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$\mathcal{H}_\infty$ State Feedback Controller Design of Stochastic Singular Systems with Discontinuities

Jamela Raouf
El-Kébir Boukas

GERAD and Mechanical Engineering Department
École Polytechnique de Montréal
P.O. Box 6079, station Centre-ville
Montréal (Québec) Canada, H3C 3A7
el-kebir.boukas@polymtl.ca

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Abstract

In this paper, the problem of $\mathcal{H}_\infty$ controller design for Markovian singular systems with discontinuities (MSSD) is investigated. Sufficient conditions to guarantee that the unforced system is regular, impulse-free and stochastically stable in mean square sense with an $\mathcal{H}_\infty$ norm bound constraint, are provided. The derived conditions are expressed in terms of a set of linear matrix inequalities (LMIs), which can be solved by using existing linear algorithms. Also, based on this result, a memoryless $\mathcal{H}_\infty$ State feedback control which ensures the piecewise regularity, the absence of impulsive behavior and the stochastic stability in mean square sense of the closed-loop systems, is proposed by combining the LMI technique, the cone complementarity approach and the sequential linear programming matrix method (SLPMM). A numerical example is given to demonstrate the effectiveness of the proposed methods.

Key Words: singular markov jump systems, stability, stabilizability, discontinuity, uncertainty, $\mathcal{H}_\infty$ control.

Résumé

Dans cet article le problème de commande $\mathcal{H}_\infty$ pour la classe des systèmes singuliers à sauts markoviens et discontinuités dans les états est considéré. Des conditions suffisantes pour garantir que le système autonome est régulier, sans impulsion et stable sont développées sous forme de LMIs. Utisant cette condition un algorithme de design d'un contrôleur par retour d'état est développé pour garantir que la boucle fermée est régulière, sans impulsion et stable. Une méthode numérique employant certains outils de la littérature de commande robuste est proposée pour le calcul des gains de ce contrôleur. Un exemple numérique est développé pour montrer l'efficacité des résultats proposés.

1 Introduction

Recently, Markovian singular systems (MSS) have received considerable interest during the past years, due to the fact that this class of systems provides a more natural description of dynamical systems subject to abrupt changes, random failures and repairs may occur, than the singular representation. This class of systems have many important applications in various fields such as robotics, electrical circuits (Boukas, 2005), economics systems (Yin and Zhang, 2002), hydraulic processes (Raouf and Boukas, 2006). This fact motivates the study of problems such as stability, stabilization via different types of controller like, a state feedback, observer-based control, guaranteed cost control and their robustness. Fore more details on this matter, we refer the reader to Boukas *et al.* (2005); Xu and Lam (2006); Yan-Ming *et al.* (2006) and the references therein. However, all these above results have been concentrated on situations in which there are no abrupt changes in the states at the transitions between modes. Except for the recent works by Raouf and Boukas (2007a,b), in which the authors studied the stochastic stability and state feedback stabilization for the case where no disturbance signal appears, to the best of our knowledge, there are no results on the problems of stochastic stability and $\mathcal{H}_\infty$ control for Markovian singular systems with discontinuities of state trajectories at jump times (MSSD).

The aim of this paper is to address the stochastic stability and the $\mathcal{H}_\infty$ stochastic stabilization of MSSD. First a sufficient condition which guarantees regularity, absence of impulses between consecutive jumps and stochastic stability of such system in mean square sense, is derived. Then, based on this condition, an $\mathcal{H}_\infty$ state feedback controller design method is addressed such that the resulting closed-loop system is regular, piecewise impulse-free and stochastically stable in mean square sense with an $\mathcal{H}_\infty$ norm bound constraint. The proposed results which will extend the $\mathcal{H}_\infty$ control problem developed for singular systems and Markovian systems, in Masubuchi *et al.* (1997) and Boukas (2005) to MSSD, are based on the LMIs technique, the cone complementarity linearization approach (Elghaoui *et al.*, 1997) and the SLPMM (Leibfritz, 2001).

The rest of this paper is organized as follows. Section 2 states the problem to be studied. In Section 3, sufficient conditions are established to check the stochastic stability of the system under consideration. In Section 4, the $\mathcal{H}_\infty$ stabilizing controller design and its algorithm are given. Finally, a numerical example is given in Section 5 to show the applicability of the proposed results.

Throughout this paper, the following notations will be used. The superscript "$^\top$" denotes matrix transposition and for symmetric matrices X and Y, the notation $X > Y$ (respectively $X < Y$) means that $(X - Y)$ is positive-definite (respectively negative-definite). $\mathbb{I}$ denotes the identity matrix with the appropriate dimension. $\mathbb{E}[.]$ stands for the mathematical expectation operator with respect to the given probabilities Γ. $|.|$ refers to the Euclidian norm for vectors. For a square matrix $A = \{a_{l,s}\}$, $\|A\| = \max \sum_{s=1}^{n} |a_{l,s}|, \forall 1 \leq i \leq n$, denotes the infinity norm for matrix A. $\|.\|$ refers to the Euclidian norm of vectors. $\mathscr{L}^2[0,T]$ stands for the space of square-integrable vector functions over the interval $[0,T]$.

$\|.\|_2 = \sqrt{\int_0^T \|.\|^2 dt}$, denotes the norm in $\mathscr{L}^2[0,T]$, and $\|.\|_{E_2} = \mathbb{E}[\|.\|_2]$. The trace of square matrix is $Tr\ (.)$. diag $[.]$ denotes a block diagonal matrix.

