Les Cahiers du GERAD ISSN: 0711–2440

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G–2007–92

December 2007

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Abstract

How can a cooperative agreement made at the start of a dynamic game can be sustained over time? Early work has avoided this question by supposing that the players sign binding agreements. This assumption is hard to accept from a theoretical perspective, and a practical one as well. Conceptually, there is no reason to believe that rational players would stick to an agreement if they can achieve a better outcome by abandoning, no matter what they have announced before. At an empirical level, it suffices to look at the number of disputes (between spouses, business partners, countries, etc.) in the courts to convince ourselves that binding agreements are not so binding. Scholars in dynamic games have followed different lines of thoughts to answer the question. This tutorial reviews one of them, namely time consistency, a concept which has also been termed dynamic individual rationality, sustainability, dynamic stability, agreeability, or acceptability.

Key Words: Time Consistency, Differential Games, Sustainability of Cooperation.

Résumé

Comment garantir qu'un accord signé au début d'un jeu dynamique demeure en place à mesure que le temps passe? Les premiers travaux dans le domaine ont contourné cette question en supposant que l'accord était contraignant (binding). Cette hypothèse est difficile à accepter aussi bien sur le plan théorique qu'empirique. En effet, pourquoi des joueurs qui sont par définition rationnels continuent à coopérer si une déviation à une autre stratégie leur permettrait de performer mieux? Au niveau empirique, il suffit de constater le nombre de procès (entre conjoints, partenaires d'affaires, pays, etc.) pour se convaincre que l'accord n'était pas si contraignant. Les chercheurs en jeux dynamiques ont traité la question posée selon plusieurs lignées. Ce tutorial revoit une d'elles, à savoir la cohérence dynamique, un concept connu aussi sous les termes de rationalité individuelle dynamique et de durabilité.

Mots clés : cohérence dynamique, jeux différentiels, coopération durable.

Acknowledgments: Some sections of this tutorial draws heavily on Jørgensen and Zaccour (2002a). I thank Steffen Jørgensen, Gilbert Laporte, Guiomar Martín-Herrán and Leon Petrosjan for their helpful comments. The usual disclaimer applies. Research supported by NSERC, Canada.

1 Prologue

This paper is an invited contribution to the special issue of *INFOR* to celebrate the 50th anniversary of the Canadian Operational Research Society. I am honored by this invitation to report on (a part of) an area, cooperative differential games, to which Canadian researchers at GERAD, jointly with colleagues from different countries, have made significant contributions. Differential games were initiated by Rufus Isaacs at the Rand Corporation in the early sixties of the last century. His book (Isaacs (1965)) is considered as the starting point of the field. Initially, the focal point of differential games scientists was military applications and antagonistic zero-sum games. The theory of differential games has developped much since then, and applications are now found in many areas, e.g., in management science (operations management, marketing, finance), economics (industrial organization, macroeconomics, resource economics, environmental economics, etc.), biology, ecology, military, etc. For an introduction to differential games (DG), the interested reader may consult one of the available textbooks on the subject, e.g., Başar and Olsder (1995), Petrosjan (1993), Dockner et al. (2000), Jørgensen and Zaccour (2004), Engwerda (2005), Yeung and Petrosjan (2005).

2 Introduction

It happens every day that players (e.g., firms, union and management, countries, spouses, etc.) agree to cooperate over a certain period of time, say $[t_0, T]$, where t_0 is the starting date of the agreement and T the end date of the contract. Cooperation means that the parties agree to coordinate their strategies in view of optimizing a collective performance index (profit, cost, welfare, happiness, etc.) Although coordination may induce some loss of freedom to the parties in terms of their choice of actions, its rationale stems, on balance, from the collective and individual gains it generates compared to noncooperation.

One interesting question is why economic and social agents sign long-term contracts, instead of keeping all of their options open by committing for only one period at the time? A first answer is that negotiating to reach an acceptable arrangement is costly (not only in terms of dollars, but also in time, emotions and feelings, etc.), and, therefore, it naturally makes sense to avoid frequent renegotiation whenever this is feasible. Second, some problems are inherently dynamic. For instance, curbing polluting emissions in the industrial and transport sectors requires investments in cleaner technologies, changes in consumption habits, etc., which clearly cannot be achieved overnight. If the players have short-planning horizons when they perform their cost-benefit analysis, they may end up constantly postponing relevant decisions concerning the future, and nothing would ever be achieved. This explains why the parties (countries, provinces, regions, etc.) typically seek long-term environmental agreements.

It also happens every day that some cooperative programs are abandoned before reaching their maturity at T. In a dynamic game setting, if an agreement breaks down before its intended end date, we say that it is *time inconsistent*. This means that some parties prefer, payoff-wise, to switch at an intermediate instant of time $\tau \in [t_0, T]$ to a noncooperative mode of play, rather than stick to the agreement. The interest in dealing with such instabilities is not in explaining why they may occur (which would be a tautology), but in attempting to design mechanisms, schemes, side payments, etc., that would help prevent breakdowns from taking place. This tutorial aims at introducing the reader to time consistency, a concept that has also been termed *sustainability of cooperation, dynamic individual rationality, dynamic stability, durability of an agreement, agreeable solution,* etc.

The remainder of the paper is organized as follows. Section 3 provides a refresher on differential games. It introduces only the elements that are needed for the sequel. Following the same philosophy, Section 4 recalls some concepts of cooperative game theory. Section 5 formally introduces the main issues related to time consistency and Section 6 some schemes to implement it. Section 7 concludes by pointing out to the reader some alternative approaches to maintaining cooperation over time.

3 Differential Games: A Refresher

The description of a deterministic differential game played on a prespecified time interval $[t_0, T]$ involves the following ingredients:

1. A set of players $I = \{1, \ldots, n\}$.
2. A vector of controls $u_i(t) \in U_i \subseteq \mathbb{R}^{m_i}, \forall i \in I$, where U_i is the set of admissible controls of player i.
3. A vector of state variables $x(t) \in X \subseteq \mathbb{R}^p$, where X is the state space. The evolution of the state variables is governed by a system of differential equations (hence the name of differential games), called the state equations:

$$\dot{x}(t) = f\left(x(t), u_1(t), \ldots, u_n(t), t\right), \quad x(t_0) = x_0. \tag{1}$$

4. A payoff functional of player i

$$J_i(u(\cdot); x_0, t_0) = \int_{t_0}^{T} g_i(x(t), u(t), t)\, dt + S_i(x(T), T) \tag{2}$$

   where $u(t) \triangleq (u_1(t), \ldots, u_n(t))$, function g_i is player i's instantaneous payoff and function S_i is the terminal payoff. Throughout the paper, I assume that the players seek to maximize their payoffs.
5. An information structure. Here one needs to specify what information is available to a player when she selects a value of her control variable $u_i(t)$. The open-loop and Markovian information structures are the most often used in applications of differential games in management science. Open-loop means that the players base their decision only on time, whereas they use the *position of the game*, i.e., (x, t), as information, in a Markovian context.

6. A strategy for each player. For player $i \in I$, a strategy φ_i is a decision rule, chosen from the outset, which selects an action as a function of the information. A *Markovian* (or feedback) strategy selects the control action according to the rule $u_i(t) = \varphi_i(x, t)$. This means that player i observes the position (x, t) of the system and chooses her action as prescribed by the decision rule φ_i. An *open-loop strategy* is a degenerate Markovian strategy, that is, the control action is selected according to the decision rule $u_i(t) = \varphi_i(t)$. (In this case, there is actually no need to distinguish between $u_i(t)$ and $\varphi_i(t)$.)

Before introducing the cooperative and noncooperative solutions, we state the following remarks.

Remark 1 *If $\sum_{i \in I} J_i(u(\cdot); x_0, t_0) = 0$ the game is zero sum. Applications of DG in management science and economics are however typically of the nonzero-sum variety.*

Remark 2 *If the game is played over an infinite horizon, then each player optimizes the discounted stream of profits*

$$J_i(u(\cdot); x_0, t_0) = \int_{t_0}^{\infty} e^{-r_i t} g_i(x(t), u(t))\, dt,$$

where r_i is the discount rate of player i. Note that g_i is independent of t, and there is no salvage value, i.e., $S_i(x(T), T) = 0$. Following the tradition in dynamic optimization, in this context, one focuses on autonomous problems, i.e., one considers functions f and g_i that do not depend explicitly on time, and confines the interest to stationary strategies: $u_i(t) = \varphi_i(x)$. The reason is that at any instant of time, the players face essentially the same game for the remaining part of the time horizon.

