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Abstract

In this paper, the problem of stability and stabilization of Markov jumping singular systems with discontinuities and saturating inputs is addressed. The design procedure via linear matrix inequality technique (LMI) and the sequential linear programming matrix methods (SLPMM), are used to determine simultaneously a state feedback control and an associated domain of safe admissible states for which the regularity, the absence of impulsive behavior and the stochastic stability in mean square sense of the closed-loop systems are guaranteed. A numerical example is provided to demonstrate the effectiveness of the proposed methods.

Key Words: Singular Markov jump systems, Stability, Stabilizability, Saturating inputs, Impulsive jump.

Résumé

Dans cet article, nous étudions le problème de stabilité et de stabilisation stochastique, de la classe des systèmes singuliers avec saturation sur la commande, perturbés par un processus à sauts markoviens, qui introduit en plus de la discontinuité inhérente à ce type de système, une discontinuité dans la partie continue de l'état de ce dernier. Basée sur les techniques d'inégalités matricielles linéaires et de la complémentarité sur le cône, l'algorithme proposé pour la conception d'un contrôleur par retour d'état garantit non seulement que le système en boucle fermée soit régulier, non impulsif par morceaux et stochastiquement stable en moyenne quadratique, mais aussi que la contrainte sur la commande est toujours satisfaite. Un exemple numérique est donné pour montrer la validité des résultats développés.

Mots clés : Système singulier, système à sauts, stabilité stochastique, stabilisation stochastique, commande avec saturation.

1 Introduction

During the past decades, singular systems has received considerable interest due to the fact that this class of systems is appropriate to model practical systems in different areas such as electrical power systems, mechanical systems, robotics, chemical systems, see for instance, [9, 10, 18, 23] and the references therein.

Some systems cannot be represented by deterministic models, since their structures vary in response to random changes, which includes for instance failures and repairs of machines in manufacturing systems, modification of the operating point of a linearized model of a nonlinear systems. Systems with these abrupt changes may be modelled by the Markov jump systems which have becoming more and more popular in describing their dynamics behavior. For practical systems modelled by this class of systems, one refers the reader to [2] and the references therein.

The stability and control theory of conventional systems with impulsive effects have been recently developed. Some interesting results for this class of systems have been proposed see [1, 24, 25, 29, 30]. For singular systems, the case with impulses at initial time has been studied in [11], [14], and very few results on singular systems with impulsive effects have been reported such as the works of [12], [13], and [15].

For the class of Markovian singular systems, when a switch occurs from one mode to another, the values of the continuous state variables that are inherited from the preceding mode, are not concistent for the new mode, and in this case a state jump will occur. Indeed, it is well known [3, 8, 20] that the following markovian singular system:

$$\begin{cases} E(r(t))\dot{x}(t) &= A(r(t))x(t), \\ x(0) &= x_0, r(0) = r_0. \end{cases} \tag{1}$$

with $u(t) = 0$ and $r(t) = i$ can be decoupled as follows:

$$\dot{\xi}_1(t) = A_1(i)\xi_1(t) + A_2(i)\xi_2(t), \tag{2}$$

$$0 = A_3(i)\xi_1(t) + A_4(i)\xi_2(t). \tag{3}$$

with $\xi_1(t) \in \mathbb{R}^{n_1}$, $\xi_2(t) \in \mathbb{R}^{n_2}$, and:

$$M(i)E(i)N(i) = \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, M(i)A(i)N(i) = \begin{bmatrix} A_1(i) & A_2(i) \\ A_3(i) & A_4(i) \end{bmatrix}, N(i)x(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}. \tag{4}$$

Suppose that $A_4(i)$ is non singular, and at the instant t_k, $k = 1, 2, \ldots$, the system switches from the mode i to the mode j, then from (2) and (3), it can be seen that:

$$\xi_1(t_k^-) = \xi_1(t_k^+), \tag{5}$$

$$\xi_2(t_k^-) = -A_4^{-1}(i)A_3(i)\xi_1(t_k^-), \tag{6}$$

On the other hand, we have:

$$\xi_2(t_k^+) \quad = \quad -A_4^{-1}(j)A_3(j)\xi_1(t_k^+). \tag{7}$$

Thus, based on (5) and (6), (7) becomes:

$$\xi_2(t_k^+) = \left(-A_4^{-1}(j)A_3(j)\right)\left(-A_4^{-1}(i)A_3(i)\right)^{\dagger}\xi_2(t_k^-). \tag{8}$$

with $(.)^{\dagger}$ is the pseudo inverse of (.). From (8), we can conclude that $\xi_2(t_k^+) \neq \xi_2(t_k^-)$, which implies that this state variable will have finite jumps at each switched mode t_k. Most of the research on singular systems driven by continuous-time Markovian process (SMS), has been concentrated on this situation where the discontinuities are inherent to the system, and there are no perturbations acting on this latter at the transitions between modes, see for instance [3], [4], [5], [21]. However, for some SMS, when a switch occurs from one mode to another, to these inherent discontinuities, will be added another discontinuity on the continuous state at these switched moments. This phenomena often occur in many singular systems such as economic systems (leontief model where the production state vector jumps when some administration effects are included [13]). This motivated us to study the class of Markovian singular systems with discontinuities and saturating inputs. To the best of the author's knowledge, the stochastic stability and stabilizability of this class of system has never been tackled.

The aim of this paper is to address the stochastic stability and stabilization of Markovian jump singular system with discontinuities and saturating inputs. A state feedback controller design method is proposed to guarantee that the resulting closed-loop system is piecewise regular, impulse-free and stochastically stable in mean square sense, even if the saturation effectively occurs. All developed results are based on LMIs and SLPMM [17]. The rest of this paper is organized as follows. Section 2 states the problem to be studied. In Section 3, sufficient conditions are established to check the stability and stabilizability of the system under consideration. Finally, a numerical example is given in Section 4 to show the applicability of the proposed results.

Throughout this paper, the following notations will be used. The superscript "$^\top$" denotes matrix transposition and for symmetric matrices X and Y, the notation $X > Y$ (respectively $X < Y$) means that $(X-Y)$ is positive-definite (respectively negative-definite). $\mathbb{I}$ denotes the identity matrix with the appropriate dimension. $\mathbb{E}[.]$ stands for the mathematical expectation operator with respect to the given probabilities Υ. $|.|$ refers to the Euclidian norm for vectors. For a square matrix $A = \{a_{l,s}\}$, $\|A\| = \max \sum_{s=1}^{n} |a_{l,s}|, \forall 1 \leq i \leq n$, denotes the infinity norm for matrix A. The trace of square matrix is *trace* (.). diag [.] denotes a block diagonal matrix. $\lambda(A)$ and $Re\ \lambda(A)$ stand for the eigenvalues and the real part of the eigenvalues of (A) respectively.

