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Abstract

The exponential stability, and the static output feedback stabilization with an α-stability constraint problems of continuous-time singular linear systems with time-varying delay in a range and subject to saturating actuators are addressed. New delay-range-dependent sufficient conditions such that the system is regular, impulse free and α-stable are developed in the linear matrix inequality (LMI) setting. An iterative LMI (ILMI) design algorithm for a static output feedback controller which guarantees that the closed-loop dynamics will be regular, impulse-free, and α-stable is proposed. Some numerical examples are employed to show the usefulness of the proposed results.

Key Words: Singular time-delay systems, delay-dependent, stability, α-stability, linear matrix inequality, stabilization, static output feedback.

Résumé

Cet article traite de la statbilité exponentielle et de la commande par retour de sortie de la classe des systèmes singuliers avec retard variant dans le temps dans une plage donnée et dont la commande est bornée. De nouvelles conditions suffisantes en forme d'inégalités matricielles pour assurer que le système en boucle fermée est régulier, sans impulsion et exponentiellement stable sont développées. Un algorithme itérative est proposé pour résoudre ces inégalites matricielles. Des exemples numériques sont présentés pour montrer l'utilité des résultats proposés.

Mots clés : systèmes singuliers avec retard, conditions dépendantes du retard, stabilité exponentielle, stabilisabilité, commande par retour de sortie.

1 Introduction

Singular time-delay systems arise in a variety of practical systems such as networks, circuits, power systems and so on [2]. Since singular time-delay systems are matrix delay differential equations coupled with matrix difference equations, the study for such systems is much more complicated than that for standard state-space time-delay systems or singular systems. The existence and uniqueness of a solution to a given singular time-delay system is not always guaranteed and the system can also have undesired impulsive behavior. Therefore, for a singular time-delay system, it is important to develop conditions which guarantee that the given singular system is not only stable but also regular and impulse-free.

Both delay-independent and delay-dependent stability conditions for singular time-delay systems have been derived by using the time domain method, see [3, 4, 5] and references therein. Recently, a free-weighting matrix method is proposed in [8], [9] and [10] to study the delay-dependent stability for time-delay systems with constant and time-varying delay, in which the bounding techniques on some cross product terms are not involved. The new method has been shown more effective in reducing conservatism entailed in previous results, especially for uncertain systems. In 2007, Zhu et al adopted this technique for singular time-delay systems [5]. Also, delay-range-dependent concept was recently studied, where the delays are considered to vary in a range and thereby more applicable in practice [13, 6]. To the best of the authors' knowledge, delay-dependent stability conditions for singular time-delay systems has been addressed only for constant delay, and delay-range-dependent stability conditions has not been addressed yet.

Formally speaking, these conditions provide only the asymptotic stability of singular time-delay systems. In [22], the global exponential stability for a class of singular systems with multiple time delays is investigated and an estimate of the convergence rate of such systems is presented. One may ask if there exists a possibility to use the LMI approach for deriving exponential estimates for solutions of singular time-delay systems.

The problem of stabilizing linear systems with saturating controls has been widely studied because of its practical interest [16]. Control saturation constraint comes from the impossibility of actuators to drive signal with unlimited amplitude or energy to the plants. However, only few works have dealt with stability analysis and the stabilization of singular linear systems in the presence of actuator saturation, see for example [17]. It is established in [17] that a singular linear system with actuator saturation is semi-globally asymptotically stabilizable by linear state feedback if its reduced system under actuator saturation is semi-globally asymptotically stabilizable by linear feedback. To the best of the authors' knowledge, the stabilization for singular time-delay systems in the presence of actuator saturation has not been fully addressed yet.

Different control saturation models are proposed in the literature, i.e. regions of saturation, differential inclusion and sector modeling. In [24], a comparative analysis of these models is presented, and concluded that the differential inclusions model lead to the least conservative design.

The static output feedback problem is probably the most important open question in control engineering. In contrast to the linear systems, there are only few papers solving the static output feedback problems for singular systems, see [21, 14]. In [21], the authors introduce an equality constraint in order to get an LMI sufficient conditions for admissibility of closed-loop systems. Yet, this equality constraints introduce conservativeness. This approach has been generalized by [7] to singular time-delay systems. In [14], singular systems is assumed to have some characteristics in advance: regularity and absence of direct action of control inputs on the algebraic variables, which is not always the case.

This paper addresses two important problems which has never been addressed before. First, delay-range-dependent exponential stability conditions for singular time-delay systems is established in terms of LMIs without using any bounding technique. The method used is based on the LyapunovKrasovskii approach. Second, an iterative LMI algorithm has been proposed in order to design static output feedback stabilizing controllers for singular time-delay systems in the presence of actuator saturation. The objective of the control design is twofold. It consists in determining both a static output feedback control law to guarantee that the system is regular, impulse-free and exponentially stable with a predefined decaying rate for the closed-loop system, and a set of safe initial conditions for which the exponential stability of the saturated closed-loop system is guaranteed. This set is maximized by the algorithm. Also, The least conservative model for the actuator saturation given in terms of differential inclusions is used here. Two numerical examples are employed to show the usefulness of the proposed results.

Notation: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "$^\mathsf{T}$" denotes matrix transposition and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\mathbb{I}$ is the identity matrices with compatible dimensions. $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote, respectively, the maximal and minimal eigenvalue of matrix P. co $\{\cdot\}$ denotes a convex hull. $C_\tau = C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. $\|\cdot\|$ refers to the Euclidean vector norm whereas $\|\phi\|_c = sup_{-\tau\leq t\leq 0}\|\phi(t)\|$ stands for the norm of a function $\phi \in C_\tau$. C_τ^v is defined by $C_\tau^v = \{\phi \in C_\tau; \|\phi\|_c < v, v > 0\}$. $\lceil x\rceil^+$ stands for the smallest integer greater than or equal to x.

2 Problem Statement and Definitions

Consider the linear singular time-delay system:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + Bsat(u(t)) \quad (1a)$$

$$y(t) = Cx(t) \quad (1b)$$

$$x(t) = \phi(t), \quad t \in [-d_2, 0] \quad (1c)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the saturating control input, $y(t) \in \mathbb{R}^q$ is the measurement, the matrix $E \in \mathbb{R}^{n \times n}$ may be singular, and we assume that $rank(E) = r \leq n$, A, A_d, B and C are known real constant matrices, $sat(u(t)) = [sat(u_1(t)), ..., sat(u(t))]$, where $-\overline{u}_i \leq sat(u_i(t)) \leq \overline{u}_i$, $\phi(t) \in C_\tau$ is a compatible vector valued continuous function and $d(t)$ is a time-varying continuous function that satisfies:

$$0 < d_1 \leq d(t) \leq d_2 \qquad \text{and} \qquad \dot{d}(t) \leq \mu < 1 \quad (2)$$

The following definitions will be used in the rest of this paper:

Definition 2.1

i. *System (1) is said to be regular if the characteristic polynomial, $det(sE - A)$ is not identically zero.*

ii. *System (1) is said to be impulse-free if $deg(det(sE - A)) = rank(E)$*

iii. *System (1) is said to be* exponentially stable *if there exist $\sigma > 0$ and $\gamma > 0$ such that, for any compatible initial conditions $\phi(t)$, the solution $x(t)$ to the singular time-delay system satisfies*

$$\|x(t)\| \leq \gamma e^{-\sigma t} \|\phi\|_c$$

iv. *System (1) is said to be exponentially admissible if it is regular, impulse-free and exponentially stable.*

Lemma 2.1 ([26]) *Suppose system (1) is regular and impulse-free, then the solution to system (1) exists and it is impulse-free and unique on $(0, \infty)$.*

Lemma 2.2 ([27]) *Given a matrix D, let a positive-definite matrix S and a positive scalar $\eta \in (0, 1)$ exist such that*

$$D^\top SD - \eta^2 S < 0$$

then, the matrix D satisfies the bound

$$\left\|D^i\right\| \leq \chi e^{-\lambda i} \text{ with } \chi = \sqrt{\frac{\lambda_{max}(S)}{\lambda_{min}(S)}} \text{ and } \lambda = -ln(\eta)$$

Now, consider the following static output feedback controller:

$$u(t) = Ky(t), \;\; K \in R^{m\times q} \tag{3}$$

Applying this controller to system (1), we obtain the closed-loop system as follows:

$$E\dot{x}(t) = Ax(t) + A_d x(t-d(t)) + Bsat(KCx(t)) \tag{4}$$

Generally, for a given stabilizing state feedback K, it is not possible to determine exactly the region of attraction of the origin with respect to system (4). Hence, a domain of initial conditions, for which the exponential stability of the system (4) is ensured, has to be determined. Thus, our problem is considered as a local stabilization problem.

