmatrix} A & BK & 0 & 0 \end{bmatrix}$$

with

$$\tilde{M}_{11} = A^\top P + P^\top A + Q + R - W_1 - W_1^\top,$$

$$\tilde{M}_{12} = W_1^\top - W_2 + PBK, \qquad \tilde{M}_{13} = -W_3 \qquad \tilde{M}_{14} = W_1^\top,$$
$$\tilde{M}_{22} = -(1-\mu)Q + W_2 + W_2^\top \qquad \tilde{M}_{23} = W_3, \qquad \tilde{M}_{24} = W_2^\top,$$
$$\tilde{M}_{33} = -R, \qquad \tilde{M}_{34} = W_3^\top, \qquad \tilde{M}_{44} = -\frac{1}{h}S.$$

If the following holds:

$$hS < \varepsilon P, \varepsilon > 0, \tag{5}$$

$\widetilde{M}$ can be rewritten as follows:

$$\widetilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} & \tilde{M}_{13} & \tilde{M}_{14} & A^\top \\ \star & \tilde{M}_{22} & \tilde{M}_{23} & \tilde{M}_{24} & (BK)^\top \\ \star & \star & \tilde{M}_{33} & \tilde{M}_{34} & 0 \\ \star & \star & \star & \tilde{M}_{44} & 0 \\ A & BK & 0 & 0 & -\frac{1}{\varepsilon}P^{-1} \end{bmatrix}$$

Let $X = P^{-1}$. Pre- and post-multiply (5) respectively by X, we get:

$$h\bar{S} < \varepsilon X, \varepsilon > 0,$$

where $\bar{S} = XSX$.

Let $Y = KX$, $\bar{Q} = XQX$, $\bar{R} = XRX$, $\bar{W}_1 = XW_1X$, $\bar{W}_2 = XW_2X$, $\bar{W}_3 = XW_3X$. Pre- and post-multiply $\widetilde{M}$ respectively by $\text{diag}\,(X, X, X, X, \mathbb{I})$, we get the following sufficient condition for the design of the memoryless state feedback controller of the form (2).

Theorem 3.2 *Let ε be a given positive scalar. There exists a state feedback controller of the form (2) such that the closed-loop system (1) is stable if there exist a symmetric and positive-definite matrix X, matrices $\bar{W}_1$, $\bar{W}_2$, $\bar{W}_3$ and symmetric and positive-definite matrices $\bar{Q}$, $\bar{R}$ and $\bar{S}$ such that the following set of coupled LMIs holds:*

$$h\bar{S} < \varepsilon X, \varepsilon > 0, \tag{6}$$

$$\begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} & \bar{M}_{13} & \bar{M}_{14} & XA^\top \\ \star & \bar{M}_{22} & \bar{M}_{23} & \bar{M}_{24} & Y^\top B^\top \\ \star & \star & \bar{M}_{33} & \bar{M}_{34} & 0 \\ \star & \star & \star & \bar{M}_{44} & 0 \\ \star & \star & \star & \star & -\frac{1}{\varepsilon}X \end{bmatrix} < 0. \tag{7}$$

where

$$\bar{M}_{11} = XA^\top + AX + \bar{Q} + \bar{R} - \bar{W}_1 - \bar{W}_1^\top,$$

$$\begin{array}{lll} \bar{M}_{12} = \bar{W}_1^\top - \bar{W}_2 + BY, & \bar{M}_{13} = -\bar{W}_3, & \bar{M}_{14} = \bar{W}_1^\top, \\ \bar{M}_{22} = -(1-\mu)\bar{Q} + \bar{W}_2 + \bar{W}_2^\top & \bar{M}_{23} = \bar{W}_3, & \bar{M}_{24} = \bar{W}_2^\top, \\ \bar{M}_{33} = -\bar{R}, & \bar{M}_{34} = \bar{W}_3^\top, & \bar{M}_{44} = -\dfrac{1}{h}\bar{S}. \end{array}$$

The stabilizing memoryless controller gain is given by $K = YX^{-1}$.

4 Numerical examples

To show the validness of our results let us consider linear time-invariant system with the following data:

$$A = \begin{bmatrix} 0.0 & 1.0 \\ 0.0 & -0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.0 \\ 0.1 \end{bmatrix}.$$

First of all notice that the matrix A of the system is stable. Fixing $\varepsilon = 1$, $h = 5.25$, $\mu = 0.2$ and solving the LMIs (6)-(7), we get:

$$X = \begin{bmatrix} 6.3019 & -0.9470 \\ -0.9470 & 0.6258 \end{bmatrix}, \qquad Y = \begin{bmatrix} -0.9264 & -0.0161 \end{bmatrix},$$

which gives $K = \begin{bmatrix} -0.1953 & -0.3212 \end{bmatrix}$. The other matrices are not of importance to compute the controller gain and we omit to give them.

Fixing now $\varepsilon = 1$, $h = 5.25$, $\mu = 2.2$ and solving the LMIs (6)-(7), we get:

$$X = \begin{bmatrix} 6.4433 & -0.9705 \\ -0.9705 & 0.6499 \end{bmatrix}, \qquad Y = \begin{bmatrix} -0.9089 & -0.0358 \end{bmatrix},$$

which gives $K = \begin{bmatrix} -0.1927 & -0.3428 \end{bmatrix}$. The other matrices are not of importance to compute the controller gain and we omit to give them.

5 Conclusion

This paper dealt with the class of continuous-time linear systems. Results on stabilizability with delayed state feedback controller are developed. The LMI framework is used to establish the different results on stability and stabilizability. The conditions we established are delay-dependent. The results we developed can easily be solved using any LMI toolbox in the marketplace.

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