2 Problem statement

Let $\{r_t, t \geq 0\}$ be a right-continuous-time Markov process, taking values in a finite state space $\mathscr{S} = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\pi_{ij})_{N\times N}$ given by:

$$\Gamma\left[r(t+\Delta) = j | r(t) = i\right] = \begin{cases} \pi_{ij}\Delta + o(\Delta) & \text{if } \ i \neq j \\ 1 + \pi_{ii}\Delta + o(\Delta) & \text{if } \ i = j \end{cases}$$

where $\Delta > 0, \lim_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$. Here $\pi_{ij} \geq 0$, $\forall i, j, i \neq j$, is the transition rate from the mode i to the mode j, while

$$\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij} \tag{1}$$

Let $\{\tau_k, k = 1, 2, \ldots\}$ be a given number sequence satisfying $\tau_1 < \tau_2 < \ldots < \tau_k < \tau_{k+1} < \ldots$, where $\tau_k > 0$ is the kth switched moment, i.e: the moment of the transition of the mode from $r(\tau_k) = i$ to $r(\tau_k^+) = j \neq i$, with $\tau_k^+ = \lim_{\Delta\to 0}(\tau_k + \Delta), \forall k > 0$.

Consider the MSSD with the following dynamics:

$$\begin{cases} E(r(t))\dot{x}(t) = A(r(t))x(t) + B(r(t))u(t) + B_w(r(t))w(t), \\ z(t) = C_z(r(t))x(t) + D_z(r(t))u(t) + B_z(r(t))w(t), t \neq \tau_k, \\ x(\tau_k^+) = R(r(\tau_k), r(\tau_k^+))x(\tau_k), t = \tau_k, \\ x(0) = x_0, r(0) = r_0. \end{cases}$$

where $x(t) \in \mathbb{R}^n$ is the state vector at time, $u(t) \in \mathbb{R}^p$ is the control input, $w(t) \in \mathbb{R}^s$ is the disturbance input which belongs to $\mathscr{L}^2[0,\infty]$, which means that the following holds:

$$\int_0^\infty w^\top(t)w(t)dt < \infty. \tag{2}$$

Also, $w(t)$ is supposed to be independent of the Markov process $\{r_t, t \geq 0\}$, $z(t) \in \mathbb{R}^q$ is the controlled output which belongs to $\mathscr{L}^2((\Omega, \mathcal{F}, \Gamma), [0, \infty])$; $A(i)$, $B(i)$ and $B_w(i)$ are real known matrices with appropriate dimensions for any $r_t = i \in \mathscr{S}$, The matrix $E(i) \in \mathbb{R}^{n\times n}$ may be singular with $rank(E(i)) = n_E \leq n$.

For $i \in \mathscr{S}$; $R(.,.)$ is a known real constant matrix that reflects the discontinuity of the state trajectory of system (2) (Bainov and Simeonov, 1989), we assume that $R(ii) = \mathbb{I}$, and there exist a set of scalars $0 < h_k \leq 1$ such that:

$$\max_{1\leq i,j\leq N} \|R(ij)\| \leq h_k \tag{3}$$

In this paper, we give an $\mathcal{H}_\infty$ feedback control design method for MSSD. The desired controller will both stochastically stabilizes the closed-loop of the considered class of systems, and reduce the effect of the disturbance input on the controlled output to a prescribed level. It will be developed in terms of the solutions of linear matrix inequality that can be easily obtained using any LMI toolbox. To this purpose, we assume that the Markov jump parameter process $r(t)$ and the system state process $x(t)$ are available for feedback for all $t \geq 0$.

Before giving our main result, we need some definitions and lemmas:

Definition 2.1 *(Dai, 1989) For any mode $i \in \mathscr{S}$, system (2) (with $u(t) \equiv 0$) is said to be:*

- *regular if $det(sE(i) - A(i))$, is not identically zero,*
- *impulse-free if $deg(det(sE(i) - A(i)))=$ rank $E(i)$,*
- *Piecewise stochastically admissible (PSA) if the system (2) is piecewise regular, piecewise impulse free and stochastically stable in mean square sense.*

Definition 2.2 *Let $\gamma > 0$ be a given positive scalar. For each $i \in \mathscr{S}$,*

- *system (2) with $u(t) \equiv 0$, for all $t \geq 0$, is said to be PSA with $\gamma-$disturbance attenuation, if this latter satisfies the following properties for any initial conditions x_0, r_0:*
  - *there exists a constant $M(x_0, r_0)$ with $M(0, r_0) = 0$ for all $r(t) \in \mathscr{S}$, such that the following holds ((2) is PSA):* $\lim_{t\to\infty} \mathbb{E}(|x(t)|^2|x_0, r_0) = 0$,
  - *the controlled output verifies:*

$$\begin{aligned} \|z\|_{E_2} &= \left[\mathbb{E}\int_0^\infty z^t(t)z(t)dt|x_0, r_0\right]^{1/2} \\ &\leq \left[\gamma^2\|w(t)\|_2^2 + M(r_0, x_0)\right]^{\frac{1}{2}} \end{aligned} \tag{4}$$

- *System (2) is said to be stabilizable, if there exists a linear state feedback*

$$u(t) = K(r(t))x(t) \tag{5}$$

  with $K(i)$ is a gain controller for each $r(t) = i \in \mathscr{S}$, such that the closed-loop system is PSA with $\gamma-$disturbance attenuation.

