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Models for Production and Maintenance Planning in Stochastic Manufacturing Systems

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Abstract

This chapter deals with the control of production systems and presents models for production and maintenance planning. The production systems are supposed to be subject to random abrupt changes in their structures that may results from breakdowns or repairs. Two categories of models are presented. The first one uses dynamic programming and some appropriate numerical methods to compute the policies, while the second category is totally based on linear programming.

Key Words: Production systems with random breakdown, Production planning, Preventive maintenance.

Résumé

Ce chapitre traite de la commande des systèmes de production et présente un certain nombre de modèles pour la planification de la production et de la maintenance. Les systèmes de production sont supposés être soumis à des changements brusques dans leur structure qui peuvent provenir des pannes ou des réparations. Deux catégories de modèles sont prśentées. La première utilise la programmation dynamique et une méthode numérique approporiée pour calculer la solution, quand à la deuxième, la programmation linéaire est employée pour trouver la solution au problème de planification.

1 Introduction

Production systems are the facilities by which we produce most of the goods we are consuming in our daily life. These goods ranges from electronics parts to cars and aircrafts. The production systems are in general complex systems and represent a challenge for the researchers from operations research and control communities. Their modeling and control are among the hardest problems we can have.

In the literature, we can found two main approaches that have been used to tackle the control problems for manufacturing systems (see [1, 6, 11, 39, 50, 57, 61, 63, 72] and the references therein). The first one supposes that the production system is deterministic (neglecting all the random events that may occur) and uses either the linear programming or dynamic programming to solve the production planning problem (see [53] and the references therein). Some attempts to include the maintenance have also been proposed. The second approach includes the random events like breakdowns, repairs, etc. that are inevitable in such systems and uses either the control theory or operations research tools to deal with the production and the maintenance planning.

In the last decades the production and maintenance planning problem has been an active area of research. The contribution on this topic can be divided into two categories. The first one ignores the production planning and considers only the maintenance planning for more details on this directions we refer the reader to [69] and the references therein, while the second category combines the production and the maintenance planning (see [54] and the references therein). For a recent review of maintenance policies that have been used for production systems we refer the reader to the recent survey on the topics by [69] and also to the reference therein.

The aim of this chapter is to propose models that provide simultaneously the production and maintenance planning for manufacturing systems with random breakdowns. Two models are covered. The first one uses the continuous-time framework and based on dynamic programming approach the policies of production and maintenance are computed. The second one use discrete-time framework and proposes a hierarchical approach with two levels with an appropriate algorithm to compute the production and maintenance. At the two levels, the problems are formulated as linear programming problems.

The rest of this chapter is organized as follows. In Section 2, the production and maintenance planning is formulated. In section 3, the approach that uses the dynamic programming is presented and the procedure to solve the problem is developed. In Section 4, the approach that uses the linear programming is presented and the hierarchical algorithm is developed to compute the production and maintenance policies.

2 Problem statement and preliminary results

Let us consider a manufacturing systems with random breakdowns. The system is assumed to be composed of m unreliable machines and producing p part types. Since the machines are unreliable, it results that the production capacity will change randomly which will make it difficult to respond in some cases to a given demand. The preventive maintenance is a way to keep the average system capacity in a desired range and therefore be able to respond to the desired demand. This requires a good planning of the maintenance and at the same time the production.

The problem we will tackle in this chapter consists of determining the production and the maintenance policies we should adopt in order to satisfy the desired demand despite the random events that may disturb the production planning. This chapter will propose two ways to deal with the production and maintenance planning. The first approach that will be developed in Section 3, uses the continuous-time framework and based on dynamic programming proposes a way to compute the solution of the production and maintenance planning. This approach unfortunately needs a lot of numerical computations. To avoid this another approach is proposed at Section 4 and it uses a hierarchical algorithm with two levels. It separates the production and the maintenance at the two levels and treats them separately as linear programming optimization problems.

Before ending this section, let us recall some results that will be used in Section 3. Mainly, we recall the piecewise deterministic problem and its dynamic programming solution and the numerical method that can be used to solve the Hamilton Jacobi Bellmann equation.

Let $\mathcal{E}$ be a countable set and ℓ be a function mapping $\mathcal{E}$ into $\mathbb{N}$, i.e. $\ell : \mathcal{E} \to \mathbb{N}$. For each $\alpha \in \mathcal{E}, E^0_\alpha$ denotes a Borel set of $\mathbb{R}^{\ell(\alpha)}$, i.e. $E^0_\alpha \subset \mathbb{R}^{\ell(\alpha)}$. Define

$$E^0 = \bigcup_{\alpha \in \mathcal{S}} E^0_\alpha = \{(\alpha, z) : \alpha \in \mathcal{E}, z \in E^0_\alpha\},$$

which is a disjoint union of E^0_α's. For each $\alpha \in \mathcal{E}$, where we assume the vector field

$$g^\alpha : E^0_\alpha \to E^0_\alpha$$

is a locally Lipschitz continuous function, determining a flow $\phi_\alpha(x)$. For each $x = (\alpha, z) \in E^0$, define

$$t_*(x) = \begin{cases} \inf\{t > 0 : \phi_\alpha(t, z) \in \partial E^0_\alpha\}, \\ \infty \quad \text{if no such time exists,} \end{cases}$$

where ∂E^0_α is the boundary of E^0_α. Thus $t_*(x)$ is the boundary hitting time for the starting point x. If $t_\infty(x)$ denotes the explosion time of the trajectory $\phi_\alpha(\cdot, z)$, then we assume that $t_\infty(x) = \infty$ when $t_*(x) = \infty$, thus effectively ruling out explosions. Now define

$$\begin{aligned} \partial^\pm E^0_\alpha &= \{z \in \partial E^0_\alpha : z = \phi_\alpha(\pm t, \xi) \text{ for some } \xi \in E^0_\alpha, t > 0\}, \\ \partial_1 E^0_\alpha &= \partial^- E^0_\alpha \backslash \partial^+ E^0_\alpha, \\ E_\alpha &= E^0_\alpha \bigcup \partial_1 E^0_\alpha. \end{aligned}$$

With these definitions, the state space and boundary of a piecewise deterministic Markov process (PDP) can be respectively defined as follows:

$$E = \bigcup_{\alpha \in \mathcal{E}} E_\alpha, \quad \text{state space,} \tag{1}$$

$$\Gamma^* = \bigcup_{\alpha \in \mathcal{E}} \partial^+ E_\alpha^0, \quad \text{boundary.} \tag{2}$$

Thus the boundary of the state space consists of all those points which can be hit by the state trajectory. The points on some ∂E_α^0 which cannot be hit by the state of the trajectory are also included in the state space. The boundary of E consists of all the active boundary points, i.e. points in ∂E_α^0 that can be hit by the state trajectory.

The evolution of a PDP taking values in E is characterized by its three local characteristics:

1. a Lipschitz continuous vector field $f^\alpha : E \to \mathbb{R}^n$, which determines a flow $\phi_\alpha(t, z)$ in E such that, for $t > 0$.

$$\frac{d}{dt}\phi_\alpha(t,z) = f^\alpha(t,z), \ \phi_\alpha(0,z) = z, \forall x = (\alpha, z) \in E.$$

2. a jump rate $q : E \to \mathbb{R}_+$, which satisfies that for each $x \in E$, there is an $\varepsilon > 0$ such that

$$\int_0^\varepsilon q(\alpha, \phi_\alpha(t,z))dt < \infty.$$

3. a transition measure $Q : E \to \mathcal{P}(E)$, where $\mathcal{P}(E)$ denote the set of probability measures on E.

By using these characteristics, a right-continuous sample path $\{x_t : t > 0\}$ starting at $x = (\alpha, z) \in E$ can be constructed as follows. Define

$$x_t \triangleq (\alpha, \phi_\alpha(t,z)), \ \text{if } 0 \leq t < \tau_1,$$

where τ_1 is the realization of the first jump time T_1 with the following generalized negative exponential distribution

$$P(T_1 > t) = exp\left(-\int_0^t q(\alpha, \phi_\alpha(s,z))ds\right).$$

Having realized $T_1 = \tau_1$, we have $x_{T_1^-} \triangleq (\alpha, \phi_\alpha(\tau_1, z)$ and the post-jump state x_{τ_1} which has the distribution given by

$$P((\alpha', z_{\tau_1}) \in A | T_1 = \tau_1) = Q(A, x_{\tau_1^-})$$

on a Borel set A in E.

Restarting the process at x_{τ_1} and proceeding recursively according to the same recipe, one obtains a sequence of jump-time realizations $\tau_1, \tau_2, \cdots$. Between each two consecutive jumps, $\alpha(t)$ remains constant and $z(t)$ follows the integral curves of f^α. Considering this construction as generic yields the stochastic process $\{x_t : t \geq 0, x_0 = x\}$ and the sequence of its jump times $T_1, T_2, \cdots$. It can be shown that x_t is a strong Markov process with right continuous, left-limited sample paths (see [29]).

Piecewise-deterministic processes include a variety of stochastic processes arising from engineering, operation research, management science, economics and inventory system etc. Examples are queuing systems, insurance analysis (see Dassios and Embrechts [28]), capacity expansion (see Davis, Dampster, Sethi and Vermes [32]), permanent health insurance model (Davis [29]), inventory control model (see Sethi and Zhang [61]), production and maintenance model (see Boukas and Haurie [11]). Due to its extensive applications, the optimal control problem has received considerable attention. Gatarek [38], Costa and Davis [27], and Davis [29] have studied the impulse control of PDPs. In the context of nonsmooth analysis, Dempster [33] developed the condition for the uniqueness of the solution to the associated HJB equation of PDPs optimal control involving Clarke generalized gradient. The existence of relaxed controls for PDPs was proved by Davis [29]. Soner [64], Lenhart and Liao [51] used the viscosity solution to formulate the optimal control of PDPs. For more information on the optimal control of PDPs, the reader is referred to Davis [29] and Boukas [9].

In this chapter, the models for the production and maintenance control in manufacturing system that we are treating here can be presented as a special class of piecewise deterministic Markov processes without active boundary points in the state space and the state jump can be represented by a function g. The model can be described as follows:

$$\dot{z}(t) = f^{\alpha(t)}(z(t), u(t)), \quad \forall t \in [T_n, T_{n+1}), \tag{3}$$
$$z(T_n) = g^{\alpha(T_n)}(z(T_n^-)), \quad n = 0, 1, 2, \ldots \tag{4}$$

where $z = [z_1, \ldots, z_p]^T \in \mathbb{R}^p, u = [u_1, \ldots, u_q]^T \in \mathbb{R}^q$ are respectively, the state and control vectors, $f^\beta = [f_1^\beta, \ldots, f_p^\beta]^T$ and $g^\beta = [g_1^\beta, \ldots, g_p^\beta]^T$ represent real valued vectors, and x^T denotes the transpose of x. The initial conditions for the state and for the jump disturbance, i.e the mode, are respectively $z(0) = z^0 \in \mathbb{R}^p$ and $\alpha(0) = \beta_0 \in \mathcal{E}$. The set $\mathcal{E}$ is referred as the index set.

