Traditional and Emergent Environmental Regulations and the Policies of the Firm

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Abstract

We study in this paper the impact of a Public Disclosure Program (PDP) as well as traditional environmental regulation (tax/subsidy) on optimal policies of the firm. A PDP aims at forcing the firm to report its emissions. This information affects its image (goodwill), and ultimately its profit. In our model, this impact is endogenous, i.e., a firm polluting less than its prescribed target would win consumer’s sympathy and raises its goodwill, whereas it is the other way around when the firm exceeds its emissions quota. The concept of goodwill (or brand equity) is inherently dynamic and so is our model. The evolution of this goodwill is assumed to depend also on advertising expenditures. We address the following research questions: (1) What are the optimal emissions, pricing and advertising policies of the firm under the different regulatory regimes? (2) How the different regulatory scenarios, i.e., PDP, tax/subsidy, both regulations, and no regulation (laissé-faire policy), compare in terms of the above policies? (3) Under which conditions, if any, a PDP can be profit improving?

Key Words: Traditional Environmental Regulation; Public Disclosure Program; Pricing; Advertising; Goodwill; Optimal Control.

Résumé

Nous examinons dans ce papier les effets des régulations émergentes et traditionnelle (taxation/subvention) sur le comportement de la firme. La régulation émergente prend la forme de programmes gouvernementaux de révélation de l’information sur les pollueurs. Cette divulgation de l’information affecte l’image publique de la firme et son profit. Cet impact est endogène dans notre modèle : si la firme dépasse le standard imposé par le régulateur, elle perdra la sympathie des consommateurs, et c’est le contraire qui aura lieu dans le cas inverse. La formation du stock de goodwill dépend dans ce modèle dynamique à la fois de la publicité et de la performance environnementale de la firme. Notre principal objectif est de présenter un modèle théorique capable de répondre à la question centrale suivante : Comment les deux régulations affectent-elles la performance environnementale de la firme, son prix, son effort de publicité et son profit? Notre démarche consiste à comparer quatre scénarios différents : laisser-faire, régulation traditionnelle pure, régulation émergente seulement, la double régulation où les deux modes sont conjointement mis en pratique.

Mots clés : Régulation environnementale traditionnelle, programmes publics de révélation de l’information, prix, publicité, contrôle optimal.

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1 Introduction

In their attempt to curb pollutant emissions, governments and their regulatory bodies have, in the past, adopted a series of rules and mechanisms that can be classified schematically into two classes: traditional regulation and emergent regulation. The former consists in monitoring firms and enforcing an improvement in their environmental performance. Monitoring is the process of verifying if the firm complies with environmental standards, whereas enforcement is the undertaking of punitive actions to push the firm to improve its environmental performance (Foulon et al. (2002)). Emergent regulation consists in a planned information strategy used by the regulator to reveal the environmental performance of the firm. The rationale here is that by making the information public, through what is known as a Public Disclosure Program (PDP), the polluters will be pushed to reduce their emissions to avoid being punished by consumers and capital markets. An implicit assumption here is that these reductions can be achieved at a lower cost than by traditional means.

There is a significant literature on monitoring and enforcement, that addresses a variety of issues. For instance, Magat and Viscusi (1990), and Laplante and Rilstone (1996) show that inspections significantly reduce absolute levels of water pollution emitted by pulp and paper plants in the United States and Canada. Gray and Deily (1996) state that an increase in enforcement actions in the US steel industry reduces noncompliance in air pollution. Nadeau (1997) obtains that monitoring and enforcement actions diminish the duration of noncompliance, and Helland (1998), that inspections encourage self-reporting. Kleit et al. (1998) predict that the penalty depends essentially on the gravity of the violation and on the firm’s previous record of environmental violations. Dion et al. (1998) show that regulators appear to monitor larger plants for visibility of their actions, but avoid enforcing them for electoral reasons. Dasgupta et al. (2001) demonstrate that at the plant level, the variation in frequency of inspections of industrial air- and water-pollution in China is a better determinant of the firms’ environmental performance than is the variation in pollution levies. Stafford (2002) shows that a rise in the maximum penalty reduces violations for waste pollutants. More recently, Shimshack and Ward (2005) find that a fine produces a decrease of about two-thirds in violation rates and that the majority of this impact can be attributed to reputation enhancement by the regulator. For a survey, see Cohen (1998).

The above references are empirical. Some papers address issues pertaining to the monitoring and enforcement of regulations in a theoretical framework. For instance, Harford and Harrington (1991), Harrington (1988), and Heyes (1996) show that optimal fines need not be maximal, which diverges from the well-known result in Becker (1968). More recently, Arguedas (2005) explores the possibility that firms and regulators achieve cooperative agreements in environmental regulation, and show that all the policies in the bargaining set induce the firm to exceed the standard.
Although the references cited report a generally good environmental record for traditional regulation, others think that the latter is socially expensive and that its results, in practice, are often obstructed by affected firms. According to EPA\textsuperscript{1} ex-Administrator William Reilly, for instance, four out of every five decisions made by EPA are contested in court (see Heyes (2000)). Further, it is recognized in the literature on environmental economics, that firms do not always fully comply with the imposed regulation. For instance, Harford (1978), states that “In the case of both air and water pollution standards, it has been the case that these standards have not always been complied with.” In the United Kingdom, for example, “published compliance rates with many key water quality standards are significantly below 100%, sometimes as low as 50%, and the true compliance rates are likely to be even lower” (Heyes (2000)). In such a context, information on firms’ environmental records has recently been seen as a supplement or an alternative to direct command-and-control regulation (Konar and Cohen (1997)).

Generally speaking, the emergent-regulation literature is mainly interested by the reaction of capital markets to the release of environmental information. Konar and Cohen (1997) obtain, for instance, that firms with the largest decline in stock price when the information is made public reduce their emissions more than their industry peers. Badrinath and Bolster (1996) find that, on average, there is a loss in value of about $14.3 million during the week of the settlement. Hamilton (1995) reports that the stock value decreases on average by $4.1 million on the day that the list of polluters is released. Lanoie et al. (1998) obtain a different result. Indeed, their analysis suggests that appearing on the British Columbia polluters’ list has no impact on a firm’s equity value. On the other hand, Klassen and McLaughlin (1996) find that market valuation increases on average by $80.5 million following the announcement of an environmental award. Similarly, Ludgren (2003) argues that “…by lowering the environmental risk via investments in abatement capital, the company lowers its systematic risk (market risk), and as a consequence its total risk. This tends to, \textit{ceteris paribus}, increase the current stock price.” Foulon et al. (2002) state that such programs do indeed create additional and strong incentives for pollution control, and improve the environmental performance of polluters.

