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Anne Mercier

GERAD and Département de mathématiques et de génie industriel
École Polytechnique de Montréal
C.P. 6079, Succ. Centre-ville
Montréal (Québec) Canada H3C 3A7
anne.mercier@gerad.ca

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Abstract

The integrated aircraft routing and crew pairing problem consists in determining a minimum-cost set of aircraft routes and crew pairings such that each flight leg is covered by one aircraft and one crew, and some side constraints are satisfied. Linking constraints impose minimum connection times for crews that depend on aircraft connections. The main solution approach for this problem consists in solving a constrained crew pairing problem iteratively, adding feasibility cuts until a solution is found where the connection set used by the crew pairings is feasible for the aircraft routing problem. The feasibility cuts can be generated by a Benders decomposition approach in which aircraft routing is handled by the subproblem, or they can be selected from a predefined family. We perform a theoretical comparison of the different types of feasibility cuts. We also propose a simple procedure to strengthen these cuts. Computational experiments performed on test instances provided by two major airlines are presented to support the theoretical results.

Key Words: aircraft routing; crew pairing; integrated planning; Benders decomposition; extreme rays; feasibility cuts.

Résumé

Le problème intégré de la création d'itinéraires d'avions et d'horaires d'équipages consiste à déterminer un ensemble d'itinéraires d'avion et de rotations d'équipage tels que chaque vol est couvert par un appareil et un équipage, tout en satisfaisant un ensemble de contraintes supplémentaires. Alors que certaines contraintes supplémentaires ne concernent que les équipages ou les avions, des contraintes liantes imposent des temps minimums de connexion pour les équipages qui dépendent des connexions utilisées par les appareils. L'approche principale proposée pour résoudre ce problème intégré consiste à résoudre, itérativement, un problème de rotations d'équipages modifié, où une coupe de réalisabilité est ajoutée au problème d'équipages jusqu'à ce que l'ensemble des connexions courtes utilisé par les équipages dans la solution courante soit réalisable pour les avions. Les coupes de réalisabilité peuvent être générées par une approche de décomposition de Benders, avec comme sous-problème le problème de routage d'avions, ou à partir d'une famille prédéfinie de coupes. Nous présentons une comparaison théorique des coupes de Benders et des autres familles de coupes. Nous proposons aussi une méthode de renforcement simple qui peut être appliquée à tous les types de coupes de réalisabilité pour le problème intégré avions-équipages.

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Introduction

The planning of airline operations is usually performed sequentially (see, *e.g.*, Yu (1998)). The airline first solves a *flight scheduling problem* to determine each flight leg to be flown during a given period, with corresponding departure and arrival times. Then, the *fleet assignment* is performed to assign an aircraft type to each flight leg, taking into account the number of available aircraft of each type and the estimated profit. An *aircraft routing problem* is then solved, for each aircraft type, to determine a sequence of flight legs to be assigned to each individual aircraft so that each leg is flown exactly once and maintenance is performed at regular interval. The airline then creates minimum-cost crew pairings by solving a *crew scheduling problem* for each aircraft type, ensuring that every flight leg is covered and that all work rules are satisfied. A *pairing* is a sequence of duty periods separated by overnight rests, and a *duty period* is a sequence of flight legs separated by smaller rest periods, called *sits* or *connections*. Finally, by solving a *crew bidding problem* or a *crew rostering problem*, pairings are combined to form personalized monthly schedules which are assigned to employees. On the one hand, because the five steps are linked together, a sequential planning procedure is likely to yield suboptimal solutions. On the other hand, completely integrating all steps would yield an intractable model. This paper focuses on the integration of the aircraft routing and the crew scheduling problems.

Aircraft routing decisions have an impact on the set of feasible pairings since the minimum connection time required between two successive flight legs covered by the same crew depends on whether the same aircraft is used on both legs. Cordeau et al. (2001b), Klabjan et al. (2002) and Cohn and Barnhart (2003) have shown that integrating the two problems yields solutions that are significantly better than those obtained by solving the problems sequentially. A connection that is not long enough to be used by a crew if the crew changes aircraft is said to be *short*. Cordeau et al. (2001b) have introduced a model where one linking constraint per short connection is added to the aircraft routing and crew pairing formulations to ensure that a crew uses a short connection only if the two connecting flights are flown by the same aircraft. To handle these linking constraints, a solution approach based on Benders decomposition is used. The latter model was further enhanced by Mercier et al. (2005) who have shown that reversing the order of the solution sequence, i.e., solving the crew pairing problem in the Benders master problem as opposed to the aircraft routing problem, yields significant improvements over the approach of Cordeau et al. (2001b). Since the aircraft routing problem is usually modeled as a feasibility problem, all costs in the integrated model are associated with the crew pairings. Hence, when solving an aircraft routing subproblem, only feasibility information is transfered to the master problem (in the form of feasibility cuts). Huang et al. (2003) have also proposed a Benders decomposition approach for the integrated problem, but instead of generating feasibility cuts from the dual subproblem variables, they generate them from primal information. Cohn and Barnhart (2003) have proposed an integrated model which includes variables representing complete solutions to the aircraft routing problem instead of incorporating the aircraft routing formulation. The authors show that only a subset of the feasible solutions needs to be included in the model. They have also proposed an approach which first solves a crew pairing problem in which all short connections are allowed. If the set of short connections used in the solution to this crew pairing problem leads to a feasible aircraft routing problem, an optimal solution has been identified. Otherwise, a feasibility cut is introduced in the crew pairing problem to forbid a minimally infeasible subset of short connections, and the process is repeated. Finally, Klabjan et al. (2002) have presented a partially integrated approach that solves a modified crew scheduling problem including additional constraints that count the number of available aircraft on the ground at any time.

The contribution of this paper is fourfold. First, we present a theoretical comparison of different families of feasibility cuts that have been proposed in the literature for the integrated aircraft routing and crew scheduling problem. We discuss their strength and the procedures used for their generation. We show that the feasibility cuts proposed in the literature are all included in the set of Benders feasibility cuts, and may also be dominated by the latter. Second, we present a comparison of Benders feasibility cuts and Benders optimality cuts through the description of the corresponding dual subproblem polyhedra. The latter cuts are generated from an aircraft routing primal subproblem that is made feasible by the introduction of artificial variables. In that case, the choice of artificial variables has an impact on the dual subproblem polyhedron, and thus, on the feasibility cuts being generated. Indeed, we show that cuts generated when a unique artificial variable is used correspond to extreme rays while it is not the case for cuts generated when other combinations of artificial variables are used. Third, we show that in the case of cuts preventing the use of a set of short connections that are individually infeasible for the aircraft routing problem, linear combinations of extreme rays give stronger cuts than extreme rays. Finally, we introduce a simple lifting procedure that can be applied to any type of feasibility cut for the integrated aircraft routing and crew scheduling problem. Computational experiments performed on test instances provided by two major airlines are presented to support all theoretical results.

The remainder of the article is organized as follows. The next section introduces some notation and a mathematical formulation of the problem. Section 2 presents the Benders feasibility cuts and a description of the dual subproblem polyhedron. Section 3 compares different approaches to bound the Benders dual subproblem polyhedron and generate optimality cuts. Some families of feasibility cuts are then introduced and compared in Section 4. Section 5 introduces two types of stronger cuts: Pareto-optimal cuts and feasibility cuts obtained from a simple lifting procedure. This is followed by computational experiments in Section 6, and by the conclusion.

1 Mathematical Formulation

We consider the *Extended Crew Pairing* (ECP) formulation proposed by Cohn and Barnhart (2003) for the integrated aircraft routing and crew pairing problem. In this formulation, variables representing complete solutions to the aircraft routing problem are added to the classical crew pairing model. All maintenance constraints can thus be eliminated and a single aircraft routing solution is chosen through the use of a *convexity constraint*. Cohn and Barnhart (2003) show that only a subset of the feasible solutions needs to be included in the model, i.e., one column for each *unique and maximal maintenance-feasible short connection set* (UM). These columns can be generated individually and sequentially, in a preprocessing step, by solving a series of aircraft routing problems with additional constraints and a modified objective function. Although the aircraft routing problem is considered to be the easiest of the airline planning problems, the modified routing problem is a difficult combinatorial optimization problem (Mercier et al. (2005)). Nevertheless, we consider the ECP formulation in this paper to facilitate the comparison between the different feasibility cuts that have been proposed in the literature. Furthermore, once the UM columns have all been generated, ECP can be used to perform sensitivity analysis on the crew pairing problem. Finally, our conclusions also apply to alternative formulations of the integrated aircraft routing and crew pairing problem, such as that of Mercier et al. (2005). They should also apply to any problem for which Benders decomposition has proved to be a successful solution method, as long as the Benders dual subproblem contains extreme rays, and especially when the Benders primal subproblem is a feasibility problem (see, e.g., Cordeau et al. (2001a), Santoso et al. (2005), Cordeau et al. (2006), and Rasmussen (2006)).