Remark 3 *Using an open-loop strategy means that the player commits at t_0 to a fixed time path for her control actions, that is, her choice of control at each instant of time is predetermined. Obviously, a Markovian strategy gives more flexibility since it involves less commitment. However, open-loop strategies are technically easier to identify than their feedback counterparts.*

3.1 Noncooperative and Cooperative Solutions

Whereas in one-player decision problems the meaning of optimality is unambiguous, in many-player decision problems, the optimal collective and individual outcomes depend on the mode of play, i.e., whether or not the players cooperate. In a cooperative game, the starting point is that the players do not face any legal, political, sociological, psychological or economic obstacles to communicating and coordinating their strategies in view of optimizing their collective payoff. One popular assumption in such context is that the players maximize their joint payoff

$$J(u(\cdot); x_0, t_0) = \sum_{i \in I} J_i(u(\cdot); x_0, t_0). \tag{3}$$

Remark 4 *A more general formulation is to suppose that the players optimize a weighted sum of their objectives, i.e.,* $J = \sum_{i\in I} \alpha_i J_i(u(\cdot); x_0, t_0)$*, where the weights reflect bargaining strength and satisfy* $\alpha_i \geq 0$ *and* $\sum_{i\in I} \alpha_i = 1$*. In the sequel, we assume that all players have equal weight.*

Denote by $u^*(t) = (u_1^*(t), \dots u_n^*(t)), t \in [t_0, T]$, the control paths that provide a solution of the optimal control problem (3), subject to the state equations in (1). Denote by $J^*(x_0, t_0)$ the solution to the joint optimization problem. If this solution is implemented throughout the game, then player i receives the payoff (before any side payment)

$$J_i^*(t_0, x_0) = \int_{t_0}^{T} g_i(x^*(t), u^*(t), t)\, dt + S_i(x^*(T), T), \tag{4}$$

where $x^*(t)$ is the solution to

$$\dot{x}(t) = f(x(t), u^*(t), t), \qquad x(t_0) = x_0. \tag{5}$$

In the absence of cooperation, the players seek an equilibrium. If the players intervene simultaneously in the game, then the fundamental solution concept is the Nash equilibrium.

Definition 1 *The n-tuple* $\varphi^N = \left(\varphi_1^N, \dots, \varphi_n^N\right)$ *is a Nash equilibrium if*

$$J_i(\varphi_1^N, \dots, \varphi_n^N) \geq J_i(\varphi_1^N, \dots, \varphi_{i-1}^N, u_i, \varphi_{i+1}^N \dots, \varphi_n^N), \forall u_i \in U_i, \forall i \in I.$$

In words, the above definition says that no player can benefit from unilaterally deviating from the equilibrium strategy profile. Note that there may be more than one Nash equilibrium. In some game situations, one player (the follower) knows the strategy of the other (the leader) when she has to design her own strategy. An equilibrium in such a sequential game is called a Stackelberg equilibrium. When the game involves $n > 2$ players, then typically one assumes that the leader announces her strategy and then the $n - 1$ followers play a simultaneous game in which a Nash equilibrium is sought.

Remark 5 *There are some papers dealing with the coincidence of open-loop and feedback Nash equilibria and on the coincidence of Nash and Stackelberg equilibria (see, e.g., Reinganum (1982), Dockner et al. (1985), Freshtman (1983), Rubio (2006)).*

The usual tools for identifying noncooperative equilibria are the Hamilton-Jacobi-Bellman and the Maximum Principle methods, both originally developed within a context of dynamic optimization.

Denote by $u^N(t) = \left(u_1^N(t), \dots, u_n^N(t)\right)$ the control path generated by the Nash equilibrium strategy $\varphi^N(x(t), t) = \left(\varphi_1^N(x(t), t), \dots, \varphi_n^N(x(t), t)\right)$. The resulting payoff for player i is given by

$$J_i^N(x_0, t_0) = \int_{t_0}^{T} g_i\left(x^N(t), u^N(t), t\right) dt + S_i\left(x^N(T), T\right), \tag{6}$$

where $x^N(t)$ is the solution of

$$\dot{x}(t) = f\left(x(t), u^N(t), t\right), \qquad x(t_0) = x_0. \tag{7}$$

Noncooperative equilibria often play the role of benchmarks in a cooperative game, i.e., they provide what players could secure for themselves if there is no agreement.

4 Sharing the (Total) Cooperative Payoff

By virtue of joint optimization, the sum of individual payoffs under cooperation is greater than or equal to for its noncooperative counterpart, i.e.,

$$J^*(x_0, t_0) = \sum_{i \in I} J_i^*(x_0, t_0) \geq \sum_{i \in I} J_i^N(x_0, t_0) \ldots$$

To allocate the total cooperative outcome, and thereby distribute the dividend of cooperation, given by $\sum_{i \in I} J_i^*(x_0, t_0) - \sum_{i \in I} J_i^N(x_0, t_0)$, one can rely on a cooperative game approach or a bargaining procedure (e.g., Nash or Kalai-Smorodinsky). The staring point here is the concept of a *characteristic function* which assigns to each possible coalition of players K $(K \subseteq I)$ a numerical value $v(K)$ to be interpreted as a measure of its power (payoff, strength). In a n-player game, one needs to compute $2^n - 1$ values. The characteristic function satisfies the condition $v(\emptyset) = 0$, i.e., a void coalition has zero value or no power at all. For the grand coalition, we have $v(I) = J^*(x_0, t_0)$, i.e., the maximal outcome that the players can achieve when they cooperate. If $v(\cdot)$ satisfies the condition

$$v(K \cup L) \geq v(K) + v(L), \qquad \forall K, L \subseteq I, \quad K \cap L = \emptyset,$$

then the characteristic function is superadditive. This means that when two coalitions join forces, they can achieve at least the same payoff as they act separately. (In minimization games, the sought-after property is subadditivity, i.e., $v(K \cup L) \leq v(K) + v(L), \forall K, L \subseteq I, K \cap L = \emptyset$).

To compute the value for coalition $K, \forall K \subseteq I$, one assumes that its members optimize their joint payoff. The optimal result that the coalition can achieve depends on the behavior of the left-out players (LOP). Here, a number of options are available. A first option is the one suggested by von Neumann and Morgenstern (1944) in their breakthrough book. They assumed that the LOP form an anticoalition whose objective is to minimize the payoff of coalition S. The reason for supposing such extreme behavior is that computing $v(K)$ then amounts to solving a zero-sum game between K and $I \backslash K$ (i.e., the complement of K in I), which is easy and, more importantly, was known at that very early stage in the development of game theory. A second option is to suppose that the LOP also form a coalition, and define $v(K)$ as the Nash equilibrium outcome of coalition K in a two-player noncooperative game between K and $I \backslash K$. A third possibility is to assume that the left-out players do not form a coalition, but play individually. The value $v(K)$ is then defined as the Nash-equilibrium outcome in the noncooperative game with $n-k+1$ players, where k is the number of players in K. In this case, which is commonly used in applications, we have $v(\{i\}) = J_i^N(x_0, t_0), \forall i \in I$, i.e., each player obtains her Nash payoff.