Lemma 2.1 *If there exists a set of matrices $P = (P(1), \ldots, P(N))$, such that the following LMI holds for every $i \in \mathscr{S}$:*

$$\begin{aligned} &P^\top(i)A(i) + A^\top(i)P(i) \\ &+ \sum_{j=1}^N \pi_{ij}R^\top(ij)E^\top(j)P(j)R(ij) < 0 \end{aligned} \tag{6}$$

under the constraint:

$$E^{\top}(i)P(i) = P^{\top}(i)E(i) \geq 0, \tag{7}$$

then the system (2) is PSA.

For the proof of this lemma, the reader is referred to Raouf and Boukas (2007a) (Lemma 3.1).

Lemma 2.2 *(Pettersson, 1987) Let Ω, F and Ξ be real matrices of appropriate dimensions with $F^T F \leq \mathbb{I}$. For any scalar $\varepsilon > 0$:*

$$\Omega F \Xi + \Xi^T F^T \Omega^T \leq \varepsilon \Omega \Omega^T + \varepsilon^{-1} \Xi^T \Xi \tag{8}$$

Lemma 2.3 *(Boukas, 2005) Let $C^{2,1}(\mathbb{R}^n \times \mathcal{S}; \mathbb{R}^+)$, denote the family of all nonnegative functions $V(x, r(t) = i)$ on $\mathbb{R}^n \times \mathscr{S}$. For each $V(x, r(t) = i) \in C^{2,1}(\mathbb{R}^n \times \mathscr{S}; \mathbb{R}^+)$, the infinitesimal generator $\mathbb{L}V$ of the Markov process $\{x(t), r(t), t \geq 0\}$, from $\mathbb{R}^n \times \mathscr{S}$ to $\mathbb{R}$ is given by:*

$$\begin{aligned} \mathbb{L}V(x(t), r(t) = i) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Big\{ \mathbb{E}\Big[V(x(t+\Delta), \\ r(t+\Delta)) | R(ij) x(t), r(t) = i \Big] - V(x, r(t) = i) \Big\} \end{aligned} \tag{9}$$

Lemma 2.4 *(Raouf and Boukas, 2007a) Select $V(x(t), i) = x^{\top}(t) E^{\top}(i) P(i) x(t), i \in \mathcal{S}$, where $P(i)$ is a non singular matrix, as the Lyapunov function for the system (2), then, for each $i \in \mathscr{S}$, and a positive scalar $0 < h_k < 1$, we have the following:*

$$\begin{aligned} \mathbb{E}\left[\int_0^T \mathbb{L}V(x(s), i_s) ds | (x_0), i_0) \right] = \mathbb{E}\big[V(x(T), i) \big] \\ + \sum_{p=1}^{l} (1 - h_p^2) \mathbb{E}\big[V(x(\tau_p), i_p) \big] - \mathbb{E}\big[V(x_0, i_0) \big], \end{aligned}$$

where l is the number of jumps on the interval $[0, T]$.

For the proof of this lemma, the reader is referred to Raouf and Boukas (2007a).

3 Stability

Before presenting the main results, we introduce the following lemmas which will play a key role in the derivation of the solution of our control problem.

Theorem 3.1 *Given a scalar $\gamma > 0$. If there exists a set of symmetric and positive-definite matrices $P = (P(1), \ldots, P(N))$ such that the following LMI holds for every $r(t) = i \in \mathscr{S}$:*

$$\Theta(i) < 0 \tag{10}$$

where:

$$\begin{aligned} \Theta(i) &= \begin{bmatrix} J_o(i) & C_z^\top(i)B_z(i) + P^\top(i)B_w(i) \\ B_z^\top(i)C_z(i) + B_w^\top(i)P(i) & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} \\ J_o(i) &= A^\top(i)P(i) + P^\top(i)A(i) + C_z^\top(i)C_z(i) \\ &+ \sum_{j=1}^{N} \pi_{ij} R^\top(ij)E^\top(i)P(j)R(ij) \end{aligned}$$

under the constraint:

$$E^\top(i)P(i) = P^\top(i)E(i) \geq 0, \tag{11}$$

then system (2) with $u(t) \equiv 0$*, for all* $t \geq 0$ *, is PSA and satisfies the following:*

$$\|z(t)\|_{E_2} \leq \left[\gamma^2 \|w(t)\|_2^2 + x_0^\top E^\top(r_0)P(r_0)x_0\right]^{\frac{1}{2}} \tag{12}$$

Proof: From (10), we get the following inequality:

$$\begin{aligned} &A^\top(i)P(i) + P^\top(i)A(i) + C_z^\top(i)C_z(i) \\ &+ \sum_{j=1}^{N} \pi_{ij} R^\top(ij)E^\top(j)P(j)R(ij) < 0 \end{aligned} \tag{13}$$

which implies the following since $C_z^\top(i)C_z(i) \geq 0$:

$$\begin{aligned} &A^\top(i)P(i) + P^\top(i)A(i) \\ &+ \sum_{j=1}^{N} \pi_{ij} R^\top(ij)E^\top(j)P(j)R(ij) < 0 \end{aligned} \tag{14}$$

By Lemma 2.1, it can be seen that the system under study (2) is PSA.