$\alpha = \{\alpha(t) : t \geq 0\}$ represents a controlled Markov process with right continuous trajectories and taking values on the finite state space $\mathcal{E}$. When the stochastic process $\alpha(t)$ jumps from mode β to mode β', the derivatives in (3) change from $f^\beta(z, u)$ to $f^{\beta'}(z, u)$. Between consecutive jump times the state of the process $\alpha(t)$ remains constant. The evolution of this process is completely defined by the jump rates $q(\beta, z, u)$ and the transition probabilities $\pi(\beta'|\beta, z, u)$. The set $\mathcal{E}$ is assumed to be finite. T_n (random variable) is the time of the occurrence of the nth jump of the process α. For each $\beta \in \mathcal{E}$, let $q(\beta, z, u)$ be a bounded and continuously differentiable function. At the jump time T_n, the state z is reset at a value $z(T_n)$ defined by Eq. (4) where $g^{\beta}(.) : \mathbb{R}^p \mapsto \mathbb{R}^p$ is, for any value $\beta \in \mathcal{E}$, a given function.

Remark 2.1 *This description of the system dynamics generalizes the control framework studied in depth by Rishel [58], Wonham [70] and Sworder and Robinson [65], etc. The generalization lies in the fact that the jump Markov disturbances are controlled, and also from the discontinuities in the z-trajectory generated by Eqs. (3)–(4).*

For each $\beta \in \mathcal{E}$, let $f^{\beta}(.,.) : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^p$ be a bounded and continuously differentiable function with bounded partial derivatives in z. Let $U(\beta)$, $\beta \in \mathcal{E}$, (a closed subset of $\mathbb{R}^q$) denotes the control constraints. Any measurable function with values in $U(\beta)$, for each $\beta \in \mathcal{E}$, is called an admissible control. Let $\mathcal{U}$ be a class of stationary control functions $u_{\beta}(z)$, with values in $U(\beta)$ defined on $\mathcal{E} \times \mathbb{R}^p$, called the class of admissible policies. The continuous differentiability assumption is a severe restriction on the considered class of optimization problems, but it is the assumption which allows the simpler exposition that was given in Boukas and Haurie [12]. Later, in the practical models, the restriction will be removed by introducing the notion of viscosity solution of Hamilton-Jacobi-Bellman equation.

The optimal control problem may now be stated as follows: given the dynamical system described by Eqs. (3)–(4), find a control policy $u_{\beta}(z) \in \mathcal{U}$ such that the expected value of the cost functional

$$J(\beta, z, u) = \mathbf{E_u}\Big\{\int_0^{\infty} e^{-\rho t} c(\alpha(t), z(t), u(t))dt|\alpha(0) = \beta, z(0) = z\Big\} \tag{5}$$

is minimized over $\mathcal{U}$.

In Equation (5), ρ ($\rho > 0$) represents the continuous discount rate, and $c(\beta, ., .)$: $\mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^+$, $\beta \in \mathcal{E}$, is the family of cost rate functions, satisfying the same assumptions as $f^{\beta}(.,.)$.

We now proceed to give more precise definition of the controlled stochastic process. Let $(\Omega, \mathcal{F})$ be a measure space. We consider a function $X(t, \omega)$ defined as:

$$X : \mathcal{D} \times \Omega \mapsto \mathcal{E} \times \mathbb{R}^p,\ \mathcal{D} \subset \mathbb{R}^+,$$
$$X(t, \omega) = (\alpha(t, \omega), z(t, \omega))$$

which is measurable with respect to $\mathcal{B}_{\mathcal{D}} \times \mathcal{F}$ ($\mathcal{B}_{\mathcal{D}}$ is a σ-field).

Let $\mathcal{F}_t = \sigma\{X(s, .) : s \leq t\}$ be the σ-field generated by the past observations of X up to time t. We now assume the following:

Assumption 2.1 *The behavior of the dynamical system (3)–(4) under an admissible control policy $u_\beta(.) \in \mathcal{U}$ is completely described by a probability measure P_u on $(\Omega, \mathcal{F}_\infty)$. Thus the process $X_u = (X(t,.), \mathcal{F}_t, P_u), t \in \mathcal{D}$, is well defined. For a given $\omega \in \Omega$ with $z(0,\omega) = z^0$ and $\alpha(0,\omega) = \beta_0$, we define:*

$$\begin{array}{rcl} T_1(\omega) &=& inf\{t > 0: \ \alpha(t,\omega) \neq \beta_0\}, \\ \beta_1(\omega) &=& \alpha(T_1(\omega), \omega), \\ &\vdots& \\ T_{n+1}(\omega) &=& inf\{t > T_n(\omega) : \alpha(t,\omega) \neq \alpha(T_n, \omega)\}, \\ \beta_{n+1}(\omega) &=& \alpha(T_{n+1}(\omega), \omega), \\ &\vdots& \end{array}$$

Assumption 2.2 *For any admissible control policy $u_\beta(.) \in \mathcal{U}$, and almost any $\omega \in \Omega$, there exists a finite number of jump times $T_n(\omega)$ on any bounded interval $[0,T]$, $T > 0$. Thus the function $X_u(t,\omega) = (\alpha_u(t,\omega), z_u(t,\omega))$ satisfies:*

$$\begin{array}{rcl} \alpha_u(0,\omega) &=& \beta_0 \\ z_u(t,\omega) &=& z^0 + \displaystyle\int_0^t f^{\beta_0}(z_u(s,\omega), u_{\beta_0}(z(s,\omega)))ds, \quad \forall t \in [0, T_1(\omega)), \\ &\vdots& \\ \alpha_u(t,\omega) &=& \beta_n(\omega) \\ z_u(t,\omega) &=& g^{\beta_n(\omega)}(z_u(T_n^-(\omega),\omega)) + \displaystyle\int_{T_n(\omega)}^t f^{\beta_n(\omega)}(z_u(s,\omega), u_{\beta_n}(z(s,\omega)))ds, \\ && \qquad \forall t \in [T_n(\omega), T_{n+1}(\omega)), \\ &\vdots& \end{array}$$

Assumption 2.3 *For any admissible control policy $u_\beta(.) \in \mathcal{U}$, we have:*

$$P_u\Big(T_{n+1} \in [t, t+dt] | \ T_{n+1} \geq T_n, \alpha(t) = \beta_n, z(t) = z\Big) = q(\beta_n, z, u_{\beta_n}(z))dt + o(dt),$$

$$P_u\Big(\alpha(t) = \beta_{n+1} | \ T_{n+1} = t, \alpha(t^-) = \beta_n, z(t^-) = z\Big) = \pi(\beta_{n+1}|\beta_n, x, u).$$

Given these assumptions and an initial state (β_0, z^0), the question which will be addressed in the rest of this section is to find a policy $u_\beta(.) \in \mathcal{U}$ that minimizes the cost functional defined by (5) subject to the dynamical system (3)–(4).

Remark 2.2 *From the theory of the stochastic differential equations and the previous assumptions on the functions f^β and g^β for each β, we recall that the system (3)–(4) admits a unique solution corresponding to each policy $u_\beta(z) \in \mathcal{U}$. Let $z^\beta(s; t, z)$ denote the value of this solution at time s.*

The class of control policies $\mathcal{U}$ is such that for each β, the mapping $u_\beta(.) : z \to U(\beta)$ is sufficiently smooth. Thus for each control law $u(.) \in \mathcal{U}$, there exists a probability measure P_u on $(\Omega, \mathcal{F})$ such that the process (α, z) is well defined and the cost (5) is finite. Let the value function $V(\beta, z)$ be defined by the following equation:

$$V(\beta, z) = \inf_{u \in \mathcal{U}} \mathbf{E}_u \Big\{ \int_0^\infty e^{-\rho\tau} c(\alpha(\tau), z(\tau), u(\tau)) d\tau | \alpha(0) = \beta, z(0) = z \Big\}.$$

Under the appropriate assumptions, the optimality conditions of the infinite horizon problem are given by the following theorem:

Theorem 2.1 *A necessary and sufficient condition for a control policy $u_\beta(.) \in \mathcal{U}$ to be optimal is that for each $\beta \in \mathcal{E}$ its performance function $V(\beta, z)$ satisfies the nonlinear partial differential equation:*

$$\begin{aligned} \rho V(\beta, z) \quad = \quad & \min_{u(.) \in U(\beta)} \Big\{ c(\beta, z, u) + \sum_{i=1}^{p} \frac{\partial}{\partial z_i} V(\beta, z) f_i^\beta(z(t), u_\beta(z)) - q(\beta, z, u_\beta(z)) V(\beta, z) \\ & + \sum_{\beta' \in \mathcal{E} - \{\beta\}} q(\beta, z, u) V(\beta', g^{\beta'}(z(t))) \pi(\beta' | \beta, z, u) \Big\}, \ \forall \beta \in \mathcal{E} \qquad (6) \end{aligned}$$

where $\frac{\partial}{\partial z_i} V(\beta, z)$ stands for the partial derivative of the value function $V(\beta, z)$ with respect to the component z_i of the state vector z.

Proof. The reader is referred to Boukas and Haurie [12] for the proof of this theorem. □

As we can see the system given by (6) is not easy to solve since it combines a set of nonlinear partial derivatives equations and optimization problem. To overcome this difficulty, we can approximate the solution by using numerical methods. In the next section, we will develop two numerical methods to solve these optimality conditions and which we believe that they can be extended to other class of optimization problems especially the nonstationary case.

To approximate the solution of the Hamilton-Jacobi-Bellman (HJB) equation corresponding to the deterministic or the stochastic optimal control problem, many approaches have been proposed. For this purpose, we refer the reader to Boukas [8] and Kushner and Dupuis [49].