The objective of this paper is to study the impact of a PDP on the optimal policies of the firm. The point of view taken here is that the information on the environmental behavior of the firm affects its image (goodwill), and ultimately, its profit. In our model, this impact is endogenous, i.e., a firm polluting less than its prescribed target would win consumer sympathy and raise its goodwill, whereas it is the other way around when the firm exceeds its emissions quota. The concept of goodwill (or brand equity) is inherently dynamic and so is our model. The evolution of this goodwill is assumed to also depend, as in standard models in this area, on advertising expenditures. The latter can be seen as a communications effort conducted by the firm to enhance its image. Although our main focus is emergent regulation, we shall also consider the presence of traditional regulation in the form of a tax/subsidy program. This will allow us to eventually look at the effect of

\textsuperscript{1}U.S. Environmental Protection Agency.
each type of regulation with and without the other. More specifically, we wish to address the following research questions:

1. What are the firm’s optimal policies regarding emissions, pricing and advertising, under the different regulatory regimes?
2. How do the four regulatory scenarios, i.e., PDP, tax/subsidy, both regulations, and no regulation (laisser-faire policy), compare in terms of the above policies?
3. Under which conditions, if any, can a PDP be profit improving with respect to a laissez-faire policy?

Our main contribution lies in the simultaneous consideration of both types of regulation within the same framework, the endogenous determination of the impact of an emissions standard on a firm’s policies, and the consideration of a link between the goodwill of the firm, its environmental record and its profit. As mentioned above, we view the firm’s goodwill as a stock (i.e., capital or state variable) fueled by two flows; one environmental (PDP) and the other, the communication effort emanating from the firm (advertising).

The remainder of the paper is organized as follows: In Section 2, the model is set up. In Section 3, we solve the firm program. In Section 4, we show the impact of the two types of regulation on the environmental performance of the firm. Finally, in Section 5, some concluding remarks are made.

2 The Model

Consider a firm producing a good at a constant unit cost $c$. Denote by $G(t)$ the goodwill (or brand equity) of the firm, and by $q(t)$ the demand at instant of time $t \in [0, \infty)$. We assume that demand equals production, i.e., that there are no inventories, and that it depends on $G(t)$ and on the price of the product $p(t)$. We adopt the following linear demand specification,

$$q(G, p) = a + G(t) - p(t),$$

where $a > 0$. The above specification assumes that the product’s market potential, i.e., the demand when the price tends towards zero, is given by a constant $a > 0$, which corresponds to an “average” or “normal” market potential, plus $G(t)$. The sign of the latter is not exogenously assumed but depends, as discussed below, on the firm’s environmental record and its advertising policy. We suppose in the sequel that $q(t)$ remains positive for all $t \in [0, \infty)$.

Denote by $e(t)$ the pollutant emissions that are an inevitable by-product of production. We suppose a simple proportional relationship between emissions and production, i.e., $e(t) = \alpha q(t)$, with $0 < \alpha < 1$. Denote by $\bar{e}$ the standard for emissions set by the regulator and by $v(t)$ the difference between the firm’s emissions and this standard, i.e., $v(t) = e(t) - \bar{e}$. Suppose that the regulator taxes (subsidizes), at a given rate $\tau \geq 0$, each unit of emissions above (below) the target or the standard assigned to the firm. Then, the quantity
\( \tau v(t) = \tau (e(t) - \bar{e}) \) represents a revenue for the firm if it pollutes below its standard (i.e., \( e(t) < \bar{e} \)), or a cost, otherwise. The scenario retained here is one of equal tax and subsidy rates. This need not to be the case in reality. Actually, the tax rate may be strictly positive, whereas the subsidy rate is zero. Note that Jones (1989) shows that if the penalty function is linear, as is the case here, then standards and taxes can achieve the first-best outcome.

Remark 1 We do not address the question of how the standard is determined. A traditional way of doing so is to assume that the regulator chooses the standard that corresponds to the socially optimal output. How \( \bar{e} \) is set does not qualitatively affect the model or the analysis.

Remark 2 There is an alternative interpretation to the costs or revenues that the firm incurs or obtains from the regulator’s tax/subsidy policy. Indeed, \( \bar{e} \) can be defined as the number of emissions permits (or quotas) freely allocated to the firm by the regulator, and \( \tau \) as the price of a permit in the competitive market. Thus, the quantity \( \tau v(t) = \tau (e(t) - \bar{e}) \) would represent the revenue the firm can obtain from selling unused permits in this market (i.e., \( e(t) < \bar{e} \)), or the cost of buying permits if it is the other way around.

The evolution of the goodwill of the firm is governed by the following differential equation,

\[
\dot{G}(t) = \theta A + \varphi (\bar{e} - e(t)) - \delta G(t), \quad G(0) = G_o, \tag{2}
\]

where \( A(t) \) is the firm’s advertising effort in appropriate media, \( \theta \) is a positive parameter measuring advertising efficiency, and \( \varphi \geq 0 \). The above specification extends the standard Nerlove and Arrow (1962) dynamics by adding the term \( \varphi (\bar{e} - e(t)) \), which is intended to capture the impact of the regulator’s public disclosure program on the firm’s goodwill. (There is an extensive literature dealing with advertising and goodwill. See the surveys by Feichtinger et al. (1994) for optimal control models and Jørgensen and Zaccour (2004) for the competitive setting). Thus, we consider that the evolution of the firm’s goodwill depends not only on the firm’s advertising effort, but also on its emissions behavior. The impact of advertising is considered, as usual, to be positive. The second impact is endogenous. If the firm exceeds (meets) the target set by the regulatory body, then it loses (attracts) consumers who are sensitive to environmental issues. The marginal effect of the difference \( (\bar{e} - e(t)) \) is measured here by a given parameter, \( \varphi \). Intuitively, its magnitude would depend on a series of elements, among them the availability of substitutes, the sensitivity of consumers to pollution, etc. When the firm meets its target, then its goodwill increases and so does, \textit{ceteris paribus}, its market potential. If the firm does not reach its target \((\bar{e} < e(t))\), then its goodwill suffers. If the latter effect is higher (in absolute value) than the positive impact of advertising, then the goodwill decreases and so does the firm’s market potential. This idea, which assumes implicitly that consumers may prefer greener products and firms, has been put forward in the literature under different names and in different contexts. For Porter (1991), it might pay to be green. For Kriström and Lundgren (2003), green goodwill can explain why firms voluntarily reduce their emissions. It is also related to the significant demand for ecolabeled apples in Blend and Ravenswaay (1999).
The advertising cost $C(A)$ is convex increasing and taken, for simplicity, quadratic, i.e., $C(A) = \frac{1}{2}A^2$. Such an assumption is made frequently in dynamic models of advertising (see, e.g., Jørgensen and Zaccour (2004)).