Let L be the set of flight legs, K the set of feasible crew pairings, R the set of feasible aircraft routing solutions, and S the set of short connections included in at least one aircraft routing solution. Denote by c_k the cost of pairing $k \in K$. For every leg $i \in L$ and every pairing $k \in K$, define a binary constant a_{ki} that takes value 1 if leg $i \in L$ is covered by pairing k. For every short connection $s \in S$ and every pairing $k \in K$ (resp. routing $r \in R$), define a binary constant d_{ks} (resp. b_{rs}) that takes value 1 if connection $s \in S$ is included in pairing $k \in K$ (resp. routing $r \in R$). Finally, let x_k and y_r be binary variables that take value 1 if and only if pairing $k \in K$ and routing $r \in R$ is used in the solution, respectively. The ECP model can be stated as follows:

$$\text{Minimize} \quad \sum_{k \in K} c_k x_k \tag{1}$$

subject to

$$\sum_{k \in K} a_{ki} x_k = 1 \qquad (i \in L) \tag{2}$$

$$\sum_{r \in R} b_{rs} y_r - \sum_{k \in K} d_{ks} x_k \geq 0 \qquad (s \in S) \tag{3}$$

$$\sum_{r \in R} y_r = 1 \tag{4}$$

$$x_k \in \{0,1\} \qquad (k \in K) \tag{5}$$

$$y_r \in \{0,1\} \qquad (r \in R). \tag{6}$$

The objective function (1) minimizes crew pairing costs. No costs are associated with aircraft routings. Constraints (2) together with constraints (5) ensure that each leg is covered by exactly one crew. Constraints (3) prevent a crew to be assigned to any two flight legs forming a short connection, unless an aircraft is also assigned to both legs. Constraint (4) together with constraints (6) ensure that exactly one solution to the aircraft routing problem is chosen.

2 Benders Decomposition

The ECP model (1)–(6) includes both crew pairing and aircraft routing variables. Benders decomposition (see, *e.g.*, Benders (1962); Nemhauser and Wolsey (1988)) can be used to reformulate the problem so as to reduce the number of variables at the expense of an increase in the number of constraints. The additional constraints can, however, be generated dynamically only when they are violated. In most applications, only a very small subset of constraints needs to be generated for an optimal solution to be identified.

Let $\boldsymbol{X}$ be the set of solutions satisfying the crew constraints (2) and (5). For a given vector $\bar{\boldsymbol{x}} \in \boldsymbol{X}$, ECP reduces to the following primal subproblem (PSP) involving only aircraft variables:

$$\text{Minimize} \quad 0 \tag{7}$$

subject to

$$\sum_{r \in R} b_{rs} y_r \geq \sum_{k \in K} d_{ks} \bar{x}_k \qquad (s \in S) \tag{8}$$

$$\sum_{r \in R} y_r = 1 \tag{9}$$

$$y_r \geq 0 \qquad (r \in R). \tag{10}$$

Observe that upper bounds on the y_r variables are not needed because of constraints (9) and that the integrality constraints on the aircraft variables y_r have been replaced with non-negativity constraints. In fact, once the crew variables x_k are given binary values, the polyhedron corresponding to (8)–(10) has integer extreme points (Cohn and Barnhart (2003)). Let $\boldsymbol{\alpha} = (\alpha_s \geq 0 \,|\, s \in S)$ and β be the dual variables associated with constraints (8) and (9), respectively. When convexity constraint (9) is multiplied by -1, the dual of (7)–(10) is the following dual subproblem (DSP):

$$\text{Maximize} \quad \sum_{k \in K} \sum_{s \in S} d_{ks} \bar{x}_k \alpha_s - \beta \tag{11}$$

subject to

$$\sum_{s \in S} b_{rs} \alpha_s - \beta \leq 0 \qquad (r \in R) \tag{12}$$

$$\alpha_s \geq 0 \qquad (s \in S). \tag{13}$$

Since $b_{rs} \geq 0$, for all $r \in R$ and $s \in S$, one can observe that $\beta \geq 0$ by (12) and (13). The dual subproblem is always feasible since the null vector $\mathbf{0}$ satisfies constraints (12)–(13). Hence, the primal subproblem is either infeasible or feasible and bounded. For the dual subproblem to be bounded (and the primal subproblem feasible), the value of its objective function (11), which is a maximization, must be non-positive for every extreme ray of its feasible region. Let $\boldsymbol{\Delta}$ denote the polyhedron defined by constraints (12)–(13). One can see that $\boldsymbol{\Delta}$ is a pointed polyhedral cone, i.e., it has a unique extreme point, the null vector, and a multitude of extreme rays. Let $R_{\boldsymbol{\Delta}}$ be the set of extreme rays of $\boldsymbol{\Delta}$. One can notice that only the objective function of the dual subproblem (11)–(13) contains crew information. This implies that the feasible region of the dual subproblem is independent of the crew pairing solution and that all the extreme rays of $\boldsymbol{\Delta}$ could be enumerated *a priori*. Model (1)–(6) can thus be reformulated as the following Benders master problem (MP):

$$\text{Minimize} \quad \sum_{k \in K} c_k x_k \tag{14}$$

subject to

$$\sum_{k \in K} a_{ki} x_k = 1 \qquad (i \in L) \tag{15}$$

$$\sum_{k \in K} \sum_{s \in S} d_{ks} \alpha_s x_k \leq \beta \qquad ((\boldsymbol{\alpha}, \beta) \in R_{\boldsymbol{\Delta}}) \tag{16}$$

$$x_k \in \{0, 1\} \qquad (k \in K). \tag{17}$$

The Benders master problem is comprised of the crew pairing problem (constraints (14), (15) and (17)) and of the set of feasibility constraints (16) which ensure that the values given to the crew pairing variables x_k lead to a bounded dual subproblem (a feasible primal aircraft routing subproblem). In general, model (14)–(17) contains more constraints than model (1)–(6) but most feasibility constraints are inactive in an optimal solution. Hence, these constraints need not be enumerated exhaustively but can instead be generated dynamically by iterating between a relaxed master problem and the subproblem. The relaxed master problem contains constraints (15) and (17) as well as a subset of the Benders cuts (16). The optimal solution of the relaxed Benders master problem is used to set up constraints (8) in the primal subproblem at every iteration. If the primal subproblem is feasible, an optimal solution to the ECP problem has been obtained. Otherwise, an extreme ray of $R_{\boldsymbol{\Delta}}$ violating one of the constraints (16) is identified. Hence, exactly one constraint is added to the relaxed Benders master problem at each iteration.

2.1 Characterization of the extreme rays

When the extreme rays of the Benders dual subproblem polyhedron have a special structure, efficient algorithms can sometimes be devised to generate Benders feasibility cuts without explicitly solving the subproblem. In the case of ECP, a single characterization of the elements of $R_{\mathbf{\Delta}}$ is not possible because the extreme rays of the DSP polyhedron may take several forms (Mercier (2006)).

Assume that $q = (\boldsymbol{\alpha}^q, \beta^q)$ is a ray of $\mathbf{\Delta}$. We now define the following additional notation:

- $S^q = \{s \in S \mid \alpha^q_s > 0\}$: the set of short connections for which the corresponding dual variable is positive in q.
- $R^q = \{r \in R \mid \sum_{s \in S^q} b_{rs}\alpha^q_s = \beta^q\}$: the set of aircraft routings for which the corresponding constraint (12) is active at q.
- $n(R^q)$: the maximum number of linearly independent elements in R^q.

In the remainder of the paper, we will refer interchangeably to a routing $r \in R$ and to the dual constraint (12) associated with it. The same is true for a short connection $s \in S$ and its corresponding dual variable α_s. For notational convenience, one can define the variables $x_s = \sum_{k \in K} d_{ks}x_k$, $\forall\, s \in S$, representing the flow on short connection s. To ease readability, the proofs of all lemmas are provided in Appendix A.

Lemma 1 *A ray q is an extreme ray of $\mathbf{\Delta}$ if and only if $n(R^q) = |S^q|$.*

The following simple example shows that the positive elements of the extreme rays of $\mathbf{\Delta}$ can take different values. This result is important since all the predefined families of feasibility cuts that have been proposed in the literature assume that the positive elements of $\boldsymbol{\alpha}$ always take the value one (see Section 4).