To prove (12), it suffices to show that the following performance function, for $T > 0$:

$$J_T = \mathbb{E}\Big[\int_0^T \big(z^\top(t)z(t) - \gamma^2 w^\top(t)w(t)\big)dt\Big] \tag{15}$$

is bounded when $T \to \infty$, i.e:

$$J_\infty \leq V(x_0, r_0) = x_0^\top E^\top(r_0)P(r_0)x_0 \tag{16}$$

Notice that by (9), we have:

$$\begin{array}{rcl}\mathbb{L}V(x(t),i) &=& x^\top(t)\Big[A^\top(i)P(i)+P^\top(i)A(i)\\ &+& \sum_{j=1}^{N}\pi_{ij}R^\top(ij)E^\top(j)P(j)R(ij)\Big]x(t)\\ &+& x^\top(t)P^\top(i)B_w(i)w(t)\\ &+& w^\top(t)B_w^\top(i)P(i)x(t),\end{array}$$

and

$$\begin{array}{rl} & z^\top(t)z(t)-\gamma^2w^\top(t)w(t)\\ = & \big[C_z(i)x(t)+B_z(i)w(t)\big]^\top\big[C_z(i)x(t)+B_z(i)w(t)\big]\\ - & \gamma^2w^\top(t)w(t)\\ = & x^\top(t)C_z^\top(i)C_z(i)x(t)+x^\top(t)C_z^\top(i)B_z(i)w(t)\\ = & w^\top(t)B_z^\top(i)C_z(i)x(t)+w^\top(t)B_z^\top(i)B_z(i)w(t)\\ - & \gamma^2w^\top(t)w(t).\end{array} \tag{17}$$

which implies:

$$\begin{array}{l} z^\top(t)z(t)-\gamma^2w^\top(t)w(t)+\mathbb{L}V(x(t),r(t))\\ =\eta^\top(t)\Theta(r(t))\eta(t)\end{array} \tag{18}$$

with: $\eta^\top(t)=\left[\begin{array}{c} x^\top(t),w^\top(t)\end{array}\right]$ and $\Theta(i)$ is given by (10). Therefore, for $t\in(\tau_k,\tau_{k+1}], k=0,1,2...$, we have:

$$\begin{array}{c} J_{\tau_{k+1}}=\mathbb{E}\big[\int_{\tau_k}^{\tau_{k+1}}(z^\top(t)z(t)-\gamma^2w^\top(t)w(t)\\ +\mathbb{L}V(x(t),r(t))dt\big]-\mathbb{E}\big[\int_{\tau_k}^{\tau_{k+1}}\mathbb{L}V(x(t),r(t)dt\big].\end{array} \tag{19}$$

Using (18), (19) becomes with $T>0$:

$$\begin{array}{rcl} J_T &=& \mathbb{E}\big[\int_0^T(\eta^\top(t)\Theta(r(t))\eta(t)dt\big]\\ &-& \mathbb{E}\big[\int_0^T\mathbb{L}V(x(t),r(t)dt|(x_0,r_0)\big].\end{array} \tag{20}$$

By Lemma 2.4, we obtain:

$$J_T = \mathbb{E}\Big[\int_0^{\top} \eta^{\top}(t)\Theta(r(t))\eta(t)dt\Big] + V(x_0, r_0)$$
$$-\mathbb{E}\left[V(x(T), r(T))\right] - \sum_{p=1}^{l}(1-h_p^2)\mathbb{E}\big[V(x(\tau_p), i_p)\big] \tag{21}$$

By (10), $\Theta(i) < 0$ for each mode i, furthermore since $0 < h_p < 1, p = 1, 2, ...l$, then the term $\mathbb{E}\big[V(x(T), i)\big] + \sum_{p=1}^{\infty}(1-h_p^2)\mathbb{E}\big[V(x(\tau_p), i(p))\big]$ is positif, thus (21) implies the following: $J_T \leq V(x_0, r_0)$. Letting T to go to infinity, and using (18) give the desired result (12). This completes the prof of the theorem.

Remark 3.1 *When $E(i) = \mathbb{I}, i \in \mathscr{S} = \{1\}$, (2) (resp. Theorem 3.1) reduces to conventional system, (Theorem 1 in Gahinet and Apkarian (1994)).*

When $E(i) = \mathbb{I},$, and $R(ij) = \mathbb{I}, i, j \in \mathscr{S}$, Theorem 3.1) reduces to Theorem 4.1.2 in Boukas (2005).

Whereas in the case $\mathscr{S} = \{1\}$, i.e: the system has one mode, Theorem 3.1 coincides with the result of Masubuchi et al. *(1997). In view of this, Theorem 3.1 in this paper can be viewed as an extension of the existing results for conventional system, singular systems, and Markovian systems.*

4 $\mathcal{H}_\infty$ controller design

In this section, we discuss the $\mathcal{H}_\infty$ control of Markovian singular system (2). For this purpose, plugging controller (5) in the dynamics (2) gives:

$$\begin{cases} E(r(t))\dot{x}(t) = \widetilde{A}(r(t))x(t) + B_w(r(t))w(t), \\ z(t) = \widetilde{C}_z(r(t))x(t) + B_z(r(t))w(t); t \neq \tau_k \\ x(\tau_k^+) = R(r_{\tau_k^+} = j, r_{\tau_k} = i)x(\tau_k), t = \tau_k. \end{cases} \tag{22}$$

when $r(t) = i \in \mathscr{S}$:

$$\widetilde{A}(i) = A(i) + B(i)K(i), \tag{23}$$
$$\widetilde{C}_z(i) = C_z(i) + D_z(i)K(i). \tag{24}$$