Denote by Y the set of imputations. A vector $y = (y_1, ..., y_n) \in Y$ is an imputation if it satisfies

$$y_i \geq v(\{i\}), \forall i \in I,$$
$$\sum_{i=1}^{n} y_i = v(I).$$

An imputation is a vector of players' payoffs. The first condition above refers to individual rationality. Individual rationality means that a player will not accept an outcome that is not at least equal to what she could secure by acting alone, as measured by her characteristic function value. Group rationality simply states that the total cooperative gain when the grand coalition I forms is fully shared. From a negotiation perspective, the set of imputations can be seen as the set of feasible agreements. This set is seldom a singleton and therefore one needs other properties to predict the final outcome of the game. This is precisely the objective pursued by the different solution concepts of cooperative games. The set of solutions include the *kernel*, the *bargaining set*, the *stable set*, the *core*, the *Shapley value* and the *nucleolus* (see, e.g., Osborne and Rubinstein (1994) or Ordeshook (1986) for an introduction to these concepts). A solution is a sharing mechanism based on a series of desirable properties (often stated as axioms), such as fairness or stability. The two widely used solutions in practice are the Shapley value and the core. The Shapley value selects a single imputation, a n-vector denoted $\phi(v) = (\phi_1(v), \ldots, \phi_n(v))$, satisfying three axioms: fairness (similar players are treated equally), efficiency ($\sum_{i=1}^{n} \phi_i(v) = v(I)$) and linearity (a rather technical axiom needed to obtain uniqueness). The Shapley value is defined by

$$\phi_i(v) = \sum_{K \ni i} \frac{(n-k)!\,(k-1)!}{n!} \left(v(K) - v(K \backslash \{i\}) \right), \qquad \forall i \in I.$$

The term $v(K) - v(K \backslash \{i\})$ corresponds to the marginal contribution of player i to coalition K. Thus, the Shapley value allocates to each player the weighted sum of her contribution.

The core is the set of undominated imputations. This confers stability onto the core: there is no coalition that can claim to offer a better deal to its members. A drawback is that the core can be empty or contain a large number of imputations. In the former case, the players must adopt another solution concept to share the dividend of their cooperation, and in the latter, they will still need to negotiate to choose one specific imputation to be implemented. For an imputation to be in the core, it must satisfy

$$\sum_{i \in K} y_i \geq v(K), \forall K \subseteq I.$$

This condition is a generalization of the concept of individual rationality to coalitional rationality.

Remark 6 *The Shapley value may not lie in the core of the cooperative game, even when the core is nonempty. However, if the game is convex, that is,*

$$v(K \cup L) + v(K \cap L) \geq v(K) + v(L), \qquad \forall k, L \subseteq I,$$

then the core is nonempty and the Shapley value corresponds to its center of gravity.

To summarize, using a cooperative game approach yields individual payoffs for the whole interval $[t_0, T]$. In terms of individual rationality, whatever the selected imputation, it has, by definition, the property that each player's payoff in the cooperative game played on $[t_0, T]$ is higher or equal to what she would get in a noncooperative game played on the same time interval. This property can be termed as **overall individual rationality** (OIR). For each player, it constitutes a *necessary global* condition, or a minimal requirement, for her to adhere to the cooperative agreement. Dutta (1995) refers to this condition as individual rationality in an *ex ante* sense.

An alternative way of looking at a payoff-sharing problem is to consider it as a bargaining situation with two or more parties making claims until they reach a solution that is acceptable to all. Skipping the negotiation process itself, Nash (1953) proposed an axiomatic approach to solve a two-player bargaining problem. To introduce the *Nash Bargaining Solution* (NBS) in a simple way, denote by Z the set of feasible solutions, and by z_1 and z_2 the utilities of player 1 and 2, respectively. Nash proved that the unique solution satisfying the six axioms defined below is

$$\begin{aligned} \max g(z_1, z_2) &= (z_1 - \tilde{z}_1)(z_2 - \tilde{z}_2), \\ \text{subject to} &: \quad z_1 + z_2 = c, \end{aligned}$$

where $(\tilde{z}_1, \tilde{z}_2)$ is the status quo point, that is, the utilities that the players obtain if they walk away from the negotiation table, and c a constant corresponding to the maximal joint payoff. Denote by (z_1^*, z_2^*) the unique solution to the above optimization problem. The axioms state that the solution must be (1) feasible, $(z_1^*, z_2^*) \in T$; (2) individually rational $z_1^* \geq \tilde{z}_1$ and $z_2^* \geq \tilde{z}_2$; (3) Pareto-optimal; (4) invariant with respect to linear transformations; (5) fair, $z_1^* - \tilde{z}_1 = z_2^* - \tilde{z}_2$; (6) independent of irrelevant alternatives.

In the notation used here, the NBS allocates to player i the payoff

$$J_i^c(x_0, t_0) = J_i^N(x_0, t_0) + \frac{1}{2} \sum_{i=1}^{2} \left(J_i^*(x_0, t_0) - J_i^N(x_0, t_0) \right). \tag{8}$$

In words, NBS allocates to each player her noncooperative outcome (status quo) plus half of the surplus or dividend of cooperation, given as previously by the optimal joint payoff minus the sum of the noncooperative payoff. This sharing rule is often referred to as the egalitarian principle (see, e.g., Moulin (1980)). The following remarks are in order here. First, the status quo point $\left(J_1^N(t_0,x_0), J_2^N(t_0,x_0)\right)$ need not necessarily be the Nash outcome. It could be given by another point representing the payoffs that the players can secure if negotiation breaks down. Second, the formula in (8) assumes that the two players have equal bargaining power. This is a specific instance of the NBS, which also deals with the general case of unequal weights. Finally, the NBS can be generalized to $n > 2$ players.

Remark 7 *The optimization of the sum of payoffs of all players yields the individual payoffs $J_i^*(x_0,t_0), i \in I$. These rewards are the ones obtained before any side payment has taken place. However, the payoff $J_i^c(x_0,t_0), i \in I$ defined in (8) corresponds to the after-side-payment payoff of player i. Similarly, the Shapley-value component $\phi_i(v)$ of player i is also the after-side-payment payoff of player i. Put differently, the $J_i^*(x_0,t_0), i \in I$ do not embed the properties of the solution of a cooperative game, but the payoffs $J_i^c(x_0,t_0), i \in I$ do.*

5 Time Consistency

The overall individual rationality criteria does not guarantee that the players will stick to cooperation as time goes by. In a two-player differential game setting, Haurie (1976) showed that an agreement that is individually rational at initial instant of time t_0 may fail to remain in place till T. He distinguished between two possible reasons for such a breakdown:

1. If the players agree to renegotiate the original agreement at time $\tau \in [t_0, T]$, it is not sure that they will wish to continue with that agreement. In fact, they will not go on with the original agreement if it is not a solution of the cooperative game that starts out at time τ.
2. Suppose that a player is considering deviating from the agreement, that is, as of time $\tau \in [t_0, T]$ she will use a strategy different from the cooperative one. Actually, a player should deviate if this gives her a payoff in the continuation game that is greater than the one she stands to receive through continued cooperative play.

Given these possible instabilities, the natural question is whether or not something can be done about them. Before dealing with this, some definitions and notation are needed.

Recall that the total before-side-payment payoff that player i collects assuming that cooperation is in place during the whole planning horizon, is given by $J_i^*(x_0,t_0)$. Depending on the cooperative-game solution that the players agree to adopt, each one of them will end up with an imputation element that corresponds to an after-side-payment outcome. Denote by $J_i^c(x_0,t_0)$, player i's imputation. By definition of an imputation, we have $J_i^c(x_0,t_0) \geq v(\{i\})$, and $\sum_{i\in I} J_i^c(x_0,t_0) = J^*(x_0,t_0) = v(I)$. To define time consistency, denote by $J_i^*(x^*(\tau),\tau)$ the cooperative payoff-to-go before side payment for player i, at position $(x^*(\tau), \tau)$, $\tau \in [t_0, T]$ of the game. This quantity is easily obtained from (4) by a restriction of the time interval to $[\tau, T]$. Note that the implicit, but important, assumption here is that cooperation has prevailed from the beginning of the game at t_0 untill τ. Denote by $J_i^c(x^*(\tau), \tau)$ the cooperative payoff-to-go after side payment for player i, at position $(x^*(\tau), \tau)$, $\tau \in [t_0, T]$ of the game. This is the amount that player i will actually pocket. The difference between $J_i^c(x(\tau), \tau)$ and $J_i^*(x(\tau), \tau)$ can assume any sign, depending on whether the player is receiving or paying a certain amount.