Example 1 *A possible set of constraints for the Benders DSP:*

$$\alpha_A + \alpha_B + \alpha_C \leq \beta \tag{18}$$
$$\alpha_A + \alpha_B + \alpha_D \leq \beta \tag{19}$$
$$\alpha_A + \alpha_C + \alpha_D \leq \beta \tag{20}$$
$$\alpha_B + \alpha_C + \alpha_D \leq \beta \tag{21}$$
$$\alpha_B + \alpha_D + \alpha_E \leq \beta \tag{22}$$
$$\alpha_A + \alpha_E \leq \beta \tag{23}$$
$$\alpha_C + \alpha_F \leq \beta \tag{24}$$
$$+\alpha_G \leq \beta \tag{25}$$
$$\alpha_A,\ \alpha_B,\ \alpha_C,\ \alpha_D,\ \alpha_E,\ \alpha_F,\ \alpha_G \geq 0. \tag{26}$$

Let $\mathbf{\Delta}_1$ denote the polyhedron defined by constraints (18)–(26) and let $R_{\mathbf{\Delta}_1}$ be the set of extreme rays of $\mathbf{\Delta}_1$. Ray $a = \{1, 1, 0, 0, 1, 0, 0, 2\}$ is an extreme ray of $\mathbf{\Delta}_1$ since $|S^a| = n(R^a) = 3$ ($S^a = \{A, B, E\}$ and $R^a = \{(18),\ (19),\ (22),\ (23)\}$, but $n(R^a) = 3$ since (18) and (19) are linearly dependent at a). One can verify that the rays $b = \{1, 1, 0, 0, 1, 2, 0, 2\}$ and $c = \{2, 1, 1, 1, 2, 3, 4, 4\}$ are also extreme rays of $\mathbf{\Delta}_1$. The extreme ray b yields the following feasibility cut: $x_A + x_B + x_E + 2x_F \leq 2$, which implies that one can either choose *connection F, or a maximum of two connections among A, B, and E. The cut exhibits coefficients of unequal value and is strictly stronger than the ones generated with extreme ray* $a = \{1,1,0,0,1,0,0,2\}$ *or ray* $d = \{1,1,0,0,1,1,0,2\}$.

3 Bounding the Benders Dual Subproblem

When artificial variables are added to constraints (8) or (9) to ensure the feasibility of the Benders PSP, the Benders DSP becomes bounded and Benders feasibility cuts (16) are no longer needed in the Benders MP. However, because a cost is associated with each artificial variable, the PSP becomes an optimization problem and Benders optimality cuts must then be considered in the MP. This type of formulation is interesting since it is generally computationally easier to generate optimality cuts than feasibility cuts. Indeed, the latter requires the identification of a non-basic dual variable which causes unboundedness. Depending on the LP solver being used, this may be a difficult task, especially if the primal form of the subproblem is being solved. There are three different ways of making the primal subproblem feasible. One can either introduce: (a) one artificial variable in each short connection linking constraint (8), (b) a unique artificial variable appearing in every constraint (8), or (c) a unique artificial variable appearing in the convexity constraint (9). Since this choice has an impact on the dual subproblem polyhedron, it may also have an impact on the cuts generated. The models will thus be individually examined below. However, model (c) will not be part of the discussion since the dual subproblem of model (c) is equivalent to the one of model (b) (Mercier (2006)).

In this section, we show that the sets of Benders cuts generated with both models include all feasibility cuts (16) generated from extreme rays of the DSP. We also show that the set of cuts generated from model (b) is, in fact, the same as the set of feasibility cuts (16). However, the set of cuts generated from model (a) also includes cuts that do not correspond to extreme rays, but to rays which are combinations of extreme rays. All cuts generated with model (a) or model (b) thus yields valid feasibility cuts, but a larger number of iterations could be required with model (a) before converging to an optimal solution. In addition, we show that model (a) can favor the generation of cuts with equal coefficients, even if these cuts are weaker. Finally, unlike feasibility cuts generated from predefined families (see Section 4), Benders cuts may include, at a given iteration, short connections that are not used in the current crew pairing solution (if they are incompatible with the ones that are used). Since these cuts are stronger than those in which such connections take the value zero, they may improve the convergence of the method. However, we show that there always exists an optimal solution of model (a) or model (b) where all dual variables corresponding to a connection not currently used are set to zero. Lifting procedures to alleviate this weakness are discussed in Section 5.

3.1 Model (a): One artificial variable in each short connection linking constraint

Recall that $\boldsymbol{X}$ is the set of solutions satisfying the crew constraints (2) and (5). For a given vector $\bar{\boldsymbol{x}} \in \boldsymbol{X}$, when model (a) is chosen, ECP reduces to the following feasible primal subproblem (FPSPa) involving only aircraft variables:

$$\text{Minimize} \quad \sum_{s \in S} c_s a_s \tag{27}$$

subject to

$$\sum_{r \in R} b_{rs} y_r + a_s \geq \sum_{k \in K} d_{ks} \bar{x}_k \qquad (s \in S) \tag{28}$$

$$-\sum_{r \in R} y_r = -1 \tag{29}$$

$$y_r \geq 0 \qquad (r \in R) \tag{30}$$

$$a_s \geq 0 \qquad (s \in S), \tag{31}$$

where a_s is an artificial variable associated with short connection $s \in S$ and c_s is the cost of using the artificial variable a_s. As is common in the first phase of the simplex algorithm, all artificial variables can be assumed to have an equal cost, $\bar{c}$. We can also assume, w.l.o.g., that $\bar{c} = 1$ since no other cost is present in the FPSPa.

Let F^t be the short connection set used by the optimal crew pairing solution at iteration t, i.e., $F^t = \{s \in S \mid \sum_{k \in K} d_{ks} \bar{x}_k = 1\}$. The optimal solution of the FPSPa chooses an aircraft routing solution that minimizes the sum of the artificial variables, and thus maximizes the number of short connections used among those in F^t.

The dual of (27)–(31) is the following bounded dual subproblem (BDSPa):

$$\text{Maximize} \quad \sum_{k \in K} \sum_{s \in S} d_{ks} \bar{x}_k \alpha_s - \beta \tag{32}$$

subject to

$$\sum_{s \in S} b_{rs} \alpha_s - \beta \leq 0 \qquad (r \in R) \tag{33}$$

$$\alpha_s \leq c_s \qquad (s \in S) \tag{34}$$

$$\alpha_s \geq 0 \qquad (s \in S). \tag{35}$$

No new dual variable is introduced, but the value of each α_s variable is now bounded by the cost of the associated artificial variable. Let $\boldsymbol{\Delta}^{B_a}$ denote the polyhedron defined by constraints (33)–(35) and let $P_{\boldsymbol{\Delta}^{B_a}}$ be the set of extreme points of $\boldsymbol{\Delta}^{B_a}$. Introducing the additional free variable z_0, the integrated problem can thus be reformulated as the following Benders master problem (MP2a):

$$\text{Minimize} \quad \sum_{k \in K} c_k x_k + z_0 \tag{36}$$

subject to

$$\sum_{k \in K} a_{ki} x_k = 1 \qquad (i \in L) \tag{37}$$

$$-z_0 + \sum_{k \in K} \sum_{s \in S} d_{ks} \alpha_s x_k \leq \beta \qquad ((\boldsymbol{\alpha}, \beta) \in P_{\boldsymbol{\Delta}^{B_a}}) \tag{38}$$

$$x_k \in \{0, 1\} \qquad (k \in K). \tag{39}$$

The value of z_0 is restricted to be larger than or equal to the optimal value of the dual subproblem, associated with an extreme point, by optimality constraints (38). One can notice that the difference between Benders feasibility cuts (16) and Benders optimality cuts (38) is the value of the cost associated with the artificial variables. This value is minimized by the Benders MP2a and is equal to zero at optimality. In this problem, Benders optimality cuts are thus in fact feasibility cuts.

3.1.1 Cuts corresponding to combinations of extreme rays

All cuts corresponding to an extreme ray of the BDSP can be generated from the extreme points of the BDSPa. However, some extreme points of the BDSPa correspond to a combination of these extreme rays.

Recall that R^q is the set of aircraft routings for which the corresponding dual constraint (12) in the DSP is active at ray q and $n(R^q)$ is the maximum number of linearly independent elements in R^q. Now, let H^q be the complete set of active constraints at q (without the non-negativity constraints) and $n(H^q)$ be the maximum number of linearly independent elements in H^q. For example, in the BDSPa, only the active dual constraints (33) which are active at q are included in R^q, and H^q is comprised of all active constraints (33) and (34).

Lemma 2 *A point p is an extreme point of $\boldsymbol{\Delta}^{B_a}$ if and only if $n(H^p) \geq |S^p| + 1$.*

Proposition 3 *Any extreme ray of $\boldsymbol{\Delta}$ corresponds to an extreme point of $\boldsymbol{\Delta}^{B_a}$, but the converse is not true, i.e., not all extreme points of $\boldsymbol{\Delta}^{B_a}$ correspond to an extreme ray of $\boldsymbol{\Delta}$.*

Proof. If $q = (\boldsymbol{\alpha}^q, \beta^q) \in R_{\boldsymbol{\Delta}}$, then, $n(R^q) = |S^q|$ in the DSP (see Lemma 1). By multiplying every element of q by a positive constant, one can rescale the extreme ray such that at least one of the constraints (34) is active at q and all the others are satisfied. Since the rescaling does not modify R^q or S^q, and the set of constraints (33) is the same as the set (12), then $n(R^q) = |S^q|$ also in the BDSPa. Hence, $n(H^q) \geq |S^q| + 1$ in the BDSPa and, from Lemma 2, $q \in P_{\mathbf{\Delta}^{B_a}}$. We now prove that the converse is not true. The null vector is the unique extreme point of $\mathbf{\Delta}$. All other feasible solutions to the DSP are rays of $\mathbf{\Delta}$. Since all extreme points of $\mathbf{\Delta}^{B_a}$ are feasible solutions to $\mathbf{\Delta}$, they thus correspond to a ray of $\mathbf{\Delta}$. However, not all extreme points of $\mathbf{\Delta}^{B_a}$ correspond to an extreme ray of $\mathbf{\Delta}$. One need only consider the point $e = \{1, 1, 0, 0, 0, 0, 0, 2\}$ in Example 1. The point e is an extreme point of $\mathbf{\Delta}_1^{B_a}$ since $n(H^e) = 3 \geq |S^e| + 1 = 3$, but it is not an extreme ray of $\mathbf{\Delta}$ since $n(R^e) = 1 < |S^e| = 2$. □

It can also be shown that, for a given set of short connections F^t, not all optimal extreme points of $\mathbf{\Delta}^{B_a}$ correspond to an extreme ray of $\mathbf{\Delta}$.