2 Problem statement

Let $\{r_t, t \geq 0\}$ be a right-continuous-time Markov process, defined on the probability space $(\Omega, \mathcal{F}, \Upsilon)$ and taking values in a finite state space $\mathcal{S} = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\pi_{ij})_{N\times N}$ given by:

$$\Gamma\left[r(t+\Delta) = j | r(t) = i\right] = \begin{cases} \pi_{ij}\Delta + o(\Delta) & \text{if } \ i \neq j \\ 1 + \pi_{ii}\Delta + o(\Delta) & \text{if } \ i = j \end{cases}$$

where $\Delta > 0, \lim_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$. Here $\pi_{ij} \geq 0$, $\forall i, j, i \neq j$, is the transition rate from the mode i to the mode j, while

$$\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij} \tag{9}$$

Let $\{\tau_k, k = 1, 2, \ldots\}$ be a given number sequence satisfying $\tau_1 < \tau_2 < \ldots < \tau_k < \tau_{k+1} < \ldots$, and consider the singular Markovian system jumping parameters with the following dynamics:

$$\begin{cases} E(r(t))\dot{x}(t) & = A(r(t))x(t) + B(r(t))sat(u(t)), t \neq \tau_k, \\ x(\tau_k^+) & = R(r(\tau_k), r(\tau_k^+))x(\tau_k), R(ii) = \mathbb{I}, t = \tau_k, \\ x(0) & = x_0, r(0) = r_0. \end{cases} \tag{10}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ represent the state and control vectors; $A(i)$, $B(i)$, are known real constant matrices of appropriate dimensions, and the matrix $E(i)$, when $r(t) = i$, may be singular, with rank $E(i) = n_E \leq n$, for $i \in \mathcal{S}$; $R(.,.)$ is a known real constant matrix that reflects the discontinuity of the state trajectory of system (10) [1], we assume that there exist a set of scalars $0 < h_k \leq 1$ such that:

$$\max_{1\leq i,j\leq N} \quad \|R(i,j)\| \leq h_k \tag{11}$$

$\tau_k > 0$ is the kth switched moment, i.e: the moment of the transition of the mode from $r(\tau_k) = i$ to $r(\tau_k^+) = j \neq i$, with $\tau_k^+ = \lim_{\Delta\to 0}(\tau_k + \Delta), \forall k > 0$.

In this work, we consider the saturated input when a state feedback controller is used, i.e:

$$sat(u(t)) = sat(K(r(t)x(t)) \tag{12}$$

where $K(i) \in \mathbb{R}^{n\times m}$ is a design matrix for each fixed mode $i \in \mathcal{S}$, and where each component is defined for $s = 1, \ldots, m$ by:

$$sat(K_s(i)x(t)) = \begin{cases} \overline{u}_s & if \quad K_s(i)x(t) > \overline{u}_s, \\ K_s(i)x(t) & if \quad -\overline{u}_s \leq K_s(i)x(t) \leq \overline{u}_s, \\ -\overline{u}_s & if \quad K_s(i)x(t) < -\overline{u}_s, \end{cases} \tag{13}$$

where $\overline{u}_s \in \mathbb{R}^+(s = 1, \ldots, m)$ are actuator limitations and $K_s(i)$ denotes the sth row of the matrix $K(i)$.

Our goal is to design a controller that renders the closed-loop system under study piecewise regular, impulse-free and stochastically stable in mean square sense, so that the saturation condition above is not violated. In the next section, we present the main result of the paper namely stabilization using a state feedback controller, by assuming that the Markov jump parameter process $r(t)$ and the system state process $x(t)$ are available for feedback for all $t \geq 0$.

To study the closed-loop stochastic stability in mean square sense under the saturating input, let us recall the following definitions and lemmas:

Definition 2.1 *[8] For any mode $i \in \mathcal{S}$, unforced system (10) (with $u(t) \equiv 0$) is said to be:*

- *regular if $det(sE(i) - A(i))$, is not identically zero.*
- *impulse-free if $deg(det(sE(i) - A(i)))$= rank $E(i)$.*

Definition 2.2 *For each $i \in \mathcal{S}$,*

- *Unforced system (10) is said to be stochastically stable in mean square sense (SSMSS), if there exists a finite positive constant $T(x_0, r_0)$ such that the following holds for any initial conditions x_0 and r_0:*

$$\mathbb{E}\left[\int_{t_0}^{\infty} |x(t)|^2 dt | x_0, r_0\right] \leq T(r_0, x_0); \tag{14}$$

- *A set $\mathscr{H} \subset \mathbb{R}^n$ is called domain of attraction in mean square sense if for any initial conditions r_0 and $x_0 \in \mathscr{H}$, the solution of (10) satisfies (14).*

The above definition can be regarded as an extension of the definition in [2, 6].

Definition 2.3 *System (10) is said to be stabilizable, if there exists a linear state feedback*

$$u(t) = K(r(t))x(t) \tag{15}$$

such that the closed-loop system is regular, piecewise impulse free and stochastically stable in mean square sense, for every x_0 and r_0.

Lemma 2.1 *[16] Let $C^{2,1}(\mathbb{R}^n \times \mathcal{S};\mathbb{R}^+)$, denote the family of all nonnegative functions $V(x, r(t) = i)$ on $\mathbb{R}^n \times \mathcal{S}$. For each $V(x, r(t) = i) \in C^{2,1}(\mathbb{R}^n \times \mathcal{S};\mathbb{R}^+)$, the infinitesimal generator $\mathbb{L}V$ of the Markov process $\{x(t), r(t), t \geq 0\}$, from $\mathbb{R}^n \times \mathcal{S}$ to $\mathbb{R}$ is given by:*

$$\mathbb{L}V(x(t), r(t) = i) \quad = \quad \lim_{\Delta \to 0} \frac{1}{\Delta}\Big\{\mathbb{E}\Big[V(x(t+\Delta), r(t+\Delta))|x(t), r(t) = i\Big] - V(x, r(t) = i)\Big\}$$

Lemma 2.2 *[22] If $V(x(t), r(t) = i) \in C^{2,1}(\mathbb{R}^n \times \mathcal{S};\mathbb{R}^+)$ for each $i \in \mathcal{S}$, then for any stopping times $0 \leq t_1 \leq t_2 < +\infty$*

$$\mathbb{E}\left[V(x(t_2), r(t_2))\right] = \mathbb{E}\left[V(x(t_1), r(t_1))\right] + \mathbb{E}\left[\int_{t_1}^{t_2} \mathbb{L}V(x(s), r(s))ds\right]$$

as long as the integration involved exists and finite.

Lemma 2.3 *[26] For any matrix $Q \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$ and a scalar ε, the matrix measure $\rho(Q)$ defined as $\rho(Q) = \lim_{\varepsilon\to 0} \frac{\|\mathbb{I}+\varepsilon Q\|-1}{\varepsilon}$, has the following properties:*

- $-\|Q\| \leq Re\ \lambda(Q) \leq \rho(Q) \leq \|Q\|$.
- $\rho(Q) = \frac{1}{2}\lambda_{max}(Q + Q^\top)$,

where $\lambda_{max} = max\ Re\ \lambda(Q + Q^\top)$.

Lemma 2.4 *[2] Let Ψ, F and Ξ be real matrices of appropriate dimensions with $F^T F \leq \mathbb{I}$. For any scalar $\varepsilon > 0$:*

$$\Psi F \Xi + \Xi^T F^T \Psi^T \leq \varepsilon \Psi \Psi^T + \varepsilon^{-1} \Xi^T \Xi \tag{16}$$

3 Main result

Before presenting the main results, we introduce the following lemmas which will play a key role in the derivation of the solution of our control problem.