Let $J_i^{nc}(x^*(\tau), \tau)$ be the noncooperative payoff-to-go at the same position $(x^*(\tau), \tau)$, $\tau \in [t_0, T]$. It is important to realize here that (i) this payoff is computed along the optimal collective trajectory $x^*(\tau)$ and (ii) it does not coincide with the equilibrium payoff computed earlier. Indeed, the restriction of (6) to the time interval $[\tau, T]$ would give to player i the following outcome:

$$J_i\left(x^N(\tau), \tau\right) = \int_\tau^T g_i\left(x^N(t), u^N(t), t\right) dt + S_i\left(x^N(T), T\right).$$

Unless the Nash equilibrium is efficient (Pareto-optimal), there is no reason to believe that $x^N(\tau)$ and $x^*(\tau)$ are the same. Hence, determining $J_i^{nc}(x^*(\tau), \tau)$ requires solving a noncooperative differential game on $[\tau, T]$ with an initial state value given by $x^*(\tau)$. With these ingredients, one can now define formally the concept of time consistency.

Definition 2 *A cooperative solution is time consistent at (x_0, t_0) if, at any position $(x^*(\tau), \tau)$, and for all $\tau \in [t_0, T]$, it holds that*

$$J_i^c(x^*(\tau), \tau) \geq J_i^{nc}(x^*(\tau), \tau), \qquad i \in I, \tag{9}$$

where $x^* \in X$ denotes the cooperative state trajectory.

Remark 8 *The concept of time consistency and its implementation in cooperative differential games was initially proposed in Petrosjan (1977) and Petrosjan and Danilov (1979, 1982, 1986). In these publications in Russian, as well as in the subsequent books in English (Petrosjan (1993), Petrosjan and Zenkevich (1996)), and in Petrosjan (1997), time consistency was termed dynamic stability.*

A stronger condition for dynamic individual rationality is that the cooperative payoff-to-go dominates (at least weakly) the noncooperative payoff-to-go, *at any position of the game*. This amounts to relaxing the assumption that the players have been following the cooperative state trajectory untill the comparison point, as is the case in time consistency. This is the *agreeability* concept developed by Kaitala and Pohjola (1990).

Definition 3 *A cooperative solution is agreeable at (x_0, t_0) if at any feasible position $(x(\tau), \tau)$, and for all $\tau \in [t_0, T]$, the following inequality holds:*

$$J_i^c(x(\tau), \tau) \geq J_i^{nc}(x(\tau), \tau), \quad i \in N.$$

There is a clear link between the **overall individual rationality** (OIR) condition and those of time consistency and agreeability. Indeed, OIR requires that the cooperative payoff-to-go dominate its noncooperative counterpart at position (x_0, t_0), i.e., OIR is a static concept. Time consistency and agreeability generalize OIR by imposing that the cooperative payoff-to-go dominance holds at any intermediate position of the game. Hence, one may refer to time consistency and agreeability as **subgame individual rationality** (SIR) conditions, i.e., at a game starting at an intermediate instant of time $\tau \in [t_0, T]$ with the appropriate initial condition for the state, i.e., $x(\tau) = x^*(\tau)$ for time consistency and an arbitrary feasible $x(\tau)$ for agreeability.

An alternative approach to the sustainability of cooperation is to look at instantaneous outcomes. Recalling that $g_i(x(t), u(t), t)$ denotes the instantaneous gain of player i at instant of time $t \in [t_0, T]$, one may impose one of the following **instantaneous individual rationality** (IIR) conditions,

$$g_i(x^*(t), u^*(t), t) \geq g_i(x^*(t), u(t), t), \quad \forall u(t) \in U_i, \quad \forall t \in [t_0, T], \tag{10}$$

$$g_i(x(t), u^*(t), t) \geq g_i(x(t), u(t), t), \quad \forall u(t) \in U_i, \quad \forall t \in [t_0, T]. \tag{11}$$

The inequality in (10) states that if player i implements her cooperative control at position $(x^*(t), t)$, then she would collect a higher instantaneous payoff than if she implemented any other feasible control. The inequality in (11) states a similar condition, but at any feasible position $(x(t), t)$. These conditions can be termed instantaneous time-consistency and instantaneous agreeability, respectively. Clearly, each IIR implies its corresponding SIR. Therefore, requiring an IIR condition to guarantee sustainability may appear, at a first glance, an appealing strategy. However, it may be impossible in practice to guarantee IIR, especially when decisions have carry-over effects as is generally the case in a differential game. As a possible scenario, consider two firms that want to merge their delivery operations. Suppose that the cooperative solution dictates that they should build a costly new depot at t_0. If in the benchmark noncooperative game such an investment is not necessary, then neither one of the two players is better off at t_0 under a cooperative regime. The implication is that imposing IIR may simply lead the players to drop the idea of coordinating their operations even if this option is profitable in the long term, both collectively and individually. This anecdotal example largely explains why the literature does not retain IIR as a workable concept and focuses on SIR.

6 Designing Transfer Schemes for Time Consistency

Solving a cooperative differential game and sustaining the agreement over time can be seen as a four-step algorithm:

Step 1 Compute the characteristic function values of the cooperative game for the whole agreement period $[t_0, T]$;

Step 2 Choose a solution concept and select one particular imputation to allocate to the players their shares in the total cooperative payoff;

Step 3 Compute the benchmark payoff entering in the definition of time consistency, i.e., $J_i^{nc}(x^*(\tau),\tau)\,,\ \forall i\in I,\forall\tau\in(t_0,T]$;

Step 4 Decompose over time the total individual cooperative payoffs, i.e., $J_i^c(x_0,t_0)\,,\ \forall i\in I$, subject to the satisfaction of the condition of time consistency.

The first two steps can be replaced by choosing a bargaining approach and its corresponding solution, e.g., a Nash bargaining solution.

6.1 Payoff Distribution Procedure

To implement Step 4 of the above algorithm, one determines a vector of time functions $\beta(t)=(\beta_1(t),\ldots,\beta_I(t))$ such that the following two conditions hold:

$$\int_{t_0}^{T}\beta_i(t)\,dt = J_i^c(x(t_0),t_0)\,,\quad i\in I,t\in[t_0,T]\,, \tag{12}$$

$$\int_{\tau}^{T}\beta_i(z)\,dz \geq J_i^{nc}(x^*(\tau),\tau)\,,\quad i\in I,z\in[t_0,T]\,. \tag{13}$$

The first equality is a feasibility condition. Indeed, it states that the sum of payments to player i over the agreement period $[t_0,T]$ must correspond to her share in the total cooperative outcome as determined in Step 2 of the algorithm. Petrosjan (1997) calls the vector $\beta(t)=(\beta_1(t),\ldots,\beta_I(t))$ a *payoff distribution procedure* (PDP). The inequality in (13) tells us that the payments that player i should receive over $[\tau,T]$ under cooperation must be greater than or equal to what she can get by switching to noncooperation at $\tau,\forall\tau\in[t_0,T]$, with the state value given by $x^*(\tau)$. (The assumption here is that if a player deviates and switches to her noncooperative strategy, she must continue to do so for the rest of the game.) Note that this is precisely the time-consistency condition, stated in terms of the time function $\beta_i(t)$, and that the left-hand side of (13) is player i's payoff-to-go at time τ. A PDP satisfying (13) is called a time-consistent PDP.

Two remarks are in order regarding the definition of a PDP. First, there is an infinite number of time functions that qualify as a PDP. This has the advantage of leaving room for adding other properties that one may wish to satisfy in the context of the problem at hand. One interpretation of the first remark is that the payment $\beta_i(t)$ may not be directly related to the instantaneous revenues and costs of player i at time t. Put differently, a PDP may have neither a particular relationship with the data of the problem beyond the fact that it does the job of ensuring the sustainability of the agreement, nor any particular economic interpretation. A natural question is then: could the time-functions $\beta_i(t)\,,i\in I$ be related to cooperative game solutions or bargaining outcomes? This is actually possible as seen in the following two examples. Second, the values of $\beta_i(t)\,,t\in[t_0,T]$ are not constrained to be nonnegative. This means that one cannot exclude a priori that some players may have to pay at some instants of time, instead of receiving money. This may be a less realistic in some applications. Petrosjan (1997) suggests some "regularization" procedures to alleviate the problem.