In this section we will give an extension of some numerical approximation techniques which were used respectively by Kushner [48], Kushner and Dupuis [49] and by Gonzales and Roffman [42] to approximate the solution of the optimality conditions corresponding to other class of optimization problems. Kushner has used his approach to solve an elliptic and parabolic partial differential system associated with a stochastic control problem with diffusion disturbances. Gonzales and Roffman have used their approach to solve a deterministic control problem. Our aim is to use these approaches to solve a combined nonlinear set of coupled partial differential equations representing the optimality conditions of the optimization problem presented in last subsection. The idea behind these approaches consists, within a finite grid G_z^h with unit cell of lengths $(h_1, ..., h_p)$ for the state vector and a finite grid G_u^h with unit cell of lengths $(y_1, ..., y_q)$ for the control vector, of using an approximation scheme for the partial derivatives of the value function $V(\beta, z)$ which will transform the initial optimization problem to an auxiliary discounted Markov decision problem. This will allow us to use the well-known techniques used for this class of optimization problems such as successive approximation or policy iteration.

Before presenting the numerical methods, let us define the discounted Markov decision process (DMDP) optimization problem. Consider a Markov process X_t which is observed at time points $t = 0, 1, 2, \dots$ to be in one of possible states of some finite state space $\mathcal{S} = \{1, 2, \dots, N\}$. After observing the state of the process, an action must be chosen from a finite space action denoted by A.

If the process X_t is in state s at time t and action a is chosen, then two things occur: i) we incur a cost $c(s, a)$ which is bounded and ii) the next state of the system is chosen according to the transition probabilities $P_{ss'}(a)$.

The optimization problem assumes a discounted factor $\delta \in (0, 1)$, and attempts to minimize the expected discounted cost. The use of δ is necessary to make the costs incurred at future dates less important than the cost incurred today. A mapping $\gamma : \mathcal{S} \to A$ is called a policy. Let $\mathcal{A}$ be set of all the policies. For a policy γ, let

$$V_\gamma(s) = \mathbf{E}_\gamma \Big[\sum_{t=0}^{\infty} \delta^t c(X_t, a_t) |\ X_0 = s \Big],$$

where $\mathbf{E}_\gamma$ stands for the conditional expectation given that the policy γ is used.

Let the optimal cost function be defined as:

$$V_\alpha(s) = \inf_\gamma V(s).$$

In the following, we will recall some known results on this class of optimization problems. The reader is referred to Haurie and L'Ecuyer [43] for more information on the topic and for the proofs of these results.

Lemma 2.1 *The expected cost satisfies the following equation:*

$$V_\alpha(s) = \min_{a \in A} \Big\{ c(s, a) + \delta \sum_{s'=1}^{N} P_{ss'}(a) V_\alpha(s') \Big\}, \quad \forall s \in \mathcal{S}.$$

Let $B(I)$ denote the set of all bounded real-valued functions defined on the state space $\mathcal{S}$. Let the mapping T_α be defined by:

$$\begin{aligned} T_\alpha &: B(I) \rightarrow B(I), \\ (T_\alpha w)(s) &= \min_{a \in A}\Big\{c(s,a) + \delta \sum_{s'=1}^{N} P_{ss'}(a) w(s')\Big\}, \quad \forall s \in \mathcal{S}. \end{aligned} \tag{7}$$

Let T_α^k be the composition of the map T_α with itself k times.

Lemma 2.2 *The mapping T_α defined by (7) is contractive.*

Lemma 2.3 *The expected cost $V_\alpha(.)$ is the unique solution of the following equation:*

$$V_\alpha(s) = \min_{a \in A}\Big\{c(s,a) + \delta \sum_{s'=1}^{N} P_{ss'}(a) V_\alpha(s')\Big\}, \quad \forall s \in \mathcal{S}.$$

Furthermore, for any $w \in B(I)$ the mapping $T_\alpha^n w$ converges to V_α as n goes to infinity.

Let us now see how we can put our optimization problem in this formalism. Since our problem has a continuous state vector z and a continuous control vector u, we need first to choose an appropriate discretization of the state space and the control space. Let G_z^h and G_u^h denote respectively the corresponding discrete state space and discrete control space and assume that they have finite elements with respectively n_z points for G_z^h and n_u points for G_u^h.

For the mode of the piecewise deterministic system, we do not need any discretization. Let $\mathcal{S}$ denote the global state space, $\mathcal{S} = \mathcal{E} \times G_z^h$ and N its number of elements. As we will see later, the constructed approximating Markov process X_t will jump between these states, $(s = (\alpha, z) \in \mathcal{S})$, with the transition probabilities $P_{ss'}(a)$, when the control action a is chosen from G_u^h. These transition probabilities are defined as:

$$P_{ss'}(a) = \begin{cases} p_h^\beta(z, z+h; a), & \text{if } z \text{ jumps} \\ \tilde{p}_h^\beta(\beta, z; \beta', a), & \text{if } \alpha \text{ jumps} \end{cases},$$

where $p_h^\beta(z, z+h; a)$ and $\tilde{p}_h^\beta(\beta, z; \beta', a)$ are the probability transition between state s when the action a is used. The corresponding instantaneous cost function $c(s,a)$ and the discount factor δ of the approximating DMDP depend on the used discretization approach. Their explicit expressions will be defined later.

Let h_i denote the finite difference interval, in the coordinate i, and e_i the unit vector in the ith coordinate direction. The approximation that we use for $\frac{\partial}{\partial z_i} V(\beta, z)$ for each $\beta \in \mathcal{E}$, will depend on the sign of $f_i^\beta(z,u)$. Let G_z^h denote the finite difference grid which is a subset of $\mathbb{R}^p$.

This approach was used by Kushner to solve some optimization problems and it consists of approximating the value function $V(\beta, z)$ by a function $V_h(\beta, z)$, and to replace the first derivative partial derivative of the value function, $\frac{\partial}{\partial z_i}V(\beta, z)$, by the following expressions:

$$\frac{\partial}{\partial z_i}V(\beta, z) = \begin{cases} \frac{1}{h_i}[V_h(\beta, z + e_i h_i) - V_h(\beta, z)], & \text{if } \dot{z}(t) \geq 0 \\ \frac{1}{h_i}[V_h(\beta, z) - V_h(\beta, z - e_i h_i)], & \text{otherwise.} \end{cases} \tag{8}$$

For each β, define the functions $p_h^\beta(.;.,.)$, $\tilde{p}_h^\beta(.,.;.,.)$ and $Q_h^\beta(.,.)$ respectively as follows:

$$\begin{aligned} Q_h^\beta(z,u) &= q(\beta, z, u) + \sum_{i=1}^{p}[|\dot{z}_i(t)|/h_i], \\ p_h^\beta(z; z \pm e_i h, u) &= f_i^\pm(z,u)/[h_i Q_h^\beta(z,u)], \\ \tilde{p}_h^\beta(\beta, z; \beta', u) &= q(\beta, z, u)\pi(\beta'|\beta, z, u)/Q_h^\beta(z,u), \\ f_i^+(z,u) &= max(0, f_i^\beta(z,u)), \\ f_i^-(z,u) &= max(0, -f_i^\beta(z,u)). \end{aligned}$$

Let $p_h^\beta(z; z \pm h, u) = 0$ for all points z not in the grid.

Putting the finite difference approximation of the partial derivatives as defined in (8) into (6), and collecting coefficients of the terms $V_h(\beta, z)$, $V_h(\beta, z \pm e_i h_i)$, yields, for a finite difference interval h applying to z,

$$V_h(\beta, z) = \Bigg\{ \frac{c(\beta, z, u)}{Q_h^\beta(z,u)\left[1 + \frac{\rho}{Q_h^\beta(z,u)}\right]} + \frac{1}{\left[1 + \frac{\rho}{Q_h^\beta(z,u)}\right]} \Bigg[\sum_{z' \in G_h} p_h^\beta(z; z', u)V_h(\beta, z') + \sum_{\beta' \in \mathcal{E} - \{\beta\}} \tilde{p}_h^\beta(\beta, z; \beta', u)V_h(\beta', g^{\beta'}(z)) \Bigg] \Bigg\}. \tag{9}$$

Let us define $c(s, u)$ and δ as follows:

$$\begin{aligned} c(s,u) &= \frac{c^\beta(z,u)}{Q_h^\beta(z,u)\left[1 + \frac{\rho}{Q_h^\beta(z,u)}\right]}, \\ \delta &= \frac{1}{1 + \frac{\rho}{Q_h^\beta(z,u)}}. \end{aligned}$$

A careful examination of Eq. (9) reveals that the coefficient of $V_h(.,.)$ are similar to transition probabilities between points of the finite set $\mathcal{S}$ since they are nonnegative and sum to, at most, unity. $c(s, u)$ is also nonnegative and bounded. δ, as defined, represents really a discount factor with values in $(0,1)$. Then, the Eq. (9) has the basic form of the cost equation of the discounted Markov decision process optimization for a given control action. The approximating optimization problem built on the finite state space $\mathcal{S}$ has then the following cost equation:

$$V_h(\beta,z) = \min_{u\in G_u^h}\Big\{ \frac{c(\beta,z,u)}{Q_h^\beta(z,u)[1+\frac{\rho}{Q_h^\beta(z,u)}]} + \frac{1}{[1+\frac{\rho}{Q_h^\beta(z,u)}]}\Big[\sum_{z'\in G_h} p_h^\beta(z;z',u)V_h(\beta,z') + \sum_{\beta'\in\mathcal{E}-\{\beta\}} \tilde{p}_h^\beta(z,\beta;\beta',u) V_h(\beta',g^{\beta'}(z))\Big]\Big\}. \tag{10}$$

Based on the results presented previously, we claim the uniqueness and the existence of the solution of the approximating optimization problem. It is plausible that the algorithms used in the discounted Markov process optimization would be helpful in computing this solution.

3 Dynamic programming approach

Let us consider a manufacturing system that has m machines and produces n part types. When staying in stock, the produced parts of type j will deteriorate with constant rate $\gamma_j, 1\le j\le n$. Suppose the machines are failure-prone and assume that every machine has p modes denoted by $S=\{1,\cdots,p\}$. The mode of machine i is denoted by $r_i(t)$ and $r(t)=(r_1(t),\cdots,r_m(t))^\top \in \mathbb{S}=S^m$ denotes the state of the system. $r_i(t)=p$ means that machine i is under repair and $r_i(t)=j\neq p$ means that machine i is in mode j. In this mode, the machine can produce any part type with an upper production capacity $\bar{u}_j$. $r_i(t)$ is assumed to be a Markov process taking values in state space S with state transition probabilities

$$P\left[r_i(t+h)=l|r_i(t)=k\right] = \begin{cases} q_{kl}h+o(h), & \text{if } l\neq k\\ 1+q_{kk}h+o(h), & \text{otherwise}\end{cases} \tag{11}$$

with $q_{kl}\ge 0$ for all $l\neq k$ and $q_{kk}=-\sum_{l\in S, l\neq k} q_{kl}$ for all $k\in S$, and $\lim_{h\to 0}\frac{o(h)}{h}=0$. Assume that $\{r_j(t), t\ge 0\}, 1\le j\le m$ are independent. From these assumptions it follows that $\{r(t), t\ge 0\}$ is a Markov process, with state space $\mathbb{S}$ and generator $\Lambda=(\lambda_{\alpha\alpha'})$, $\alpha=(\alpha_1,\cdots,\alpha_m), \alpha'=(\alpha'_1,\cdots,\alpha'_m)\in\mathbb{S}$. These jump rates can be computed from the individual jump rates of the machines.