The two problems to be solved in this paper can be summarized as follows:

- Find delay-range-dependent LMI conditions that guarantees the exponential admissibility of system (1) with a predefined minimum decaying rate.
- Find a static output feedback law of the form (3) and a set of initial conditions such that the closed-loop system (1) is exponentially admissible with a predefined minimum decaying rate.

3 Main Results

3.1 Delay-Range-Dependent Exponential Stability

Theorem 3.1 *Given scalars $0 < d_1 < d_2$, $\mu < 1$ and α, system (1) with time-varying delay $d(t)$ satisfying (2) is exponentially admissible with $\sigma = \alpha$ if there exist a nonsingular matrix $P \in \mathbb{R}^{n\times n}$, $n\times n$ symmetric and positive-definite matrices Q_1, Q_2, Q_3, Z_1 and Z_2, and $n \times n$ matrices M_i, N_i and S_i, $i = 1, 2$ such that the following LMIs hold:*

$$\begin{bmatrix}
\Pi_{11} & \Pi_{12} & e^{\alpha d_1}M_1E & -e^{\alpha d_2}S_1E & \frac{e^{2\alpha d_2}-1}{2\alpha}N_1 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}S_1 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}M_1 & \Pi_{18}\\
\star & \Pi_{22} & e^{\alpha d_1}M_2E & -e^{\alpha d_2}S_2E & \frac{e^{2\alpha d_2}-1}{2\alpha}N_2 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}S_2 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}M_2 & A_d^\top U\\
\star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0\\
\star & \star & \star & -Q_2 & 0 & 0 & 0 & 0\\
\star & \star & \star & \star & -\frac{e^{2\alpha d_2}-1}{2\alpha}Z_1 & 0 & 0 & 0\\
\star & \star & \star & \star & \star & -\frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}(Z_1+Z_2) & 0 & 0\\
\star & \star & \star & \star & \star & \star & -\frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}Z_2 & 0\\
\star & \star & \star & \star & \star & \star & \star & -U
\end{bmatrix} < 0 \tag{5}$$

$$E^\top P = P^\top E \geq 0 \tag{6}$$

where

$$\begin{aligned}
\Pi_{11} &= P^\top A + A^\top P + \sum_{i=1}^{3} Q_i + N_1 E + (N_1 E)^\top + 2\alpha E^\top P \\
\Pi_{12} &= P^\top A_d + (N_2 E)^\top - N_1 E + S_1 E - M_1 E \\
\Pi_{22} &= -(1-\mu)e^{-2\alpha d_2} Q_3 + S_2 E + (S_2 E)^\top - N_2 E - (N_2 E)^\top - M_2 E - (M_2 E)^\top \\
d_{12} &= d_2 - d_1, \quad U = d_2 Z_1 + d_{12} Z_2, \quad \Pi_{18} = A^\top U
\end{aligned}$$

Proof. First, we will show that the system is regular and the impulsive-free. For this purpose, choose two nonsingular matrices R, L such that

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{A} = RAL = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{7}$$

Now, let

$$\bar{A}_d = RA_dL = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix} \qquad \bar{P} = R^{-\top} PL = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{8}$$

$$\bar{N}_i = L^\top N_i R^{-1} = \begin{bmatrix} N_{i11} & N_{i12} \\ N_{i21} & N_{i22} \end{bmatrix} \qquad \bar{Q}_i = L^\top Q_i L = \begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i21} & Q_{i22} \end{bmatrix} \tag{9}$$

From (6) and (7), , we conclude that $P_{12} = 0$ and $P_{11} > 0$.

Also, from (5), we get $\Pi_{11} < 0$ which gives $P^\top A + A^\top P + \sum_{j=1}^{3} Q_i + N_1 E + (N_1 E)^\top < 0$.

Based on (7)-(9), pre- and post-multiply this inequality by $L^\top$ and L, respectively, and noting that $Q_i > 0$, we have

$$\bar{P}^\top \bar{A} + \bar{A}^\top \bar{P} + \bar{N}_1 \bar{E} + (\bar{N}_1 \bar{E})^\top < 0$$

Noting that

$$\bar{N}_1 \bar{E} = \begin{bmatrix} N_{111} & 0 \\ N_{121} & 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} \star & \star \\ \star & A_{22}^\top P_{22} + P_{22}^\top A_{22} \end{bmatrix} < 0 \qquad \text{that implies in turn that} \qquad A_{22}^\top P_{22} + P_{22}^\top A_{22} < 0.$$

Therefore A_{22} is nonsingular, which implies in turn that system (1) is regular and impulse-free (see [26]). Next, we show the exponential stability of system (1). Since system (1) is regular, there exist two other matrices R, L such that (see [26])

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{A} = RAL = \begin{bmatrix} A_1 & 0 \\ 0 & \mathbb{I}_{n-r} \end{bmatrix} \tag{10}$$

Define $\bar{A}_d$, $\bar{P}$, $\bar{N}_i$, $\bar{Q}_i$ in a similar manner as before, $\bar{M}_i$, $\bar{S}_i$ similar to $\bar{N}_i$, and $\bar{Z}_i = R^{-\top} Z_i R^{-1}$. Using (5) and Shur complement, we get

$$\begin{bmatrix} \Pi_{11} & \Pi_{21} \\ \star & \Pi_{22} \end{bmatrix} < 0$$

Substitute (10) into the previous inequality, pre- and post multiply by $diag\{L^\top, L^\top\}$, $diag\{L, L\}$ and using Schur complement argument, we have

$$\begin{bmatrix} P_{22}^\top + P_{22} + \sum_{j=1}^{3} Q_{i22} & P_{22}^\top A_{d22} \\ A_{d22}^\top P_{22} & -(1-\mu)e^{-2\alpha d_2} Q_{322} \end{bmatrix} < 0$$

Pre- and post-multiplying by $\left[-A_{d22}{}^\top \mathbb{I}\right]$ and its transpose, and noting that $Q_i > 0$ and $\mu \geq 0$ (since if $\mu < 0$, the first condition in (2) will be violated), we get

$$A_{d22}{}^\top Q_{322} A_{d22} - e^{-2\alpha d_2} Q_{322} < 0 \qquad \text{which implies} \qquad \rho(e^{\alpha d_2} A_{d22}) < 1 \tag{11}$$

Let $\zeta(t) = L^{-1}x(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}$, where $\zeta_1(t) \in \mathbb{R}^r$ and $\zeta_2(t) \in \mathbb{R}^{n-r}$. Then, system (1) becomes equivalent to the following one

$$\dot{\zeta}_1(t) = A_1\zeta_1(t) + A_{d11}\zeta_1(t-d(t)) + A_{d12}\zeta_2(t-d(t)), \tag{12}$$

$$0 = \zeta_2(t) + A_{d21}\zeta_1(t-d(t)) + A_{d22}\zeta_2(t-d(t)). \tag{13}$$

Now, Choose the Lyapunov functional as follows:

$$V(\zeta_t) = \zeta(t)^\top \bar{E}^\top \bar{P}\zeta(t) + \sum_{i=1}^{2} \int_{t-d_i}^{t} \zeta(s)^\top e^{2\alpha(s-t)} \bar{Q}_i \zeta(s) ds + \int_{t-d(t)}^{t} \zeta(s)^\top e^{2\alpha(s-t)} \bar{Q}_3 \zeta(s) ds$$
$$+ \int_{-d_2}^{0} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_1 \bar{E}\dot{\zeta}(s) ds d\theta + \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_2 \bar{E}\dot{\zeta}(s) ds d\theta$$

where $\zeta_t = \zeta(t-\beta)$, $\beta \in (-d_2, 0]$. Then, the time-derivative of $V(\zeta_t)$ along the solution of (12) and (13) is given by

$$\begin{aligned}
\dot{V}(\zeta_t) &= 2\zeta(t)^\top \bar{P}^\top \bar{E}\dot{\zeta(t)} + \sum_{i=1}^{2}\left\{\zeta(t)^\top \bar{Q}_i\zeta(t) - \zeta(t-d_i)^\top e^{-2\alpha d_i}\bar{Q}_i\zeta(t-d_i)\right\} \\
&+ \zeta(t)^\top \bar{Q}_3\zeta(t) - (1-\dot{d}(t))\zeta(t-d(t))^\top e^{-2\alpha d(t)}\bar{Q}_3\zeta(t-d(t)) \\
&+ d_2(\bar{E}\dot{\zeta}(t))^\top \bar{Z}_1\bar{E}\dot{\zeta}(t) - \int_{t-d_2}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)ds \\
&+ (d_2-d_1)(\bar{E}\dot{\zeta}(t))^\top \bar{Z}_2\bar{E}\dot{\zeta}(t) - \int_{t-d_2}^{t-d_1}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_2\bar{E}\dot{\zeta}(s)ds \\
&- 2\alpha\sum_{i=1}^{2}\int_{t-d_i}^{t}\zeta(s)^\top e^{2\alpha(s-t)}\bar{Q}_i\zeta(s)ds - 2\alpha\int_{t-d(t)}^{t}\zeta(s)^\top e^{2\alpha(s-t)}\bar{Q}_3\zeta(s)ds \\
&- 2\alpha\int_{-d_2}^{0}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)dsd\theta \\
&- 2\alpha\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_2\bar{E}\dot{\zeta}(s)dsd\theta
\end{aligned} \tag{14}$$