3.1.2 Cuts that may only include short connections taken by the crew pairings

In the BDSPa, points corresponding to cuts only including short connections taken by the current crew pairing solution have an equal objective function value with respect to stronger cuts including short connections not currently used.

Lemma 4 *If p is an extreme point of $\mathbf{\Delta}^{B_a}$, then $\alpha_s^p = \bar{c}$ for at least one short connection $s \in S^p$, and $\beta^p = \max_{r \in R} \sum_{s \in S^p} b_{rs} \alpha_s^p$.*

Lemma 5 *If, for a given set F^t, p is an optimal extreme point of $\mathbf{\Delta}^{B_a}$, then there exists $r_i \in R^p$ such that $\sum_{s \in F^t} b_{r_i s} \alpha_s^p = \beta^p$, i.e., there exists at least one active dual constraint (33) in the BDSPa in which all contributions come from the variables α from F^t ($b_{r_i s} = 0$, for all $s \in (S^p \backslash F^t)$).*

Since $F^t = \{s \in S \mid \sum_{k \in K} d_{ks} \bar{x}_k = 1\}$, let $v_a(p, F^t) = \sum_{s \in F^t} \alpha_s^p - \beta^p$ be the value of the objective function (32) of the BDSPa at the point p for a given set F^t of short connections, and let $v_a^*(F^t)$ be the optimal value of (32) for the same problem. Let $\bar{P}_{\mathbf{\Delta}^{B_a}}^{F^t}$ be the set of extreme points of $\mathbf{\Delta}^{B_a}$ that maximize the value of (32) for a given set F^t, i.e., $\bar{P}_{\mathbf{\Delta}^{B_a}}^{F^t} = \{p \in P_{\mathbf{\Delta}^{B_a}} \mid v_a(p, F^t) = v_a^*(F^t)\}$.

Proposition 6 *If, for a given set F^t, p is an optimal extreme point of $\mathbf{\Delta}^{B_a}$ such that there exists a short connection $s \in S \backslash F^t$ with $\alpha_s^p > 0$, then there exists another feasible point p', where $\alpha_s^{p'} = \alpha_s^p$, for all $s \in F^t$, and $\alpha_s^{p'} = 0$, for all $s \in S \backslash F^t$, such that $v_a(p', F^t) = v_a(p, F^t)$, i.e., there always exists an optimal solution of the BDSPa where all short connections not currently used are set to zero.*

Proof. From Lemma 5, if, for a given set F^t, $p \in \bar{P}_{\mathbf{\Delta}^{B_a}}^{F^t}$, then $\exists\, r_i \in R^p \mid \sum_{s \in F^t} b_{r_i s} \alpha_s^p = \beta^p$. Hence, even if $\alpha_s^{p'} \neq \alpha_s^p$, $\forall\, s \in S \backslash F^t$, $\beta^{p'} = \beta^p$. From that, and since $\alpha_s^{p'} = \alpha_s^p$, $\forall\, s \in F^t$, $v_a(p', F^t) = \sum_{s \in F^t} \alpha_s^p - \beta^p = v_a(p, F^t)$. □

3.1.3 Cuts with equal positive coefficients

In the BDSPa, points corresponding to cuts having equal positive coefficients might have a better objective function value with respect to stronger cuts with unequal coefficients.

Let $p_1 = (\boldsymbol{\alpha}^{p_1}, \beta^{p_1})$ be an extreme point of $\boldsymbol{\Delta}^{B_a}$. From Lemma 4, $\alpha_s^{p_1} = \bar{c}$ for at least one short connection $s \in S^{p_1}$. Let $p_2 = (\boldsymbol{\alpha}^{p_2}, \beta^{p_2})$ be a point in $\boldsymbol{\Delta}^{B_a}$ where (i) $\alpha_s^{p_2} = \max_{j \in S} \alpha_j^{p_1} = \bar{c}, \forall\, s \in (S^{p_1} \cap F^t)$, (ii) $\alpha_s^{p_2} = \alpha_s^{p_1}, \forall\, s \in (S^{p_1} \backslash F^t)$, and (iii) $\beta^{p_2} = \max_{r \in R} \sum_{s \in S^{p_2}} b_{rs} \alpha_s^{p_2}$. Let $S^p = S^{p_1} = S^{p_2}$ be the common set of positive variables. One can observe that p_2 is feasible since it satisfies all constraints (33)–(35) from the BDSPa. For a given set F^t, recall that $v_a(p, F^t)$ is the value of the objective function (32) of the BDSPa at the point p.

Lemma 7 *If, for a given set F^t, p_1 is an extreme point of $\boldsymbol{\Delta}^{B_a}$ and p_2 is a different point which has the same set of positive variables as p_1, but in which the values of all variables $\alpha_s, s \in (S^{p_2} \cap F^t)$, are equal to their common upper bound, $\bar{c}$, then, $v_a(p_2, F^t) \geq v_a(p_1, F^t)$. In addition, if no aircraft routing $r \in R^{p_1}$ contains all short connections $s_i \in F^t$ such that $\alpha_{s_i}^{p_1} < 1$, then $v_a(p_2, F^t) > v_a(p_1, F^t)$.*

To illustrate Lemma 7, recall that ray $b = \{1, 1, 0, 0, 1, 2, 0, 2\}$, from Example 1, is an extreme ray of $\boldsymbol{\Delta}_1$. However, b is not a feasible point of $\boldsymbol{\Delta}_1^{B_a}$ since it violates one of the upper bound constraints (34), but the equivalent extreme ray $l = \{1/2, 1/2, 0, 0, 1/2, 1, 0, 1\}$ corresponds to an an extreme point of $\boldsymbol{\Delta}_1^{B_a}$. Nevertheless, one can see that l is not always an optimal solution of the BDSPa. For example, when $F^t = \{A, B, E, F\}$, the value of the objective function (32) at l is lower than it is at the point $d = \{1, 1, 0, 0, 1, 1, 0, 2\}$ ($v_a(l, F^t) = 3/2 < v_a(d, F^t) = 2$), but d gives a weaker cut than l. Recall that d is not an extreme ray of $\boldsymbol{\Delta}_1$. It is, in fact, a linear combination of the extreme rays l and $a = \{1, 1, 0, 0, 1, 0, 0, 2\}$.

Proposition 8 *If, for a given set F^t, p is an optimal extreme point of $\boldsymbol{\Delta}^{B_a}$, then $\alpha_s^p = \bar{c}$, for all $s \in S^p$, or there exists another feasible point q, where $\alpha_s^q = \bar{c}$, for all $s \in S^q$, such that $v_a(q, F^t) = v_a(p, F^t)$, i.e., in at least one of the optimal points of the BDSPa, all positive variables α take the value of their common upper bound, $\bar{c}$.*

Proof. From Lemma 7, if $p \in P_{\boldsymbol{\Delta}^{B_a}}$ and, for a given set F^t, there exists a short connection $s_i \in (S^p \cap F^t)$ with $\alpha_{s_i}^p < \bar{c}$, then, there exists another feasible point p' where (i) $\alpha_s^{p'} = \bar{c}, \forall\, s \in (S^p \cap F^t)$, (ii) $\alpha_s^{p'} = \alpha_s^p, \forall\, s \in (S^p \backslash F^t)$, and (iii) $v_a(p', F^t) \geq v_a(p, F^t)$. From that, and since p is optimal, $v_a(p', F^t) = v_a(p, F^t)$. On the one hand, if there are no short connections $s \in S \backslash F^t$ with $\alpha_s^p > 0$, then $(S^p \cap F^t) = S^p$ (or $(S^{p'} \cap F^t) = S^{p'}$). On the other hand, if there exists a short connection $s \in S \backslash F^t$ with $\alpha_s^p > 0$, then, from Proposition 6, since p (or p') is an optimal point in $\boldsymbol{\Delta}^{B_a}$, there exists another optimal point p'' where $\alpha_s^{p''} = 0, \forall\, s \in S \backslash F^t$. Hence, if $p \in \bar{P}^{F^t}_{\boldsymbol{\Delta}^{B_a}}$, then $\alpha_s^p = \bar{c}, \forall\, s \in S^p$, or there exists another feasible point q, where $\alpha_s^q = \bar{c}$, $\forall\, s \in S^q$, such that $v_a(q, F^t) = v_a(p, F^t)$. □

3.2 Model (b): A unique artificial variable appearing in every short connection linking constraint

When model (b) is used, only one artificial variable is added in the primal subproblem and the bounded dual subproblem (BDSPb) is the following:

$$\text{Maximize} \quad \sum_{k \in K} \sum_{s \in S} d_{ks} \bar{x}_k \alpha_s - \beta \tag{40}$$

subject to

$$\sum_{s \in S} b_{rs} \alpha_s - \beta \leq 0 \qquad (r \in R) \tag{41}$$

$$\sum_{s \in S} \alpha_s \leq 1 \tag{42}$$

$$\alpha_s \geq 0 \qquad (s \in S). \tag{43}$$

No new dual variable is introduced, but the sum of the α_s variables is now bounded by the value 1. Let $\mathbf{\Delta}^{B_b}$ denote the polyhedron defined by constraints (41)–(43) and let $P_{\mathbf{\Delta}^{B_b}}$ be the set of extreme points of $\mathbf{\Delta}^{B_b}$.