Suppose the demand rates of the products are constants and denoted by $d = (d_1, \cdots, d_n)^\top$. Let $u_{ij}(t)$ be the production rate of part type j on machine i and write

$$u(t) = \begin{pmatrix} u_{11}(t) & \cdots & u_{m1}(t) \\ u_{12}(t) & \cdots & u_{m2}(t) \\ \vdots & \vdots & \vdots \\ u_{1n}(t) & \cdots & u_{mn}(t) \end{pmatrix}$$

which are the control variables in this paper. To complete our model, let us give some notations. For any $x \in \mathbb{R}, x^+ = max(x, 0), x^- = max(-x, 0)$. For any $x \in R^n$, let

$$x^\oplus = (x_1^+, \cdots, x_n^+)^\top, \; x^\ominus = (x_1^-, \cdots, x_n^-)^\top, |x| = (|x_1|, \cdots, |x_n|)^\top$$

and $\|x\|$ denote the Euclidian norm.

Under above assumptions, the differential equation that describes the evolution of the inventory of our facility is therefore given by:

$$\dot{x}(t) = f(x(t), u(t), r(t)), x(0) = x_0, r(0) = \alpha, \tag{12}$$

where

$$f(x(t), u(t), r(t)) = -\gamma x^\oplus(t) + u(t)e - d, \tag{13}$$

with $\gamma = diag\{\gamma_1, \cdots, \gamma_n\}$ and $e = (1, \cdots, 1)^\top \in \mathbb{R}^m$. In (12) $u(t) \in \mathbb{R}_+^{n\times m}$ is the control vector which is assumed to satisfy the following constraints

$$u(t) \in U(r(t)) \triangleq \left\{u(t) : 0 \leq bu(t) \leq \bar{u}_{r(t)}\right\} \tag{14}$$

where $\bar{u}_{r(t)} = (\bar{u}_{r_1(t)}, \cdots, \bar{u}_{r_m(t)})$ is the production capacity of the system and $b = (b_1, \cdots, b_n)$ with $b_l \geq 0$ is a constant scalar.

Our objective in this paper is to seek a control law that minimizes the following cost function:

$$J(x_0, \alpha, u(\cdot)) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} g(x(t), r(t)) dt | x(0) = x_0, r(0) = \alpha\right], \tag{15}$$

where ρ ($\rho \geq 0$) is the discount factor and $\mathbb{E}$ stands for the mathematical expectation operator, $g(x(t), r(t)) = [c^+ x^\oplus(t) + c^- x^\ominus(t)]$ with $c^+ \in \mathbb{R}_+^{1\times n}$ being the inventory holding cost and $c^- \in \mathbb{R}_+^{1\times n}$ is the shortage cost.

This optimization problem falls into the framework of the optimization of the class of systems with Markovian jumps. This class of systems has been studied by many authors and many contributions have been reported to the literature. Among them, we quote

Krasovskii and Lidskii 1960, Rishel 1975, Boukas 1993, Sethi and Zhang 1994 and the references therein.

The goal of the rest of this section is to determine what would be the optimal production rate $u(t)$ that minimizes the cost function (15). Before determining this control, let us introduce some useful definitions.

Definition 3.1 *A control $u(\cdot) = \{u(t) : t \geq 0\}$ with $u(t) \in R_+^{n\times m}$ is said to be admissible if: (i) $u(\cdot)$ is adapted to the σ-algebra generated by the random process $r(\cdot)$, denoted as $\sigma\{r(s) : 0 \leq s \leq t\}$, and (ii) $u(t) \in U(r(t))$ for all $t \geq 0$.*

Let $\mathcal{U}$ denote the set of all admissible controls of our control problem.

Definition 3.2 *A measurable function $u(x(t), r(t)) : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n\times m}$ is an admissible feedback control, or simply the feedback control, if (i) for any given initial continuous state x and discrete mode α, the following equation has an unique solution $x(\cdot)$:*

$$\dot{x}(t) = -\gamma x^{\oplus}(t) + u(x(t), r(t))e - d,\ x(0) = x \tag{16}$$

and (ii) $u(\cdot) = u(x(\cdot), r(\cdot)) \in \mathcal{U}$.

Let the value function $v(x(t), r(t))$ be defined by:

$$v(x(t), r(t)) = \min_{u(t)} J(x(t), r(t), u(\cdot)). \tag{17}$$

Using the dynamic programming principle (see Boukas, 1993), we have

$$v(x(t), r(t)) = \min_{u(t)} \mathbb{E}\left[\int_t^{\infty} e^{-\rho(s-t)} g(x(s), r(s)) ds | x(t), r(t)\right]. \tag{18}$$

Formally, the Hamilton-Jacobi-Bellman equation can be given by the following:

$$\min_{u(t)\in U(r(t))} \left[(\mathcal{A}_u v)(x(t), r(t)) + g(x(t), r(t))\right] = 0, \tag{19}$$

where $(\mathcal{A}_u v)(x(t), r(t))$ is defined as follows:

$$(\mathcal{A}_u v)(x(t), r(t)) \quad = \quad f^{\top}(x(t), u(t), r(t))\frac{\partial v}{\partial x}(x(t), r(t)) + \sum_{\beta\in\mathcal{S}} \lambda_{r(t)\beta} v(x(t), \beta). \tag{20}$$

To characterize the optimal control, let us establish some properties of the value function.

Theorem 3.1 *For any control $u(\cdot) \in \mathcal{U}$, the state trajectory of (12) has the following properties.*

(i) Let x_t be the state trajectory with initial state x_0, then there exists $C_1 \in \mathbb{R}^n_+$ such that

$$|x_t| \leq |x_0| + C_1 t. \tag{21}$$

(ii) Let x_t^1, x_t^2 be the state trajectories corresponding to $(x_1, u(\cdot))$ and $(x_2, u(\cdot))$ respectively, then there exists a constant $C_2 > 0$ such that

$$|x_t^1 - x_t^2| \leq C_2 |x_1 - x_2|, \tag{22}$$

implying

$$\|x_t^1 - x_t^2\| \leq C_2 \|x_1 - x_2\|.$$

Proof. For the proof of this theorem, we refer the reader to Boukas and Liu [14]. □

Theorem 3.2 *(i) For each $r(t) \in \mathcal{S}$, the value function, $v(x(t), r(t))$, is convex;*
(ii) There exists a constant C_3, such that

$$v(x(t), r(t)) \leq C_3(1 + \|x(t)\|);$$

(iii) For each $r(t) \in \mathcal{S}$, the value function, $v(x(t), r(t))$ is Lipschitz.

Proof. For the proof of this theorem, we refer the reader to Boukas and Liu [14]. □

Theorem 3.3 *Suppose that there is a continuously differentiable function $\hat{v}(x(t), r(t))$ which satisfies the Hamilton-Jacobi-Bellman equation (19). If there exists $u^\star(\cdot) \in \mathcal{U}$, for which the corresponding $x^\star(t)$ satisfies (12) with $x^\star(0) = x$, and*

$$\min_{u \in U(r(t))} \Big[(\mathcal{A}_u \hat{v})(x^*(t), r(t)) \Big] = (\mathcal{A}_{u^*} \hat{v})(x^*(t), r(t)) \tag{23}$$

almost everywhere in t with probability one, then $\hat{v}(x, \alpha)$ is the optimal value function and $u^\star(.)$ is optimal control, i.e.

$$\hat{v}(x, \alpha) = v(x, \alpha) = J(x, \alpha, u^\star(.)).$$

Proof. For the proof of this theorem we refer the reader to Boukas and Liu [14]. □

This discussion shows that solving the optimal control problem involves solving HJB equation (19), which often doesn't have closed form solution in the general case. However, in the simplest case, Theorem 3.3 reveals that the optimal control has some special structure, which may be helpful to design the controller. In the sequel of this paper, we will restrict our study to the case of one machine that has two modes and produces one part type, i.e. $m = 1, p = 2, n = 1, S = \{1, 2\}$. In this case, the deteriorating rate, production capacity and demand are denoted by $\gamma, \bar{u}$ and d respectively.

Let us also assume that the value function is continuously differentiable with respect to the continuous arguments. Using the expressions for the functions $f(\cdot)$ and $g(\cdot)$ and the HJB equation given by Eq. (19), one has

$$\begin{aligned}\rho v(x,1) &= \min_u \Big[\big(-\gamma x^+ + u - d\big)\, v_x(x,1) + q_{11} v(x,1) \\ &\qquad + q_{12} v(x,2) + c^+ x^+ + c^- x^-\Big], \qquad (24)\\ \rho v(x,2) &= \Big[\big(-\gamma x^+ - d\big)\, v_x(x,2) + q_{21} v(x,1) \\ &\qquad + q_{22} v(x,2) + c^+ x^+ + c^- x^-\Big]. \qquad (25)\end{aligned}$$

Based on the structure of the optimality conditions, it results that the optimal control law is given by:

$$u^\star(t) = \begin{cases} \bar{u}, & \text{if } v_x(x(t),1) < 0 \text{ and } r(t) = 1, \\ \gamma x^+(t) + d, & \text{if } v_x(x(t),1) = 0 \text{ and } r(t) = 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (26)$$

Moreover, by the convexity of $v(x,1)$ we have

$$u^\star(t) = \begin{cases} \bar{u}, & \text{if } x < x^*(t), \text{ and } r(t) = 1 \\ \gamma I_{\{x \geq 0\}} x^* + d, & \text{if } x = x^*(t) \text{ and } r(t) = 1 \\ 0, & \text{otherwise} \end{cases} \qquad (27)$$

where x^* is the minimal point of $v(x,1)$, i.e. $v_x(x^*,1) = 0$.