And adding these terms

$$2\left[\zeta(t)^\top \bar{N}_1 + \zeta(t-d(t))^\top \bar{N}_2\right] \times \left[\bar{E}\zeta(t) - \bar{E}\zeta(t-d(t)) - \int_{t-d(t)}^{t}\bar{E}\dot{\zeta}(s)ds\right] = 0$$

$$2\left[\zeta(t)^\top \bar{S}_1 + \zeta(t-d(t))^\top \bar{S}_2\right] \times \left[\bar{E}\zeta(t-d(t)) - \bar{E}\zeta(t-d_2) - \int_{t-d_2}^{t-d(t)}\bar{E}\dot{\zeta}(s)ds\right] = 0$$

$$2\left[\zeta(t)^\top \bar{M}_1 + \zeta(t-d(t))^\top \bar{M}_2\right] \times \left[\bar{E}\zeta(t-d_1) - \bar{E}\zeta(t-d(t)) - \int_{t-d(t)}^{t-d_1}\bar{E}\dot{\zeta}(s)ds\right] = 0$$

to (14) gives

$$\begin{aligned}
\dot{V}(\zeta_t) &\le \sum_{i=1}^{9}\Psi_i + (\bar{E}\dot{\zeta}(t))^\top\left[d_2\bar{Z}_1 + d_{12}\bar{Z}_2\right](\bar{E}\dot{\zeta}(t)) \\
&- 2\left[\zeta(t)^\top \bar{N}_1 + \zeta(t-d(t))^\top \bar{N}_2\right]\int_{t-d(t)}^{t}\bar{E}\dot{\zeta}(s)ds - \int_{t-d(t)}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)ds \\
&- 2\left[\zeta(t)^\top \bar{S}_1 + \zeta(t-d(t))^\top \bar{S}_2\right]\int_{t-d_2}^{t-d(t)}\bar{E}\dot{\zeta}(s)ds \\
&\quad - \int_{t-d_2}^{t-d(t)}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\left(\bar{Z}_1 + \bar{Z}_2\right)\bar{E}\dot{\zeta}(s)ds
\end{aligned}$$

$$
\begin{aligned}
&- 2\left[\zeta(t)^\top \bar{M}_1 + \zeta(t-d(t))^\top \bar{M}_2\right] \int_{t-d(t)}^{t-d_1} \bar{E}\dot{\zeta}(s)ds - \int_{t-d(t)}^{t-d_1} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_2 \bar{E}\dot{\zeta}(s)ds \\
&- 2\alpha \sum_{i=1}^{2} \int_{t-d_i}^{t} \zeta(s)^\top e^{2\alpha(s-t)} \bar{Q}_i \zeta(s)ds - 2\alpha \int_{t-d(t)}^{t} \zeta(s)^\top e^{2\alpha(s-t)} \bar{Q}_3 \zeta(s)ds \\
&- 2\alpha \int_{-d_2}^{0} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_1 \bar{E}\dot{\zeta}(s)dsd\theta \\
&- 2\alpha \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_2 \bar{E}\dot{\zeta}(s)dsd\theta
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_1 &= \zeta(t)^\top \left[\bar{P}^\top \bar{A} + \bar{A}^\top \bar{P} + \sum_{i=1}^{3} \bar{Q}_i + \bar{N}_1 \bar{E} + (\bar{N}_1 \bar{E})^\top\right] \zeta(t) \\
\Psi_2 &= 2\zeta(t)^\top \left[\bar{P}^\top \bar{A}_d + (\bar{N}_2 \bar{E})^\top - \bar{N}_1 \bar{E} + \bar{S}_1 \bar{E} - \bar{M}_1 \bar{E}\right] \zeta(t-d(t)) \\
\Psi_3 &= \zeta(t-d(t))^\top \big[- (1-\mu) e^{-2\alpha d_2} \bar{Q}_3 + \bar{S}_2 \bar{E} + (\bar{S}_2 \bar{E})^\top - \bar{N}_2 \bar{E} \\
&\qquad - (\bar{N}_2 \bar{E})^\top - \bar{M}_2 \bar{E} - (\bar{M}_2 \bar{E})^\top \big] \zeta(t-d(t)) \\
\Psi_4 &= 2\zeta(t)^\top \bar{M}_1 \bar{E} \zeta(t-d_1) \Psi_5 = -2\zeta(t)^\top \bar{S}_1 \bar{E} \zeta(t-d_2) \\
\Psi_6 &= 2\zeta(t-d(t))^\top \bar{M}_2 \bar{E} \zeta(t-d_1) \qquad \Psi_7 = -2\zeta(t-d(t))^\top \bar{S}_2 \bar{E} \zeta(t-d_2) \\
\Psi_8 &= -\zeta(t-d_1)^\top e^{-2\alpha d_1} \bar{Q}_1 \zeta(t-d_1) \quad \Psi_9 = -\zeta(t-d_2)^\top e^{-2\alpha d_2} \bar{Q}_2 \zeta(t-d_2)
\end{aligned}
$$

Noting that $\bar{Z}_1 > 0$ and $\bar{Z}_2 > 0$, adding and subtracting these terms:

$$
\begin{aligned}
&+ \int_{t-d_2}^{t} \left[\zeta(t)^\top \bar{N}_1 + \zeta(t-d(t))^\top \bar{N}_2\right] \bar{Z}_1^{-1} e^{-2\alpha(s-t)} \left[\zeta(t)^\top \bar{N}_1 + \zeta(t-d(t))^\top \bar{N}_2\right]^\top ds \\
&- \int_{t-d(t)}^{t} \left[\zeta(t)^\top \bar{N}_1 + \zeta(t-d(t))^\top \bar{N}_2\right] \bar{Z}_1^{-1} e^{-2\alpha(s-t)} \left[\zeta(t)^\top \bar{N}_1 + \zeta(t-d(t))^\top \bar{N}_2\right]^\top ds \\
&+ \int_{t-d_2}^{t-d_1} \left[\zeta(t)^\top \bar{S}_1 + \zeta(t-d(t))^\top \bar{S}_2\right] \left(\bar{Z}_1 + \bar{Z}_2\right)^{-1} e^{-2\alpha(s-t)} \left[\zeta(t)^\top \bar{S}_1 + \zeta(t-d(t))^\top \bar{S}_2\right]^\top ds \\
&- \int_{t-d_2}^{t-d(t)} \left[\zeta(t)^\top \bar{S}_1 + \zeta(t-d(t))^\top \bar{S}_2\right] \left(\bar{Z}_1 + \bar{Z}_2\right)^{-1} e^{-2\alpha(s-t)} \left[\zeta(t)^\top \bar{S}_1 + \zeta(t-d(t))^\top \bar{S}_2\right]^\top ds \\
&+ \int_{t-d_2}^{t-d_1} \left[\zeta(t)^\top \bar{M}_1 + \zeta(t-d(t))^\top \bar{M}_2\right] \bar{Z}_2^{-1} e^{-2\alpha(s-t)} \left[\zeta(t)^\top \bar{M}_1 + \zeta(t-d(t))^\top \bar{M}_2\right]^\top ds \\
&- \int_{t-d(t)}^{t-d_1} \left[\zeta(t)^\top \bar{M}_1 + \zeta(t-d(t))^\top \bar{M}_2\right] \bar{Z}_2^{-1} e^{-2\alpha(s-t)} \left[\zeta(t)^\top \bar{M}_1 + \zeta(t-d(t))^\top \bar{M}_2\right]^\top ds
\end{aligned}
$$