Lemma 3.1 *Select $V(x(t), i) = x^\top(t)E^\top(i)P(i)x(t), i \in \mathcal{S}$, where $P(i)$ is a non singular matrix, as the Lyapunov function for the system (10), then, for each $i \in \mathcal{S}$, and a positive scalar $0 < h_k < 1$, we have the following:*

- *a)* $\mathbb{E}\left[V(x(t), i)\right] - \mathbb{E}\left[V(x(\tau_k^+), i_k)\right] = \mathbb{E}\left[\int_{\tau_k}^t \mathbb{L}V(x(s), i(s))ds|(x(\tau_k), i_k)\right]$.
- *b)* $\mathbb{E}\big[V(x(\tau_k^+), i(k))\big] \leq h_k^2 \mathbb{E}\big[V(x(\tau_k), i(k))\big]$.
- *c)* $\mathbb{E}\left[\int_{t_0}^T x^\top(s)x(s)ds|(x_0, i_0)\right] = \mathbb{E}\big[V(x(T), i)\big] - \mathbb{E}\big[V(x_0, i_0)\big] + \sum_{p=1}^l (1 - h_p^2)$ $\mathbb{E}\big[V(x(\tau_p), i(p))\big]$, *where l is the number of jumps on the interval $[t_0, T]$.*

Proof: a) The first relation in lemma 3.1, can be obtained by a direct application of the Dynkin formula, given in lemma 2.2, on the interval $(\tau_k, t]$.
b) At the switched moment τ_k, the expectation of the Lyapunov function for the mode $i(k)$ is given by:

$$\begin{aligned}
\mathbb{E}\big[V(x(\tau_k^+), i(k))\big] &= \mathbb{E}\big[x^\top(\tau_k^+)E^\top(i_k)P(i_k)x(\tau_k^+)\big] \\
&= \mathbb{E}\big[x(\tau_k)R^\top(i_k i_{k^+})E^\top(i_k)P(i_k)R(i_k, i_k^+)x(\tau_k)\big] \\
&\leq \mathbb{E}\big[\|E^\top(i_k)P(i_k)R(i_k, i_k^+)x(\tau_k)\|^2\big] \\
&\leq \mathbb{E}\big[\|E^\top(i_k)P(i_k)\|^2\|R(i_k, i_k^+)\|^2\|x(\tau_k)\|^2\big] \\
&\leq \mathbb{E}\big[\|E^\top(i_k)P(i_k)\|^2 max\|R(i_k, i_{k^+})\|^2\|x(\tau_k)\|^2\big],
\end{aligned}$$

Then by using (11), we obtain:

$$\begin{aligned}
\mathbb{E}\big[V(x(\tau_k^+), i(k))\big] &\leq h_k^2\mathbb{E}\big[\|E^\top(i_k)P(i_k)x(\tau_k)\|^2\big] \\
&= h_k^2\mathbb{E}\big[x^\top(\tau_k)E^\top(i_k)P(i_k)x(\tau_k)\big].
\end{aligned}$$

Notice that the last term under the expectation is the Lyapunov function expression of the mode i_k at the moment τ_k, thus we obtain b).

c) For t in the interval $[t_0, T]$, we have:

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_0}^{T} \mathbb{L}V(x(s), i_s)ds|(x_0, i_0)\right] = \mathbb{E}\left[\int_{t_0}^{\tau_1} \mathbb{L}V(x(s), i_s)ds|(x_0, i_0)\right] \\
&+\left[\int_{\tau_1}^{\tau_2} \mathbb{L}V(x(s), i_s)ds|(x_1, i_1)\right] + \ldots + \left[\int_{\tau_k - 1}^{\tau_k} \mathbb{L}V(x(s), i_s)ds|(x_{\tau_{k-1}}, i_{k-1})\right] \\
&+\mathbb{E}\left[\int_{\tau_k}^{T} \mathbb{L}V(x(s), i_s)ds|(x_{\tau_k}, i_k)\right],
\end{aligned}$$

then by applying the first relation $a)$ in lemma 3.1, we get:

$$\begin{aligned}
&\mathbb{E}\left[\int_{0}^{T} \mathbb{L}V(x(s), i_s)ds|(x_0, i_0)\right] = \mathbb{E}\big[V(x(\tau_1), i_1)\big] - \mathbb{E}\big[V(x_0, i_0)\big] \\
&+\mathbb{E}\big[V(x(\tau_2), i_2)\big] - \mathbb{E}\big[V(x(\tau_1^+), i_1)\big] + \ldots + \\
&\mathbb{E}\big[V(x(\tau_k), i_k)\big] - \mathbb{E}\big[V(x(\tau_{k-1}^+), i_{k-1})\big] + \mathbb{E}\big[V(x(T), i)\big] - \mathbb{E}\big[V(x(\tau_k^+), i_k)\big]
\end{aligned}$$

And using the second relation $b)$ of lemma 3.1, yields:

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_0}^{T} \mathbb{L}V(x(s), i_s)ds|(x_0, i_0)\right] = \mathbb{E}\big[V(x(\tau_1), i_1)\big] - \mathbb{E}\big[V(x_0, i_0)\big] \\
&+\mathbb{E}\big[V(x(\tau_2), i_2)\big] - h_1^2\mathbb{E}\big[V(x(\tau_1), i_1)\big] + \ldots +
\end{aligned}$$

$$\mathbb{E}\big[V(x(\tau_k),i_k)\big] - h_{k-1}^2\mathbb{E}\big[V(x(\tau_{k-1}),i_{k-1})\big] + \mathbb{E}\big[V(x(T),i)\big] - h_{k-1}^2\mathbb{E}\big[V(x(\tau_k),i_k)\big]$$
$$= \sum_{p=1}^{l}(1-h_p^2)\mathbb{E}\big[V(x(\tau_p),i(p))\big] + \mathbb{E}\big[V(x(T),i)\big] - \mathbb{E}\big[V(x_0,i_0)\big].$$

Lemma 3.2 *If there exists a set of nonsingular matrices $P(i), i \in \mathcal{S}$, such that the following LMI holds:*

$$P^\top(i)A(i) + A^\top(i)P(i) + \sum_{j=1}^{N}\pi_{ij}R^\top(ij)E^\top(j)P(j)R(ij) < 0, \tag{17}$$

with the constraint:

$$E^\top(i)P(i) = P^\top(i)E(i) \geq 0, \tag{18}$$

then, the system (10) is piecewise regular, impulse-free and stochastically stable in mean square sense.

Proof: Suppose that there exists a set of nonsingular matrices $P(i), i \in \mathcal{S}$, such that (17) and (18) hold. Under this condition, we first show the regularity and the absence of impulses of the singular Markovian jump system with discontinuities in (10) between consecutive jumps, which ensure the existence and the uniqueness of its solution. To this end, the same proof given in Theorem 1 in [28] holds here. Indeed, let $M(i)$ and $N(i)$ be two nonsingular matrices such that [8]:

$$\begin{aligned}\bar{E}(i) &= M(i)E(i)N(i), \bar{A}(i) = M(i)A(i)N(i), &(19)\\ \bar{P}(i) &= M^{-T}(i)P(i)N(i) = \begin{bmatrix} P_1(i) & P_3(i) \\ P_2(i) & P_4(i)\end{bmatrix}, &(20)\\ \bar{R}(i) &= N^{-1}(i)R(ij)N(i) = \begin{bmatrix} R_1(ij) & R_3(ij) \\ R_2(ij) & R_4(ij)\end{bmatrix}. &(21)\end{aligned}$$