Assuming a profit maximizing behavior, and denoting by $r$ the discount rate, the objective functional of the firm then reads as follows,

$$\max_{p,A} \pi = \int_0^\infty e^{-rt} \left\{ (p(t) - c - \tau \alpha) (a + G(t) - p(t)) - \frac{1}{2}A^2(t) + \tau \bar{e} \right\} dt. \quad (3)$$

By (2)-(3) we have defined an infinite-horizon dynamic optimization problem with one state variable ($G(t)$) and two controls ($p(t) \geq 0, A(t) \geq 0$).

We shall assume in the sequel that $a - c - \alpha \tau \geq 0$. This assumption ensures that the firm produces a positive quantity when its goodwill is zero. To save on notation, we let $m = a - c - \alpha \tau$, and eliminate the time argument when no confusion may arise.

### 3 The Optimal Solution

We denote by $x(G; \varphi, \tau)$ the optimal value of decision variable $x$ for $\varphi$ and $\tau$ positive. Similarly, $x(G; 0, \tau)$, $x(G; \varphi, 0)$ and $x(G; 0, 0)$ respectively denote the optimal value in the absence of a PDP ($\varphi = 0$), in the absence of tax/subsidy ($\tau = 0$), and with a laisser-faire policy ($\varphi = \tau = 0$). The following proposition characterizes the optimal solution in the general case.

**Proposition 1** Assuming an interior solution, the optimal pricing and advertising policies and the value function are given by

$$p(G; \varphi, \tau) = \frac{a + c + \tau \alpha + G(\varphi k_1 + 1) + \varphi k_2}{2}, \quad (4)$$
$$A(G; \varphi, \tau) = \theta (k_1 G + k_2), \quad (5)$$
$$V(G; \varphi, \tau) = \frac{1}{2} k_1 G^2 + k_2 G + k_3, \quad (6)$$

where

$$k_1 = \frac{r + 2 \delta + \alpha \varphi - \sqrt{(r + 2 \delta + \alpha \varphi)^2 - (2 \theta^2 + \alpha^2 \varphi^2)}}{2 \theta^2 + \alpha^2 \varphi^2}, \quad (7)$$
$$k_2 = \frac{m (1 - \alpha \varphi k_1) + 2 \varphi k_1 \bar{e}}{r + \sqrt{(r + 2 \delta + \alpha \varphi)^2 - (2 \theta^2 + \alpha^2 \varphi^2)}}, \quad (8)$$
$$k_3 = \frac{1}{4r} \left( k_2^2 (2 \theta^2 + \alpha^2 \varphi^2) + 2 \varphi k_2 (2 \bar{e} - \alpha m) + 4 \bar{e} \alpha \tau + m^2 \right) \quad (9)$$
Proof. See Appendix.

The Proposition states that the optimal pricing and advertising policies are both linear in the goodwill. This is expected in view of the linear-quadratic structure of the problem. Recalling that the advertising cost is \( C(A) = \frac{1}{2}A^2 \), the policy in (5) then states that the level of advertising is chosen so that the marginal cost is equal to the marginal revenue. Further, the pricing and advertising policies are increasing in \( G \). Indeed, for an interior solution, \( \frac{\partial p}{\partial G} \) and \( \frac{\partial A}{\partial G} \) are clearly positive. Simple observation tells us that well-established brands, in terms of quality, consumer perception, etc., command a high price, and are usually heavily advertised, precisely to reinforce the brand positioning. In that sense, advertising acts as a complementary device to pricing. Indeed, it renders consumers less sensitive to price, or to put it differently, it increases their willingness-to-pay. We have here the same phenomenon with the addition that the goodwill depends on the advertising and environmental policies of the firm, and it has a direct influence on market size. The advertising message can be different depending on the sign of the term \((\bar{e} - e(t))\). If it is positive, then the message would put forward the idea that the firm is environmentally responsible and is a good citizen. Otherwise, it would attempt to provide an explanation about why it is so difficult to meet the target. Note that the pricing result is similar to the one in Kriström and Lundgren (2003) who argue that “If consumers prefer to buy products from a greener firm, then the cost of being environmentally friendly may be justified by higher revenues.”

The emissions are given by

\[
e(G; \varphi, \tau) = \alpha q(G; \varphi, \tau) = \alpha (a + G - p(G; \varphi, \tau)),
\]

\[
= \alpha \left( a - c + \alpha \tau + G (1 - \varphi \alpha k_1) - \varphi \alpha k_2 \right).
\]

It is shown in the Appendix that \( 1 - \varphi \alpha k_1 > 0 \), and therefore that emissions are increasing in goodwill. Actually, here we have an interesting circular relationship between goodwill, price and demand (and hence emissions). Increasing the price, \textit{ceteris paribus}, makes the demand shifts downward, resulting in lower emissions. This leads to higher goodwill, which shifts output upward, thanks to the demand by consumers having a preference for green products, and this in turn leads to higher emissions, and so forth.

**Proposition 2** The goodwill steady state is given by

\[
G_{ss}(\varphi, \tau) = \frac{(2\theta^2 + \alpha^2 \varphi^2) k_2 - \varphi (\alpha m - 2\bar{e})}{-r + \sqrt{(r + 2\delta + \alpha \varphi)^2 - (2\theta^2 + \alpha^2 \varphi^2)}} \quad (10)
\]

and is globally asymptotically stable if and only if

\[
r - \sqrt{(r + 2\delta + \alpha \varphi)^2 - (2\theta^2 + \alpha^2 \varphi^2)} < 0. \quad (11)
\]
Proof. Inserting the optimal values for $A$ and $p$ in the state dynamics yields

$$\dot{G} = \frac{G}{2} \left(2\theta^2 k_1 - 2\delta - \alpha\varphi (1 - \alpha\varphi k_1)\right) + \theta k_2 - \frac{\alpha}{2} (m - \alpha\varphi k_2) - 2\bar{e}.$$ 

Substituting for $k_1$, equating to zero and solving leads to the following expression for the steady state:

$$G_{ss} = \frac{(2\theta^2 + \alpha^2\varphi^2) k_2 - \varphi (\alpha m - 2\bar{e})}{-r + \sqrt{(r + 2\delta + \alpha\varphi)^2 - (2\theta^2 + \alpha^2\varphi^2)}}.$$ 

The steady state is globally asymptotically stable if and only if the coefficient of $G$ in $\dot{G}$ above is negative. After substitution for $k_1$ and straightforward calculations, this is equivalent to

$$r - \sqrt{(r + 2\delta + \alpha\varphi)^2 - (2\theta^2 + \alpha^2\varphi^2)} < 0.$$ 

Global asymptotic stability means that the goodwill trajectory will converge to its steady-state value for any initial condition. The condition derived in the above proposition states that the higher is the discount rate ($r$), the decay rate ($\delta$), the marginal impact of emissions on goodwill ($\varphi$) and the factor emissions to production ($\alpha$), then the easier is the realization of the global asymptotical stability. The optimal state trajectory is given by

$$G^* = (G_o - G_{ss}) e^{\lambda t} + G_{ss},$$

where

$$\lambda = \frac{1}{2} \left((2\theta^2 + \alpha^2\varphi^2) k_1 - 2\delta - \alpha\varphi\right)$$

$$= \frac{1}{2} \left(r - \sqrt{(r + 2\delta + \alpha\varphi)^2 - (2\theta^2 + \alpha^2\varphi^2)}\right).$$

From the global stability condition, we have $\lambda < 0$. Convergence of the goodwill trajectory to the steady state is from above if $G_o > G_{ss}$, and from below if $G_o < G_{ss}$.