Then, by Theorem 3.1, this closed-loop system is PSA with $\gamma-$disturbance attenuation and satisfies (12) if the following inequality holds:

$$\begin{bmatrix} J_o(i) & \begin{bmatrix} \widetilde{C}_z^{\top}(i)B_z(i) \\ +P^{\top}(i)B_w(i) \end{bmatrix} \\ \begin{bmatrix} B_z^{\top}(i)\widetilde{C}_z(i) \\ +B_w^{\top}(i)P(i) \end{bmatrix} & B_z^{\top}(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} < 0 \tag{25}$$

with:

$$\begin{aligned} J_o(i) &= \widetilde{A}^\top(i)P(i) + P^\top(i)\widetilde{A}(i) + \widetilde{C}_z^\top(i)\widetilde{C}_z(i) \\ &+ \sum_{j=1}^{N} \pi_{ij} R^\top(ij)E^\top(i)P(j)R(ij). \end{aligned}$$

To synthesize the controller gain, notice that (25) can be transformed as follows:

$$\begin{aligned} &\begin{bmatrix} \widetilde{J}_o(i) & \begin{bmatrix} \widetilde{C}_z^\top(i)B_z(i) \\ +P^\top(i)B_w(i) \end{bmatrix} \\ \begin{bmatrix} B_z^\top(i)\widetilde{C}_z(i) \\ +B_w^\top(i)P(i) \end{bmatrix} & B_z^\top(i)B_z(i) - \gamma^2\mathbb{I} \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{J}_1(i) & P^\top(i)B_w(i) \\ B_w^\top(i)P(i) & -\gamma^2\mathbb{I} \end{bmatrix} \\ &+ \begin{bmatrix} \widetilde{C}_z^\top(i) \\ B_z^\top(i) \end{bmatrix} \begin{bmatrix} \widetilde{C}_z(i), B_z(i) \end{bmatrix} \end{aligned}$$

where:

$$\begin{aligned} \widetilde{J}_1(i) &= \widetilde{A}^\top(i)P(i) + P^\top(i)\widetilde{A}(i) \\ &+ \sum_{j=1}^{N} \pi_{ij} R^\top(ij)E^\top(i)P(j)R(ij). \end{aligned}$$

This together with using Schur complement, gives:

$$\begin{bmatrix} \widetilde{J}_1(i) & P^\top(i)B_w(i) & \widetilde{C}_z^\top(i) \\ B_w^\top(i)P(i) & -\gamma^2\mathbb{I} & B_z^\top(i) \\ \widetilde{C}_z(i) & B_z(i) & -\mathbb{I} \end{bmatrix} < 0, \tag{26}$$

The previous inequality is nonlinear. However, it is not suitable for the derivation of LMI conditions of the controller synthesis since $\widetilde{A}(i)$ is nonlinear in $K(i)$ and $P(i)$, thus it can not be solved by using existing linear algorithms. To transform it into an LMI, we need to the following lemma:

Lemma 4.1

- *For any positif scalar* $\{\varepsilon(ij), j = 1, j \neq i, i \in \mathscr{S}\}$, *the following inequality is satisfied:*

$$\sum_{j=1, j\neq i}^{N} \lambda_{ij} R^\top(ij)E^\top(j)P(j)R(ij)$$

$$
\begin{aligned}
&\le \sum_{j=1,j\neq i}^{N} \frac{1}{4}\pi_{ij}^2 \varepsilon^{-1}(ij)\mathbb{I} \\
&+ \sum_{j=1,j\neq i}^{N} \varepsilon(ij)\left[R^\top(ij)E^\top(j)P(j)R(ij)\right]^\top \\
&\times\left[R^\top(ij)E^\top(j)P(j)R(ij)\right].
\end{aligned}
\tag{27}
$$

- *For each mode $i \in \mathscr{S}$, given any symmetric and positive-definite matrix $V_P(i)$ such that:*

$$
\begin{aligned}
&\sum_{j=1,j\neq i}^{N} \varepsilon(ij)\left[R^\top(ij)E^\top(j)P(j)R(ij)\right]^T \\
&\left[R^\top(ij)E^\top(j)P(j)R(ij)\right] \le V_P(i).
\end{aligned}
\tag{28}
$$

then we have:

$$
\begin{bmatrix} V_P(i) & W(i) \\ W^\top(i) & \mathscr{P}(i) \end{bmatrix} \ge 0 \tag{29}
$$

where:

$$
\begin{aligned}
W(i) &= \left[R^\top(i1)E^\top(1)P(1)R(i1), \ldots,\right. \\
&\quad R^\top(ii-1)E^\top(i-1)P(i-1)R(ii-1), \\
&\quad R^\top(ii+1)E^\top(ii+1)P(i+1)R(ii+1), \ldots, \\
&\quad \left.R^\top(iN)E^\top(N)P(N)R(iN)\right], \qquad (30) \\
\mathscr{P}(i) &= \mathrm{diag}\left[\varepsilon^{-1}(1)\mathbb{I}, \ldots, \varepsilon^{-1}(i-1)\mathbb{I}, \varepsilon^{-1}(i+1)\mathbb{I},\right. \\
&\quad \left.\ldots, \varepsilon^{-1}(N)\mathbb{I}\right]. \qquad (31)
\end{aligned}
$$

Proof : The inequality (27) can be deduced by a direct application of Lemma 2.2, whereas the LMI (29) can be obtained by using Schur complement to (28).