gives

$$
\begin{aligned}
&\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) \leq \\
&\eta(t)^\top [\Pi + \widetilde{A}^\top \left(d_2\bar{Z}_1 + d_{12}\bar{Z}_2\right)\widetilde{A} + \frac{e^{2\alpha d_2}-1}{2\alpha}\widetilde{N}\bar{Z}_1{}^{-1}\widetilde{N}^\top \\
&+ \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{S}\left(\bar{Z}_1+\bar{Z}_2\right)^{-1}\widetilde{S}^\top + \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{M}\bar{Z}_2{}^{-1}\widetilde{M}^\top]\eta(t) \\
&- \int_{t-d(t)}^{t}\left[\eta(t)^\top\widetilde{N} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_1\right]e^{-2\alpha(s-t)}\bar{Z}_1{}^{-1}\left[\eta(t)^\top\widetilde{N} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_1\right]^\top ds \\
&- \int_{t-d_2}^{t-d(t)}\left[\eta(t)^\top\widetilde{S} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\left(\bar{Z}_1+\bar{Z}_2\right)\right]e^{-2\alpha(s-t)}\left(\bar{Z}_1+\bar{Z}_2\right)^{-1} \\
&\qquad\qquad\qquad\qquad \left[\eta(t)^\top\widetilde{S} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\left(\bar{Z}_1+\bar{Z}_2\right)\right]^\top ds \\
&- \int_{t-d(t)}^{t-d_1}\left[\eta(t)^\top\widetilde{M} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_2\right]e^{-2\alpha(s-t)}\bar{Z}_2{}^{-1}\left[\eta(t)^\top\widetilde{M} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_2\right]^\top ds \\
&\leq \eta(t)^\top [\Pi + \widetilde{A}^\top \left(d_2\bar{Z}_1 + d_{12}\bar{Z}_2\right)\widetilde{A} + \frac{e^{2\alpha d_2}-1}{2\alpha}\widetilde{N}\bar{Z}_1{}^{-1}\widetilde{N}^\top \\
&\quad + \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{S}\left(\bar{Z}_1+\bar{Z}_2\right)^{-1}\widetilde{S}^\top + \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{M}\bar{Z}_2{}^{-1}\widetilde{M}^\top]\eta(t)
\end{aligned}
$$

where

$$
\eta(t) = \begin{bmatrix} \zeta(t) \\ \zeta(t-d(t)) \\ \zeta(t-d_1) \\ \zeta(t-d_2) \end{bmatrix} \quad \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \bar{M}_1\bar{E} & -\bar{S}_1\bar{E} \\ \star & \Pi_{22} & \bar{M}_2\bar{E} & -\bar{S}_2\bar{E} \\ \star & \star & -e^{-2\alpha d_1}\bar{Q}_1 & 0 \\ \star & \star & 0 & -e^{-2\alpha d_2}\bar{Q}_2 \end{bmatrix}
$$

$$
\widetilde{N} = \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ 0 \\ 0 \end{bmatrix} \quad \widetilde{M} = \begin{bmatrix} \bar{M}_1 \\ \bar{M}_2 \\ 0 \\ 0 \end{bmatrix} \quad \widetilde{S} = \begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \\ 0 \\ 0 \end{bmatrix} \quad \widetilde{A} = \begin{bmatrix} \bar{A}^\top \\ \bar{A}_d{}^\top \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{aligned}
\Pi_{11} &= \bar{P}^\top\bar{A} + \bar{A}^\top\bar{P} + \sum_{i=1}^{3}\bar{Q}_i + \bar{N}_1\bar{E} + (\bar{N}_1\bar{E})^\top + 2\alpha\bar{E}^\top\bar{P} \\
\Pi_{12} &= \bar{P}^\top\bar{A}_d + (\bar{N}_2\bar{E})^\top - \bar{N}_1\bar{E} + \bar{S}_1\bar{E} - \bar{M}_1\bar{E} \\
\Pi_{22} &= -(1-\mu)e^{-2\alpha d_2}\bar{Q}_3 + \bar{S}_2\bar{E} + (\bar{S}_2\bar{E})^\top - \bar{N}_2\bar{E} - (\bar{N}_2\bar{E})^\top - \bar{M}_2\bar{E} - (\bar{M}_2\bar{E})^\top
\end{aligned}
$$

Pre- and post-multiply (5) by *diag* $\left\{L^\top, L^\top, e^{-\alpha d_1}L^\top, e^{-\alpha d_2}L^\top, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}\right\}$ and its transpose, then using Schur complement implies

$$
\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) \leq 0 \qquad \text{which leads to} \qquad V(\zeta_t) \leq e^{-2\alpha t}V(\phi(t))
$$

Then, the following estimation is obtained

$$\lambda_1 |\zeta_1(t)|^2 \leq V(\zeta_t) \leq e^{-2\alpha t} V(\phi(t)) \leq \lambda_2 e^{-2\alpha t} \|\phi\|_c$$

which leads to

$$|\zeta_1(t)| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c e^{-\alpha t} \tag{15}$$

where

$\lambda_1 = \lambda_{min}(\bar{P}_{11}) > 0,$

$\lambda_2 > 0,$ is sufficiently large and can be found since $V(\phi(t))$ is a bounded quadratic functional of $\phi(t)$.

Define,

$$\begin{aligned} t_0 &= t \\ t_i &= t_{i-1} - d(t_{i-1}) \end{aligned}$$

From (13), we get,

$$\begin{aligned} \zeta_2(t) &= -A_{d21}\zeta_1(t_1) - A_{d22}\zeta_2(t_1) \\ &= -A_{d21}\zeta_1(t_1) - A_{d22}[-A_{d21}\zeta_1(t_2) - A_{d22}\zeta_2(t_2)] \\ &= -A_{d21}\zeta_1(t_1) - A_{d22}[-A_{d21}\zeta_1(t_2) - A_{d22}[-A_{d21}\zeta_1(t_3) - A_{d22}\zeta_2(t_3)]] \end{aligned}$$

Note that there exist positive integers $k_1 = \left\lceil \frac{t}{d_1} \right\rceil^+$ and $k_2 = \left\lceil \frac{t}{d_2} \right\rceil^+$ such that

$$t - k_2 d_2 \in (-d_2, 0], \qquad t - k_1 d_1 \in (-d_2, 0]$$

and note also that

$$t - i d_2 \leq t_i = t - \sum_{j=0}^{i-1} d(t_j) \leq t - i d_1$$

Therefore, there exists a positive integer $k(t)$, where $k_2 \leq k(t) \leq k_1$, such that

$$\zeta_2(t) = (-A_{d22})^{k(t)} \zeta_2(t_{k(t)}) - \sum_{i=0}^{k(t)-1} (-A_{d22})^i A_{d21} \zeta_1(t_{i+1})$$

and

$$t_{k(t)} \in (-d_2, 0]$$

Therefore, from (15), (11), Lemma 2.2 and noting that

$$k(t)d_2 \geq t, \qquad t_i = t - \sum_{j=0}^{i-1} d(t_j) \geq t - id_2$$

we get,

$$\begin{aligned}
|\zeta_2(t)| &\leq \|A_{d22}{}^{k(t)}\| \, \|\phi\|_c + \|A_{d21}\| \sum_{i=0}^{k(t)-1} \left\|A_{d22}{}^{i}\right\| \left|\zeta_1(t_{(i+1)})\right| \\
&\leq \chi e^{-\alpha d_2 k(t)} \|\phi\|_c + \|A_{d21}\| \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c \sum_{i=0}^{k(t)-1} \left\|A_{d22}{}^{i}\right\| e^{-\alpha(t-(i+1)d_2)} \\
&\leq \left[\chi \|\phi\|_c + \|A_{d21}\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha d_2} \|\phi\|_c \sum_{i=0}^{k(t)-1} \|A_{d22}\|^{i} e^{i\alpha d_2}\right] e^{-\alpha t} \\
&\leq \left[\chi + \|A_{d21}\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha d_2} M\right] \|\phi\|_c \, e^{-\alpha t}
\end{aligned}$$

where

$$M = \frac{1}{1 - \|A_{d22} e^{\alpha d_2}\|}, \qquad \chi = \sqrt{\frac{\lambda_{max}(Q_{322})}{\lambda_{min}(Q_{322})}}$$