Remark 3 Under the global asymptotic stability condition, the denominator of $G_{ss}$ is strictly positive. Thus, the steady state has the same sign as its numerator, i.e.,

$$\text{sign}(G_{ss}) = \text{sign} \left((2\theta^2 + \alpha^2\varphi^2) k_2 - \varphi (\alpha m - 2\bar{e})\right).$$

The following three corollaries summarize the results in the above two propositions for the different special scenarios.
Corollary 1 Assuming an interior solution, in the absence of a public disclosure program, the optimal pricing and advertising policies and the value function are given by

\[ p(G;0,\tau) = \frac{a + G + c + \alpha \tau}{2}, \]  \hfill (12) \\
\[ A(G;0,\tau) = \theta \left( \tilde{k}_1 G + \tilde{k}_2 \right), \]  \hfill (13) \\
\[ V(G;0,\tau) = \frac{1}{2} \tilde{k}_1 G^2 + \tilde{k}_2 G + \tilde{k}_3, \]  \hfill (14)

where

\[ \tilde{k}_1 = \frac{r + 2\delta - \sqrt{(r + 2\delta)^2 - 2\theta^2}}{2\theta^2}, \]  \hfill (15) \\
\[ \tilde{k}_2 = \frac{m}{r + \sqrt{(r + 2\delta)^2 - 2\theta^2}}, \]  \hfill (16) \\
\[ \tilde{k}_3 = \frac{1}{4r} \left( 2\tilde{k}_2^2 \theta^2 + 4\bar{c} \alpha \tau + m^2 \right). \]  \hfill (17)

The steady-state goodwill is given by

\[ G_{ss}(0,\tau) = \frac{2\theta^2 m}{-r + \sqrt{(r + 2\delta)^2 - 2\theta^2}} \sqrt{(r + 2\delta)^2 - 2\theta^2}, \]  \hfill (18)

and is globally asymptotically stable if and only if

\[ r - \sqrt{(r + 2\delta)^2 - 2\theta^2} < 0. \]  \hfill (19)

Proof. It suffices to set \( \varphi = 0 \) in Propositions 1 and 2 to get the result.

The interesting result here is that, in the absence of a PDP, the globally asymptotic steady-state goodwill is positive (recall that \( m \) is positive by assumption). The firm’s advertising effort is pure addition to goodwill, and hence, to market potential. In the general case, part of this advertising is done to (possibly) offset the negative environmental record. This is especially the case when the standard is “too” restrictive, and therefore, very costly to meet.

Corollary 2 Assuming an interior solution, in the absence of tax/subsidy program, the optimal pricing and advertising policies and the value function are given by
\[ p(G; \varphi, 0) = \frac{a + c + G(\varphi\alpha k_1 + 1) + \varphi\alpha \hat{k}_2}{2}, \] (20)
\[ A(G; \varphi, 0) = \theta \left( k_1 G + \hat{k}_2 \right), \] (21)
\[ V(G; \varphi, 0) = \frac{1}{2} k_1 G^2 + \hat{k}_2 G + \bar{k}_3, \] (22)

where
\[ \hat{k}_2 = \frac{(a - c) (1 - \alpha \varphi k_1) + 2\varphi k_1 \bar{e}}{r + \sqrt{(r + 2\delta + \alpha \varphi)^2 - (2\theta^2 + \alpha^2 \varphi^2)}}, \] (23)
\[ \bar{k}_3 = \frac{1}{4r} \left( \hat{k}_2^2 (2\theta^2 + \alpha^2 \varphi^2) - \varphi \hat{k}_2 (2\alpha (a - c) - 4\bar{e}) + (a - c)^2 \right). \] (24)

**Proposition 3** The globally asymptotically stable steady state is given by
\[ G_{ss} (\varphi, 0) = \frac{(2\theta^2 + \alpha^2 \varphi^2) \hat{k}_2 - \varphi (\alpha (a - c) - 2\bar{e})}{-r + \sqrt{(r + 2\delta + \alpha \varphi)^2 - (2\theta^2 + \alpha^2 \varphi^2)}}. \]

**Proof.** It suffices to set \( \tau = 0 \) in Propositions 1 and 2 to get the result. \( \square \)

Note that \( k_1 \) and the global asymptotic stability condition are the same as in the general case. Also, one again, the sign of the steady-state goodwill is the sign of its numerator, i.e.,
\[ \text{sign} \left( G_{ss} (\varphi, 0) \right) = \text{sign} \left( (2\theta^2 + \alpha^2 \varphi^2) \hat{k}_2 - \varphi (\alpha (a - c) - 2\bar{e}) \right). \]

**Corollary 3** Assuming an interior solution, in the laisser-faire case, the optimal pricing and advertising policies and the value function are given by
\[ p(G; 0, 0) = \frac{a + c + G}{2}, \] (25)
\[ A(G; 0, 0) = \theta \left( \bar{k}_1 G + \bar{k}_2 \right), \] (26)
\[ V(G; 0, 0) = \frac{1}{2} \bar{k}_1 G^2 + \bar{k}_2 G + \bar{k}_3, \] (27)

where
\[ \bar{k}_2 = \frac{a - c}{r + \sqrt{(r + 2\delta)^2 - 2\theta^2}}, \] (28)
\[ \bar{k}_3 = \frac{1}{4r} \left( 2\theta^2 \bar{k}_2^2 + (a - c)^2 \right). \] (29)
Proposition 4 The steady state is given by
\[ G_{ss}(0, 0) = \frac{2\theta^2\bar{k}_2}{-r + \sqrt{(r + 2\delta)^2 - 2\theta^2}}, \]
and is positive and globally asymptotically stable if and only if
\[ r - \sqrt{(r + 2\delta)^2 - 2\theta^2} < 0. \]

Proof. It suffices to set \( \tau = \varphi = 0 \) in Propositions 1 and 2 to get the result. Note that \( \tilde{k}_1 \) and the global asymptotic stability condition are the same as for \( G_{ss}(0, \tau) \).