3.2.1 Cuts corresponding to extreme rays

The set of possible cuts generated with the BDSPb corresponds exactly to the set generated with the DSP.

Lemma 9 *If p is an extreme point of $\mathbf{\Delta}^{B_b}$, then $\sum_{s \in S^p} \alpha_s^p = 1$ (constraint (42) is active), and there are exactly $|S^p|$ linearly independent active constraints (41) at p.*

Lemma 10 *If q is an extreme ray of $\mathbf{\Delta}$, then the point p, with (i) $\alpha_s^p = \alpha_s^q / \sum_{s \in S^q} \alpha_s^q$, $\forall s \in S^q$, (ii) $\alpha_s^p = 0$, $\forall s \in S \backslash S^q$, and (iii) $\beta^p = \beta^q / \sum_{s \in S^q} \alpha_s^q$, is an extreme point of $\mathbf{\Delta}^{B_b}$, i.e., every extreme ray of $\mathbf{\Delta}$ corresponds to an extreme point of $\mathbf{\Delta}^{B_b}$, which is a rescaling of the ray such that constraint (42) is satisfied as an equality.*

To illustrate Lemma 10, recall that the ray $a = \{1, 1, 0, 0, 1, 0, 0, 2\}$, from Example 1, is an extreme ray of $\mathbf{\Delta}_1$ and it corresponds to the feasibility cut $x_A + x_B + x_E \leq 2$. From the rescaling described in the lemma, one can construct point $a' = \{1/3, 1/3, 0, 0, 1/3, 0, 0, 2/3\}$. One can verify that the same set of dual constraints (41) are active at a and a', and that constraint (42) is satisfied as an equality at a'. Point a' corresponds to the cut $1/3\, x_A + 1/3\, x_B + 1/3\, x_E \leq 2/3$, which is, in fact, exactly the same as the cut generated from a.

Proposition 11 *Any extreme point of $\mathbf{\Delta}^{B_b}$ corresponds to an extreme ray of $\mathbf{\Delta}$, and conversely.*

Proof. Every extreme ray of $\mathbf{\Delta}$ corresponds to an extreme point of $\mathbf{\Delta}^{B_b}$ (see Lemma 10). Hence, $R_{\mathbf{\Delta}} \subset P_{\mathbf{\Delta}^{B_b}}$. From Lemma 9, if $p \in P_{\mathbf{\Delta}^{B_b}}$, then $n(R^p) = |S^p|$ in the BDSPb. Since the set of constraints (12) is the same as the set (41), $n(R^p) = |S^p|$ also in the DSP and $P_{\mathbf{\Delta}^{B_b}} \subset R_{\mathbf{\Delta}}$. Hence, $P_{\mathbf{\Delta}^{B_b}} = R_{\mathbf{\Delta}}$. □

3.2.2 Cuts that may only include short connections taken by the crew pairings

One can easily see that Proposition 6, from model (a), can be generalized to model (b) since the set of dual variables, the set of dual constraints (33) and the dual subproblem objective function remain the same. In the BDSPb, points corresponding to cuts only including short connections taken by the current crew pairing solution thus also have an equal objective function value with respect to stronger cuts including short connections not currently used.

3.2.3 Stronger cuts with unequal positive coefficients

One can easily see that Proposition 8 does not hold for model (b). From Proposition 11, only extreme rays are generated from model (b). A cut corresponding to ray $d = \{1, 1, 0, 0, 1, 1, 0, 2\}$, from Example 1, could thus not be generated instead of the stronger cut corresponding to $b = \{1, 1, 0, 0, 1, 2, 0, 2\}$.

3.3 Comparison of the different bounding approaches for the Benders dual subproblem

From the previous results, one can say that model (a) is likely to generate weaker cuts than model (b). Theoretically, then, it is preferable to use model (b), but the computational results found in Section 6 show that it is not the case in practice. Combinations of extreme rays can actually give cuts that are much stronger than those generated from the extreme rays themselves. A particular case is described in the following section.

3.3.1 Individually infeasible short connections

Recall that S was defined, in Section 1, as the set of short connections included in at least one aircraft routing solution. When the ECP formulation (1)–(6) is chosen and complete aircraft routing solutions are included in the formulation, removing from the set of short connections those that are individually infeasible for the aircraft routing problem is an easy task. Indeed, all the arcs corresponding to a short connection s such that $b_{rs} = 0$, for all $r \in R$, can be removed from the crew networks in a preprocessing step. However, getting all feasible routings is, by itself, a very difficult combinatorial optimization problem (Mercier et al. (2005)). When complete routing solutions are not available and an explicit formulation is used for the integrated problem, determining the individually infeasible short connection (IISC) set requires solving a series of aircraft routing problems, imposing the use of every short connection, one at a time, and verify the feasibility of the problem. If one wishes to overlook this time-consuming step, then the set of short connections may include a subset of IISC. In that case, some feasibility cuts forbidding their use in the crew pairings will be generated. These cuts have a null right-hand-side ($\beta = 0$). Interestingly, model (a) has the ability to generate cuts forbidding, at the same time, the use of all IISC taken by the crews at a given iteration, whereas model (b) does not. Therefore, model (a) may dominate model (b) in practice. In fact, the numerical results reported in Section 6 show that the number of feasibility cuts generated with model (b) is smaller than with model (a) only when the IISC arcs are removed. When an explicit formulation is used for the integrated problem, the time needed to identify the set of IISC is actually even larger than the total time needed to solve model (a).

Let $\hat{S}$ be the set of short connections not included in any aircraft routing solution. When S includes a subset of infeasible short connections, i.e., $|S \cap \hat{S}| \neq 0$, constraints (33) and (41) from the BDSPa and the BDSPb are in fact:

$$\sum_{s \in S \setminus \hat{S}} b_{rs}\alpha_s - \beta \leq 0 \qquad (r \in R). \tag{44}$$

Recall that, for a given set F^t, (i) $v_a(p, F^t) = \sum_{s\in F^t} \alpha_s^p - \beta^p$ is the value of the objective function (32) of the BDSPa at the point p, (ii) $v_a^*(F^t)$ is the optimal value of the BDSPa, and (iii) $\bar{P}^{F^t}_{\mathbf{\Delta}^{B_a}} = \{p \in P_{\mathbf{\Delta}^{B_a}} \mid v_a(p, F^t) = v_a^*(F^t)\}$ is the set of optimal extreme points of $\mathbf{\Delta}^{B_a}$. Also recall that we can assume, w.l.o.g., that the cost of using an artificial variable in an aircraft routing solution is 1, i.e., $\bar{c} = 1$.

Proposition 12 *For a given set of short connections F^t, if p is an optimal point of $\mathbf{\Delta}^{B_a}$ where $\beta^p = 0$, then $\alpha_s^p = 1, \forall\, s \in (\hat{S} \cap F^t)$, and p is an extreme point of $\mathbf{\Delta}^{B_a}$, i.e., if there is an optimal solution to the BDSPa in which $\beta = 0$, then all variables from F^t corresponding to an infeasible short connection take the value 1 in the solution and it corresponds to an extreme point.*

Proof. When $\beta^p = 0$, then $\alpha_s^p = 0, \forall\, s \in (S \setminus \hat{S})$, and $v_a(p, F^t) = \sum_{s\in F^t} \alpha_s^p - \beta^p = \sum_{s\in F^t} \alpha_s^p = \sum_{s\in(\hat{S}\cap F^t)} \alpha_s^p$. Hence, increasing the value of the variables corresponding to the short connections $s \in (\hat{S} \cap F^t)$ directly improves the value of the objective function, and this, without violating any dual constraints (44). From that, and since all variables $\alpha_s^p, s \in \hat{S}$, are individually bounded by $\bar{c} = 1$ (constraints (34)), then, $\alpha_s^p = 1, \forall\, s \in (\hat{S} \cap F^t)$, when $\beta^p = 0$ and $v_a(p, F^t) = v_a^*(F^t)$. In addition, since the total number of linearly independent active constraints at p ($|S^p|$ constraints (34) as well as $|S| - |S^p| + 1$ non-negativity constraints) is equal to the total number of variables ($|S| + 1$), then p is an extreme point of $\mathbf{\Delta}^{B_a}$. □