$\bar{E}(i)$ and $\bar{A}(i)$ are defined as in (4). Pre-and post-multiplying (17) and (18) by $N^\top(i)$ and $N(i)$, respectively, then by using the expressions of $\bar{P}(i)$ and $\bar{R}(i)$ given in (20-21), we have:

$$\bar{E}^\top(i)\bar{P}(i) = \bar{P}^\top(i)\bar{E}(i) \geq 0, \tag{22}$$
$$\bar{P}^\top(i)\bar{A}(i) + \bar{A}^\top(i)\bar{P}(i) + \sum_{j=1}^{N}\pi_{ij}\bar{R}^\top(ij)\bar{E}^\top(j)\bar{P}(j)\bar{R}(ij) < 0. \tag{23}$$

¿From (22), it is easy to show that $P_3(i) = 0$, therefore, for each mode $i \in \mathcal{S}$ the inequality (23) is equivalent to:

$$\begin{bmatrix} \mathscr{A}_1(i) & \mathscr{A}_2(i) \\ \mathscr{A}_3(i) & \mathscr{A}_4(i)\end{bmatrix} < 0, \tag{24}$$

with:

$$\begin{array}{rcl}\mathscr{A}_1(i) & = & A_1^\top(i)P_1(i) + P_1^\top(i)A_1(i) + A_3^\top(i)P_2(i) + P_2^\top(i)A_3(i) + R_1^\top(i)P_1(i)R_1(i),\\ \mathscr{A}_2(i) & = & A_3^\top(i)P_4(i) + P_1^\top(i)A_2(i) + P_2^\top(i)A_4(i) + R_1^\top(i)P_1(i)R_3(i),\\ \mathscr{A}_3(i) & = & A_2^\top(i)P_1(i) + A_4^\top(i)P_2(i) + P_3^\top(i)A_3(i) + R_3^\top(i)P_1(i)R_1(i),\\ \mathscr{A}_4(i) & = & P_4^\top(i)A_4(i) + A_4^\top(i)P_4(i) + R_3^\top(i)P_1(i)R_3(i).\end{array}$$

Then by (24), we have

$$P_4^\top(i)A_4(i) + A_4^\top(i)P_4(i) + R_3^\top(i)P_1(i)R_3(i) < 0. \tag{25}$$

Since $R_3^\top(i)P_1(i)R_3(i) \geq 0$, we get:

$$P_4^\top(i)A_4(i) + A_4^\top(i)P_4(i) < 0, \tag{26}$$

This, by Lemma 2.3 gives:

$$Re\ \lambda\Big(P_4^\top(i)A_4(i)\Big) \leq \rho\Big(P_4^\top(i)A_4(i)\Big) = \frac{1}{2}\lambda_{max}\Big(P_4^\top(i)A_4(i) + A_4^\top(i)P_4(i)\Big) < 0,$$

then $P_4^\top(i)A_4(i)$ is invertible, which implies that $A_4(i)$ is nonsingular. Therefore, from Definition 2.1, it is easy to see that the singular Markovian system with discontinuities is piecewise regular and impulse-free.

Next, we will show the stochastic stability. To this end, let us consider the unforced system (10), then the infinitesimal operator $\mathbb{L}$ of the Markov process $\{x(t), r(t), t \geq 0\}$ can be evaluated as:

$$\begin{array}{rcl}\mathbb{L}V(x(t), r(t) = i) & = & \dot{x}^\top(t)E^\top(i)P(i)x(t) + x^\top(t)E^\top(i)P(i)\dot{x}(t)\\ & + & \displaystyle\sum_{j=1, j\neq i}^{N} \pi_{ij}\big[V(R(ij)x(t), j) - V(x(t), i)\big]\\ & = & x^\top(t)\Theta(i)x(t)\end{array}$$

with: $\Theta(i) = A^\top(i)P(i) + P^\top(i)A(i) + \sum_{j=1}^{N}\pi_{ij}R^\top(ij)E^\top(j)P(j)R(ij)$.
Note that from the lemma 3.2, $\Theta(i) < 0$ for each mode i, therefore, we have:

$$\mathbb{L}V(x(t), i) \quad \leq \quad -min_{i\in\mathcal{S}}\lambda_{min}\left(-\Theta(i)\right)x^\top(t)x(t), \tag{27}$$

which, combined with lemma 3.1 (relation c), gives:

$$\begin{array}{rcl}\displaystyle\min_{i\in\mathcal{S}}\{\lambda_{min}(-\Theta(i))\}\mathbb{E}\left[\int_{t_0}^{T} x^\top(s)x(s)ds|(x_0, i_0)\right] & \leq & \mathbb{E}\big[V(x_0, i_0)\big] - \mathbb{E}\big[V(x(T), i)\big]\\ & - & \displaystyle\sum_{p=1}^{l}(1 - h_p^2)\mathbb{E}\big[V(x(\tau_p), i(p))\big].\end{array}$$

Let T goes to infinity, and since $0 < h_k < 1$, then the term $\mathbb{E}\big[V(x(T), i)\big] + \sum_{p=1}^{\infty}(1 - h_p^2)\mathbb{E}\big[V(x(\tau_p), i(p))\big]$ is positif, this yields to the following:

$$\mathbb{E}\left[\int_{t_0}^{\infty} x^\top(s)x(s)ds|(x_0, i_0)\right] \leq T(x_0, i_0), \tag{28}$$

with $T(x_0, i_0) = \frac{\mathbb{E}[V(x_0,i_0)]}{\min_{i\in\mathcal{S}}\{\lambda_{min}(-\Theta(i))\}}$. This complete the proof of the Lemma 3.2.

Based on the mean square stochastic stability condition cited above, for Markovian singular systems with discontinuities, first of all, we develop sufficient conditions via LMIs and BMIs, that allows us to synthesize the unconstrained state feedback controller that assures that the closed-loop system is piecewise regular, impulse-free and stochastically stable. The following Theorem summarizes this result.

Theorem 3.1 *If there exist a set of nonsingular matrices $P = (P(1), \ldots, P(N))$ and $X = (X(1), \ldots, X(N))$, a set of symmetric and positive-definite matrices $V_P = (V_P(1), \ldots, V_P(N))$ and $Z = (Z(1), \ldots, Z(N))$, a matrix $Y = (Y(1), \ldots, Y(N))$ and a positive scalar $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(N))$, such that the following inequalities hold for each $i \in \mathcal{S}$:*

$$\begin{bmatrix} \Pi(i) & X^\top(i) & S^\top(i) \\ X(i) & -Z(i) & \mathbf{0} \\ S(i) & \mathbf{0} & -\mathscr{X}(i) \end{bmatrix} < 0 \tag{29}$$

$$\begin{bmatrix} V_P(i) & W(i) \\ W^\top(i) & \mathscr{J}(i) \end{bmatrix} \geq 0 \tag{30}$$

with equality constraints:

$$E^\top(i)P(i) = P^\top(i)E(i) \geq 0, \tag{31}$$

$$P(i)X(i) = \mathbb{I}, \tag{32}$$

$$V_P(i)Z(i) = \mathbb{I}, \tag{33}$$

where:

$$\Pi(i) = X^\top(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i) + \pi_{ii}X^\top(i)E^\top(i), \tag{34}$$

$$W(i) = \big[R^\top(i1)E^\top(1)P(1)R(i1), \ldots, R^\top(ii-1)E^\top(i-1)P(i-1)R(ii-1), \tag{35}$$

$$R^\top(ii+1)E^\top(i+1)P(i+1)R(ii+1), \ldots, R^\top(iN)E^\top(N)P(N)R(iN)\big], \tag{36}$$