6.1.1 Decomposition Over Time of Shapley Value

Petrosjan and Zaccour (2003) have analyzed an infinite-horizon differential game of pollution control where each player (country) minimizes the cost of emissions reduction. Recall that the position of the game is defined by the pair $(x(t), t)$, where $x(t)$ is the state value at t. In this particular application, $x(t)$ represents the pollution stock. Denote by $\Gamma(x(t), t)$ the subgame starting at date t with a stock of pollution $x(t)$. Denote by $x^*(t)$ the optimal trajectory of the pollution stock under full cooperation, that is, when the grand coalition forms and minimizes the sum of all players' joint costs. Let $\Gamma(x^*(t), t)$ denote a subgame that starts along the cooperative trajectory of the state. The characteristic function value for a coalition $S \subseteq I$ in subgame $\Gamma(x^*(t), t)$ is defined to be its minimal cost and is denoted $v(S, x^*(t), t)$. With this notation, the total cooperative cost to be allocated among the players is then $v(I, x_0, t_0)$, that is, the minimal cost for the grand coalition I, as given by its characteristic function value in the game $\Gamma(x_0, t_0)$. Let $\phi(v, x(t), t) = (\phi_1(v, x(t), t), ..., \phi_n(v, x(t), t))$ denote the Shapley value in subgame $\Gamma(x(t), t)$. In this setting of cost minimization, $\beta_i(t)$ denotes the cost to be allocated to player i at instant of time t. Denote by r the common discount rate of all players.

Definition 4 *The vector $\beta(t) = (\beta_1(t), \cdots, \beta_n(t))$ is an Imputation Distribution Procedure (IDP) if*

$$\phi_i(v, x_0, t_0) = \int_{t_0}^{\infty} e^{-rt}\beta_i(t)dt, \qquad i = 1, ..., n. \tag{14}$$

Note that here the authors use the term *imputation* instead of payoff, to stress the relationship with the cooperative solution. The interpretation of the above definition is obvious: a time-function $\beta_i(t)$ qualifies as an IDP if it decomposes over time the total cost of player i as given by her Shapley-value component for the whole game $\Gamma(x_0, t_0)$, i.e., the sum of discounted instantaneous costs is equal to $\phi_i(v, x_0, t_0)$.

Definition 5 *The vector $\beta(t) = (\beta_1(t), \cdots, \beta_n(t))$ is a time-consistent IDP if at $(x^*(\tau), \tau)$, $\forall \tau \in [t_0, \infty)$, the following condition holds,*

$$\phi_i(v, x_0, t_0) = \int_{t_0}^{\tau} e^{-rt}\beta_i(t)dt + e^{-r\tau}\phi_i(v, x^*(\tau), \tau). \tag{15}$$

To interpret condition (15), assume that the players wish to renegotiate the initial agreement reached in the game $\Gamma(x_0, t_0)$ at (any) intermediate instant of time τ. At this moment, the state of the system is $(x^*(\tau), \tau)$, meaning that cooperation has prevailed from the initial time untill τ, and that each player i would have been allocated a stream of monetary amounts given by the first right-hand-side term. Now, if the subgame $\Gamma(x^*(\tau), \tau)$, starting with initial condition $x(\tau) = x^*(\tau)$, is played cooperatively, then player i will get her Shapley-value component in this game given by the second right-hand-side term of (15). If what she has been allocated until τ and what she will be allocated from this date onward add up to her cost in the original agreement, i.e., her Shapley value $\phi_i(v, x_0, t_0)$, then a renegotiation would leave the original agreement unaltered. If one can find an IDP $\beta(t) = (\beta_1(t), \cdots, \beta_n(t))$ such that (15) holds true, then this IDP is time consistent. To obtain the value $\beta_i(t), t \in [t_0, \infty)$, it suffices to differentiate (15), that is

$$\beta_i(t) = r\phi_i(v, x^*(\tau), \tau) - \frac{d}{dt}\phi_i(v, x^*(\tau), \tau). \tag{16}$$

This formula has an interesting economic interpretation. It allocates at instant of time t, to player i, a cost corresponding to the interest payment (interest rate time her cost-to-go under cooperation given by her Shapley value) minus the variation over time of this cost-to-go. This example shows that it is possible to choose a meaningful payoff-distribution procedure (or imputation-distribution procedure).

Remark 9 *The decomposition of the Shapley value in (16) is independent of the context considered in Petrosjan and Zaccour (2003), and therefore, the result is general.*

6.1.2 Decomposition of NBS Outcomes

Suppose that two players agree to cooperate over $[t_0, T]$ and that there is no salvage value, i.e., $S_i(x(T), T) = 0$. Suppose further that the players adopt the Nash bargaining solution to share the joint Pareto-optimal payoff. Recall that, for the whole game, player i's share is given by

$$J_i^c(x_0, t_0) = J_i^N(x_0, t_0) + \frac{1}{2}\left[J^*(x_0, t_0) - J^N(x_0, t_0)\right],$$

where $J^N(x_0, t_0) = \sum_{i=1}^2 J_i^N(x_0, t_0)$. Following the same philosophy as above, one defines

$$J_i^c(x_0, t_0) = \int_{t_0}^{\tau} \beta_i(t)dt + J_i^c(x^*(\tau), \tau), \tag{17}$$

where the last term corresponds to player i's payoff-to-go under a continuation of NBS given by

$$J_i^c(x^*(\tau), \tau) = J_i^N(x^*(\tau), \tau) + \frac{1}{2}\left[J^*(x^*(\tau), \tau) - J^N(x^*(\tau), \tau)\right].$$

Differentiating (17) with respect to time leads to

$$\begin{aligned} \beta_i(t) &= -\frac{d}{dt}\left(J_i^c(x^*(t), t)\right) \\ &= -\frac{d}{dt}\left(J_i^N(x^*(t), t) + \frac{1}{2}\left[J^*(x^*(t), t) - J^N(x^*(t), t)\right]\right). \end{aligned} \tag{18}$$

To show that the time function defined above is indeed a PDP, one needs to show that the following equality holds,

$$\int_{t_0}^{T} \beta_i(t)dt = J_i^c(x_0, t_0).$$

Integrating (18) yields

$$\begin{aligned}\int_{t_0}^{T}\beta_i(t)dt &= -\left[J_i^N\left(x^*\left(\tau\right),\tau\right)+\frac{1}{2}\left[J^*\left(x^*\left(\tau\right),\tau\right)-J^N\left(x^*\left(\tau\right),\tau\right)\right]\right]_{t_0}^{T}\\ &= -J_i^N\left(x^*\left(T\right),T\right)+J_i^N\left(x^*x_0,t_0\right)-\frac{1}{2}J^*\left(x^*\left(T\right),T\right)\\ &\quad+\frac{1}{2}J^*\left(x^*x_0,t_0\right)+\frac{1}{2}J^N\left(x^*\left(T\right),T\right)-\frac{1}{2}J^N\left(x^*x_0,t_0\right).\end{aligned}$$

Noting that all terms at $(x^*(T),T)$ are equal to zero, the above equation becomes

$$\begin{aligned}\int_{t_0}^{T}\beta_i(t)dt &= J_i^N\left(x_0,t_0\right)+\frac{1}{2}\left(J^*\left(x_0,t_0\right)-J^N\left(x_0,t_0\right)\right)\\ &= J_i^c(x_0,t_0).\end{aligned}$$

¿From (18), the interpretation of $\beta_i(t)$ is straightforward. Indeed, it corresponds to the negative of the variation of the NBS payoff-to-go along the Pareto-optimal state trajectory. If the value function is decreasing over time (which is intuitive in deterministic maximization problems), then $\beta_i(t)$ is non-negative.

Remark 10 *It is easy to verify that in the presence of a salvage value, one needs to make the following correction,*

$$\int_{t_0}^{T}\beta_i(t)dt+S_i\left(x^*\left(T\right),T\right)=J_i^c(x_0,t_0).$$

In an infinite-horizon game, one introduces discounting and follows the same reasoning as in the Shapley-value example to get the appropriate formula.