Proposition 13 *For a given set of short connections F^t, if $|\hat{S} \cap F^t| \neq 0$ and p is an optimal extreme point of $\mathbf{\Delta}^{B_b}$ where $\beta^p = 0$, then $|S^p| = 1$, i.e., only one variable from F^t corresponding to an infeasible short connection takes a positive value.*

Proof. When $\beta^p = 0$, then $\alpha_s^p = 0, \forall\, s \in (S \backslash \hat{S})$, and $v_b(p, F^t) = \sum_{s \in F^t} \alpha_s^p - \beta^p = \sum_{s \in F^t} \alpha_s^p = \sum_{s \in (\hat{S} \cap F^t)} \alpha_s^p$. Hence, increasing the value of the variables corresponding to the short connections $s \in (\hat{S} \cap F^t)$ directly increases the value of the objective function without violating constraints (44), but, by constraint (42), $\sum_{s \in S} \alpha_s^p \leq 1$. From that, and since $v_b(p, F^t) = v_b^*(F^t)$, then $\sum_{s \in (\hat{S} \cap F^t)} \alpha_s^p = 1$ and $v_b(p, F^t) = \sum_{s \in (\hat{S} \cap F^t)} \alpha_s^p = 1$. One can observe that p is an extreme point of $\mathbf{\Delta}^{B_b}$ only if $|S^p| = 1$, since the total number of active constraints (constraint (42) as well as $|S| - |S^p| + 1$ non-negativity constraints) is equal to the total number of variables ($|S| + 1$) only in that case. □

Proposition 12 implies that, for a given crew pairing solution, a single cut simultaneously forbidding all maintenance infeasible short connections from the set chosen by the crew pairings, F^t, is generated by model (a). In contrast, Proposition 13 indicates that as many as $|\hat{S} \cap F^t|$ cuts are generated by model (b) to forbid every short connection from the same set. One can easily see that a cut with a zero right-hand-side generated by model (b) corresponds to the extreme ray of $\mathbf{\Delta}$ in the direction of the non-negativity constraint of the corresponding positive dual variable. Similarly, the cut generated by model (a) corresponds to a linear combination of extreme rays of $\mathbf{\Delta}$ in the direction of the non-negativity constraint of all the corresponding positive dual variables. Hence, in the case of cuts preventing the use of a set of short connections that are individually infeasible for the aircraft routing problem, linear combinations of extreme rays give stronger cuts than extreme rays. In a Benders decomposition method, the subproblem formulation does not have to be identical at every iteration. One bounding method could thus be used in the first iterations to generate all the IISC cuts, and then, the Benders subproblem could shift to another bounding method.

4 Some Predefined Families of Feasibility Cuts

Without the use of a Benders decomposition method, one can still solve the integrated problem iteratively, by adding feasibility cuts to the crew pairing problem until the set of short connections used by the pairings is feasible for the aircraft routing problem. This solution process is called the *Constrained Crew Pairing* (CCP) by Cohn and Barnhart (2003), who have proposed three families of feasibility cuts. The first family simply forbids the current crew pairing solution:

$$\sum_{k \in K^t} x_k \leq |K^t| - 1, \tag{45}$$

where $K^t = \{k \in K \,|\, x_k^t = 1\}$ and $\boldsymbol{x}^t$ is the optimal crew pairing solution at iteration t. The authors remark, however, that this type of cut is not very efficient since it prohibits a maintenance infeasible set of pairings while there may exist other sets of pairings using the same set of short connections. Hence, the second type of feasibility cut that is proposed prohibits the short connection set F^t used by the optimal solution at iteration t:

$$\sum_{k\in K}\sum_{s\in F^t} d_{ks}x_k \leq |F^t| - 1. \tag{46}$$

The authors observe that this type of cut may also be inefficient if a subset of incompatible short connections from F^t is attractive for the crew pairing problem and is likely to be chosen in several successive iterations. For example, if $F^t = \{A, B, C, D, E\}$, where A, B and C are incompatible, the next iterations could yield solutions with short connection sets $\{A, B, C, D\}$, $\{A, B, C, E\}$ and $\{A, B, C\}$, which are all maintenance infeasible.

MIS feasibility cuts

To circumvent this weakness, the authors finally propose a family of cuts that directly prohibit a *Minimally Infeasible Subset* (MIS) $\tilde{F}^t$ of F^t. An MIS is an infeasible set of short connections such that the removal of any element from the set yields a feasible subset. The resulting cut is the following:

$$\sum_{k\in K}\sum_{s\in \tilde{F}^t} d_{ks}x_k \leq |\tilde{F}^t| - 1. \tag{47}$$

To determine a set $\tilde{F}^t$, one may solve the following integer problem denoted by (PMIS):

$$\text{Minimize} \quad \sum_{s\in F^t} f_s \tag{48}$$

subject to

$$\sum_{s\in F^t\setminus S(r)} f_s \geq 1 \qquad (r \in R) \tag{49}$$

$$f_s \in \{0,1\} \qquad (s \in F^t), \tag{50}$$

where f_s is a binary variable indicating whether the short connection $s \in F^t$ is included in $\tilde{F}^t$ and $S(r)$ is the set of short connections used in routing $r \in R$. Constraints (49) require that for every feasible aircraft solution $r \in R$, there be at least one element of $\tilde{F}^t$ that is not in $S(r)$. The latter constraints ensure that the chosen subset $\tilde{F}^t$ is maintenance infeasible. The objective function (48) finds the smallest maintenance infeasible subset of F^t.

PSP feasibility cuts

When the dual subproblem is bounded with model (a), a feasibility cut can easily be generated with the primal subproblem (PSP) information. Recall that $S(r)$ is the set of short connections used in routing r, and let $\bar{r}^t$ be the FPSPa aircraft routing solution at iteration t. Huang et al. (2003) have proposed a family of cuts generated from $S(\bar{r}^t)$ as follows:

$$\sum_{k \in K} \sum_{s \in F^t} d_{ks} x_k \leq |S(\bar{r}^t) \cap F^t|. \tag{51}$$

Since $|S(\bar{r}^t) \cap F^t|$ is the maximum number of short connections that a feasible aircraft routing solution can use among those chosen by the current crew pairing solution, the PSP is infeasible when $|S(\bar{r}^t) \cap F^t| < |F^t|$, and cuts (51) are valid feasibility cuts. One can notice that the PSP cuts are stronger than the feasibility cuts (46).

4.1 Comparison of MIS and Benders feasibility cuts

In this section, we show that the set of MIS feasibility cuts is included in the set of Benders feasibility cuts. In addition, we show that some MIS cuts can be lifted, and the resulting cut corresponds to a Benders cut. The latter cuts can thus dominate the MIS cuts.

Proposition 14 *Any MIS feasibility cut (47) corresponds to an extreme ray of* $\mathbf{\Delta}$*, but the converse is not true, i.e., not all extreme rays of* $\mathbf{\Delta}$ *correspond to a MIS cut.*

Proof. Let $c = (\boldsymbol{\alpha}^c, \beta^c)$ be a point corresponding to an MIS cut formed from the short connection set $\tilde{F}^t$, where $\alpha_s^c = 1, \forall\, s \in \tilde{F}^t$, and $\beta^c = |\tilde{F}^t| - 1$. Since no routing contains more than $|\tilde{F}^t| - 1$ short connections from $\tilde{F}^t$ ($\tilde{F}^t$ is a minimally infeasible subset) and $\alpha_s^c = 0, \forall\, s \in S \backslash \tilde{F}^t$, each constraint (12) contains at most $|\tilde{F}^t| - 1$ positive variables (of value 1) and is satisfied. The point c is thus feasible since the constraints (13) are satisfied from the definition of c. One can also observe that νc is in $\mathbf{\Delta}, \forall\, \nu > 0$, and that c is thus a ray of $\mathbf{\Delta}$. Since there exists a feasible aircraft routing for each subset of $\tilde{F}^t$, there exists a feasible routing for each of the $|\tilde{F}^t|$ subsets of size $|\tilde{F}^t| - 1$. Those distinct subsets (and only those) each correspond to a constraint in the Benders dual subproblem that is active at c, since $\beta^c = |\tilde{F}^t| - 1$ and $\alpha_s^c = 1, \forall\, s \in \tilde{F}^t$. There are thus exactly $|\tilde{F}^t|$ linearly independent active constraints at c. Hence, $n(R^c) = |\tilde{F}^t| = |S^c|$ and c is an extreme ray of $\mathbf{\Delta}$. To prove that the converse is not true, one only has to consider the extreme ray $b = \{1, 1, 0, 0, 1, 2, 0, 2\}$, from Example 1. It cannot correspond to an MIS since $\alpha_F > 1$. □

Furthermore, the following simple example shows an extreme ray of $\mathbf{\Delta}$ that does not correspond to an MIS even if, in this case, $\alpha_s = 1, \forall\, s \in S$, and $S = F^t$.