$$\mathscr{J}(i) = diag\big[\varepsilon^{-1}(1)\mathbb{I}, \ldots, \varepsilon^{-1}(i-1)\mathbb{I}, \varepsilon^{-1}(i+1)\mathbb{I}, \ldots, \varepsilon^{-1}(N)\mathbb{I}\big], \tag{37}$$

$$S(i) = \frac{1}{2}\big[\pi_{i1}X^\top(i), \ldots, \pi_{ii-1}X^\top(i), \pi_{ii+1}X^\top(i), \ldots, \pi_{iN}X^\top(i)\big], \tag{38}$$

$$\mathscr{X}(i) = diag\big[\varepsilon(1)\mathbb{I}, \ldots, \varepsilon(i-1)\mathbb{I}, \varepsilon(i+1)\mathbb{I}, \ldots, \varepsilon(N)\mathbb{I}\big]. \tag{39}$$

then the closed-loop system is piecewise regular, impulse-free and SSMSS. In this case, the stabilizing controller gain is given by:

$$K(i) = Y(i)X^{-1}(i). \tag{40}$$

Proof: For this purpose, plugging controller (15) in the dynamics (10) gives:

$$E(r(t))\dot{x}(t) = A_c(r(t))x(t) \tag{41}$$

with $A_c(i) = A(i)+B(i)K(i)$, when $r(t) = i$. Then, by Lemma 3.2, this closed-loop system is piecewise regular, impulse free and SSMSS if the following LMI holds:

$$A_c^\top(i)P(i) + P^\top(i)A_c(i) + \sum_{j=1}^{N} \pi_{ij} R^\top(ij)E^\top(j)P(j)R(ij) < 0 \tag{42}$$

with the constraint (18).

Now, replace $A_c(i)$ by its expression, then the second equation in (42) becomes:

$$\begin{aligned} & A^\top(i)P(i) + P^\top(i)A(i) + P(i)B(i)K(i) + K^\top(i)B^\top(i)P^\top(i) \\ & + \sum_{j=1}^{N} \pi_{ij} R^\top(ij)E^\top(j)P(j)R(ij) < 0 \end{aligned}$$

which is equivalent to:

$$\begin{aligned} & A^\top(i)P(i) + P^\top(i)A(i) + P(i)B(i)K(i) + K^\top(i)B^\top(i)P^\top(i) + \pi_{ii}E^\top(i)P(i) \\ + & \sum_{j=1,j\neq i}^{N} \frac{1}{2}\pi_{ij} R^\top(ij)E^\top(j)P(j)R(ij) + \frac{1}{2}\pi_{ij} R^\top(ij)E^\top(j)P(j)R(ij) < 0 \end{aligned} \tag{43}$$

Then by using Lemma 2.4 to (43), we have:

$$\begin{aligned} & A^\top(i)P(i) + P^\top(i)A(i) + P(i)B(i)K(i) \\ & \quad + K^\top(i)B^\top(i)P^\top(i) + \pi_{ii}E^\top(i)P(i) + \sum_{j=1,j\neq i}^{N} \frac{1}{4}\pi_{ij}^2 \varepsilon^{-1}(j)\mathbb{I} \\ & \quad + \sum_{j=1,j\neq i}^{N} \varepsilon(j)\left[R^\top(ij)E^\top(j)P(j)R(ij)\right]^\top \left[R^\top(ij)E^\top(j)P(j)R(ij)\right] < 0 \end{aligned} \tag{44}$$

with $\varepsilon(j), j \in \mathcal{S}$, is any positive scalar.

For all $i \in \mathcal{S}$, suppose that there exists a symmetric and positive-definite matrix $V_P(i)$, such that:

$$\sum_{j=1,j\neq i}^{N} \varepsilon(j)\left[R^\top(ij)E^\top(j)P(j)R(ij)\right]^T \left[R^\top(ij)E^\top(j)P(j)R(ij)\right] \leq V_P(i) \tag{45}$$

Then, by applying the Schur complement to the last inequality, we obtain (30).

Now, taking into account the inequality (45), and pre-and post-multiplying both sides of (44) by $P^{-\top}(i)$ and $P^{-1}(i)$, yields:

$$\begin{aligned} & P^{-\top}(i)A^{\top}(i) + A(i)P^{-1}(i) + B(i)K(i)P^{-1}(i) + P^{-\top}(i)K^{\top}(i)B^{\top}(i) \\ +\pi_{ii}P^{-\top}(i)E^{\top}(i) + & \sum_{j=1, j\neq i}^{N} \frac{1}{4}\pi_{ij}P^{-\top}(i)\varepsilon_{ij}^{-1}P^{-1}(i)\pi_{ij} + P^{-\top}(i)V_P(i)P^{-1}(i) < 0 \end{aligned}$$

Then, applying the change of variables $X(i) = P^{-1}(i)$, $Y(i) = K(i)X(i)$ and $Z(i) = V_p^{-1}(i)$, where $Z(i)$ is a symmetric and positive-definite matrix, we obtain:

$$\begin{aligned} X^{\top}(i)A^{\top}(i) + A(i)X(i) + Y^{\top}(i)B^{\top}(i) + B(i)Y(i) + \pi_{ii}X^{\top}(i)E^{\top}(i) \\ + \sum_{j=1, j\neq i}^{N} \frac{1}{4}\pi_{ij}X^{\top}(i)\varepsilon^{-1}(j)X(i)\pi_{ij} + X^{\top}(i)V_P(i)X(i) < 0 \end{aligned}$$

If we define $S(i)$ and $\mathscr{X}(i)$ as in (38) and (39), then by applying Schur complement to the above inequality, we obtain (29). This completes the proof of the Theorem.

Remark 3.1 *It is evident that the conditions in Theorem 3.1 are no longer LMIs because of the term $\varepsilon^{-1}(i)$ in (37), furthermore, (29) and (30) are two coupled LMIs and the solution of one should be the inverse of the other to satisfy the coupling constraints (32) and (33). Thus, the problem is not convex. As a result, we can not solve the conditions in Theorem 3.1 by using convex optimization algorithms. For this purpose, firstly, let $\beta(i) = \varepsilon^{-1}(i)$, this will convert the problem into a combination of a linear and bilinear problem. Secondly, instead of solving the nonconvex problem directly, we use the SLPMM proposed by Leibfreitz in [17] and used in [27]. This algorithm consist in:*

- *linearizing the bilinear part of the objective functional by weakening the equality constraints cited above to semi-definite programming conditions as follows:*

$$\begin{bmatrix} \beta(i) & \mathbb{I} \\ \mathbb{I} & \varepsilon(i) \end{bmatrix} \geq 0, \begin{bmatrix} P(i) & \mathbb{I} \\ \mathbb{I} & X(i) \end{bmatrix} \geq 0, \begin{bmatrix} V_P(i) & \mathbb{I} \\ \mathbb{I} & Z(i) \end{bmatrix} \geq 0, \qquad (46)$$

- *minimizing successively the resulting LMI constrained semi-definite programming problems.*

Moreover, it should be noted that for computational purposes, we prefer to have closed sets (form more details, see [17] and [27]). Thus we introduce a positive scalar $\upsilon = (\upsilon(1), \ldots, \upsilon(N))$, $i \in \mathcal{S}$ to replace the nonconvex and open set $\Xi \triangleq \Big\{(X, Z, Y, \varepsilon)$, such that (29-33) are satisfied$\Big\}$, by a closed and convex one as it will be summarized in the following theorem:

Theorem 3.2 *If there exist a set of non-singular matrices $P = (P(1), \dots, P(N))$, and $X = (X(1), \dots, X(N))$, a set of symmetric and positive definite matrices $V_P = (V_P(1), \dots, V_P(N))$ and $Z = (Z(1), \dots, Z(N))$, a matrix $Y = (Y(1), \dots, Y(N))$, and a set of positive scalars $\varepsilon = (\varepsilon(1), \dots, \varepsilon(N))$, $\beta = (\beta(1), \dots, \beta(N))$ and $\upsilon = (\upsilon(1), \dots, \upsilon(N))$ satisfying the following problem, for all $i \in \mathcal{S}$:*

$$\min \operatorname{trace}\Big(P(i)X(i) + V_P(i)Z(i) + \beta(i)\varepsilon(i)\Big)$$

subject to LMIs (46) and:

$$\begin{bmatrix} \Pi(i) & X^\top(i) & S^\top(i) \\ X(i) & -Z(i) & \mathbf{0} \\ S(i) & \mathbf{0} & -\mathscr{X}(i) \end{bmatrix} \leq 0 \tag{47}$$

$$\begin{bmatrix} V_P(i) & W(i) \\ W^\top(i) & \mathscr{G}(i) \end{bmatrix} \geq 0, \tag{48}$$

with the equality constraint (18) and the matrices $W(i), S(i), \mathscr{X}(i)$ are given by (35), (38), (39), while $\Pi(i)$ and $\mathscr{G}(i)$ are as follows:

$$\Pi(i) = X^\top(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i) + \pi_{ii}X^\top(i)E^\top(i) + \upsilon(i)\mathbb{I}, \tag{49}$$

$$\mathscr{G}(i) = \operatorname{diag}\big[\beta(1)\mathbb{I}, \dots, \beta(i-1)\mathbb{I}, \beta(i+1)\mathbb{I}, \dots, \beta(N)\mathbb{I}\big], \tag{50}$$

then the resulting closed-loop system is regular, piecewise impulse free and stochastically stable, with the controller gain given in (40).

Remark 3.2 *Theorem 3.2 gives a sufficient condition on the existence of an unconstrained state feedback controller for system (10), which can not guarantee that the saturation condition (13) will be not violated. Hence additional conditions for which the regularity, the absence of impulses and the stochastic stability of the closed-loop system are assured when control saturations effectively occur, have to be determined.* △

For this purpose, first we introduce some notations. For non singular matrix $X(i) \in \mathbb{R}^{n\times n}, i \in \mathcal{S}$, we define an ellipsoid as:

$$\mathscr{F}(X(i), E(i)) = \{x \in \mathbb{R}^n : x^\top(t)E^\top(i)X^{-1}(i)x(t) \leq 1\}$$

Also let $\mathscr{L}(K(i), u_s)$ denote the subspace of the state space $\mathbb{R}^n$ in which the state feedback controller (15) satisfies the constraints, i.e:

$$\mathscr{L}(K(i), u) = \{x \in \mathbb{R}^n : \|K_s(i)x(t)\| \leq \ \overline{u}_s, s = 1, \dots, m\}$$

Now, to verify that the initial state x_o is in domain of attraction $\mathscr{F}(X(i), E(i))$ in mean square, we should add the following condition to the result stated in Theorem 3.2:

$$x_o^\top E^\top(i)X^{-1}(i)x_o \leq 1 \tag{51}$$

then by using Schur complement, this inequality is equivalent to:

$$\begin{bmatrix} 1 & x_o^\top \\ x_o & H(i) \end{bmatrix} \geq 0, \tag{52}$$

where $H(i)$ is a symmetric positive-definite matrix such that:

$$E^\top(i)X^{-1}(i) \leq H^{-1}(i). \tag{53}$$

Let $L(i) = H^{-1}(i)$ a symmetric positive-definite matrix, this implies that:

$$E^\top(i)P(i) \leq L(i), \tag{54}$$

furthermore, in order to assure that $\|K_s(i)x(t)\| \leq \ \overline{u}_s, \forall x(t) \in \bigcap_{i=1}^N \mathscr{F}(X(i), E(i))$, (i.e: $\bigcap_{i=1}^N \mathscr{F}(X(i), E(i)) \subset \mathscr{L}(K(i), u_s))$, the following inequality should be satisfied for all $x(t) \neq 0, s = 1, \ldots, m$ [7, 19]:

$$\frac{1}{\overline{u}^2} x^\top(t)K_s^\top(i)K_s(i)x(t) \leq x^\top(t)E^\top(i)X^{-1}(i)x(t),$$

this is equivalent to:

$$\begin{bmatrix} E^\top(i)X^{-1}(i) & K_s^\top(i) \\ K_s(i) & \overline{u}_s^2 \end{bmatrix} \geq 0. \tag{55}$$

Then by using Schur complement, pre-and post-multiplying (55) with $diag(X(i), \mathbb{I})$ and denoting $y_s(i) = K_s(i)X(i)$ the sth row of the matrix $Y(i)$, we have:

$$\begin{bmatrix} X^\top(i)E^\top(i) & y_s^\top(i) \\ y_s(i) & \overline{u}_s^2 \end{bmatrix} \geq 0, \tag{56}$$

This result is summarized in the following Theorem:

Theorem 3.3 *If there exist a set of non-singular matrices $P = (P(1), \ldots, P(N))$, and $X = (X(1), \ldots, X(N))$, a set of symmetric and positive-definite matrices $V_P = (V_P(1), \ldots, V_P(N))$, $Z = (Z(1), \ldots, Z(N))$, $H = (H(1), \ldots, H(N))$, and $L = (L(1), \ldots, L(N))$, a matrix $Y = (Y(1), \ldots, Y(N))$, and a set of positive scalars $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(N))$, $\beta = (\beta(1), \ldots, \beta(N))$ and $\upsilon = (\upsilon(1), \ldots, \upsilon(N))$, satisfying the following problem, for all $i \in \mathcal{S}$:*

$$\mathcal{P}_1 : \quad \min \operatorname{trace}\Big(P(i)X(i) + V_P(i)Z(i) + H(i)L(i) + \beta(i)\varepsilon(i)\Big)$$

subject to LMIs (46), (47), (48), (52), (54), (56), under the constraint (18), with $W(i)$, $S(i), \mathscr{X}(i), \Pi(i)$, and $\mathscr{G}(i)$, are given by (35), (38), (39), (49) and (50) respectively, then the initial states are in the domain of attraction $\mathscr{F}(X(i), E(i))$ in mean square sense and the controller gain is given in (40).