Remark 4 The global asymptotic stability condition for \( G_{ss}(0, 0) \) and \( G_{ss}(0, \tau) \), i.e.,
\[ r - \sqrt{(r + 2\delta)^2 - 2\theta^2} < 0, \]
or equivalently,
\[ 2\delta (r + \delta) > \theta^2, \tag{30} \]
implies the condition established for \( G_{ss}(\varphi, \tau) \) and \( G_{ss}(\varphi, 0) \), i.e.,
\[ r - \sqrt{(r + 2\delta + \alpha \varphi)^2 - (2\theta^2 + \alpha^2 \varphi^2)} < 0. \]
We shall assume in the sequel that the condition in (30) is satisfied.

4 Comparison

The aim of this section is to compare the firm’s environmental, pricing and advertising strategies under the following regulatory scenarios: traditional regulation (exemplified by taxation/subsidy), emergent regulation (exemplified by the PDP); and a combination of the two regimes. We are also interested in comparing the steady state reached in the different scenarios, as well as the consumer surplus and profit achieved by the firm.

Proposition 5 Emissions in the different scenarios compare as follows:
\[ e(G; \varphi, \tau) < e(G; 0, \tau) < e(G; 0, 0), \]
\[ e(G; \varphi, \tau) < e(G; \varphi, 0) < e(G; 0, 0), \]
\[ e(G; 0, \tau) > e(G; \varphi, 0) \iff \varphi (k_1 G + \tilde{k}_2) > \tau. \]
Proof. Recall that
\[ e(G; \varphi, \tau) = \alpha (a + G - p(G; \varphi, \tau)) . \]

After substitution for the price, we get
\[
e(G; \varphi, \tau) - e(G; 0, \tau) = -\alpha^2 \varphi \left( \frac{k_1 G + k_2}{2} \right) < 0 , \\
e(G; 0, \tau) - e(G; 0, 0) = \frac{-\tau \alpha^2}{2} < 0 , \\
e(G; \varphi, \tau) - e(G; \varphi, 0) = -\alpha^2 \left( \frac{\tau + \varphi \left( k_2 - \hat{k}_2 \right)}{2} \right) < 0 , \\
e(G; \varphi, 0) - e(G; 0, 0) = -\alpha^2 \left( \frac{k_1 G + \hat{k}_2}{2} \right) < 0 , \\
e(G; 0, \tau) - e(G; \varphi, 0) = \frac{\alpha^2}{2} \left( -\tau + \varphi \left( k_1 G + \hat{k}_2 \right) \right) .
\]

Note that regardless of the sign of \( G \), we have \((k_1 G + k_2) > 0\), and \((k_1 G + \hat{k}_2) > 0\), otherwise the advertising strategy would be negative, which is excluded by the interior solution assumption.

As for the environment, the Proposition shows that a dual regulation is better than any one regulation taken separately, which in turn is better than the laissez-faire policy. The comparison of \( e(G; 0, \tau) \) and \( e(G; \varphi, 0) \) does not lead to a clearcut result. The inequality states that pollution under a traditional regulation regime is higher than under an emerging regulation regime if the marginal loss due to information is higher than the marginal rate of tax/subsidy. Note that this difference (i.e., \( e(G; 0, \tau) - e(G; \varphi, 0) \)) depends on all model parameters (through \( k_1 \) and \( \hat{k}_2 \)) and, since \( k_1 > 0 \), it is increasing in the level of goodwill. Thus, the higher the goodwill of the firm, the higher is the prospect of witnessing a higher emissions level under a traditional regulation than under an emergent one. The policy implication here is that firms with well-established brands would prefer a traditional regulation (which they can actually afford thanks to their high price, or equivalently to a high consumer willingness-to-pay) to an emergent one, which may hurt their prestige.

Since the emissions are proportional to the outputs, the inequalities in the above proposition hold true for the quantities produced under the different scenarios.

Proposition 6 Prices in the different scenarios compare as follows:
\[
p(G; \varphi, \tau) > p(G; 0, \tau) > p(G; 0, 0) , \\
p(G; \varphi, \tau) > p(G; \varphi, 0) > p(G; 0, 0) , \\
p(G; 0, \tau) < p(G; \varphi, 0) \Leftrightarrow \varphi \left( Gk_1 + \hat{k}_2 \right) > \tau.
\]
**Proof.** Straightforward algebraic manipulations lead to the above inequalities.

Given the negative relationship between quantity and price, the results in the above proposition are expected, and actually mirror the previous ones, a dual regulation leads to a higher price to consumer than does any individual regulation, which in turn induces a higher price than does the *laisser-faire* scenario. A simple explanation is that the firm is shifting to the consumer the cost increase that results from regulation. Note that the comparison between emergent and traditional regulations involves the same condition as in the previous proposition. Further, as a direct consequence of the above two propositions, the consumer surplus $\text{CS}(G; \varphi, \tau)$ in the different scenarios compares as follows:

$$\text{CS}(G; \varphi, \tau) < \text{CS}(G; 0, \tau) < \text{CS}(G; 0, 0),$$

$$\text{CS}(G; \varphi, \tau) < \text{CS}(G; \varphi, 0) < \text{CS}(G; 0, 0),$$

$$\text{CS}(G; 0, \tau) > \text{CS}(G; \varphi, 0) \Leftrightarrow \varphi \left( \hat{k}_1 G + \hat{k}_2 \right) > \tau.$$

Based on the above, the consumer would prefer *laisser-faire* policy to one regulation, and the latter to both regulations.

Table 1 presents the comparative results of the advertising strategies.

Note that the conditions given in the above table are only sufficient. A first result is that, for a given $\varphi$ (either positive or zero), implementing a traditional regulation leads to less advertising, i.e.,

$$A(G; \varphi, \tau) < A(G; \varphi, 0),$$

$$A(G; 0, \tau) < A(G; 0, 0).$$

This can be explained as follows: introducing a traditional regulation induces a reduction in emissions, and in turn, a reduction in the deviation (difference between the standard and actual emissions). Consequently, less advertising is needed to achieve the same goodwill.