Let the optimal control be $u^\star$ and define:

$$\begin{aligned} q_{12} &= \lambda, \qquad (28)\\ q_{21} &= \mu, \qquad (29)\\ V(x) &= \begin{bmatrix} v(x,1) \\ v(x,2) \end{bmatrix}. \qquad (30)\end{aligned}$$

With these definitions and if we let $x^\star$ (Without loss of generality, we assume x^* to be greater than 0, other cases can be handled similarly) denote the minimum of the value function at mode 1, the optimality conditions become:

- $x > x^\star$, then $u^\star = 0$ and the optimality conditions become:

$$V_x(x) = \begin{bmatrix} -\frac{\lambda+\rho}{\gamma x+d} & \frac{\lambda}{\gamma x+d} \\ \frac{\mu}{\gamma x+d} & -\frac{\mu+\rho}{\gamma x+d} \end{bmatrix} V(x) + \begin{bmatrix} \frac{c^+ x}{\gamma x+d} \\ \frac{c^+ x}{\gamma x+d} \end{bmatrix} \qquad (31)$$

- $x = x^\star$, then $\gamma x + d = u^\star$ and the optimality conditions become:

$$v(x,1) = \left[\frac{\lambda}{\rho+\lambda}\right] v(x,2) + \left[\frac{c^+ x}{\rho+\lambda}\right] \tag{32}$$

and

$$v_x(x,2) = \left[-\frac{\rho(\rho+\lambda+\mu)}{(\rho+\lambda)(\gamma x+d)}\right] v(x,2) + \left[\frac{\rho+\mu+\lambda}{\rho+\lambda}\right] c^+ x \tag{33}$$

- $0 \le x < x^\star$ then $u^\star = \bar{u}$ and the optimality conditions become:

$$V_x(x) = \begin{bmatrix} -\frac{\lambda+\rho}{\gamma x+d-\bar{u}} & \frac{\lambda}{\gamma x+d-\bar{u}} \\ \frac{\mu}{\gamma x+d} & -\frac{\mu+\rho}{\gamma x+d} \end{bmatrix} V(x) + \begin{bmatrix} \frac{c^+ x}{\gamma x+d-\bar{u}} \\ \frac{c^+ x}{\gamma x+d} \end{bmatrix} \tag{34}$$

- $x \le 0$ then $u^\star = \bar{u}$ and the optimality conditions become:

$$V_x(x) = \begin{bmatrix} -\frac{\lambda+\rho}{d-\bar{u}} & \frac{\lambda}{d-\bar{u}} \\ \frac{\mu}{d} & -\frac{\mu+\rho}{d} \end{bmatrix} V(x) - \begin{bmatrix} \frac{c^- x}{d-\bar{u}} \\ \frac{c^- x}{d} \end{bmatrix} \tag{35}$$

To solve the HJB equations, we can use the numerical method used in Boukas 1995. This method consists of transforming the optimization problem to a Markov decision problem (MDP) with all the nice properties that guarantee the existence and the uniqueness of the solution. The key point of this technique is first to discretize the state space $\mathbb{R}$ and control space $[0,\bar{u}]$ to get a discrete state space $G_x = [-\underline{x}, -\underline{x}+h_x, \cdots, \bar{x}]$ with $\underline{x}, \bar{x}$ great enough and a discrete control space $G_u = [0, h_u, \cdots, \bar{u}]$, and then define a function $v_h(x,i)$ on $G_x \times S$ by letting $v_h(x,i) = v(x,i)$. By replacing $v_x(x,i)$ by

$$\begin{cases} \frac{1}{h_x}\left[v(x+h_x,i) - v(x,i)\right], & \text{if } f(x,u,i) \ge 0, \\ \frac{1}{h_x}\left[v(x,i) - v(x-h_x,i)\right], & \text{otherwise} \end{cases}$$

and substituting $v_h(x,i)$ into (24) and (25) gives the following MDP problem:

$$\begin{aligned} v_h(x,1) = \min_{u \in G_u} \Bigg[c(x,1) + \frac{1}{1+\frac{\rho}{Q^1_{h_x}}} \Bigg(& \frac{(-\gamma x^+ + u - d)^+}{h_x Q^1_{h_x}} v_h(x+h_x,1) \\ & + \frac{(-\gamma x^+ + u - d)^-}{h_x Q^1_{h_x}} v_h(x-h_x,1) + \frac{q_{12}}{Q^1_{h_x}} v_h(x,2) \Bigg) \Bigg], \end{aligned} \tag{36}$$

$$v_h(x,2) = c(x,2) + \frac{1}{1+\frac{\rho}{Q^2_{h_x}}} \left[\frac{|-\gamma x^+ - d|}{h_x Q^2_{h_x}} v_h(x-h_x,2) + \frac{q_{21}}{Q^2_{h_x}} v_h(x,1) \right], \tag{37}$$

where h_x is the discretization step for the x, $c(x,\alpha)$, Q^1_h and Q^2_h are defined by:

$$c(x,\alpha) = \frac{c^+ x^+ + c^- x^-}{Q^\alpha_{h_x}\left[1+\frac{\rho}{Q^\alpha_{h_x}}\right]}, \text{ for all } \alpha \in S, \tag{38}$$

$$Q^1_{h_x} = \frac{|\gamma x^+ - u + d|}{h_x} + |q_{11}|, \quad (39)$$

$$Q^2_{h_x} = \frac{|\gamma x^+ + d|}{h_x} + |q_{22}|. \quad (40)$$

The successive approximation technique and the policy iteration technique can be used to find an approximation of the optimal solution. For more information on these techniques, we refer the reader to Bertsekas 1987, Boukas 1995 or Kushner and Dupuis 1992 and the references therein.

Remark 3.1 *By the same argument as in Boukas et al. 1996, it is easy to prove that* $\lim_{h\to 0} v_h(x,i) = v(x,i), \forall i \in S$, *which establishes the convergence of the approximation algorithm.*

4 Linear programming approach

In the previous section we developed an approach to plan the production and maintenance using a continuous-time model. With this model we were able to compute simultaneously the production and maintenance. But this approach requires a lot of computations before the solution can be obtained. To overcome this, we propose a new approach that uses linear programming and an hierarchical algorithm for this purpose. To show how this approach works, we will restrict ourself to one machine one part type, but we have to keep in mind that the model we propose here is valid for any number of machines and part types. For this purpose, let us consider a production system with one machine that produces one part type and assume that the system must satisfy a given demand $d(k)$, $k = 0, 1, 2, \cdots$ that can be constant or time varying. Let the dynamics of the production system be described by the following difference equation:

$$x(k) = x(k-1) + u(k) - d(k), x(0) = x_0 \quad (41)$$

where $x(k) \in \mathbb{R}$, $u(k) \in \mathbb{R}$ and $d(k) \in \mathbb{R}$ represent respectively the stock level, the production and the demand at period kT, $k = 0, \cdots, N$.

The stock level, $x(k)$ and the production $u(k)$ must satisfy at each period kT the following constraints:

$$0 \leq u(k) \leq \bar{u} \quad (42)$$

$$x(k) \geq 0 \quad (43)$$

where $\bar{u}$ is known positive constant that represents the maximum production the system can have.

Remark 4.1 *The upper bound constraint on the production represents the limitation of the capacity of the manufacturing system, while the one of the stock level means that we do not tolerate the negative stock. Notice that we can also include an upper bound of the stock level.*

The objective is to plan the production in order to satisfy the given demand during a finite horizon. Since the capacity may change with time in a random way, it is therefore required to include the preventive maintenance and combine it to the production planning problem. By performing maintenance we keep the capacity in average in a certain acceptable values.

To solve the simultaneous production and maintenance planning problem, we use the following hierarchical approach with two levels

1. at level one we plan the preventive maintenance
2. and at level two, using the results of level one, we try satisfy the demand during the periods the machine is up

To present each level in this algorithm let:

- T be the time period that can be one hour, one day, one month, etc.
- $x(k)$ be the stock level at time kT
- $u(k)$ be production at time kT
- $d(k)$ be the demand at time kT
- T_{up_i} be the amount of units of time during which the machine is working before the ith maintenance takes place (T_{up_i} is a multiple of T)
- T_d be the amount of units of time of the ith maintenance takes (T_d is a multiple of T and it is assumed to be the same for all the interventions)
- NT be the total time for the planning (N is a positive integer)
- v be the upper bound of T_{up_i}
- μ be the number of preventive-maintenance taking place in NT
- $w_i(k)$ the number of deferred items at time kT for i period
- av be the availability of the machine
- $\bar{u}$ be the upper bound of $u(k)$

The algorithm we will adopt is summarized as follows:

1. Initialization: Choose the data N, T, μ, T_d
2. Solve a LP problem that gives the dates of the preventive intervention during the interval of time $[0, NT]$
3. Test: If the problem is feasible go to Step 4, otherwise increase μ and go Step 2
4. Solve the LP problem for production planning to the determine the decision variables
5. Test: If the problem is feasible stop otherwise the interval of time $[0, NT]$ is not enough to respond to the demand and no feasible solution can be obtained. We can increase the interval and repeat the steps.

The problem at level one tries to divide the planning interval $[0, NT]$ in successive periods for production, T_{upk} and maintenance T_{dk} (T_{dk} is supposed to be constant here),

$k = 1, 2, \cdots, \mu$, that sum to a time that is less or equal to NT. It is also considered that the availability of the machine should be greater or to a given av. We should also notice that T_{upk} is between 0 and v for any k. The formulation of the optimization problem at this level is given by:

$$\begin{cases} \min\max\left[\sum_{k=1}^{\mu}\left[T_{up_k} + T_d\right]\right] \\ s.t.: \\ \sum_{k=1}^{\mu}\{T_{up_k} + T_d\} \leq NT \\ \frac{\sum_{k=1}^{\mu} T_{up_i}}{\sum_{k=1}^{\mu} T_{up_i} + \mu T_d} \geq av \\ 0 \leq T_{up_i} \leq v \end{cases} \tag{44}$$

That can be transformed to:

$$P1: \begin{cases} \min Z \\ s.t.: \\ \sum_{k=1}^{\mu} T_{up_i} \leq Z - \mu T_d \\ \sum_{k=1}^{\mu}\left[T_{up_k} + T_d\right] \leq N \\ \frac{\sum_{k=1}^{\mu} T_{up_i}}{\sum_{k=1}^{\mu} T_{up_i} + \mu T_d} \geq av \\ 0 \leq T_{up_i} \leq v \end{cases} \tag{45}$$

which is a linear programming problem that can be easily solved using the powerful existing tools for this purpose.