Remark 3.3 *For any $i \in \mathcal{S}$, when $R(ij) = \mathbb{I}$, system (10) reduces to the state space Markovian singular system with actuator saturation, then the Theorem 3.3 have the following corollary:*

Corollary 3.1 *If there exist a set of non singular matrices $P = (P(1), \ldots, P(N))$, and $X = (X(1), \ldots, X(N))$, a set of symmetric and positive-definite matrices $V_P = (V_P(1), \ldots, V_P(N))$, $Z = (Z(1), \ldots, Z(N))$, $H = (H(1), \ldots, H(N))$, and $L = (L(1), \ldots, L(N))$, a matrix $Y = (Y(1), \ldots, Y(N))$ and a set of positive scalars $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(N))$, $\beta = (\beta(1), \ldots, \beta(N))$ and $\upsilon = (\upsilon(1), \ldots, \upsilon(N))$ satisfying the following problem, for all $i \in \mathcal{S}$:*

$$\mathcal{P}_2 : \quad \min \operatorname{trace}\Big(P(i)X(i) + V_P(i)Z(i) + H(i)L(i) + \beta(i)\varepsilon(i)\Big)$$

subject to LMIs (46), (47), (48), (52), (54), (56), under the constraint (18), with $S(i)$, $\mathscr{X}(i), \Pi(i)$, and $\mathscr{G}(i)$, are given by (38), (39), (49) and (50) respectively, while $W(i)$ is as follows:

$$W(i) = \left[E^\top(1)P(1), \ldots, E^\top(i-1)P(i-1), E^\top(i+1)P(i+1), \ldots, E^\top(N)P(N)\right]$$

then the initial states are in the domain of attraction $\mathscr{F}(X(i), E(i))$ in mean square sense and the controller gain is given in (40).

Remark 3.4 *When $N = 1$, then $R(ii) = \mathbb{I}$, and the system (10) reduces to the singular system with actuator saturation, therefore the Theorem 3.3 have the following corollary:*

Corollary 3.2 *If there exist a set of non-singular matrices P and X, set of symmetric and positive-definite matrices H, and L, a matrix Y and a positive scalar υ satisfying the following problem:*

$$\mathcal{P}_3 : \quad \min \operatorname{trace}\Big(PX + HL\Big)$$

subject to LMIs:

$$\begin{array}{l} E^\top P = P^\top E \geq 0 \\ E^\top P \leq L \\ X^\top A^\top + AX + Y^\top B^\top + BY + \upsilon\mathbb{I} \leq 0 \\ \begin{bmatrix} 1 & x_o^\top \\ x_o & H \end{bmatrix} \geq 0, \\ \begin{bmatrix} X^\top E^\top & y_s^\top \\ y_s(i) & \overline{u}_s^2 \end{bmatrix} \geq 0, \end{array}$$

then the initial states are in the domain of attraction $\mathscr{F}(X, E)$ in mean square sense and the controller gain is given by $K = YX^{-1}$. △

In the following section, one will demonstrate the validity of the results by considering a numerical example.

4 Numerical example

let us suppose that the generator matrix Γ, the matrix $E(i), i = 1, 2$, and the system data are given by:

$$\Gamma = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \quad E(1) = E(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1.1 & 0 \\ 0 & 1 & -4 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix}, \quad \overline{u}_s = 2, r_o = 1,$$

$$R(1) = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0.2 \end{bmatrix}, \quad R(2) = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0 & 0.2 \end{bmatrix}, \quad x_0 = [1; 1; -1/2].$$

Then to show the effectiveness of results, let us consider the following cases:

- 1. singular system with discontinuities at the switched instants and constrained input,
- 2. singular system without discontinuities at the switched instants but subject to constrained input,
- 3. deterministic system without discontinuities and constrained input.

Solving the problem $\mathcal{P}_1$ for the case 1, one gets the following gain matrices:

$$K(1) = \begin{bmatrix} -2.2540 & 0.1918 & 0.0000 \end{bmatrix}, \quad K(2) = \begin{bmatrix} -2.0929 & 0.1015 & 0.0000 \end{bmatrix}.$$

The simulation results using these gains are illustrated by Figs 1-2. From these figures, we can see that the states'system go to zero when time goes to infinity. The input control saturates at the beginning and remain always between the imposed bounds. It is also worthed to notice that the unconstrained system behaves better than the constrained one (faster response for almost all states).

Solving the problem $\mathcal{P}_2$ for the case 2, we get the following gains:

$$K(1) = \begin{bmatrix} -9.1291 & 1.6346 & 3.8248 \end{bmatrix}, \quad K(2) = \begin{bmatrix} -5.6433 & 2.7324 & -0.0527 \end{bmatrix}.$$

The simulation results using these gains are illustrated by Figs 3-4. From these figures, we can see that the states'system go to zero when time goes to infinity. The input control saturates at the beginning and remain always between the imposed bounds. It is also worthed to notice that the unconstrained system behaves better than the constrained one (faster response for almost all states). Furthermore, we can see that even if $R(1,2) = R(2,1) = \mathbb{I}$, the third state variable of the system incorporates jumps at switched moments as it was explained in the introduction.

Finally, solving the problem $\mathcal{P}_3$ for case 3, we get the following gains:

$$K(1) \;=\; \left[\begin{array}{ccc} -1.5071 & 0.1191 & -0.0000 \end{array}\right].$$

The simulation results using these gains are illustrated by Fig 5. From these figures, we can see that the states'system go to zero when time goes to infinity. The input control saturates at the beginning and remain always between the imposed bounds $\overline{u} = 1$. It is also worthed to notice that the unconstrained system behaves better than the constrained one (faster response for almost all states).

Thus we can conclude that the proposed results can be used to stabilize a wide class of systems subject to saturating controls.

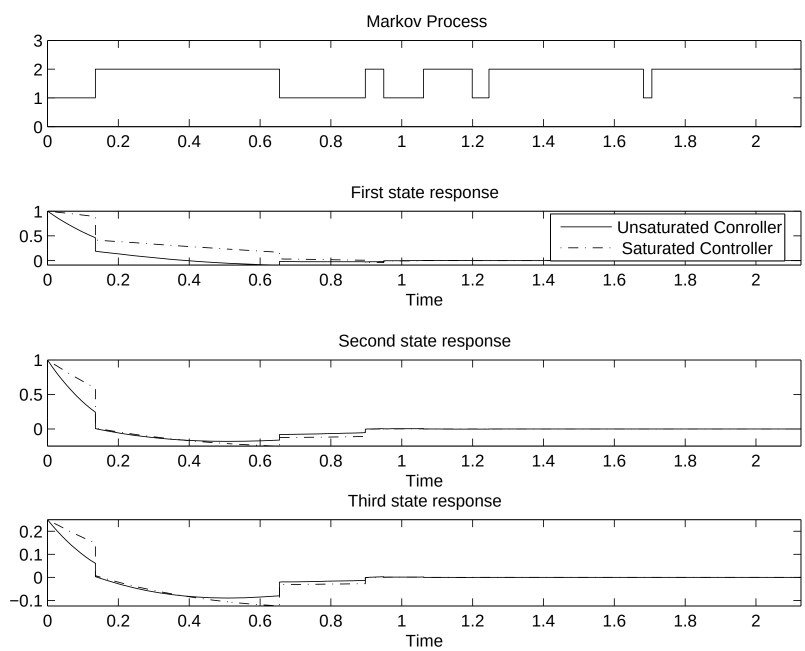

Figure 1: $r(t), x_1(t), x_2(t)$ and $x_3(t)$ versus time for saturated and unsaturated control