By Lemma 4.1, it can be seen that (26) can be transformed into (29) and:

$$
\begin{bmatrix} \widetilde{J}_2(i) & P^\top(i)B_w(i) & \widetilde{C}_z^\top(i) \\ B_w^\top(i)P(i) & -\gamma^2\mathbb{I} & B_z^\top(i) \\ \widetilde{C}_z(i) & B_z(i) & -\mathbb{I} \end{bmatrix} < 0 \tag{32}
$$

with:

$$
\begin{aligned}
&\widetilde{J}_2(i) = \widetilde{A}^\top(i)P(i) + P^\top(i)\widetilde{A}(i) + \pi_{ii}E^\top(i)P(i) \\
&+V_P(i) + \sum_{j=1,j\neq i}^{N} \frac{1}{4}\pi_{ij}^2\varepsilon^{-1}(j)\mathbb{I}
\end{aligned}
$$

Pre- and post-multiply (32) by $diag(X^\top(i),\mathbb{I},\mathbb{I})$, $diag(X(i),\mathbb{I},\mathbb{I})$ where $X(i) = P^{-1}(i)$, which gives:

$$\begin{bmatrix} \widetilde{J}_3(i) & B_w(i) & X^\top(i)\widetilde{C}_z^\top(i) \\ B_w^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) \\ \widetilde{C}_z(i)X(i) & B_z(i) & -\mathbb{I} \end{bmatrix} < 0 \tag{33}$$

with:

$$\widetilde{J}_3(i) = X^\top(i)\widetilde{A}^\top(i) + \widetilde{A}(i)X(i) + \pi_{ii}X^\top(i)E^\top(i)$$
$$+X^\top(i)V_P(i)X(i) + \sum_{j=1,j\neq i}^{N} \frac{1}{4}\pi_{ij}X^\top(i)\varepsilon^{-1}(j)X(i)\pi_{ij}$$

Substituting the expressions in (23), (24) to (33), then by applying changes of variable, $Y(i) = K(i)X(i)$, $V_P(i) = Z^{-1}(i)$ and using the Schur complement to (33), this latter becomes:

$$\begin{bmatrix} \Pi(i) & B_w(i) & \begin{bmatrix} X^\top(i)C_z^\top(i) \\ +Y^\top(i)D_z^\top(i) \end{bmatrix} & X^\top(i) & \mathcal{S}(i) \\ B_w^\top(i) & -\gamma^2\mathbb{I} & B_z^\top(i) & 0 & 0 \\ \begin{bmatrix} C_z(i)X(i) \\ +D_z(i)Y(i) \end{bmatrix} & B_z(i) & -\mathbb{I} & -Z(i) & 0 \\ X(i) & 0 & 0 & 0 & -\mathscr{X}(i) \\ \mathcal{S}^\top(i) & 0 & 0 & & \end{bmatrix} < 0, \tag{34}$$

where $\mathcal{S}(i)$ and $\mathscr{X}(i)$ are given by:

$$\begin{aligned} \Pi(i) &= X^\top(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) \\ &\quad +B(i)Y(i) + \pi_{ii}X^\top(i)E^\top(i), \qquad (35) \\ \mathcal{S}(i) &= \frac{1}{2}\big[\pi_{i1}X^\top(i), \ldots, \pi_{ii-1}X^\top(i), \pi_{ii+1}X^\top(i), \\ &\quad \ldots, \pi_{iN}X^\top(i)\big], \qquad (36) \\ \mathscr{X}(i) &= \text{diag}\big[\varepsilon(i1)\mathbb{I}, \ldots, \varepsilon(ii-1)\mathbb{I}, \varepsilon(ii+1)\mathbb{I}, \\ &\quad \ldots, \varepsilon(iN)\mathbb{I}\big]. \qquad (37) \end{aligned}$$

The following theorem summarizes this sufficient condition for solvability of the $\mathcal{H}_\infty$ control problem:

Theorem 4.1 *Given a scalar $\gamma > 0$. If there exist a set of nonsingular matrices $P = (P(1), \ldots, P(N))$, and $X = (X(1), \ldots, X(N))$, a set of symmetric and positive-definite* *matrices* $Z = (Z(1), \ldots, Z(N))$*, and* $V_P = (V_P(1), \ldots, V_P(N))$*, a matrix* $Y = (Y(1), \ldots, Y(N))$ *and a set of positive scalar* $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(N))$*, such that the LMIs (29) and (34), holds for every* $i \in \mathscr{S}$*, under the constraints (11) and:*

$$\begin{aligned} P(i)X(i) &= \mathbb{I}, & (38)\\ V_P(i)Z(i) &= \mathbb{I}, & (39) \end{aligned}$$

then the system (2) under the controller (5) with $K(i) = Y(i)X^{-1}(i)$*, is PSA and moreover the closed-loop system satisfies the disturbance rejection of level* γ*.*

Remark 4.1 *For each mode* $i \in \mathscr{S}$*, when* $E(i) = \mathbb{I}$*, and* $R(ii) = \mathbb{I}$*, then Theorem 4.1) reduces to Theorem 4.2.2 in Boukas (2005). Thus Theorem 4.1) in this paper presents a more general result on the stochastic stability in mean square sense than the existing ones for Markovian systems.*