The optimization problem at level two consists of performing the production planning inside the time during which the machine is up in order to satisfy the demand and all the system constraints by penalizing the stock level and the production with appropriate unit costs. This problem is given by:

$$P2: \begin{cases} \min\sum_{k=1}^{N}\left[c^x x(k) + c^u u(k)\right] \\ s.t.: \\ x(k) = x(k-1) + u(k) - d(k), x(0) = x_0 \\ u(k) \leq \bar{u} \\ u(k) \geq 0 \\ x(k) \geq 0 \end{cases} \tag{46}$$

and which is also a linear programming optimization problem.

Both the problems at the two levels are linear which make them easier to solve with the existing tools and for high dimensions problems. This can include production systems with multiple machines multiple part types.

To show the validness of the approach of this section, let us consider the system with the data of Table 1.

Table 1: System data

T	N	T_d	v	μ	c^x	c^u	$\bar{u}$
1	20	1	7	4	2	3	4

Solving the previous optimization problems following the proposed algorithm with these data, we get the results of Figures 1–5.

Figure 1 gives the solution of the optimization problem at level one and it illustrates the sequence of the phases up and down for the considered machine. Since we don't impose conditions on the state of the machine when the age grows, the results at level one shows that we can perform periodic preventive maintenance that will take constant time as we did in this example.

Figure 2 shows the results of the solution of the optimization problem at level two for a given deterministic demand. This figure shows that the cumulative stock levels tracks well the given time-varying demand.

Figure 3 illustrated the production at each period obtained by the solution of the optimization problem at level two. As it can be seen from these figures all the constraints are satisfied.

With the same data, we have generated randomly the time-varying demand and solved the two levels optimization problems and the solution is illustrated by Figures 4–5.

In some circumstances due to reduction in the system capacity, we may defer the demand by some periods and pay a penalty cost. As first extension of the previous model let us now add the ability of deferring some items in the demand to the next period and see how to solve the production and maintenance planning for this case. Firstly notice that the optimization problem at level one will not change since it is independent on the demand. The changes will affect mainly the second optimization problem, more specifically the cost function should take care of the cost incurred by the deferred items and the dynamics that must be changed to include the deferred items. The rest of the constraints on the stock level and the production stay unchanged. The changes in this case are:

- the previous cost function can be changed by the following: $\sum_{k=1}^{N}[c^x x(k) + c^u u(k) + c^w w(k)]$
- and the new dynamics is: $x(k) = x(k-1) + u(k) + w(k) - w(k-1) - d(k)$

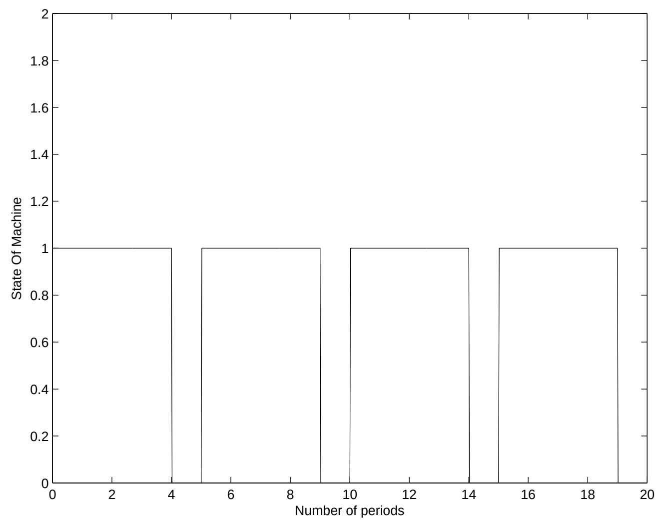

Figure 1: State of the machine

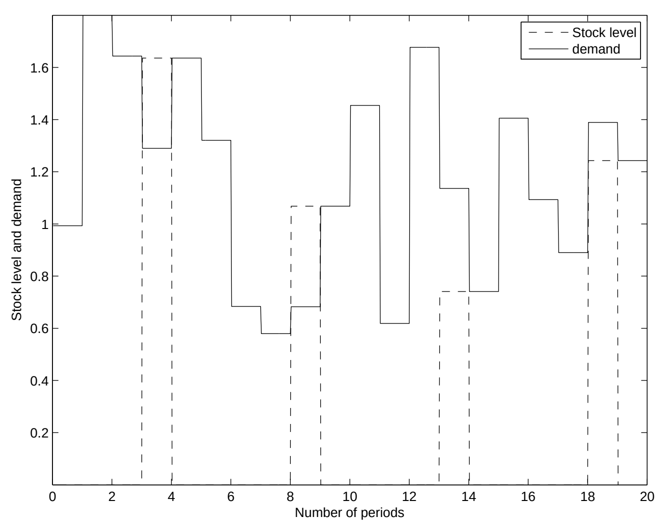

Figure 2: Stock level and demand (deterministic case)

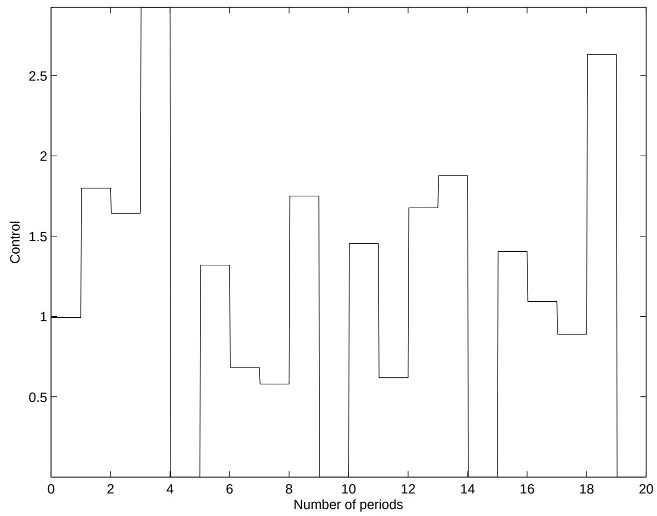

Figure 3: Production Rate (deterministic case)

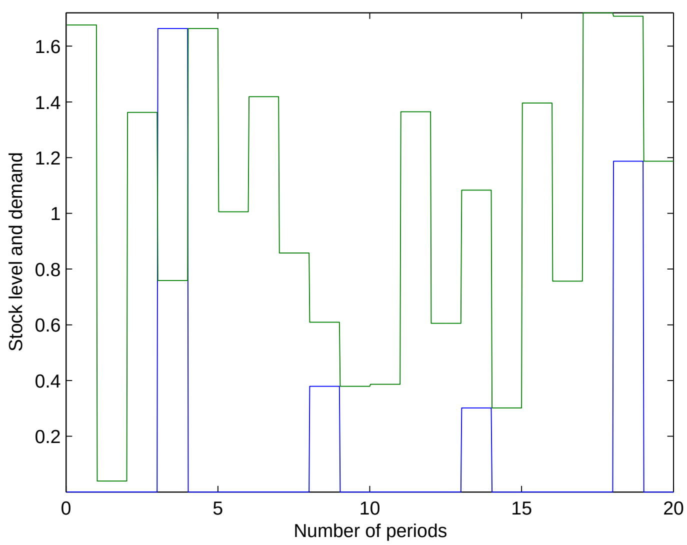

Figure 4: Stock level and demand (stochastic case)

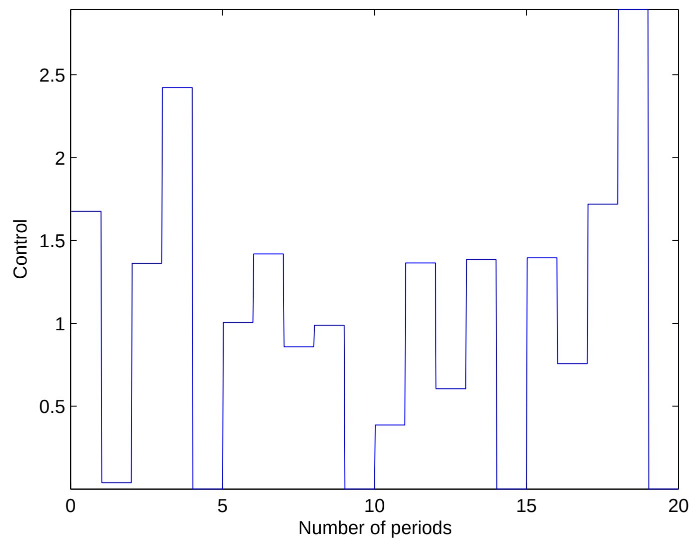

Figure 5: Production Rate (stochastic case)

The optimization problem at level two becomes:

$$P2' \begin{cases} \min \sum_{k=1}^{N} \left[c^x x(k) + c^u u(k) + c^w w(k)\right] \\ s.t. : \\ x(k) = x(k-1) + u(k) + w(k) - w(k-1) - d(k), x(0) = x_0 \\ u(k) \leq \bar{u} \\ u(k) \geq 0 \\ x(k) \geq 0 \end{cases} \tag{47}$$

With the same date of the Table 1 solving the optimization problems ate the two level, we get the results illustrated by Figures 6–11. Figure 6 gives the same results as for the case without deferred items. The other figures give the stock level and the production at different periods for the deterministic case and the stochastic one as we did for the previous model.