Thus, system in (12) and (13) is exponentially stable with a minimum decaying rate $= \alpha$. Finally, as we have shown that this system is also regular and impulse-free, by Definition (2.1), we then have that the system is exponentially admissible. This completes the proof. □

Remark 3.1 *It is noted that the condition in (6) is non-strict LMI, which contains equality constraints; this may result in numerical problems when checking such non-strict LMI conditions since equality constraints are fragile and usually not satisfied perfectly. Therefore, strict LMI conditions are more desirable than non-strict ones from the numerical point of view [26]. Considering this, Eqs. (5) and (6) can be combined into a single strict LMI. Let $P > 0$ and $S \in R^{n\times(n-r)}$ be any matrix with full column rank and satisfies $E^\top S$ = 0. Changing P to $PE + SQ$ in (5) yields the strict LMI.*

Remark 3.2 *Taking the limits of the elements of (5) as $\alpha \to 0$, Theorem 3.1 yields an admissibility conditions for singular time-delay systems. Moreover, when $E = \mathbb{I}$, the singular delay system in (1) reduces to a state-space delay system and the result of Theorem 3.1 as $\alpha \to 0$ coincides exactly with the result in [13].*

Remark 3.3 *If d_2 is not known precisely and we want to find the maximum d_2 such that our system remains exponentially stable with decay rate α, we replace d_2 with $d_1 + d_{12}$. In this case, (5) is not LMI anymore since d_{12} is a variable. Thus, a searching method similar to the one in [15] will be provided briefly as follows: Step (1) For $d_{12} = 0$, find a feasible solution to (5) and (6) as (P_0, Q_{j0}, Z_{i0}, M_{i0}, N_{i0}, S_{i0}) and set $k = 0$; Step (2) For (P_k, Q_{jk}, Z_{ik}, M_{ik}, N_{ik}, S_{ik}), find d_{12k} and set $k = k + 1$; Step (3) check to see if $\left|d_{12(k+1)} - d_{12k}\right| \leq \epsilon$, with $\epsilon > 0$. If not, return to step (2).*

3.2 Static Output Feedback Stabilization

Let us write the saturation term as [23]

$$sat(Kx(t)) = D(\rho(x))KCx(t), \quad D(\rho(x)) \in R^{m\times m}$$

where $D(\rho(x))$ is a diagonal matrix for which the diagonal elements $\rho_i(x)$ are defined for $i = 1, ..., m$ as

$$\rho_i(x) = \begin{cases} -\frac{\overline{u}_i}{(KC)_i x} & \text{if } (KC)_i x \leq -\overline{u}_i \\ 1 & \text{if } -\overline{u}_i < (KC)_i x < \overline{u}_i \\ \frac{\overline{u}_i}{(KC)_i x} & \text{if } (KC)_i x \geq \overline{u}_i \end{cases}$$

and $0 \leq \rho_i(x) \leq 1$.

Then, system (4) can then be written in the equivalent form:

$$E\dot{x}(t) = (A + BD(\rho(x))KC)x(t) + A_d x(t - d(t)) \tag{16}$$

The coefficient $\rho_i(x)$ can be viewed as an indicator of the degree of saturation of the ith entry of the control vector. In fact, the smaller $\rho_i(x)$, the farther is the state vector from the region of linearity.

Let $0 \leq \underline{\rho}_i \leq 1$ be a lower bound to $\rho_i(x)$, and define a vector $\underline{\rho} = [\underline{\rho}_1, ..., \underline{\rho}_m]$. The vector $\underline{\rho}$ is associated to the following region in the state space:

$$S(K, \overline{u}^\rho) = \{x \in R^n \mid -\overline{u}^\rho \leq KCx \leq \overline{u}^\rho\}$$

where every component of the vector $\overline{u}^\rho$ is defined by $\overline{u}_i/\underline{\rho}_i$.

Define now matrices A_j, $j = 1, ..., 2^m$, as follows:

$$A_j = A + BD(\gamma_j)KC$$

where $D(\gamma_j)$ is a diagonal matrix of positive scalars $\gamma_{j(i)}$ for $i = 1, ..., m$, which arbitrarily take the value one or $\underline{\rho}_i$. Note that the matrices A_j are the vertices of a convex polytope of matrices. If $x(t) \in S(K, \overline{u}^\rho)$, it follows that $(A + BD(\rho(x))KC) \in co\{A_1, ..., A_{2^m}\}$. We conclude that if $x(t) \in S(K, \overline{u}^\rho)$, then $E\dot{x}(t)$ can be determined from the following polytopic model:

$$E\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_{j,t} A_j x(t) + A_d x(t - d(t)) \tag{17}$$

with $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$ and $\lambda_{j,t} \geq 0$

Remark 3.4 *Notice that the trajectories of the polytopic system (17) includes all trajectories of the saturated system (4), but the converse is not necessarily true. This means that the stability of system (17) is only a sufficient condition to the stability of system (4). Thus, some unavoidable conservativeness is introduced.*

Remark 3.5 *Using this saturation model, the problem of controlling the nonlinear system (16) is transformed to the problem of controlling the linear time-variant system (17). Yet, the time-variant matrix $A(t)$ evolves with time inside a convex polyhedron of matrices. Now, the interesting question is as follows: if we proof the stabilizability of the 2^m linear time-invariant systems that uses the vertices of that convex polyhedron as its A's matrices, does this imply the stabilizability of the linear time-variant system? The answer is yes, and this will be the result of the next theorem.*

Theorem 3.2 *Consider the continuous singular time-delay system (1). Given scalars $0 < d_1 < d_2$, $\mu < 1$ and α, suppose that there exist symmetric and positive-definite matrix $P \in \mathbb{R}^{n\times n}$, a matrix $Q \in \mathbb{R}^{(n-r)\times n}$, $n \times n$ symmetric and positive-definite matrices Q_1, Q_2, Q_3, Z_1 and Z_2, $n \times n$ matrices M_i, N_i and S_i, $i = 1, 2$, a matrix $K \in \mathbb{R}^{m\times q}$ and a positive scalar κ such that*

$$\begin{bmatrix} \Pi_{j11} & \Pi_{j12} & e^{\alpha d_1} M_1 E & -e^{\alpha d_2} S_1 E & \frac{e^{2\alpha d_2}-1}{2\alpha} N_1 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha} S_1 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha} M_1 & \Pi_{j18} \\ \star & \Pi_{j22} & e^{\alpha d_1} M_2 E & -e^{\alpha d_2} S_2 E & \frac{e^{2\alpha d_2}-1}{2\alpha} N_2 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha} S_2 & \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha} M_2 & A_d^\top U \\ \star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\frac{e^{2\alpha d_2}-1}{2\alpha} Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}(Z_1+Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha} Z_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix} < 0 \tag{18}$$

$$j = 1, ..., 2^m$$

$$\begin{bmatrix} E^\top (PE + SQ) & \underline{\rho}_i (KC)_i{}^\top \\ \underline{\rho}_i (KC)_i & \kappa \overline{u}_i{}^2 \end{bmatrix} \geq 0, \quad i = 1, ..., m \tag{19}$$

where

$$\begin{aligned}
\Pi_{j11} &= (PE+SQ)^\top A + A^\top (PE+SQ) + \sum_{i=1}^{3} Q_i + N_1E + (N_1E)^\top \\
&\quad + (PE+SQ)^\top BD(\gamma_i)KC + \Big((PE+SQ)^\top BD(\gamma_i)KC\Big)^\top + 2\alpha E^\top (PE+SQ) \\
\Pi_{j12} &= (PE+SQ)^\top A_d + (N_2E)^\top - N_1E + S_1E - M_1E \\
\Pi_{j22} &= -(1-\mu)e^{-2\alpha d_2} Q_3 + S_2E + (S_2E)^\top - N_2E - (N_2E)^\top - M_2E - (M_2E)^\top \\
d_{12} &= d_2 - d_1, \quad U = d_2Z_1 + d_{12}Z_2, \quad \Pi_{j18} = A^\top U + (BD(\gamma_i)KC)^\top U
\end{aligned}$$