Example 2 *A possible set of constraints for the Benders DSP:*

$$\alpha_A \leq \beta \tag{52}$$

$$\alpha_B \leq \beta \tag{53}$$

$$\alpha_C \leq \beta \tag{54}$$

$$\alpha_A, \alpha_B, \alpha_C \geq 0. \tag{55}$$

If the current crew pairing solution uses all short connections, i.e., $F^t = \{A, B, C\}$*, the aircraft routing problem is infeasible and there are three possible MIS:* $\{A, B\}$*,* $\{A, C\}$ *and* $\{B, C\}$*. For instance, if* $\tilde{F}^t = \{A, B\}$*, then the MIS cut is:* $x_A + x_B \leq 1$*. There are four non-trivial extreme rays, of which three correspond to the three MIS. The last one,* $\{1, 1, 1, 1\}$ *does not correspond to an MIS and it gives the strongest cut:* $x_A + x_B + x_C \leq 1$*.*

Proposition 15 *An MIS feasibility cut (47) generated at iteration* t *can be lifted by adding to the corresponding MIS a short connection that is incompatible with all feasible subsets in* $\tilde{F}^t$*. If such a short connection exists, the resulting cut is stronger and it corresponds to an extreme ray of* $\mathbf{\Delta}$*.*

Proof. From Proposition 14, a ray q with $\alpha_s = 1, \forall\, s \in \tilde{F}^t$, and $\beta = |\tilde{F}^t| - 1$ is an extreme ray of $\mathbf{\Delta}$ corresponding to the MIS $\tilde{F}^t$. Recall that $n(R^q)$ is the maximum number of linearly independent active dual constraints (12) at q, and $S^q = \tilde{F}^t$ is the set of positive variables in q. Since q is an extreme ray, $n(R^q) = |S^q|$. If there exists a short connection $s_1 \in S \backslash S^q$ which is incompatible with all feasible subsets of $\tilde{F}^t$, then a new ray $g = (\boldsymbol{\alpha}^g, \beta^g)$ is obtained, where $\alpha_{s_1} \geq 1$, $\alpha_s = 1, \forall\, s \in S^q$, and $\beta^g = \beta^q = |\tilde{F}^t| - 1$. If s_1 exists, then there exists a feasible routing using s_1 while using a strictly smaller number of short connections from $\tilde{F}^t$ than the value of β^g. Therefore, $\alpha^g_{s_1} \geq 1$ and the cut corresponding to g is valid and stronger than the one corresponding to q. In addition, it is always possible to choose for α_{s_1} a value for which a new dual constraints (12) is active, without modifying any other value. When $\alpha^g_{s_1} = \min_{r \in R \backslash R^q} \beta^g - \sum_{s \in S^q} b_{rs}$, a new dual constraints (12) is satisfied at equality and $n(R^g) = n(R^q) + 1 = |S^q| + 1 = |S^g|$. The ray g is thus an extreme ray of $\mathbf{\Delta}$. □

The lifted MIS cut can be stated as follows:

$$\sum_{k \in K} \sum_{s \in \tilde{F}^t} d_{ks} x_k + \sum_{k \in K} l_{s_1} d_{ks_1} x_k \leq |\tilde{F}^t| - 1, \tag{56}$$

where s_1 is a short connection $s \in S \backslash \tilde{F}^t$ that is incompatible with all feasible subsets of $\tilde{F}^t$ and l_{s_1} is the value taken by α_{s_1} in the corresponding extreme ray. An MIS cut can be lifted repeatedly, as long as the added short connections are incompatible with each other.

4.2 Comparison of PSP feasibility cuts and Benders cuts

In this section, we show that PSP cuts do not always correspond to an extreme ray of $\mathbf{\Delta}$, but to an optimal extreme point of $\mathbf{\Delta}^{B_a}$ that is likely to yield a weak cut.

Proposition 16 *A PSP Feasibility cut (51) does not necessarily correspond to an extreme ray of* $\mathbf{\Delta}$*.*

Proof. If $F^t = \{A, B, E, F\}$ in Example 1, then the PSP cut is $x_A + x_B + x_E + x_F \leq 2$. One can easily see that the corresponding ray $r = \{1, 1, 0, 0, 1, 1, 0, 2\}$ is not an extreme ray of $\mathbf{\Delta}_1$ since $|S^r| = 4 \neq n(R^r) = 3$. □

Lemma 17 *For a given short connection set F^t, the PSP cut (51) corresponds to an optimal extreme point of $\mathbf{\Delta}^{B_a}$.*

Proposition 18 *PSP cuts are the weakest of the cuts corresponding to optimal solutions of the BDSPa, according to the ratio violation/right-hand-side (v/rhs).*

Proof. For a given set F^t, define E as the set of cuts corresponding to an optimal solution of the BDSPa. Let $lhsf_e$ be the sum of the coefficients of the short connections $s \in F^t$ in feasibility cut $e \in E$, and rhs_e be the value of the right-hand-side of cut e. According to the ratio v/rhs, the strength of the cut e is $rv_e = (lhsf_e + add_e - rhs_e)/rhs_e$, where add_e is the sum of the coefficients of $s \in S \backslash F^t$ in e. Let the element $e_1 \in E$ coincide with the PSP cut (see Lemma 17). Since $add_{e_1} = 0$, $rv_{e_1} = (|F^t| - rhs_{e_1})/rhs_{e_1}$ (all coefficients are equal to 1 in e_1 and $lhsf_e = |F^t|$). Recall that $v_a^*(F^t) = lhsf_e - rhs_e, \forall\, e \in E$. Since all objective values for solutions corresponding to a cut $e \in E$ are equal, then, for a given $e \in E$, $v_a^*(F^t) = lhsf_e - rhs_e = |F^t| - rhs_{e_1}$. From that, $rhs_e = rhs_{e_1} - |F^t| + lhsf_e$ and $rv_e = (lhsf_e + add_e - (rhs_{e_1} - |F^t| + lhsf_e))/(rhs_{e_1} - |F^t| + lhsf_e) = (add_e + |F^t| - rhs_{e_1})/(rhs_{e_1} - (|F^t| - lhsf_e))$. Since (i) $add_e \geq 0$, (ii) $add_{e_1} = 0$, (iii) $|F^t| \geq lhsf_e, \forall\, e \in E$, and (iv) $|F^t| = lhsf_{e_1}$, then $rv_e \geq rv_{e_1}, \forall\, e \in E$. □

To illustrate Proposition 18, consider Example 1. If $F^t = \{C, E, F\}$, then the PSP cut is $j_1 : x_C + x_E + x_F \leq 2$, but the following cuts also correspond to optimal solutions of the BDSPa in that case: $j_2 : x_C + x_E \leq 1$, $j_3 : x_E + x_F \leq 1$, $j_4 : x_C + x_E + x_F + x_G \leq 2$, $j_5 : x_C + x_E + x_G \leq 1$ and $j_6 : x_E + x_F + x_G \leq 1$. All these cuts correspond to a solution to the BDSPa with an objective value of $v_a^*(F^t) = 1$ but with different strengths according to the ratio v/rhs since $rv_1 = 1/2$, $rv_2 = 1$, $rv_3 = 1$, $rv_4 = 1$, $rv_5 = 2$ and $rv_6 = 2$. One can notice that j_2, j_3, j_5 and j_6 are extreme rays of $\mathbf{\Delta}_1$ and j_2 and j_3 are MIS feasibility cuts (47).

5 Strengthening the Feasibility Cuts

By definition, MIS and PSP feasibility cuts never include short connections which are not part of the set used by the current crew pairing solution, even if they are incompatible with those included. By Proposition 6, even though it is possible to include such connections when generating a cut with BDSPa or BDSPb, neither formulation favors them since these connections do not contribute to their objective function. For this reason, and to allow unequal coefficients in MIS, PSP and BDSPa cuts, this section proposes two ways of strengthening the feasibility cuts. The first procedure can be applied to Benders cuts generated with BDSPa or BDSPb, whereas the second one can be applied to any feasibility cut for the integrated aircraft routing and crew pairing problem.

5.1 Pareto-optimal cuts

Whenever the primal subproblem (27)–(30) is degenerate (and this is often the case in practice), there may exist more than one optimal solution to the bounded dual subproblem. Although any of these points leads to a valid optimality cut, some may yield stronger cuts than others. The cut generated from the extreme point $(\boldsymbol{\alpha}^1, \beta^1)$ dominates the cut generated from the extreme point $(\boldsymbol{\alpha}^2, \beta^2)$ if and only if

$$\sum_{k\in K}\sum_{s\in S} d_{ks}\alpha_s^1 x_k - \beta^1 \geq \sum_{k\in K}\sum_{s\in S} d_{ks}\alpha_s^2 x_k - \beta^2$$

for all $\bar{\boldsymbol{x}} \in \boldsymbol{X}$ with strict inequality for at least one point. A cut is said to be Pareto-optimal if no other cuts dominate it (Magnanti and Wong, 1981).