As a second extension, let us now add the ability of deferring some items of the demand up to three periods. For this case the changes we have to make to our second optimization problem at level two concern the cost and dynamics. These changes are:

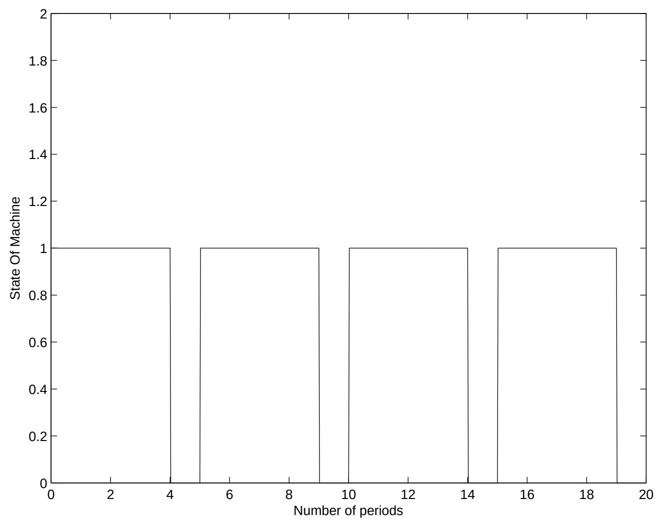

Figure 6: Stock level and demand (deterministic case

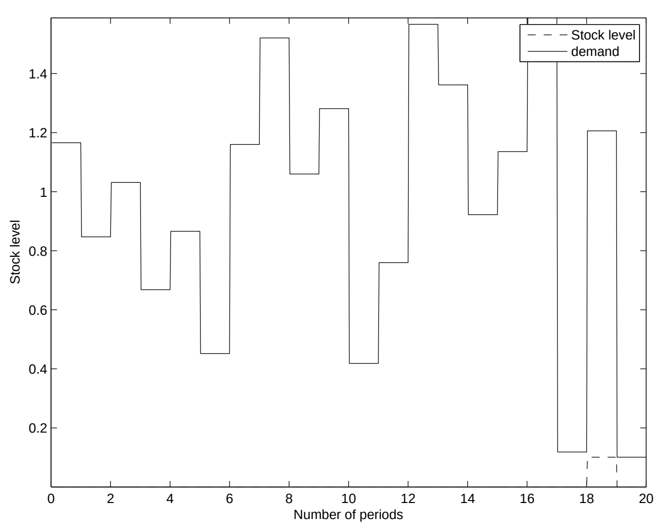

Figure 7: Production Rate (deterministic case)

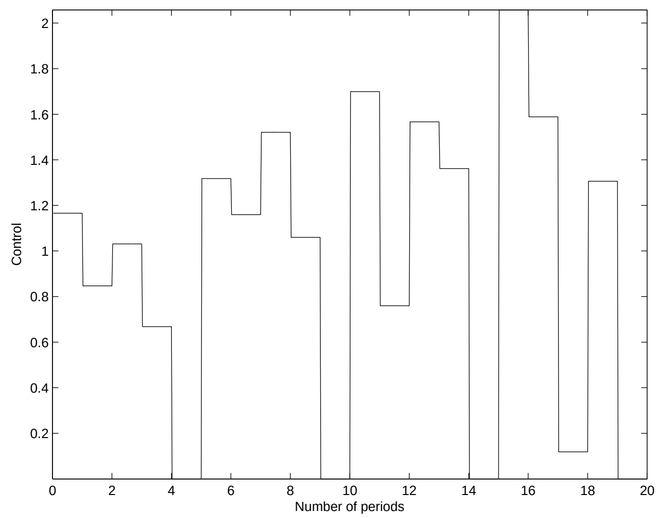

Figure 8: Deferred items (deterministic case)

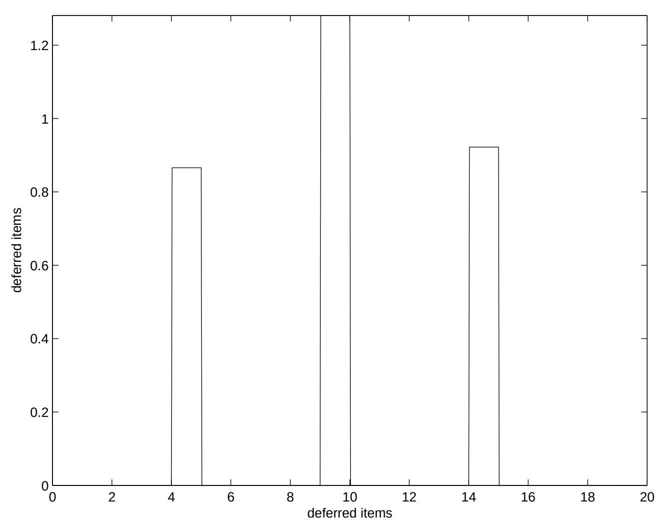

Figure 9: Stock level and demand (stochastic case)

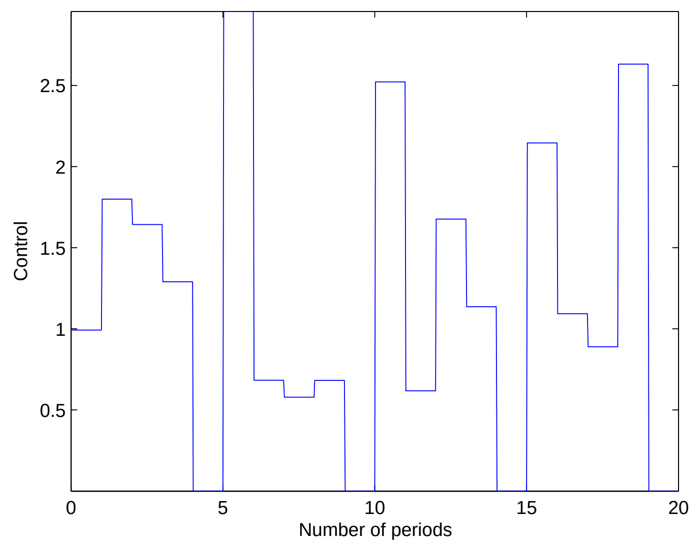

Figure 10: Production Rate (stochastic case)

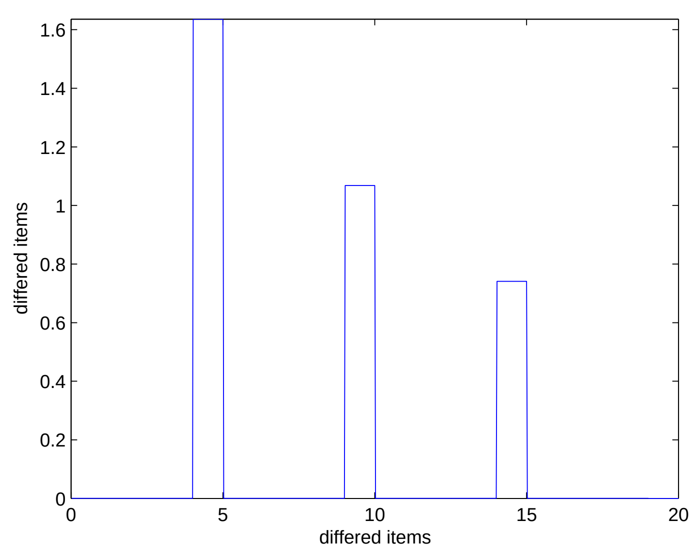

Figure 11: Deferred items (stochastic case)

- the cost function becomes:

$$\sum_{k=1}^{N} [c^x x(k) + c^u u(k) + c^{w_1} w_1(k) + c^{w_2} w_2(k) + c^{w_3} w_3(k)]$$

- the dynamics become:

$$\begin{aligned} x(k) = x(k-1) + u(k) + w_1(k) - w_1(k-1) + w_2(k) - w_2(k-2) \\ +w_3(k) - w_3(k-3) - d(k) \end{aligned}$$

The optimization problem at level two becomes:

$$P3' \begin{cases} \min \sum_{k=1}^{N} [c^x x(k) + c^u u(k) + c^{w_1} w_1(k) + c^{w_2} w_2(k) + c^{w_3} w_3(k)] \\ s.t. : \\ x(k) = x(k-1) + u(k) + w_1(k) - w_1(k-1) + w_2(k) - w_2(k-2) \\ \quad +w_3(k) - w_3(k-3) - d(k), x(0) = x_0 \\ u(k) \leq \bar{u} \\ u(k) \geq 0 \\ x(k) \geq 0 \end{cases} \tag{48}$$

With the same data of the Table 1 solving the optimization problems at the two levels, we get the results illustrated by Figures 12–18. Figure 12 gives the same results as for the case without deferred items. The other figures give the stock level and the production at different periods.

We can make more extensions for our model to include the following facts:

1. model with depreciation
2. model with depreciation after some periods of time
3. model with setups

5 Conclusion

In this chapter, we tackled the production and preventive maintenance control problem for manufacturing system with random breakdowns. This problem is formulated as a stochastic optimal control problem where the state of the production system is modeled as a Markov chain, the demand is constant and the produced items are assumed to deteriorate with a given rate γ. With some assumptions, the optimal production rate is still hedging point policy with some changes at the hedging point $x^\star$. The production and preventive maintenance problem has also been solved using a hierarchical approach with two levels. The level one determines the instants when the maintenance has to be performed. The level two determines the production to track the demand. Some extensions of this model have been proposed.

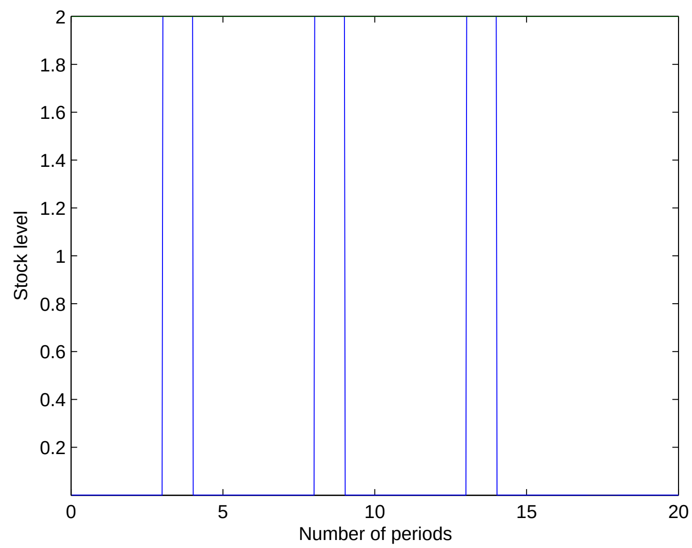

Figure 12: Stock level and demand (deterministic case)

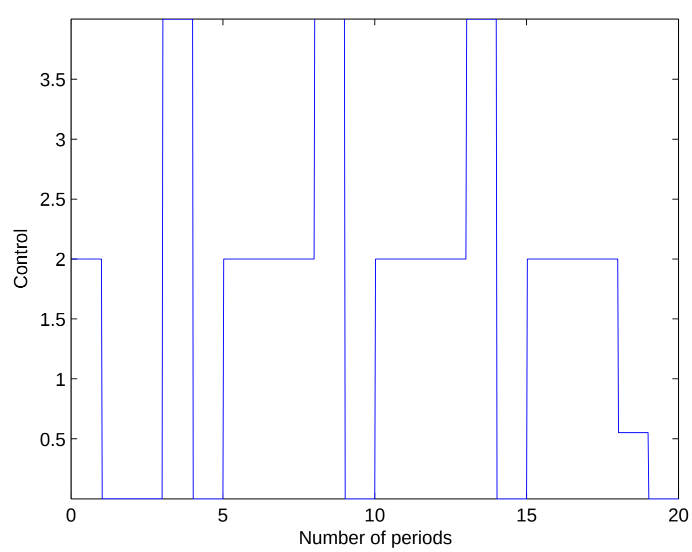

Figure 13: Production Rate (deterministic case)