$S \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^\top S = 0$. Then, there exists a static output feedback controller (3) such that the closed-loop system (4) is locally exponentially admissible with $\sigma = \alpha$ for any compatible initial condition satisfying:

$$\Omega(\nu_1,\nu_2) = \{\phi \in C^v_{d_2} \quad : \quad \frac{\|\phi\|_c^2}{\nu_1} + \frac{\|\dot{\phi}\|_c^2}{\nu_2} \le 1\} \tag{20}$$

where

$$\begin{aligned}
\nu_1 &= \frac{\kappa^{-1}}{\chi_1}, \qquad\qquad \nu_2 = \frac{\kappa^{-1}}{\chi_2} \\
\chi_1 &= \lambda_{max}(E^\top PE) + \sum_{i=1}^{2} \lambda_{max}(Q_i)\frac{1-e^{-2\alpha d_i}}{2\alpha} + \lambda_{max}(Q_3)\frac{1-e^{-2\alpha d_2}}{2\alpha} \\
\chi_2 &= \lambda_{max}(Z_1)\lambda_{max}(E^\top E)\frac{2\alpha d_2 - 1 + e^{-2\alpha d_2}}{4\alpha^2} \\
&\quad + \lambda_{max}(Z_2)\lambda_{max}(E^\top E)\frac{2\alpha d_{12} - e^{-2\alpha d_1} + e^{-2\alpha d_2}}{4\alpha^2}
\end{aligned}$$

Proof. Assume that $E\dot{x}(t)$ can be determined from the polytopic system (17). Applying Remark 3.1 to (5)-(6) in Theorem 3.1 yields a single matrix inequality. Then, if we apply this matrix inequaity 2^m times to the parameters A_j with $j = 1,\ldots,2^m$, A_d, E, d_1, d_2 and μ, we will have (18). Now, proceeding in a similar way as the proof of Theorem 3.1, yields A_{j22} to be nonsingular matrices. Using the fact that $\lambda_{j,t} \ge 0$,

$$\sum_{j=1}^{2^m} \lambda_{j,t} A_{j22} \text{ is nonsingular} \quad \forall t \in (0,\infty)$$

which implies that system (17) is regular and impulse-free. Now, choose a Lyapunov functional as in the previous theorem, and proceeding in a similar manner as before, then

$$\begin{aligned}&\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) \leq \\ &\leq \eta(t)^\top [\Pi + \widetilde{A}^\top \left(d_2 \bar{Z}_1 + d_{12}\bar{Z}_2\right) \widetilde{A} + \frac{e^{2\alpha d_2} - 1}{2\alpha} \widetilde{N} {\bar{Z}_1}^{-1} \widetilde{N}^\top \\ &\quad + \frac{e^{2\alpha d_2} - e^{2\alpha d_1}}{2\alpha} \widetilde{S} \left(\bar{Z}_1 + \bar{Z}_2\right)^{-1} \widetilde{S}^\top + \frac{e^{2\alpha d_2} - e^{2\alpha d_1}}{2\alpha} \widetilde{M} {\bar{Z}_2}^{-1} \widetilde{M}^\top]\eta(t)\end{aligned}$$

with all the variables as defined in Theorem 3.1 and A substituted by $\sum_{j=1}^{2^m} \lambda_{j,t} A_j$. Then, by convexity, condition (18) and noting that $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$ and $\lambda_{j,t} \geq 0$

$$\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) \leq 0$$

Completing the proof in a similar manner as in theorem (3.1), yields the exponential stability result.

Now, by virtue of condition (19), the ellipsoid defined by $\Gamma = \{x \in \mathbb{R}^n : x^\top E^\top (PE+SQ) x \leq \kappa^{-1}\}$ is included in the set $S(K, \overline{u}^\rho)$ [23]. Suppose now that the initial condition $\phi(t)$ satisfies (20), and conditions (18)-(19) hold. Then, from the definition of $V(t)$, it follows that $x(0)^\top E^\top (PE + SQ)x(0) \leq V(0) \leq \chi_1 \|\phi\|_c^2 + \chi_2 \|\dot{\phi}\|_c^2 \leq \kappa^{-1}$ and, in this case, one has $x(0) \in \Gamma \subset S$. Now, since $\dot{V}(0) < 0$, we conclude that $x(t)^\top E^\top (PE + SQ)x(t) \leq V(t) \leq V(0) \leq \chi_1 \|\phi\|_c^2 + \chi_2 \|\dot{\phi}\|_c^2 \leq \kappa^{-1}$, which means that $x(t) \in S$, $\forall t > 0$. Therefore, $E\dot{x}(t)$ can be determined from the polytopic system (17). This completes the proof. □

It is obvious that (18) is a BMI. Thus, an ILMI approach similar to [25] and [18] will be proposed. The derivation of the algorithm is similar to [25] and [18] and will be omitted for brevity. This algorithm has the same disadvantage as those in [25] and [18], i.e. based on a sufficient condition. The following is the proposed algorithm and the explanation is given later.

Iterative Linear Matrix Inequality (ILMI) Algorithm.

- Step 1. Solve the following optimization problem for $P^0 > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_p^0 > 0$, M_p, N_p, S_p, $p = 1, 2$, β^0 and κ
  OP1: Minimize β^0 subject to the LMI constraints in step 2 with $K^0 = 0$ and $X^0 = E$. Denote Z_{11} and Z_{21} as the values of Z_1^0 and Z_2^0 that minimizes β_0.
- Step 2. Set $i = 1$, $X_1 = E$, Solve the following optimization problem for $P_i > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, M_p, N_p, S_p, $p = 1, 2$, K, β_i and κ

OP2: Minimize β_i subject to the following LMI constraints:

$$\left[\begin{array}{ccccc} \Pi_{j11} & \left[\begin{array}{c}(B^\top T_i \\ +D(\gamma_j)FC)^\top\end{array}\right] & \Pi_{j12} & e^{-\beta_i d_1}M_1E & -e^{-\beta_i d_2}S_1E \\ \left[\begin{array}{c}(B^\top T_i \\ +D(\gamma_j)FC)\end{array}\right] & -\mathbb{I} & 0 & 0 & 0 \\ \star & 0 & \Pi_{j22} & e^{-\beta_i d_1}M_2E & -e^{-\beta_i d_2}S_2E \\ \star & 0 & \star & -Q_1 & 0 \\ \star & 0 & \star & \star & -Q_2 \\ \star & 0 & \star & \star & \star \\ \star & 0 & \star & \star & \star \\ \star & 0 & \star & \star & \star \\ \star & 0 & \star & \star & \star \end{array}\right.$$

$$\left.\begin{array}{cccc} \frac{e^{-2\beta_i d_2}-1}{-2\beta_i}N_1 & \frac{e^{-2\beta_i d_2}-e^{-2\beta_i d_1}}{-2\beta_i}S_1 & \frac{e^{-2\beta_i d_2}-e^{-2\beta_i d_1}}{-2\beta_i}M_1 & \Pi_{j18} \\ 0 & 0 & 0 & 0 \\ \frac{e^{-2\beta_i d_2}-1}{-2\beta_i}N_2 & \frac{e^{-2\beta_i d_2}-e^{-2\beta_i d_1}}{-2\beta_i}S_2 & \frac{e^{-2\beta_i d_2}-e^{-2\beta_i d_1}}{-2\beta_i}M_2 & A_d{}^\top U \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{e^{-2\beta_i d_2}-1}{-2\beta_i}Z_{1i} & 0 & 0 & 0 \\ \star & -\frac{e^{-2\beta_i d_2}-e^{-2\beta_i d_1}}{-2\beta_i}(Z_{1i}+Z_{2i}) & 0 & 0 \\ \star & \star & -\frac{e^{-2\beta_i d_2}-e^{-2\beta_i d_1}}{-2\beta_i}Z_{2i} & 0 \\ \star & \star & \star & -U \end{array}\right] < 0$$

$$j = 1, ..., 2^m$$

$$\left[\begin{array}{cc} E^\top T_i & \underline{\rho}_r(KC)_r{}^\top \\ \underline{\rho}_r(KC)_r & \kappa\overline{u}_r{}^2 \end{array}\right] \geq 0, \quad r = 1, ..., m$$

where

$$\begin{aligned}
\Pi_{j11} &= T_i^\top A + A^\top T_i + \sum_{i=1}^{3} Q_i + N_1E + (N_1E)^\top \\
&\quad -X_iBB^\top T_i - (X_iBB^\top T_i)^\top + X_iBB^\top X_i - 2\beta_i E^\top T_i \\
\Pi_{j18} &= A^\top U + (BD(\gamma_j)KC)^\top U \\
\Pi_{j22} &= -(1-\mu)e^{2\beta d(\beta)}Q_3 + S_2E + (S_2E)^\top - N_2E - (N_2E)^\top - M_2E - (M_2E)^\top \\
T_i &= (P_iE + SQ) \qquad d(\beta) = \begin{cases} d_1 & \text{if } \beta > 0 \\ d_2 & \text{if } \beta < 0 \end{cases}
\end{aligned}$$

and the other variables as defined previously. Denote $\beta_i{}^*$ and K^* as the minimized value of β_i and the value of K that minimizes β_i, respectively.