Remark 4.2 *It should be noted that the conditions in Theorem 4.1 are nonconvex feasibility problem since* $\varepsilon^{-1}(i)$ *appears in (31), furthermore, (29) and (34) are two coupled LMIs and the solution of one should be the inverse of the other to satisfy the coupling constraints (38) and (39). As a result, we can not solve the conditions in Theorem 4.1 by using convex optimization algorithms. For this purpose, let* $\beta(ij) = \varepsilon^{-1}(ij), i, j \in,$ *and uses the same procedure given in Elghaoui* et al. *(1997), which consists in weakening the equality constraints:*

$$\beta(ij)\varepsilon^{-1}(ij) = \mathbb{I}, \tag{40}$$

(38) and (39) to semi-definite programming conditions. Thus the non convex problem of finding an $\mathcal{H}_\infty$ *state feedback controller such that the system (2) is PSA with disturbance rejection of level* γ*, can be converted on the following cone complementary problem involving LMIs conditions: for all* $i \in \mathcal{S}$*:* $\mathcal{P}_f: \quad \min \sum_{i=1}^{N} \mathrm{Tr}\Big(P(i)X(i) + V_P(i)Z(i) + \beta(ij)\varepsilon(ij)\mathbb{I}\Big)$ *subject to LMIs (34), and:*

$$\begin{bmatrix} V_P(i) & W(i) \\ W^\top(i) & \mathscr{G}(i) \end{bmatrix} \geq 0, \begin{bmatrix} \beta(i) & \mathbb{I} \\ \mathbb{I} & \varepsilon(i) \end{bmatrix} \geq 0, \tag{41}$$

$$\begin{bmatrix} P(i) & \mathbb{I} \\ \mathbb{I} & X(i) \end{bmatrix} \geq 0, \begin{bmatrix} V_P(i) & \mathbb{I} \\ \mathbb{I} & Z(i) \end{bmatrix} \geq 0, \tag{42}$$

with the equality constraint (11), and the matrices $\Pi(i)$*,* $W(i)$*,* $\mathcal{S}(i)$*, and* $\mathscr{X}(i)$ *are given by (35), (30), (36) and (37), while* $\mathscr{G}(i)$ *is as follows:*

$$\mathscr{G}(i) \quad = \quad \mathrm{diag}\big[\beta(1)\mathbb{I}, \ldots, \beta(i-1)\mathbb{I}, \beta(i+1)\mathbb{I}, \ldots, \beta(N)\mathbb{I}\big].$$

The resolution of this problem is given via the SLPMM algorithm proposed in Leibfritz (2001). This iterative algorithm is presented in the following.

Algorithm 4.1

1) *Find a feasible solution* $\{P^o(i),\ X^o(i),\ V^o(i),\ Z^o(i),\ \beta^o(ij)\ ,\ \varepsilon^o(ij)\}$*, which verify (34), (41) and (42). If there are none, exist. Otherwise let* $P(i) = P^o(i), X(i) = X^o(i), V_P(i) = V^o(i), Z(i) = Z^o(i), \beta(ij) = \beta^o(ij), \varepsilon(ij) = \varepsilon^o(ij)$*, set* k*,* $k = 0$ *and go to step (2).*
2) *Solve the following convex optimization problem for the variables* $(P, X, Z, V, \beta, \varepsilon)$ *:*

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} Tr\big[P(i)X^k(i) + P^k(i)X(i)\big] \\ & + Tr\big[V_P(i)Z^k(i) + V_P^k(i)Z(i)\big] \\ & + Tr\big[\beta^k(ij)\varepsilon(ij)\mathbb{I} + \beta(ij)\varepsilon^k(ij)\mathbb{I}\big] \end{aligned}$$

subject to LMIs (34), (41) and (42).

3) *Let* $\mathscr{P}^k(i) = P(i),\ \mathscr{X}^k(i) = X(i), \mathscr{V}^k(i) = V_P(i), \mathscr{Z}^k(i) = Z(i),\ \omega^k(ij) = \beta(ij), \sigma^k(ij) = \varepsilon^k(ij)$*,*
4) *For* $\eta \in \mathbb{R}^+$*, If* $|\sum_{i=1}^{l} Tr(\mathscr{P}^k(i)X^k(i) + P^k(i)\mathscr{X}^k(i)$ $+ \mathscr{V}^k(i)Z^k(i) + V_P^k(i)\mathscr{Z}^k(i) + \sigma^k(ij)\beta^k(ij)\mathbb{I} + \omega^k(ij)\varepsilon^k(ij)\mathbb{I} - 2\sum_{i=1}^{l} Tr(P^k(i)X^k(i)$ $+ V_P^k(i)Z^k(i) + \beta^k(ij)\varepsilon^k(ij)\mathbb{I})| < \eta$*, then go to step (7).*
5) *Compute the step* $\alpha \in [0, 1]$ *by solving the LMI linearized problem* $\min_\alpha \quad h(\alpha)$*, where :*

$$\begin{aligned} h(\alpha) \quad = \quad & \sum_{i=1}^{N} Tr[(P^k(i) + \alpha(\mathscr{P}^k(i) - P^k(i))) \\ & \times(X^k(i) + \alpha(\mathscr{X}^k(i) - X^k(i))) \\ & +(V_P^k(i) + \alpha(\mathscr{V}^k(i) - V_P^k(i))) \\ & \times(Z^k(i) + \alpha(\mathscr{Z}^k(i) - Z^k(i))) \\ & +(\varepsilon^k(ij)\mathbb{I} + \alpha\mathbb{I}(\sigma^k(ij) - \varepsilon^k(ij))) \\ & \times(\beta^k(ij)\mathbb{I} - \alpha\mathbb{I}(\omega^k(ij) - \beta^k(ij))] \end{aligned}$$