6.2 Other Schemes

Gao et al. (1989) analyze a two-player differential game where the players maximize their joint payoff over a finite horizon $[t_0,T]$, assuming zero-salvage values, $S_i\left(x\left(T\right),T\right)=0$. The optimal value is given by

$$J^*\left(x_0,t_0\right)=\sum_{i=1}^{2}J_i^*\left(x_0,t_0\right),$$

where $J_i^*\left(x_0,t_0\right)$ is the before-side payment of player i. The authors suggest allocating at $\tau\in[t_0,T]$ to player i the following cooperative payoff-to-go:

$$J_i^c\left(x^*\left(\tau\right),\tau\right)=\frac{J_i^N\left(x^*\left(\tau\right),\tau\right)}{\sum_{i=1}^{2}J_i^N\left(x^*\left(\tau\right),\tau\right)}\sum_{i=1}^{2}J_i^*\left(x^*\left(\tau\right),\tau\right),\tag{19}$$

in which $J_i^N\left(x^*\left(\tau\right),\tau\right)$ is the Nash outcome of player i in the subgame $\Gamma\left(x^*\left(\tau\right),\tau\right)$. The following remarks are in order concerning this scheme:

1. The allocated amounts satisfy

$$\begin{aligned}\sum_{i=1}^{2} J_i^c\left(x^*\left(\tau\right),\tau\right) &= \sum_{i=1}^{2} J_i^*\left(x^*\left(\tau\right),\tau\right), \quad \forall\tau\in\left[t_0,T\right],\\ J_i^c\left(x^*\left(\tau\right),\tau\right) &\geq J_i^N\left(x^*\left(\tau\right),\tau\right), \quad \forall\tau\in\left[t_0,T\right],\end{aligned}$$

   which shows feasibility and time consistency.
2. The fraction of the total efficient payoff allocated to player i varies over time and is determined as the player's "market share" of the total disagreement payoff-to-go.
3. Given that player i obtains $J_i^*\left(x^*\left(\tau\right),\tau\right)$ in the joint optimal solution, side payments will generally be necessary to reach $J_i^c\left(x^*\left(\tau\right),\tau\right)$.

It is interesting to note that the above scheme can be easily generalized to more than two-player settings and to infinite-horizon problems.

Haurie and Zaccour (1986) introduce the concept of dynamic side payment to allocate the dividend of cooperation over time. The setup is a two-player differential game where the players agree to optimize their joint payoff, and to use the Nash bargaining solution to share the total efficient payoff. In the absence of cooperation, player i implements at t the noncooperative control $u_i^{nc}(t)$, which results in the state value $x^{nc}(t)$ and instantaneous payoff $g_i\left(x^{nc}(t),u^{nc}(t),t\right)$. In the authors' setting, the players are electric utilities and their payoffs are independent in the noncooperative game. Hence, in the latter, the authors solve a pair of optimal control problems, not a game. The need for a dynamic side payment stems from the fact that whereas the total dividend of cooperation $J^*\left(x_0,t_0\right)-J^{nc}\left(x_0,t_0\right)$ is nonnegative, this does not necessarily hold true at each instant of time. Indeed, there is no reason a priori to believe that the quantity

$$\sum_{i=1}^{2}\left[g_i\left(x^*(t),u^*(t),t\right)-g_i\left(x^{nc}(t),u^{nc}(t),t\right)\right]$$

is positive for all $t\in[t_0,T]$. The authors define an extended instantaneous dividend (EID), and show, under some concavity conditions, that this quantity is nonnegative at any instant of time. More precisely, let $\lambda^*(t)\in\mathbb{R}^p$ denote the costate vector associated with the solution of the joint-optimal-control problem (3), subject to the state dynamics (1). The reader who is not familiar with the concept of costate variables in optimal control theory may wish to consult a textbook, e.g., Kamien and Schwartz (1991), Sethi and Thompson (2000), Léonard and Long (1992). A costate (or adjoint) variable is the shadow price of the corresponding state variable. Denote by $G_i\left(x(t),u(t),t\right)$ the *instantaneous extended payoff* of player i:

$$G_i\left(x(t),u(t),t\right)=g_i\left(x(t),u(t),t\right)+\frac{d}{dt}\left(\lambda^*(t)x(t)\right),$$

that is, the current payoff plus the variation in the state, valued at the optimal shadow price. The extended instantaneous dividend is the difference between the extended instantaneous efficient and noncooperative payoffs, i.e.,

$$\begin{aligned} & \sum_{i=1}^{2}\left[G_i\left(x^*(t),u^*(t),t\right)-G_i\left(x^{nc}(t),u^{nc}(t),t\right)\right] \\ = & \sum_{i=1}^{2}\left[g_i\left(x^*(t),u^*(t),t\right)-g_i\left(x^{nc}(t),u^{nc}(t),t\right)\right] \\ & +\frac{d}{dt}\left(\lambda^*(t)\left[x^*(t)-x^{nc}\left(t\right)\right]\right), \end{aligned}$$

which happens to be nonnegative for concave control problems. Thus at each instant of time, the players have a positive dividend to share. The dynamic side payment is given by

$$\begin{aligned} sp(t) = & \frac{1}{2}\frac{d}{dt}\{\lambda^*(t)\left[x^*(t)-x^{nc}(t)\right]\}+\frac{1}{2}[g_1\left(x^{nc}(t),u^{nc}(t),t\right) \\ & -g_1\left(x^*(t),u^*(t),t\right)+g_2\left(x^*(t),u^*(t),t\right)-g_2\left(x^{nc}(t),u^{nc}(t),t\right)]. \end{aligned} \tag{20}$$

The above rule is based on two principles. First, the instantaneous cooperative surplus is divided according to the egalitarian principle. This induces a side-payment rule at any instant of time. Second, this payment is modified by adding the rate of change of the imputed value of the (instantaneous) deviation between the cooperative and the noncooperative states. The imputed value is calculated along the cooperative path. In this setting, the conditions for time consistency can be written as follows:

$$\begin{aligned} \int_{\tau}^{T}\left(g_1\left(x^*(t),u^*(t),t\right)-sp\left(t\right)\right)dt & \geq J_1^{nc}\left(x^*\left(\tau\right),\tau\right),\forall\tau\in\left[t_0,T\right], \\ \int_{\tau}^{T}\left(g_2\left(x^*(t),u^*(t),t\right)+sp\left(t\right)\right)dt & \geq J_2^{nc}\left(x^*\left(\tau\right),\tau\right),\forall\tau\in\left[t_0,T\right]. \end{aligned}$$

These conditions may not necessarily be satisfied from the outset. The interest of the approach followed here is that it links time consistency with some important ingredients of a dynamic bargaining problem, namely, current payoffs, instantaneous dividend of cooperation, and side payments.

7 Concluding Remarks

The following sections comment on (i) the literature on time consistency in differential games; (ii) the concept of time consistency in noncooperative games; and (iii) the other approaches proposed in the literature to sustain cooperation over time.

7.1 Additional Readings

We provide here some aditional references to the literature that use the mechanisms dealt with in this tutorial. Haurie and Zaccour (1986, 1991) introduce the notion of dynamic side payments and applied it in the context of power exchange between interconnected utilities. Kaitala and Pohjola (1988) determine a transfer payment rule based on the steady state stock level in an infinite-horizon differential game in fisheries. Kaitala and Pohjola (1995) and Jørgensen and Zaccour (2001) analyze a pollution differential game where a vulnerable player is located downstream a polluting player. In both papers, dynamic side payments are derived to sustain cooperation over time between the two players. Kaitala and Pohjola (1990) introduce the concept of agreeability in a game of capitalists versus workers. Simply stated, this is a classical problem of sharing the proceeds of collective effort by different groups in a firm. Yeung and Petrosjan (2001) provide a proportional time-consistent solution for cooperative differential games.

¿From the definitions of time consistency and agreeability, it is clear that the latter implies the former. In the class of *linear-state differential games* (LSDG), Jørgensen et al. (2003) show that, if the cooperative solution is time consistent, then it is also agreeable. The class of LSDG has the specific feature that the instantaneous payoff, salvage-value function of player $i, i \in I$, and the state dynamics are linear in the state $x(t)$. Further, Jørgensen et al. (2005) show that there is also equivalence between time consistency and agreeability in the class of *homogenous linear-quadratic differential games* (HLQDG). Such games have the following two characteristics: (i) The instantaneous-payoff function $g_i(\cdot)$ and the salvage-value function $S_i(\cdot)$ are quadratic with no linear terms in the state and control variables for $i \in I$; (ii) the function $f(\cdot)$ that describes the state evolution is linear in the state and control variables.