- Step 3. If $\beta_i{}^* \le -\alpha$, where α is a prescribed decay rate. K^* is a stabilizing static output feedback gain, go to step 7. Otherwise, go to step 4.
- Step 4. Solve the following optimization problem for $P_i > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_{pi} > 0$, M_p, N_p, S_p, $p = 1, 2$, and κ
  OP3: Minimize $tr(E^\top T_i)$ subject to the previous LMIs with $\beta_i = \beta_i{}^*$ and $K = K^*$. Denote $T_i{}^*$, ${Z^*}_{1i}$ and ${Z^*}_{2i}$ as the values of T_i, Z_{1i} and Z_{2i}, respectively, that minimizes $tr(E^\top P)$.
- Step 5. If $\|X_iB - {T^*}_iB\| < \epsilon$, go to step 6. Else set $i = i + 1$, $X_i = T_{i-1}{}^*$, $Z_{1i} = {Z^*}_{1(i-1)}$ and $Z_{2i} = {Z^*}_{2(i-1)}$, then go to step 2.
- Step 6. The system may not be stabilizable via static output feedback. Stop.
- Step 7. Solve the following optimization problem for $P_i > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, Z_p, M_p, N_p, S_p, $p = 1, 2$, K, and κ
  OP4: Minimize $w_1\left(\delta_1 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_2 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_3 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_4\right)$
  $+w_2\left(\lambda_{max}(E^\top E)\frac{2\alpha d_2-1+e^{-2\alpha d_2}}{4\alpha^2}\delta_5 + \lambda_{max}(E^\top E)\frac{2\alpha d_{12}-e^{-2\alpha d_1}+e^{-2\alpha d_2}}{4\alpha^2}\delta_6\right) + w_3\kappa$
  subject to the previous LMIs and the following LMIs:

$$\delta_1\mathbb{I} \ge E^\top PE \qquad \delta_2\mathbb{I} \ge Q_1 \qquad \delta_3\mathbb{I} \ge Q_2 \tag{21}$$

$$\delta_4\mathbb{I} \ge Q_3 \qquad \delta_5\mathbb{I} \ge Z_1 \qquad \delta_6\mathbb{I} \ge Z_2 \tag{22}$$

  with $\beta_i = \alpha$, where w_1, w_2 and w_3 are weighting factors. We solve this problem iteratively in two steps as follows:
  a) Fix K, and solve for $P_i > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_p > 0$, M_p, N_p, S_p, $p = 1, 2$, and κ.
  b) Fix Z_1 and Z_2, and solve for $P_i > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, M_p, N_p, S_p, $p = 1, 2$, K and κ. Set $X = T$.
  The set (20) is calculated from the matrices that solve this optimization problem.

Remark 3.6 *The core of this algorithm is in OP2 and OP3. As shown in [25], OP2 guarantees the progressive reduction of β_i while OP3 guarantees the convergence of the algorithm. Yet, in [25], only X needs to be fixed in order to get LMIs, while in our case, we have also to fix either Z_1 and Z_2 or K to get LMIs. Thus, we will fix Z_1 and Z_2 in OP2, and K in OP3. This way of solving this problem will not affect the convergence of the algorithm.*

Remark 3.7 *If β is positive, this corresponds to a negative decaying rate, i.e. $\|x(t)\| \le \gamma e^{\beta t}\|\phi\|_c$. It can be shown easily, similar to proof of Theorem 3.1, that after introducing the function $d(\beta)$, the results in Theorem 3.1 and Theorem 3.2 will be generalized to include negative decaying rates. This means that as β decreases in the algorithm, this is nothing but an increasing decaying rate, and as β becomes negative, the system becomes exponentially* *stable. This fact resembles the facts in [25] and [18] that all eigenvalues of* $(A - BKC)$ *are shifted progressively toward the left-half-plane through the reduction of* β*.*

Remark 3.8 *In order to start the algorithm, OP2 should have a solution for* $i = 1$*. Yet, the solution depends on the initial matrix* X*. In [18], some Riccati equation is proposed in order to select an initial* X *for the descriptor version of this algorithm. In [19], it has been proved that this Riccati equation may not have a solution and an initial value of* $X = \mathbb{I}$ *is proposed instead. Actually, the identity matrix may not do the job for even some simple systems, an example of such systems is*

$$(A, B, C) = (\mathbb{I}, \mathbb{I}, \mathbb{I}), \qquad E = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix}$$

Our numerical experience indicates that an initial choice of $X_1 = E$ *always leads to a convergent result. Yet, this is not rigorously proved. With this* X_1*, OP1 is used here to get an initial values of* Z_1 *and* Z_2*.*

Remark 3.9 *The minimization of* β *in OP1 and OP2 should be done using the bisection method. The lower bound of the bisection method can be any value less than* $-\alpha$ *since we are not interested in minimizing* β *less than these values. Also, noting Remark 3.7, it is easy to find an upper bound. This upper and lower bounds should be chosen only once and can be fixed throughout the algorithm.*

Remark 3.10 *OP4 is used in order to maximize the set of initial conditions (20). The satisfaction of (21)-(22) means that* $\chi_1 \leq \delta_1 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_2 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_3 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_4$ *and* $\chi_2 \leq \lambda_{max}(E^\top E)\frac{2\alpha d_2 - 1 + e^{-2\alpha d_2}}{4\alpha^2}\delta_5 + \lambda_{max}(E^\top E)\frac{2\alpha d_{12} - e^{-2\alpha d_1} + e^{-2\alpha d_2}}{4\alpha^2}\delta_6$*. Therefore, because* $\nu_i = \frac{\kappa^{-1}}{\chi_i}$*, if we minimize the criterion as defined in OP4, then greater the bounds on* $\|\phi\|_c^2$ *and* $\|\dot{\phi}\|_c^2$ *tend to be. In other words, by using OP4, we orient the solutions of LMIs (18)-(19) in a sense to obtain the set* $\Omega(\nu_1, \nu_2)$ *as large as possible. For more discussion on this topic, we refer the reader to [23].*

4 Examples

4.1 Example 1:

Consider the singular time-delay system studied in [5] with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.5 & 0 \\ -1 & -1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

We have known from [5] that this system is asymptotically stable for constant delay $\tau < \tau^*$ and unstable for constant delay $\tau > \tau^*$, where $\tau = 1.2092$. Now, allowing time-varying delay, the exponential stability of this system will be investigated using Theorem 3.1. For various d_2, the maximum allowable decay rates α, which guarantee the exponential stability for given lower bound d_1 and derivative bound μ, are listed in Table 1. As it is clear from the table, if we increase d_2, then we obtain smaller decay rates α. Figure 1 gives the simulation results of x_1 and x_2 as compared to $e^{-0.3t}$ when $d(t) = 0.4$ and the initial function is $\phi(t) = [1\ \ -1]^\top, t \in [-0.4, 0]$. From Figure 1, we can see that the states x_1 and x_2 exponentially converge to zero with a decay rate more than 0.3.