To shed a light on the other inequalities, first note that

$$\frac{\alpha (a - c - \alpha \tau)}{2} = e(0; 0, \tau),$$

$$\frac{\alpha (a - c)}{2} = e(0; 0, 0).$$

<table>
<thead>
<tr>
<th>$A(G; 0, \tau)$</th>
<th>$A(G; \varphi, 0)$</th>
<th>$A(G; 0, 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(G; \varphi, \tau)$</td>
<td>$&lt; \bar{e} &lt; \frac{\alpha (a - c - \alpha \tau)}{2}$</td>
<td>$&lt; \bar{e} &lt; \frac{\alpha (a - c - \alpha \tau)}{2}$</td>
</tr>
<tr>
<td>$A(G; 0, \tau)$</td>
<td>$= \bar{e} &lt; \frac{\alpha (a - c)}{2} - \frac{\alpha \tau}{2 \hat{k}}$</td>
<td>$&lt; \bar{e} &lt; \frac{\alpha (a - c)}{2}$</td>
</tr>
<tr>
<td>$A(G; \varphi, 0)$</td>
<td>$= \bar{e} &lt; \frac{\alpha (a - c)}{2}$</td>
<td>$&lt; \bar{e} &lt; \frac{\alpha (a - c)}{2}$</td>
</tr>
</tbody>
</table>
Therefore, if the standard $\bar{e}$ is lower than what the firm would emit when $G = \varphi = 0$, then it advertises at a lower level when there is a PDP ($\varphi > 0$) than in the absence of a PDP ($\varphi = 0$). This provides the firm with incentive to ride on its environmental record to build up its goodwill. As in the case of price and emissions, the comparison of advertising strategies in the two regulatory regimes depends on a condition not easily interpreted. While the order of the advertising strategies depends on the model parameters, their slopes do not.

**Proposition 7** The slopes of the advertising strategies in the different scenarios compare as follows:

$$A'(G; \varphi, \tau) = A'(G; \varphi, 0) < A'(G; 0, \tau) = A'(G; 0, 0).$$

Further, the steady-state values satisfy

$$G_{ss}(G; \varphi, \tau) < G_{ss}(G; 0, \tau) < G_{ss}(G; 0, 0),$$

$$G_{ss}(G; \varphi, \tau) < G_{ss}(G; \varphi, 0) \Leftrightarrow \varphi \alpha (r + \delta) < \theta^2,$$

$$G_{ss}(G; 0, \tau) < G_{ss}(G; 0, 0).$$

**Proof.** Recalling that the advertising strategies are given by

$$A(G; \varphi, \tau) = \theta (k_1 G + k_2), \quad A(G; 0, \tau) = \theta (\tilde{k}_1 G + \tilde{k}_2),$$

$$A(G; \varphi, 0) = \theta (k_1 G + \hat{k}_2), \quad A(G; 0, 0) = \theta (\tilde{k}_1 G + \hat{k}_2),$$

it suffices to compare $k_1$ and $\hat{k}_1$ to get the results. Straightforward algebraic manipulations lead to the results concerning the steady-state values. \[\square\]

The above proposition shows that, for a given $\varphi$, the slope of the advertising strategy is independent of the value of $\tau$, and that the steady state achieved under a dual regulatory regime is lower than under a traditional one, which in turn is lower than under a *laisser-faire* policy. Again comparing the two regulations is inconclusive. Indeed, we obtain the following uninterpretable condition,

$$G_{ss}(G; \varphi, 0) < G_{ss}(G; 0, \tau) \Leftrightarrow$$

$$\frac{2\delta (r + \delta) - \theta^2}{2\delta (r + \delta) - \theta^2 + \varphi \alpha (r + 2\delta)} < \frac{\theta^2 (a - c - \alpha \tau)}{(\theta^2 - \varphi \alpha (r + \delta)) (a - c) + \varphi \bar{e} (2r + 2\delta + \alpha \varphi)}.$$

Comparing the value functions used to rank the profits under the different regulatory regimes was inconclusive. This is not surprising in view of the previously witnessed difficulty in obtaining clearcut results when comparing advertising strategies (which are, up to a scaling factor, the derivatives of the value functions), and in view of the complexity of the expressions of the constant terms in the value functions (see (9), (17), (24) and (29)). Still,
as stated in the introduction, we wish to determine if there are conditions under which a PDP is profit improving. The difference between a firm’s payoffs with and without a PDP is given by:

\[ D(G) = V(G; \varphi, \tau) - V(G; 0, \tau) = \frac{1}{2} \left( k_1 - \tilde{k}_1 \right) G^2 + \left( k_2 - \tilde{k}_2 \right) G + \left( k_3 - \tilde{k}_3 \right). \]

We establish in the proof of Proposition 1 (see Appendix) that \( k_1 \) can be written as

\[ k_1 = \frac{1}{r + 2 \delta + \alpha \varphi + \sqrt{(r + 2 \delta)^2 - 2 \theta^2 + 2 \alpha \varphi (r + 2 \delta)}}. \]

Clearly \( k_1 \) is decreasing in \( \varphi \), and hence \( k_1 < \tilde{k}_1 \). This shows that \( D(G) \) is concave with \( \lim_{G \to \pm \infty} D(G) = -\infty \). The maximum of \( D(G) \) is given by

\[ D'(G) = \left( k_1 - \tilde{k}_1 \right) G + \left( k_2 - \tilde{k}_2 \right) = 0, \]

\[ \iff G = \bar{G} = -\frac{\left( k_2 - \tilde{k}_2 \right)}{\left( k_1 - \tilde{k}_1 \right)}. \]

At \( \bar{G} \), we have

\[ D(\bar{G}) = -\frac{1}{2} \left( \frac{k_2 - \tilde{k}_2}{k_1 - \tilde{k}_1} \right)^2 - 2 \left( k_1 - \tilde{k}_1 \right) \left( k_3 - \tilde{k}_3 \right). \]

Let

\[ \Delta = \left( k_2 - \tilde{k}_2 \right)^2 - 2 \left( k_1 - \tilde{k}_1 \right) \left( k_3 - \tilde{k}_3 \right). \] (31)

The following cases may arise:

1. \( \Delta \leq 0 \iff D(\bar{G}) \leq 0 \). In this case, \( D(G) \leq 0, \forall G \) and the PDP has a non-positive impact on the value function.
2. \( \Delta > 0 \iff D(\bar{G}) > 0 \). In this case, the equation \( D(G) = 0 \) has the following two roots:

\[ G_1 = -\frac{\left( k_2 - \tilde{k}_2 \right)}{k_1 - \tilde{k}_1} + \frac{\sqrt{\left( k_2 - \tilde{k}_2 \right)^2 - 2 \left( k_1 - \tilde{k}_1 \right) \left( k_3 - \tilde{k}_3 \right)}}{k_1 - \tilde{k}_1}, \]

\[ G_2 = -\frac{\left( k_2 - \tilde{k}_2 \right)}{k_1 - \tilde{k}_1} - \frac{\sqrt{\left( k_2 - \tilde{k}_2 \right)^2 - 2 \left( k_1 - \tilde{k}_1 \right) \left( k_3 - \tilde{k}_3 \right)}}{k_1 - \tilde{k}_1}, \]

between which \( D(G) \) is positive. In particular, if the initial goodwill \( G_0 \) is such that

\[ G_1 \leq G_0 \leq G_2, \]

then the firm benefits from a PDP; otherwise, such a program is profit deteriorating.
In sum, we have shown that the relationship between a PDP and the firm’s goodwill, as given by $D(G)$, is an inverted U-shaped function. To get an additional insight into this characterization, we provide a numerical illustration. Note that in order to have positive price, quantity and advertising policies, we require

$$G > \max \left( \frac{a + c + \tau \alpha + \phi \alpha k_2}{1 + \phi \alpha k_1}, -\frac{(a - c - \tau \alpha - \phi \alpha k_2)}{1 - \phi \alpha k_1}, -\frac{k_2}{k_1} \right).$$