This paper has focused on deterministic differential cooperative games. There is a developing literature using stochastic models. Yeung and Petrosjan (2004, 2005) and Yeung et al. (2007) deal with the consistency of cooperative solutions in stochastic differential games with, and without, transferable payoffs. Yeung and Petrosjan (2006) establish conditions to obtain dynamically stable joint ventures between firms in both deterministic and stochastic settings. The theory and some applications of cooperative stochastic differential games are developed in the book by Yeung and Petrosjan (2005).

7.2 Time Consistency in Noncooperative Games

The previous section dealt with time consistency in the context of a cooperative game. The concept originates in optimal control and noncooperative differential games. In optimal control applications it was particularly explored in the context of macroeconomic planning (Kydland and Prescott (1977), Miller and Salmon (1985)). Time consistency here means that, if a public decision-maker reconsiders her originally decided policy at time τ, when the economic system has reached state $x^*(\tau)$ on its optimal trajectory, she will find no reason to replace the continuation of the original policy with any other policy.

In a noncooperative differential game, the idea of time consistency is the same. In such a game we have an equilibrium strategy profile and its associated state trajectory. If the players reconsider their strategy choices at any intermediate point on the equilibrium trajectory, the equilibrium profile is time consistent if its restriction to the remaining time interval (also) provides an equilibrium in the subgame. (Başar and Olsder (1995) use the term "weakly time consistent"). A stronger notion is subgame, or Markov, perfectness. If the game is reconsidered at *any* position, that is, also at all feasible points *off* the equilibrium trajectory, an equilibrium profile is perfect if its restriction to the remaining time interval induces an equilibrium in the subgame starting out at the intermediate position. (Başar and Olsder (1995) use the term "strongly time consistent"). The requirement here is that equilibrium strategies should induce optimal behavior not only along the equilibrium state trajectory, but also off this trajectory. The following results are well known (see Dockner et al. (2000)):

1. Subgame perfectness implies time consistency.
2. Every Markovian (and hence every open-loop) Nash equilibrium of a noncooperative game is time consistent.
3. Feedback Stackelberg equilibrium is time consistent, but open-loop Stackelberg equilibrium is generally not. In the latter case, we have that (i) the leader is better off revising the originally announced plan; and (ii) the follower should not believe the leader's announcement. Note that in games having some special structures, it happens that the open-loop Stackelberg equilibrium is time consistent. For an example, see Martín-Herrán et al. (2005).

7.3 Other Approaches to Sustainability

In the framework of time consistency (including agreeability), the players compare at intermediate instants of time cooperative payoffs-to-go to their non cooperative counterparts.

Another line of research to enforce an efficient outcome is to design agreements which enjoy the property of being an equilibrium. Since in an equilibrium no player can improve her payoff by deviating unilaterally from the agreement, the latter is then sustained during the whole period. There are (rare) cases in which a cooperative outcome "by construction" is in equilibrium. This occurs if a game has a Nash equilibrium which is also efficient. However, there are only very few differential games having this property. The fishery game of Chiarella et al. (1984) is an example. Rincón-Zapatero et al. (2000) state conditions for Markov perfect equilibria to be Pareto optimal in a special class of differential games. Martín-Herrán and Rincón-Zapatero (2005) characterize efficient Nash equilibria in fishery differential games

The idea of constructing a cooperative equilibrium has been heavily studied in repeated games. One "Folk Theorem" result states that any individually rational payoff vector can be supported as a Nash equilibrium outcome in an infinitely repeated game if players are sufficiently far-sighted (Friedman (1986)). The implication here is that cooperative outcomes may be supported by noncooperative equilibrium strategies, but it raises the question if individually rational outcomes exist. In repeated games with complete information and perfect monitoring, the answer is yes since the players face the same game at every stage. In state-space games the situation is different. In a discrete-time setup, a stochastic game includes a state variable that evolves over time, as a product of the initial conditions, the players' actions, and a transition law. The latter may be deterministic, in which case the game sometimes is called a dynamic game (a difference game). A Folk Theorem for stochastic games is given in Dutta (1995), but there seem to be no general theorems for differential games. Particular results exist for situations in which Pareto-optimal outcomes are supported by trigger strategies (Tolwinski, Haurie and Leitmann (1986), Haurie and Pohjola (1987), Haurie et al. (1994), Dockner et al. (2000, Ch. 6)). Such strategies embody (effective) punishments that deprive any player the benefits of a defection, and the threats of punishments are credible which ensures that it is in the best interest of the player(s) who did not defect to implement a punishment.

In two-player differential games, another option is to support the cooperative solution by incentive strategies (see, e.g., Ehtamo and Hämäläinen (1986, 1989, 1993) and Jørgensen and Zaccour (2002b, 2003), Breton et al. (2007), Martín-Herrán and Zaccour (2005, 2008)). Informally, incentive strategies are functions which depend on the possible deviation of the other player relative to the coordinated solution. If this deviation is null, then the incentive strategy will prescribe to the player to choose the cooperative control. Although such strategies are relatively easy to construct, one concern is their credibility. By this we mean that it is in the best interest of each player to implement her incentive strategy, and not the coordinated solution, if she observes that the other one has deviated from the coordinated solution.

References

[1] Başar, T. and Olsder, G.J. (1995). *Dynamic Noncooperative Game Theory*, Academic Press, New York.

[2] Breton, M., Sokri A. and Zaccour, G. (2008). "Incentive equilibrium in an overlapping-generations environmental game," *European Journal of Operational Research* 185, 687–699.

[3] Chiarella, C., Kemp, M.C., Long, N.V. and Okuguchi, K. (1984). "On the Economics of International Fisheries," *International Economic Review* 25, 85–92.

[4] Dockner, E., Feichtinger, G. and Jørgensen, S. (1985). "Tractable Classes of Nonzero-Sum Open-Loop Nash Differential Games," *Journal of Optimization Theory and Applications* 45, 179–198.

[5] Dockner, E., Jørgensen, S., Van Long, N. and Sorger, G. (2000). *Differential Games in Economics and Management Science*, Cambridge University Press, Cambridge.

[6] Dutta, P.K. (1995). "A Folk Theorem for Stochastic Games," *Journal of Economic Theory* 66, 1–32.

[7] Ehtamo, H. and Hämäläinen, R.P. (1986). "On affine incentives for dynamic decision problems," *Dynamic Games and Applications in Economics*, T. Basar, Editor, Springer, Berlin, 47–63.

[8] Ehtamo, H. and Hämäläinen, R.P. (1989). "Incentive strategies and equilibria for dynamic games with delayed information," *Journal of Optimization Theory and Applications* 63, 355–370.

[9] Ehtamo, H. and Hämäläinen, R.P. (1993). "A Cooperative incentive equilibrium for a resource management problem," *Journal of Economic Dynamics and Control* 17, 659–678.

[10] Engwerda, J. (2005). *Linear-Quadratic Dynamic Optimization and Differential Games*, Wiley, New York.

[11] Fershtman, C. (1987) "Identification of Classes of Differential Games for Which the Open Loop is a Degenerate Feedback Nash Equilibrium," *Journal of Optimization Theory and Applications* 55, 217–231.

[12] Friedman, J.W. (1986). *Game Theory with Applications to Economics*, Oxford University Press, Oxford.

[13] Gao, L., Jakubowski, A., Klompstra, M.B. and Olsder, G.J. (1989). "Time-Dependent Cooperation in Games," *Differential Games and Applications*, T.S. Başar and P. Bernhard, Editors, Springer-Verlag, Berlin.

[14] Haurie, A., (1976). "A Note on Nonzero-sum Differential Games with Bargaining Solution," *Journal of Optimization Theory and Applications* 18, 31–39.

[15] Haurie, A., Krawczyk, J.B. and Roche, M. (1994). "Monitoring Cooperative Equilibria in a Stochastic Differential Game," *Journal of Optimization Theory and Applications* 81, 79–95.

[16] Haurie, A. and Pohjola, M. (1987). "Efficient Equilibria in a Differential Game of Capitalism," *Journal of Economic Dynamics and Control* 11, 65–78.