Table 1: Maximum allowable decay rates α for different d_2 with $d_1 = 0.2$ and $\mu = 0.5$

d_2	0.5	0.6	0.7	0.8	0.9	1	1.1
α	0.3239	0.3014	0.2816	0.2642	0.2411	0.1323	0.0290

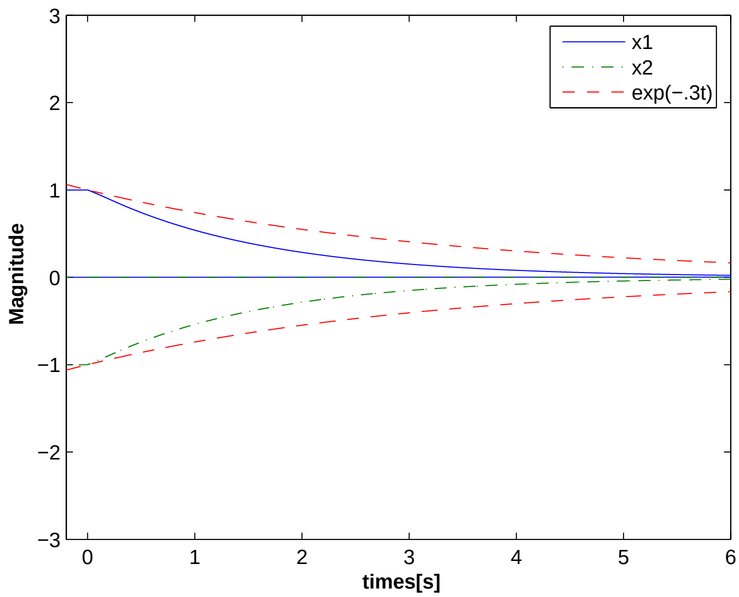

Figure 1: Simulation Results of x_1 and x_2 as compared to $e^{-0.3t}$

4.2 Example 2:

Consider the singular time-delay system described by:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 0.4 & 0 \\ 0.2 & 0.3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 \\ 0.1 & 0.3 \\ 0.1 & -0.3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This system is originally unstable for all values of delay. Now, allowing time-varying delay, the exponential stabilizability of this system will be investigated using Theorem 3.2 and the iterative algorithm. Letting $d_1 = 0.2$, $d_2 = 0.6$, $\mu = 0.5$, $\overline{u} = 7$ and $\alpha = 0.3$, the ILMI algorithm gives after 14 iterations

$$K = \begin{bmatrix} -1.4186 & -1.2682 \\ 1.3943 & 0.8652 \end{bmatrix}, \qquad \nu_1 = 14.8960 \qquad \nu_2 = 82.6586$$

Figures 2 and 3 gives the simulation results for the closed-loop system when $d(t) = 0.5$ and the initial function is $\phi(t) = [5\ 12\ 9.6]^\top, t \in [-0.5, 0]$. Changing the control amplitude saturation level, Figure 4 presents the functional dependence of ν_1 and ν_2 on the level of control saturation $\overline{u}$.

For various α, the values ν_1 and ν_2 for which we guarantee the exponential admissibility of the saturated system are listed in Table 2. The number of iterations are also listed in the table.

Table 2: Computation results of Example 2 with $\overline{u} = 15$

α	0.001	0.2	0.4	0.6	0.8	1	1.2
$\nu 1$	192.1172	97.0467	48.7601	25.8165	14.0812	7.9295	5.5883
$\nu 2$	967.1209	509.6311	268.5460	165.2845	90.6967	37.1311	28.2688
Iterations	11	13	14	15	16	17	18

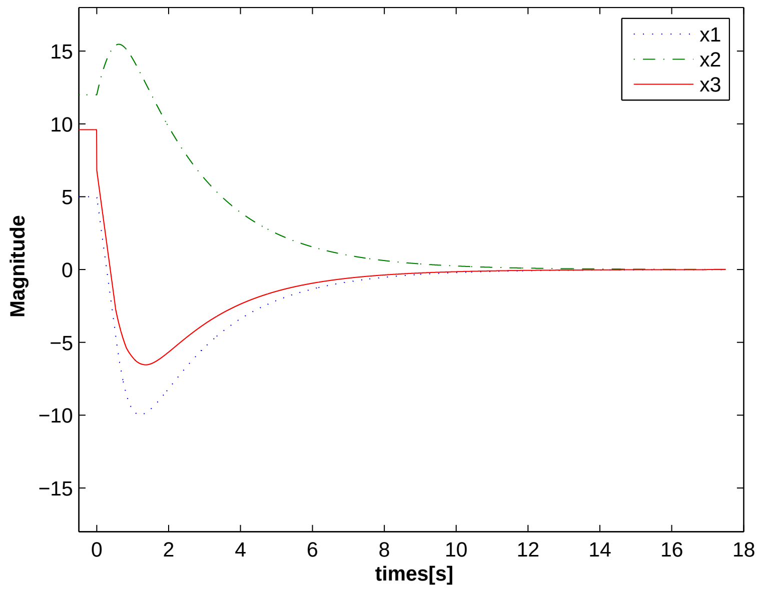

Figure 2: Simulation Results of x_1, x_2 and x_3

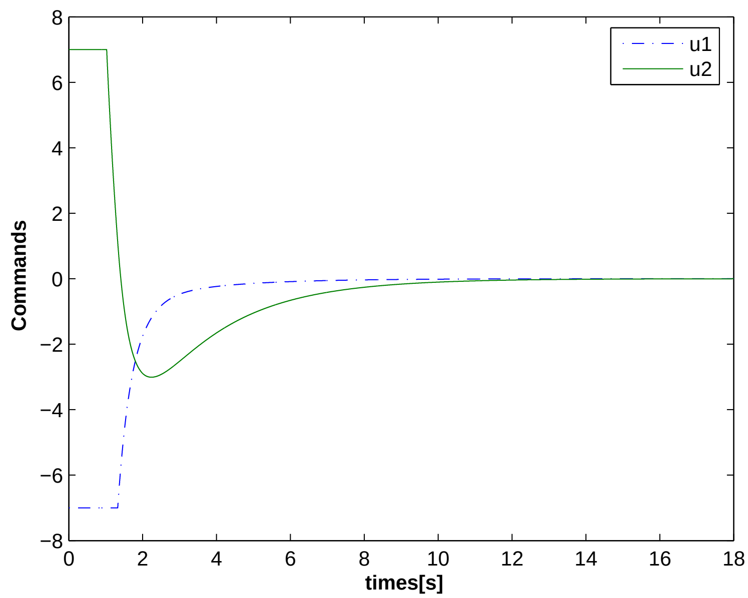

Figure 3: Simulation Results of the controllers

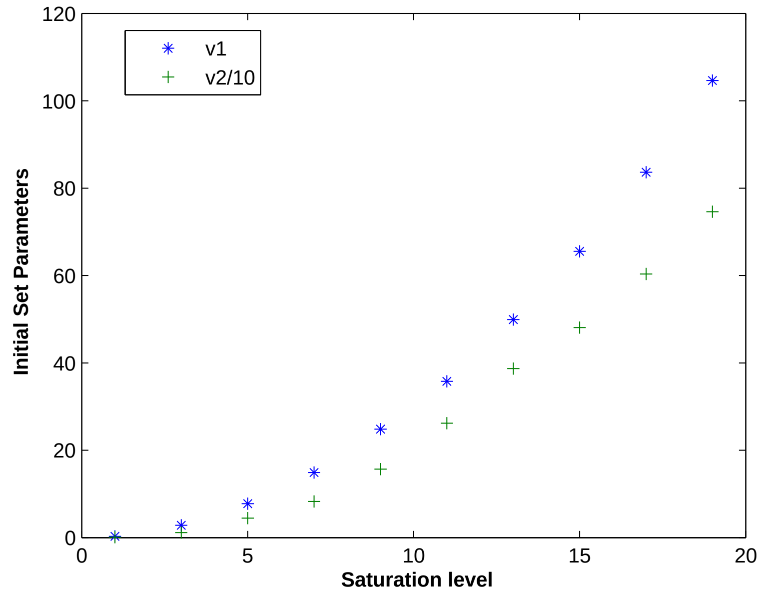

Figure 4: ν_1 and ν_2 for which the exponential admissibility is guaranteed as a function of the control amplitude saturation level

5 Conclusion

This paper dealt with the stability and the stabilization of the class of singular time-delay systems. A delay-range-dependent exponential stability conditions has been developed for singular time-delay systems. Also, delay-range-dependent static output feedback controller with input saturation has been designed for singular time-delay systems and an ILMI algorithm has been proposed to compute the controller gains. The effectiveness of the results has been illustrated through examples. As a future work, the following items can be considered

- The problem of robust stabilization may be addressed. A similar approach to [32] can be adopted to deal with uncertainties of the polytopic type.
- In [33], two recent proposed simple modifications/generalizations of static output feedback are investigated. Namely, introducing time-delay in the control law and making the gain time-varying. Both approaches has been shown to be complementary and existing results are brought together in a unifying framework. Motivated by this work, the generalization of our controller should be the subject of a forthcoming publication.

- Considering the transfer delays of sensor-to-controller and controller-to-actuator that appear in many control systems. More attention has been paid to the study of stability and stabilization of systems with control input delay. This problem has not been fully addressed for singular time-delay systems.

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