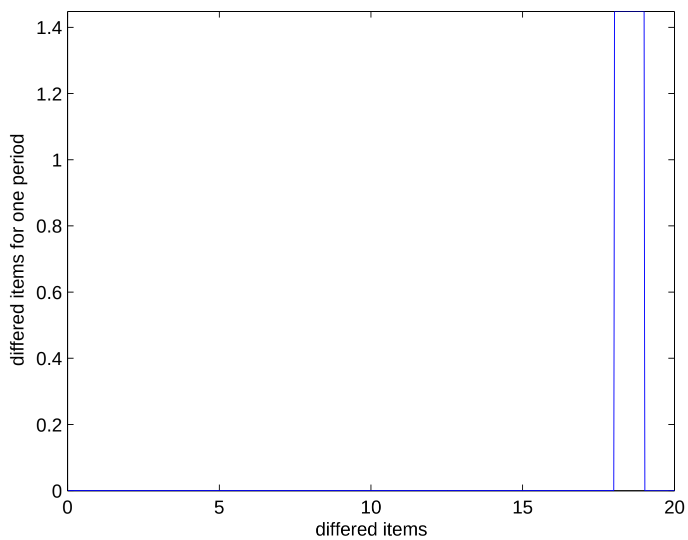

Figure 14: Deferred items for one period (deterministic case)

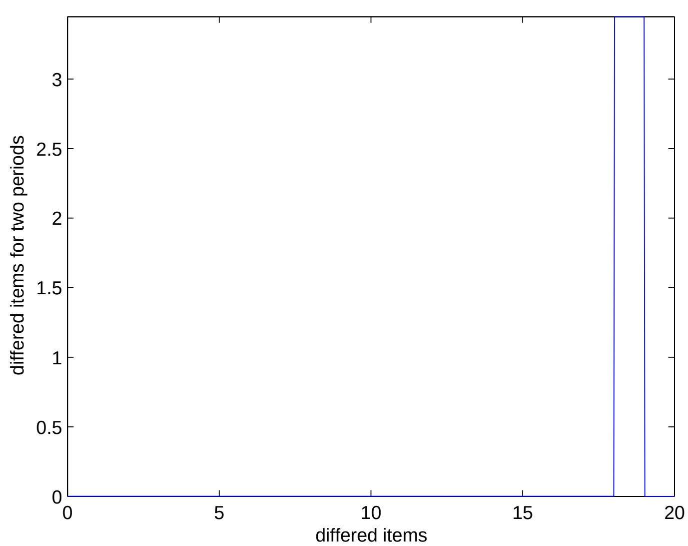

Figure 15: Deferred items for two period (deterministic case)

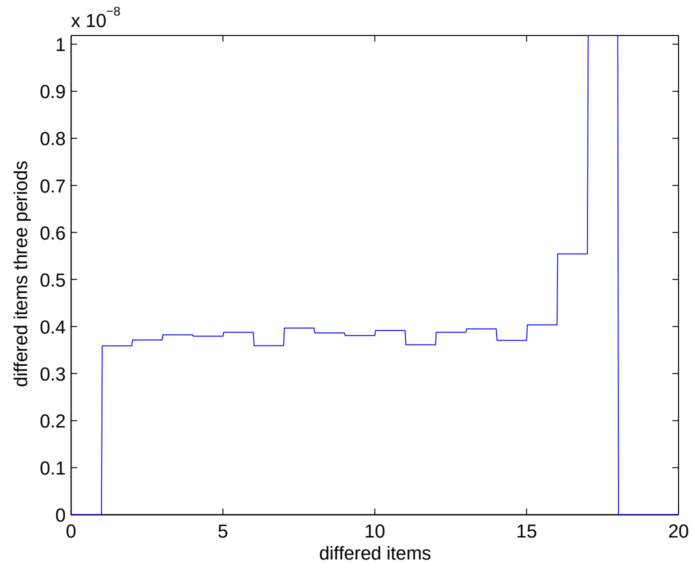

Figure 16: Deferred items for three period (deterministic case)

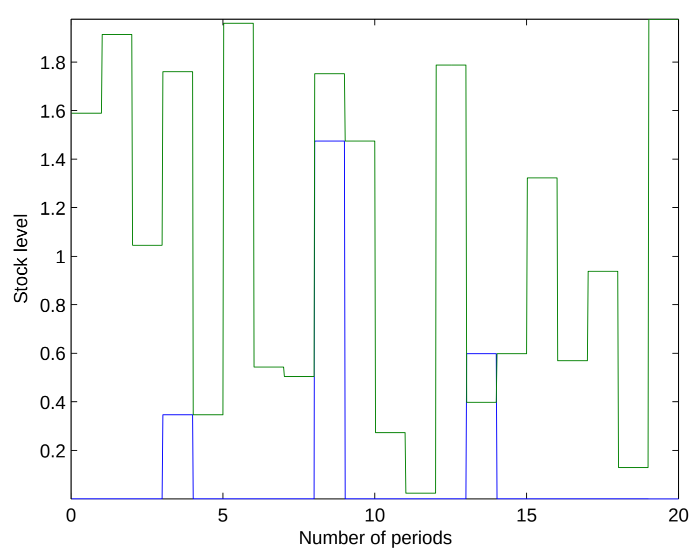

Figure 17: Production rate (stochastic case)

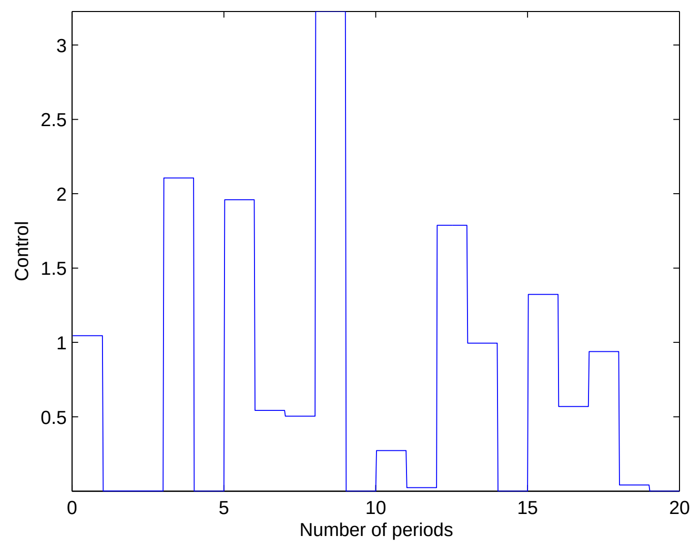

Figure 18: Stock level and demand (stochastic case)

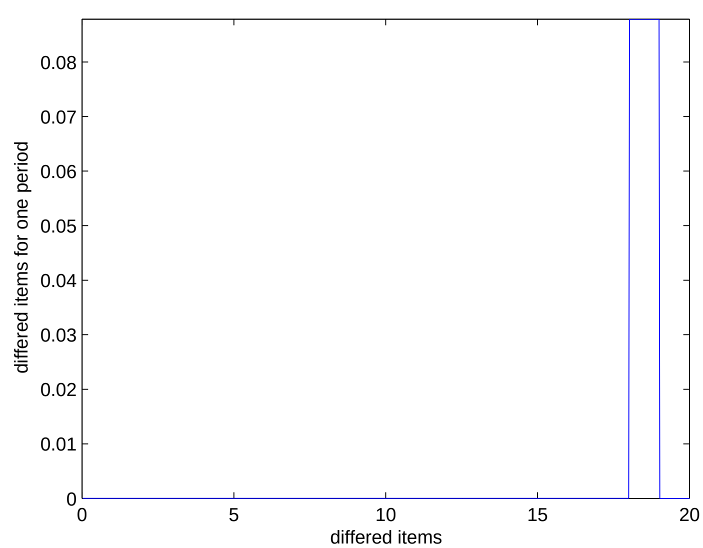

Figure 19: Deferred items for one period (stochastic case)

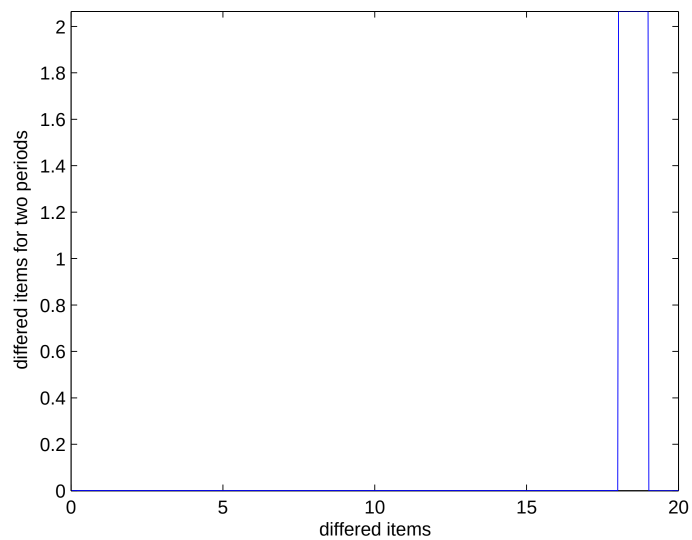

Figure 20: Deferred items for two period (stochastic case)

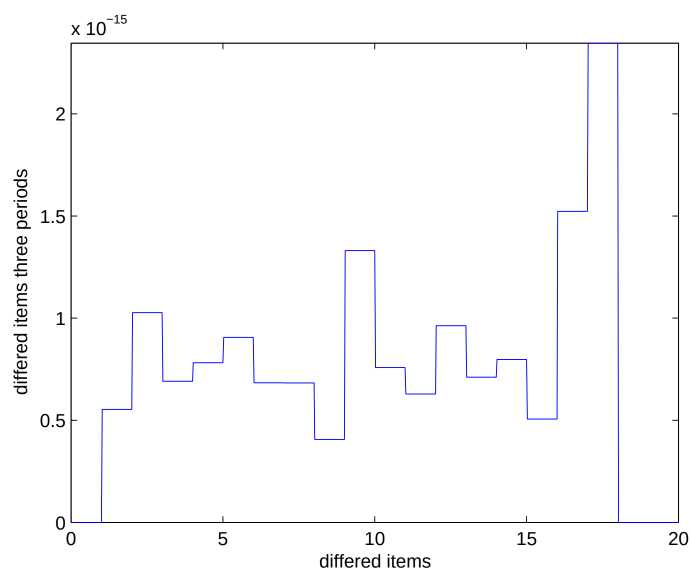

Figure 21: Deferred items for three period (stochastic case)

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