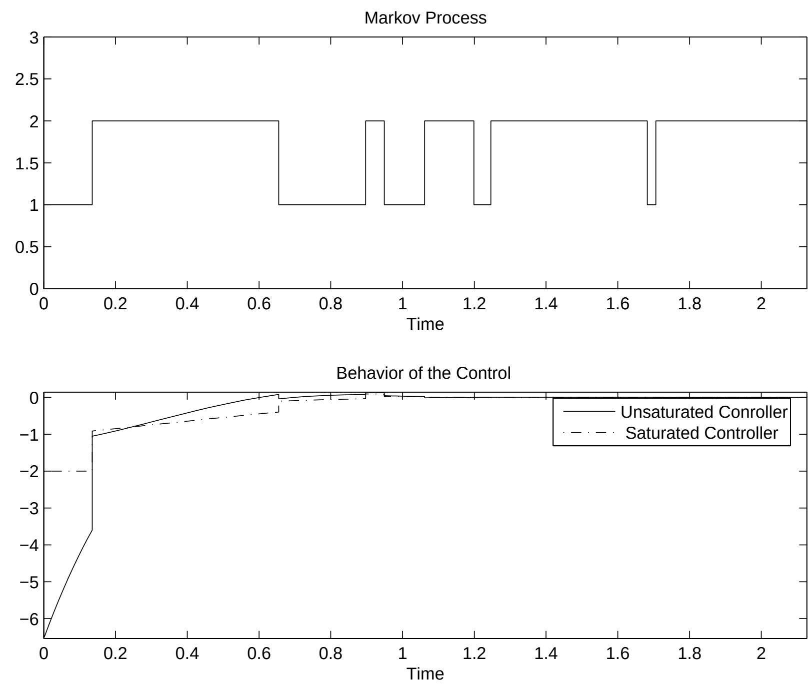

Figure 2: $r(t)$ and $u(t)$ versus time for saturated and unsaturated control

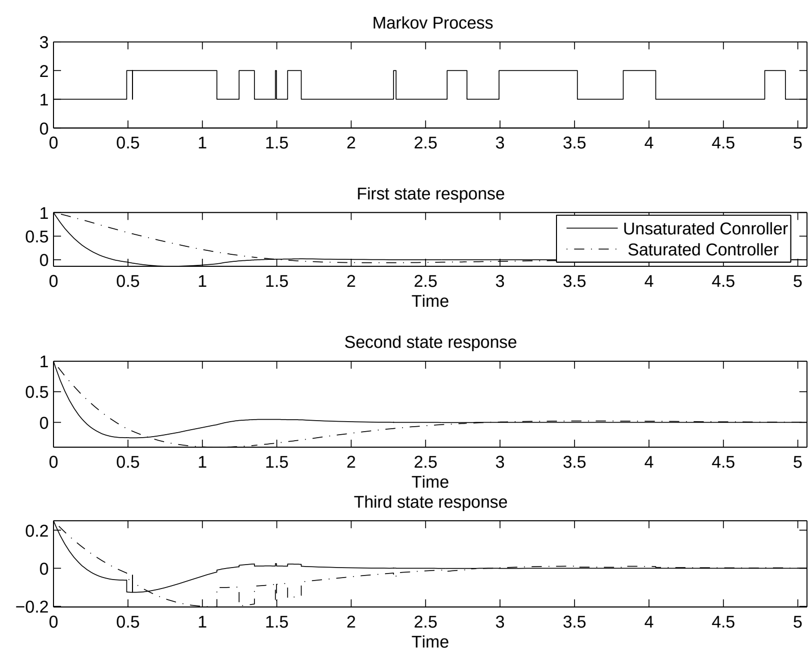

Figure 3: $r(t), x_1(t), x_2(t)$ and $x_3(t)$ versus time for saturated and unsaturated control

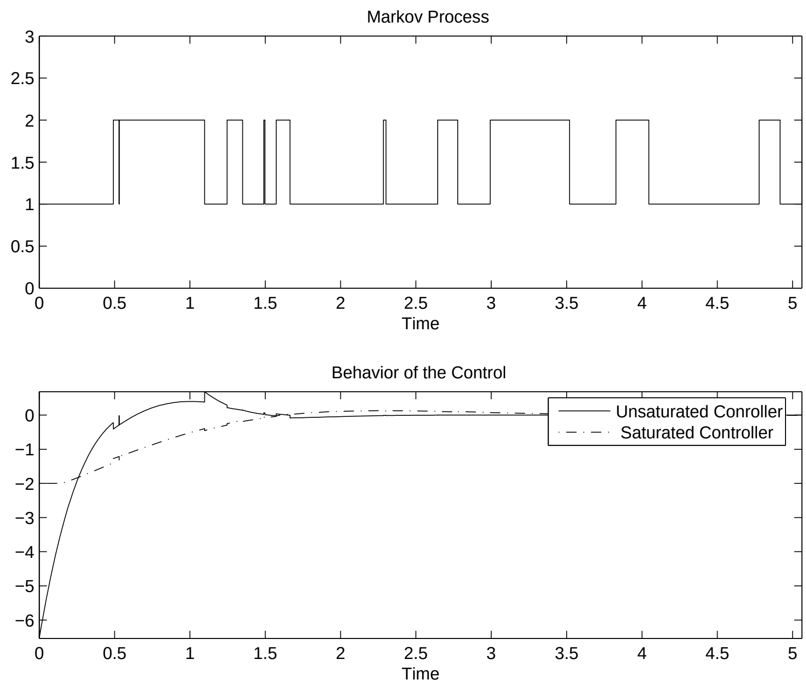

Figure 4: $r(t)$ and $u(t)$ versus time for saturated and unsaturated control

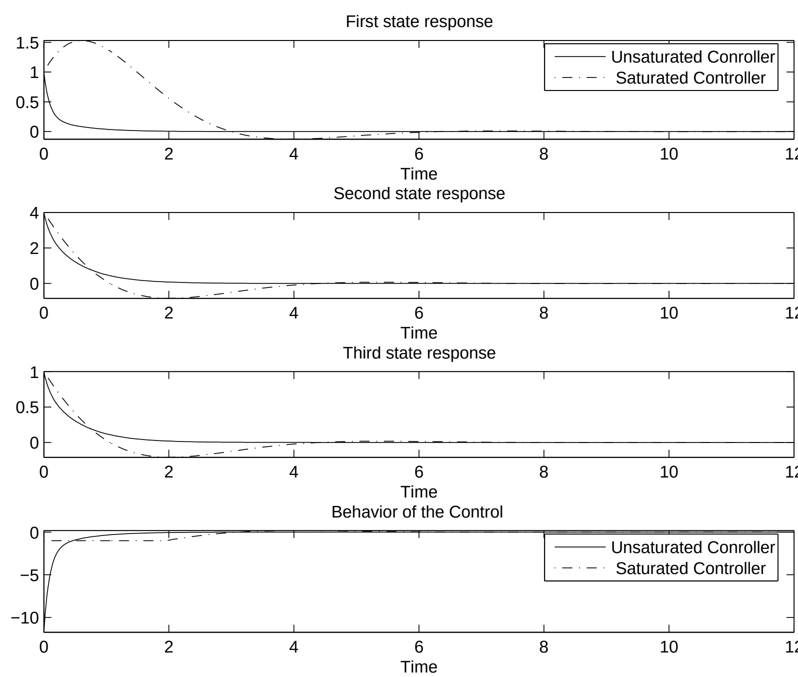

Figure 5: $x_1(t), x_2(t), x_3(t)$ and $u(t)$ versus time for saturated and unsaturated control

5 Conclusion

This paper dealt with the stochastic stabilization problem of Markovian singular systems with both discontinuities and actuator saturation. Based on the Lyapunov theory and the LMI technique, sufficient conditions for the stochastic stability and the stochastic stabilizability have been presented. Also, a new approach based on the sequential linear programming matrix methods, has been developed to design stabilizing state feedback controller which guarantees that the closed-loop system was piecewise regular, impulse free and stochastically stable in mean square sense even if the saturation effectively occurs.

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