[17] Haurie, A. and Zaccour, G. (1986). "A Differential Game Model of Power Exchange between Interconnected Utilities," *Proceedings of the 25th IEEE Conference on Decision and Control*, Athens, 262–266.

[18] Haurie, A. and Zaccour, G. (1991). "A Game Programming Approach to Efficient Management of Interconnected Power Networks," *Differential Games - Developments in Modelling and Computation*, R.P. Hämäläinen and H. Ehtamo, Editors, Springer-Verlag, Berlin.

[19] Isaacs R. (1965), *Differential Games*, Wiley, New York.

[20] Jørgensen, S., Martín-Herrán, G. and Zaccour, G. (2003). "Agreeability and Time-Consistency in Linear-State Differential Games," *Journal of Optimization Theory and Applications* 119, 49–63.

[21] Jørgensen, S., Martín-Herrán, G. and Zaccour, G. (2005). "Sustainability of Cooperation Overtime in Linear-Quadratic Differential Game," *International Game Theory Review* 7, 395–406.

[22] Jørgensen, S. and Zaccour, G. (2001). "Time Consistent Side Payments in a Dynamic Game of Downstream Pollution," *Journal of Economic Dynamics and Control* 25, 1973–1987.

[23] Jørgensen, S. and Zaccour, G. (2002a). "Time Consistency in Cooperative Differential Game," *Decision and Control in Management Science: In honor of Professor Alain Haurie*, G. Zaccour, Editor, Kluwer, Boston, 349–366.

[24] Jørgensen, S. and Zaccour, G. (2002b). "Channel Coordination over Time: Incentive Equilibria and Credibility," *Journal of Economic Dynamics and Control* 27, 801–822.

[25] Jørgensen, S. and Zaccour, G. (2003). "A Differential Game of Retailer Promotions," *Automatica* 39, 1145–1155.

[26] Jørgensen, S. and Zaccour, G. (2004). *Differential Games in Marketing*, Kluwer, Boston.

[27] Kaitala, V. and Pohjola, M. (1988). "Optimal Recovery of a Shared Resource Stock: A Differential Game Model with Efficient Memory Equilibria," *Natural Resource Modeling* 3, 91–118.

[28] Kaitala, V. and Pohjola, M. (1990). "Economic Development and Agreeable Redistribution in Capitalism: Efficient Game Equilibria in a Two-class Neoclassical Growth Model," *International Economic Review* 31, 421–437.

[29] Kaitala, V. and Pohjola, M. (1995). "Sustainable International Agreements on Greenhouse Warming: A Game Theory Study," *Annals of the International Society of Dynamic Games* 2, 67–87.

[30] Kamien, M.I. and Schwartz, N. (1991). *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, North-Holland, Amsterdam.

[31] Kydland, F.E. and Prescott, E.C. (1977). "Rules Rather Than Discretion: The Inconsistency of Optimal Plans," *Journal of Political Economy* 85, 473–493.

[32] Léonard, D. and van Long, N. (1992). *Optimal Control Theory and Static Optimization in Economics*, Cambridge University Press, Cambridge.

[33] Martín-Herrán, G. and Rincón-Zapatero, J.P. (2005). "Efficient Markov Perfect Nash Equilibria: Theory and Application to Dynamic Fishery Games," *Journal of Economics Dynamics and Control* 29, 1073–1096.

[34] Martín-Herrán, G., Taboubi, S. and Zaccour, G. (2005). "A time-consistent Open-loop Stackelberg Equilibrium of Shelf-Space Allocation," *Automatica* 41, 971–982.

[35] Martín-Herrán, G. and Zaccour, G. (2005). "Credibility of Incentive Equilibrium Strategies in Linear-State Differential Games," *Journal of Optimization Theory and Applications* 126, 1–23.

[36] Martín-Herrán, G. and Zaccour, G. (2008). "Credible Linear-Incentive Equilibrium Strategies in Linear-Quadratic Differential Games," *Annals of the International Society of Dynamic Games*, to appear.

[37] Miller, M. and Salmon, M. (1985). "Dynamic Games and the Time Inconsistency of Optimal Policy in Open Economies," *The Economic Journal*, Supplement, 95, 124–137.

[38] Moulin, H. (1988). *Axioms of Cooperative Decison Making*, Cambridge University Press, Cambridge.

[39] Nash, J. (1953). "Two Person Cooperative Games," *Econometrica* 21, 128–140.

[40] Ordeshook, P.C. (1986). *Game theory and Political Theory*, Cambridge University Press, Cambridge, UK.

[41] Osborne, M.J. and Rubinstein, A. (1994). *A Course in Game Theory*, MIT Press, Cambridge, MA.

[42] Petrosjan, L. (1977). "Stable Solutions of Differential Games with Many Participants," *Viestnik of Leningrad University* 19, 46–52.

[43] Petrosjan, L. (1993). *Differential Games of Pursuit*, World Scientific, Singapore, pp. 270–282.

[44] Petrosjan, L. (1997). "Agreeable Solutions in Differential Games," *International Journal of Mathematics, Game Theory and Algebra* 7, 165–177.

[45] Petrosjan, L. and Danilov, N.N. (1979). "Stability of solutions in nonzero sum differential games with transferable payoffs," *Journal of Leningrad University* N1, 52–59 (in Russian).

[46] Petrosjan, L. and Danilov, N.N. (1982). *Cooperative Differential Games and Their Applications*, Tomsk University Press, Tomsk.

[47] Petrosjan, L. and Danilov, N.N. (1986). "Classification of dynamically stable solutions in cooperative differential games," Isvestia of high school, 7, 24–35 (in Russian).

[48] Petrosjan, L. and Zaccour, G. (2003). "Time-Consistent Shapley Value of Pollution Cost Reduction," *Journal of Economic Dynamics and Control* 27, 381–398.

[49] Petrosjan, L. and Zenkevich, N.A. (1996). *Game Theory*, World Scientific, Singapore.

[50] Reinganum, J.F. (1982). "A Class of Differential Games For Which the Open-Loop and Closed-Loop Equilibria Coincide," *Journal of Optimization Theory and Applications* 36, 453–462.

[51] Rincón-Zapatero, J.P., Martín-Herrán, G. and Martinez, J. (2000). "Identification of Efficient Subgame-perfect Nash Equilibria in a Class of Differential Games," *Journal of Optimization Theory and Applications* 104, 235–242.

[52] Rubio, S. (2006). "On the coincidence of feedback Nash equilibria and Stackelberg equilibria in economic applications of differential games," *Journal of Optimization Theory and Applications* 128, 203–221.

[53] Sethi, S.P. and Thompson, G.L. (2000). *Optimal Control Theory: Applications to Management Science and Economics*, Kluwer, Boston.

[54] Tolwinski, B., Haurie A. and Leitmann, G. (1986). "Cooperative Equilibria in Differential Games," *Journal of Mathematical Analysis and Applications* 119, 182–202.

[55] Von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ.

[56] Yeung, D.W.K. and Petrosjan, L. (2005). *Cooperative Stochastic Differential Games*, Springer, New York, NY.

[57] Yeung, D.W.K. and Petrosjan, L. (2001). "Proportional Time-Consistent Solutions in Differential Games," *Proceedings of International Conference on Logic, Game Theory and Applications*, Saint Petersburg, 254–256.

[58] Yeung, D.W.K. and Petrosjan, L. (2004). "Subgame Consistent Cooperative Solutions in Stochastic Differential Games," *Journal of Optimization Theory and Applications* 120, 651–666.

[59] Yeung, D.W.K. and Petrosjan, L. (2005). "Consistent Solution of a Cooperative Stochastic Differential Game with Nontransferable Payoffs," *Journal of Optimization Theory and Applications* 124, 701–724.

[60] Yeung, D.W.K. and Petrosjan, L. (2006). "Dynamically Stable Corporate Joint Ventures," *Automatica* 42, 365–370.

[61] Yeung, D.W.K., Petrosjan, L. and Yeung, P.M. (2007). "Subgame Consistent Solutions for a Class of Cooperative Stochastic Differential Games with Nontransferable Payoffs," *Annals of the International Society of Dynamic Games* 9, 153–170.