The model has nine parameters, namely $a, c, r, \alpha, \delta, \theta, \bar{e}, \tau$ and $\phi$. We shall fix once for all the values of all parameters but $\tau$ and $\phi$ as follows:

$$a = 100, c = 5, r = 0.10, \alpha = 0.05, \delta = 0.05, \theta = 0.10, \bar{e} = 2.$$  

Recalling that the tax rate $\tau$ appears in the objective and the PDP parameter $\phi$ appears in the goodwill dynamics, we organize the numerical simulations taking into account the “comparable” parameters, i.e., the production cost $c$ (for $\tau$) and the marginal impact of advertising $\theta$ (for $\phi$). Each parameter of interest can take then one of four possible values as follows:

$$\tau = 0, \tau < c, \tau = c, \tau > c,$$

$$\phi = 0, \phi < \theta, \phi = \theta, \phi > \theta,$$

which leads in total to 16 scenarios. To simplify the interpretation, the numerical results are reported at the steady state of each scenario. They are given in Table 2 and allow for the following observations:

1. In all the scenarios, the result is that regulation, whatever the form it takes, i.e., tax, PDP or both, is detrimental to the firm’s profits. Further, increasing the value of either $\phi$ or $\tau$, or both, leads to lower total profit evaluated at steady state.
2. Increasing the value of either $\phi$ or $\tau$, or both, leads to a lower steady-state goodwill.
3. The impact of $\phi$, on both the steady-state goodwill and profit, is much more pronounced then the impact of $\tau$. This can be attributed to the fact that $\phi$ hurts the market potential, whereas $\tau$ is “only” an additional cost. For instance, multiplying by 10 the value of $\tau$ when $\phi$ is equal to 0.05 (i.e., considering scenarios $S_6$ and $S_8$) leads to a decrease in total profit, evaluated at the steady state, and in $G_{ss}$ of less than 1%. However, multiplying by 2 the value of $\phi$ when $\tau$ is equal to 1 (i.e., considering scenarios $S_6$ and $S_{10}$) lowers the steady state value and profit by approximatively 10%.
Table 2: Results of the Numerical Simulations

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$\phi$</th>
<th>$\tau$</th>
<th>$G_{ss}$</th>
<th>$V(G_{ss}, \ldots)$</th>
<th>$G_1$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0</td>
<td>0</td>
<td>190.00</td>
<td>$1.5794 \times 10^3$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0</td>
<td>1</td>
<td>189.90</td>
<td>$1.5777 \times 10^3$</td>
<td>–155.00</td>
<td>–155.00</td>
</tr>
<tr>
<td>$S_3$</td>
<td>0</td>
<td>5</td>
<td>189.50</td>
<td>$1.5711 \times 10^3$</td>
<td>–149.51</td>
<td>–149.51</td>
</tr>
<tr>
<td>$S_4$</td>
<td>0</td>
<td>10</td>
<td>189.00</td>
<td>$1.5629 \times 10^3$</td>
<td>–149.28</td>
<td>–149.28</td>
</tr>
<tr>
<td>$S_5$</td>
<td>0.05</td>
<td>0</td>
<td>171.75</td>
<td>$1.3907 \times 10^3$</td>
<td>–190.27</td>
<td>–49.329</td>
</tr>
<tr>
<td>$S_6$</td>
<td>0.05</td>
<td>1</td>
<td>171.66</td>
<td>$1.3885 \times 10^3$</td>
<td>–168.15</td>
<td>–79.371</td>
</tr>
<tr>
<td>$S_7$</td>
<td>0.05</td>
<td>5</td>
<td>171.31</td>
<td>$1.3828 \times 10^3$</td>
<td>–177.30</td>
<td>–102.31</td>
</tr>
<tr>
<td>$S_8$</td>
<td>0.05</td>
<td>10</td>
<td>170.87</td>
<td>$1.3757 \times 10^3$</td>
<td>–198.49</td>
<td>–121.51</td>
</tr>
<tr>
<td>$S_9$</td>
<td>0.10</td>
<td>0</td>
<td>156.63</td>
<td>$1.2422 \times 10^3$</td>
<td>–167.64</td>
<td>–71.51</td>
</tr>
<tr>
<td>$S_{10}$</td>
<td>0.10</td>
<td>1</td>
<td>156.55</td>
<td>$1.2409 \times 10^3$</td>
<td>–168.41</td>
<td>–74.796</td>
</tr>
<tr>
<td>$S_{11}$</td>
<td>0.10</td>
<td>5</td>
<td>156.24</td>
<td>$1.2359 \times 10^3$</td>
<td>–171.92</td>
<td>–87.309</td>
</tr>
<tr>
<td>$S_{12}$</td>
<td>0.10</td>
<td>10</td>
<td>155.85</td>
<td>$1.2308 \times 10^3$</td>
<td>–201.0</td>
<td>–78.511</td>
</tr>
<tr>
<td>$S_{13}$</td>
<td>0.20</td>
<td>0</td>
<td>133.07</td>
<td>$1.0284 \times 10^3$</td>
<td>–168.87</td>
<td>–69.203</td>
</tr>
<tr>
<td>$S_{14}$</td>
<td>0.20</td>
<td>1</td>
<td>133.01</td>
<td>$1.0274 \times 10^3$</td>
<td>–169.2</td>
<td>–70.884</td>
</tr>
<tr>
<td>$S_{15}$</td>
<td>0.20</td>
<td>5</td>
<td>132.77</td>
<td>$1.0235 \times 10^3$</td>
<td>–170.77</td>
<td>–77.374</td>
</tr>
<tr>
<td>$S_{16}$</td>
<td>0.20</td>
<td>10</td>
<td>132.47</td>
<td>$1.0186 \times 10^3$</td>
<td>–173.15</td>
<td>–85.068</td>
</tr>
</tbody>
</table>