6) *for* $i \in \mathscr{S}$*, let* $P^{k+1}(i) = (1-\alpha)P^k(i) + \alpha\mathscr{P}^k(i), X^{k+1}(i) = (1-\alpha)X^k(i) + \alpha\mathscr{X}^k(i)$*,* $V^{k+1}(i) = (1-\alpha)V_P^k(i) + \alpha\mathscr{V}^k(i), Z^{k+1}(i) = (1-\alpha)Z^k(i) + \alpha\mathscr{Z}^k(i), \varepsilon^{k+1}(ij) = (1-\alpha)\varepsilon^k(ij) + \alpha(ij)\sigma^k(ij), \beta^{k+1}(ij) = (1-\alpha)\beta^k(ij) + \alpha\omega^k(ij)$*, and* $k = k + 1$*. If* $k < l$*, with* l *is the maximal number of iterations, go to step (2), else go to step (7).*
7) *If (38), (39) and (40) are satisfied, then a solution is found, else exit.*

5 Numerical example

To illustrate the effectiveness of the above results, let us consider the following example

Example 5.1 *Consider an uncertain system (2) with three operation modes. The system data are given by:*

$$\Lambda = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -3 \end{bmatrix}, \qquad E(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A(1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad B(1) = \begin{bmatrix} 4. & 1.0 \\ 0 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix}, \qquad B(2) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A(3) = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad B(3) = \begin{bmatrix} -3 & 0 \\ 0 & 4 \\ -1 & 1 \end{bmatrix},$$

$$R(1) = R(2) = R(3) = \begin{bmatrix} 0.1 & 0 & 0.5 \\ 0.2 & 0.3 & 0 \\ 0.2 & 0 & 0.6 \end{bmatrix}, \qquad E(2) = E(1)$$

$$E(3) = E(1), \varepsilon(1) = 0.9296, \varepsilon(2) = 1.0045, \varepsilon(3) = 0.9179,$$
$$\beta(3) = 1.0894, \beta(2) = 0.9956, \beta(1) = 1.0758, \gamma = 1.6578.$$

Solving the problem $\mathcal{P}_f$, one gets the following gain matrices:

$$\begin{aligned} K(1) &= \begin{bmatrix} -1.0804 & 0.0379 & -1.2062 \end{bmatrix}, \\ K(2) &= \begin{bmatrix} -4.8128 & 0.4118 & -0.1475 \end{bmatrix}, \\ K(3) &= \begin{bmatrix} -7.1479 & 2.2199 & -0.5984 \end{bmatrix}. \end{aligned}$$

The simulation results using these gains are illustrated by Figure 1, from which, we can see that at the jump times, discontinuities on the state trajectories appear. Also, note that the states'system go to zero when time goes to infinity. Thus, we can conclude that the proposed H_∞ controller with the computed gains, can be used to stabilize this kind of systems with a desired H_∞ norm bound γ.

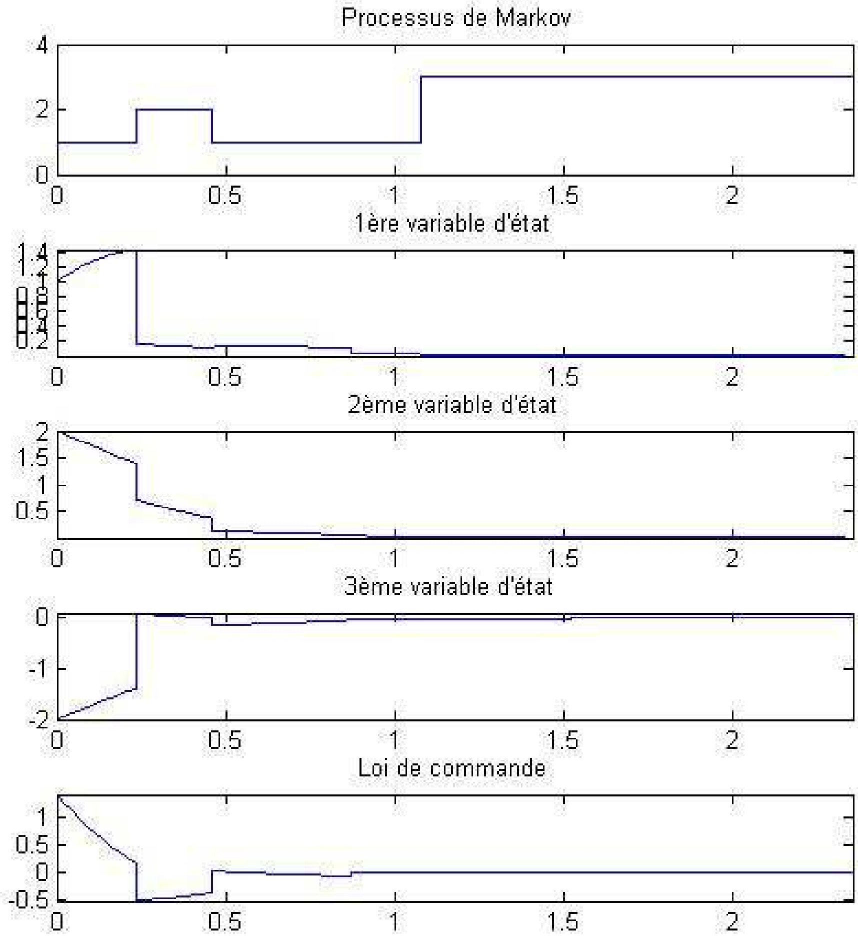

Figure 1: States and control input of the closed-loop system

6 Conclusion

The problem of H_∞ control for MSSD has been studied. Since the derived control design conditions are nonconvex problems, both cone complementarity method and SLPMM algorithm have been developed based on LMI technique, to construct a controller which guarantees that the closed-loop system is PSA and satisfies a prescribed H_∞ performance level for all finite discontinuities. A numerical example have been provided to verify the effectiveness of the proposed approach.

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