5 Conclusion

We considered in this paper a setting where a regulator supplements the traditional tax/subsidy regulation by a public disclosure program. The latter affects positively or negatively the brand image and the market potential. The main conclusion of this study is that a PDP allows the regulator to achieve a better environmental result, i.e., it leads to lower emissions. However, it implies a higher price with respect to its absence and hurts the consumer surplus. The result regarding the payoff of the firm is not clearcut and depends on the parameters’ values. However, if our simulations are a good indication of what might be a general result, then this study would not confirm the Porter hypothesis (Porter (1991), Porter and van der Linde (1995)), stating that environmental regulation may have a positive effect on firm performance. The supporters of this hypothesis argue that firms have become more sensitive to their reputation and customers have been increasingly aware of the environmental risks. This situation creates a financial benefit for greener firms (Konar and Cohen, 1997). In this vein, the PDP creates an incentive for the firm to improve its public image and portray itself as being environmentally friendly by reducing its pollution level. This is due to the fact that the violation of environmental standards is prone to penalties. This could have a harmful effect on the reputation and hence on the future profitability of the firm. On the contrary, a respectful firm of the environment can also more easily have access to the financing sources to ensure its growth.
(Lanoie and Laplante (1992)). All this tends to say that environmental and financial performance may go hand-in-hand. The opponents of this hypothesis retort by saying that firms are perfectly rational and do not need the regulator to help them in being so (see, e.g., Palmer et al. (1995)).

A natural extension to our work is to consider an oligopolistic industry where firms compete for consumers having a preference for greener products and to take abatement capital into account.

6 Appendix

6.1 Proof of Proposition 1

To derive the optimal solution, we denote by \( V(G) \) the value function of the firm and write down its Hamilton-Jacobi-Bellman (HJB) equation:

\[
rV(G) = \max_{p,A} \left\{ (p - c - \tau\alpha) (a + G - p) - \frac{1}{2} A^2 + \tau\alpha\bar{e} + V'(G) (\theta A - \varphi (a (a + G - p) - \bar{e}) - \delta G) \right\}
\]

(32)

Assuming an interior solution and performing the maximization on the right-hand side, we obtain the following strategies

\[
p(G) = \frac{a + G + c + \tau\alpha + \varphi\alpha V'}{2},
\]

(33)

\[
A(G) = \theta V'.
\]

(34)

Inserting \( p(G) \) and \( A(G) \) from above into (32) leads to

\[
rV(G) = \left( \frac{m + G + \varphi\alpha V'}{2} \right) \left( \frac{m + G - \varphi\alpha V'}{2} \right)
+ \frac{1}{2} (\theta V')^2 + \tau\alpha\bar{e} + V' \left( -\varphi \left( \alpha \left( \frac{m + G - \varphi\alpha V'}{2} \right) - \bar{e} \right) - \delta G \right)
\]

(35)

Postulating a quadratic value function

\[
V(G) = \frac{1}{2} k_1 G^2 + k_2 G + k_3,
\]

and substituting in (35) leads to

\[
r \left( \frac{1}{2} k_1 G^2 + k_2 G + k_3 \right) = G^2 \left( \frac{\theta^2}{2} + \frac{\alpha^2 \varphi^2}{4} \right) - k_1 \left( \delta + \frac{\alpha \varphi}{2} \right) + \frac{1}{4}
+ G \left( k_1 k_2 \left( \frac{\theta^2}{2} + \frac{\alpha^2 \varphi^2}{4} \right) - k_2 \left( \delta + \frac{\alpha \varphi}{2} \right) - \varphi k_1 \left( \frac{\alpha m}{2} - \bar{e} \right) + \frac{m^2}{2} \right)
+ k_2^2 \left( \frac{\theta^2}{2} + \frac{\alpha^2 \varphi^2}{4} \right) - \varphi k_2 \left( \frac{\alpha m}{2} - \bar{e} \right) + \bar{e} \alpha \tau + \frac{m^2}{4}
\]
By identification, we get the following system to be solved in the three unknowns $k_1, k_2, k_3$:

\[
\begin{align*}
\frac{1}{2} r k_1 &= \frac{1}{4} \left( k_1^2 \left( 2\theta^2 + \alpha^2 \varphi^2 \right) - k_1 \left( 4\delta + 2\alpha \varphi \right) + 1 \right), \\
r k_2 &= \frac{1}{2} \left( k_1 k_2 \left( 2\theta^2 + \alpha^2 \varphi^2 \right) - k_2 \left( 2\delta + \alpha \varphi \right) - \varphi k_1 \left( \alpha m - 2\bar{e} \right) + m \right), \\
r k_3 &= \frac{1}{2} \left( k_2^2 \left( \theta^2 + \frac{\alpha^2 \varphi^2}{2} \right) - \varphi k_2 \left( \alpha m - 2\bar{e} \right) + 2\bar{e} \alpha \tau + \frac{m^2}{2} \right)
\end{align*}
\]

Solving the first equation gives

\[
k_1 = \frac{r + 2\delta + \alpha \varphi \pm \sqrt{(r + 2\delta + \alpha \varphi)^2 - (2\theta^2 + \alpha^2 \varphi^2)}}{2\theta^2 + \alpha^2 \varphi^2}.
\]

To have a real solution, we assume that the term under the square root is nonnegative. Note that both roots are positive and we choose, for stability, to retain the smallest one, i.e., the root with the negative sign. By straightforward successive substitutions one obtains easily the expressions of $k_2$ and $k_3$ given in the Proposition.

To show that the solution is interior (i.e., $p(G) > 0$ and $A(G) > 0$), denote $X = r + 2\delta$.

Then, $k_1$ becomes:

\[
k_1 = \frac{X + \alpha \varphi - \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X - 2\theta^2}}{2\theta^2 + \alpha^2 \varphi^2}.
\]

Multiplying the numerator and the denominator by $(X + \alpha \varphi) + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X}$, we get

\[
k_1 = \frac{(X + \alpha \varphi) - \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X}}{(X + \alpha \varphi) + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X}} \cdot \frac{(X + \alpha \varphi) + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X}}{(X + \alpha \varphi) + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X}}.
\]

\[
k_1 = \frac{(X + \alpha \varphi)^2 - (X^2 - 2\theta^2 + 2\alpha \varphi X)}{(2\theta^2 + \alpha^2 \varphi^2)((X + \alpha \varphi) + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X})}.
\]

\[
k_1 = \frac{2\theta^2 + \alpha^2 \varphi^2}{(2\theta^2 + \alpha^2 \varphi^2)((X + \alpha \varphi) + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X})}.
\]

\[
k_1 = \frac{1}{(X + \alpha \varphi) + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X}}.
\]

Compute

\[
1 - \alpha \varphi k_1 = 1 - \frac{\alpha \varphi}{(X + \alpha \varphi + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X})} = \frac{X + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X}}{X + \alpha \varphi + \sqrt{X^2 - 2\theta^2 + 2\alpha \varphi X}} > 0.
\]

Hence $k_2 > 0$, and therefore we have $p(G) > 0$ and $A(G) > 0.$
References