Let $\boldsymbol{X}^{LP}$ be the polyhedron defined by (2) and constraints $x_k \geq 0\,(k \in K)$, and let $ri(\boldsymbol{X}^{LP})$ denote the relative interior of $\boldsymbol{X}^{LP}$. Pareto-optimal cuts are generated from extreme points, one thus has to ensure that the dual subproblem is bounded for any solution of the relaxed master problem. For a given solution $\bar{\boldsymbol{x}} \in \boldsymbol{X}^{LP}$, let $v(\bar{\boldsymbol{x}})$ denote the optimal value of the primal subproblem. To identify an optimal solution to the bounded dual subproblem that yields a Pareto-optimal cut, one must solve the following auxiliary bounded dual subproblem (ABDSP), where $\boldsymbol{x^0}$ is a point chosen in $ri(\boldsymbol{X}^{LP})$:

$$\text{Maximize} \quad \sum_{k\in K}\sum_{s\in S} d_{ks} x_k^0 \alpha_s - \beta \tag{57}$$

subject to

$$\sum_{k\in K}\sum_{s\in S} d_{ks}\bar{x}_k \alpha_s - \beta = v(\bar{\boldsymbol{x}}) \tag{58}$$

$$(\boldsymbol{\alpha}, \beta) \in \boldsymbol{\Delta}^B. \tag{59}$$

Besides including the same bounded dual subproblem constraints (59) ($\boldsymbol{\Delta}^B = \boldsymbol{\Delta}^{B_a}$ or $\boldsymbol{\Delta}^{B_b}$), the ABDSP contains an additional constraint (58) to ensure that one will choose an extreme point from the set of optimal solutions to the original bounded dual subproblem (with a value equal to the original optimal value $v(\bar{\boldsymbol{x}})$). The auxiliary problem is solved at every iteration of the Benders decomposition algorithm, after the BDSP has been solved. The objective function of the auxiliary problem (57) compares all possible cuts at a point $\boldsymbol{x^0}$ in the feasible region of the master problem. It therefore tries to maximize the strength of the cut to be added to the Benders master problem.

When $\boldsymbol{\Delta}^B = \boldsymbol{\Delta}^{B_a}$ in (59), the weakness highlighted by Proposition 8 for cuts generated directly from the BDSPa is also true for cuts generated from the ABDSP because the upper bound constraints on the dual variables are still present in the ABDSP. However, since $\boldsymbol{x^0}$ is a point chosen in $ri(\boldsymbol{X}^{LP})$, all dual variables $\boldsymbol{\alpha}$ may have a positive coefficient in the objective function (57), not only those corresponding to connections used in the current crew pairing solution. Finding the Pareto-optimal cuts can thus reduce the number of cuts needed before the Benders decomposition algorithm converges.

5.2 Lifting procedure

One can use the lifting procedure described in this section to strengthen all types of feasibility cuts. MIS and PSP feasibility cuts, as well as those generated from extreme rays of $\boldsymbol{\Delta}$ or from extreme points of $\boldsymbol{\Delta}^{B_a}$ or $\boldsymbol{\Delta}^{B_b}$, all correspond to feasible solutions of the DSP since they all satisfy the dual constraints (12) and (13). The lifting procedure presented here consists of first generating a feasibility cut with a chosen method and then lifting it by solving a modified version of the DSP. Let $u = (\bar{\boldsymbol{\alpha}}^u, \bar{\beta}^u)$ be the solution corresponding to a cut that one wishes to strengthen. From u, a modified Benders dual subproblem can be formulated as follows (MDSP(u)):

$$\text{Maximize} \quad \sum_{s\in S} \alpha_s \tag{60}$$

subject to

$$\sum_{s\in S} b_{rs}\alpha_s \leq \bar{\beta}^u - \sum_{s\in S} b_{rs}\bar{\alpha}_s^u \qquad (r \in R) \tag{61}$$

$$\alpha_s \geq 0 \qquad (s \in S). \tag{62}$$

Constraints (61) limit the lifting of the dual variables α_s to the slack of the constraints (12) of the DSP at u. The right-hand-sides of constraints (61) are thus all positive and bounded. This makes the MDSP(u) a bounded problem (without the use of specific upper bounds on the variables) and also a feasible problem since the null vector, representing no strengthening, is always feasible. One can observe that the dual variable β is not included in the MDSP(u). The objective function (60) maximizes the sum of all variables $\alpha_s, s \in S$. The coefficient of a variable in the lifted cut is the sum of its value in u and its value in an optimal solution of the MDSP(u). Let $\boldsymbol{\Delta}^L$ denote the polyhedron defined by constraints (61)–(62) and let $P_{\boldsymbol{\Delta}^L}$ be the set of extreme points of $\boldsymbol{\Delta}^L$. Let $v = (\bar{\boldsymbol{\alpha}}^v)$ be an optimal solution to the MDSP(u), and $w = (\bar{\boldsymbol{\alpha}}^u + \bar{\boldsymbol{\alpha}}^v, \bar{\beta}^u)$ be the constructed point from which the lifted cut is generated. The cut corresponding to w is always stronger than or equal to the cut corresponding to u since the value of both right-hand-sides is equal and the coefficient of every short connection $s \in S$ in the left-hand-side of the lifted cut is larger than or equal to its corresponding value in the cut generated from u ($\bar{\alpha}_s^v \geq 0, \forall s \in S$).

One can observe that since all variables have a positive coefficient in the objective function (60), the MDSP(u) favors the stronger cuts which include short connections $s \in S \backslash F^t$. In addition, the problem highlighted by Proposition 8 is not present in the MDSP(u) because the upper bound constraints are removed. Lifted cuts can thus contain variables with different coefficients. To illustrate this last statement, recall that the ray $a = \{1, 1, 0, 0, 1, 0, 0, 2\}$, from Example 1, is an extreme ray of $\boldsymbol{\Delta}_1$ and it corresponds to the cut $x_A + x_B + x_E \leq 2$. When the lifting procedure is applied to a, i.e., when $u = b$, the MDSP(u) is the following:

$$\text{Maximize} \quad \alpha_A + \alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_F + \alpha_G \tag{63}$$

subject to

$$\alpha_A + \alpha_B + \alpha_C \leq 0 \tag{64}$$
$$\alpha_A + \alpha_B + \alpha_D \leq 0 \tag{65}$$
$$\alpha_A + \alpha_C + \alpha_D \leq 1 \tag{66}$$
$$\alpha_B + \alpha_C + \alpha_D \leq 1 \tag{67}$$
$$\alpha_B + \alpha_D + \alpha_E \leq 0 \tag{68}$$
$$\alpha_A + \alpha_E \leq 0 \tag{69}$$
$$\alpha_C + \alpha_F \leq 2 \tag{70}$$
$$+\alpha_G \leq 2 \tag{71}$$
$$\alpha_A,\ \alpha_B,\ \alpha_C,\ \alpha_D,\ \alpha_E,\ \alpha_F,\ \alpha_G \geq 0. \tag{72}$$

One can observe that the problem can be simplified since variables α_A, α_B, α_C, α_D, α_E can be removed, as well as constraints (64)–(69). The optimal solution to the modified problem is $v = \{0, 0, 0, 0, 0, 2, 2, 0\}$, i.e., $\alpha_F = 2$ and $\alpha_G = 2$. One can construct the ray $w = (\bar{\boldsymbol{\alpha}}^u + \bar{\boldsymbol{\alpha}}^v, \bar{\beta}^u) = \{1, 1, 0, 0, 1, 2, 2, 2\}$. The corresponding lifted cut, $x_A + x_B + x_E + 2x_F + 2x_G \leq 2$, is stronger than the original one, and, since $n(R^w) = 5 = S^w$, w is an extreme ray of $\boldsymbol{\Delta}_1$.

Proposition 19 *When the lifting procedure is applied to a cut corresponding to an extreme ray of $\boldsymbol{\Delta}$, the resulting cut also corresponds to an extreme ray of $\boldsymbol{\Delta}$.*

Proof. If u is an extreme ray of $\boldsymbol{\Delta}$, then, from Lemma 1, $n(R^u) = |S^u|$. From that, and since there is no slack in an active constraint, it is easy to see that $|S^u \cap S^v| = 0$, i.e., the variables that are positive in u take the value 0 in v. Hence, $\alpha_s^w = \alpha_s^u, \forall\, s \in S^u$, and $|S^w| = |S^u| + |S^v|$. Since v is an extreme point of $\boldsymbol{\Delta}^L$ and there are no upper bound constraints and no variable β in the model, $n(R^v) = |S^v|$. Recall that $\sum_{s\in S} b_{rs}\bar{\alpha}_s^w = \sum_{s\in S} b_{rs}\bar{\alpha}_s^v + \sum_{s\in S} b_{rs}\bar{\alpha}_s^u$ and $\bar{\beta}^w = \bar{\beta}^u$. Then, for a given routing $r \in R$, $\sum_{s\in S} b_{rs}\bar{\alpha}_s^w - \bar{\beta}^w = 0$ if and only if $\sum_{s\in S} b_{rs}\bar{\alpha}_s{}^v = \bar{\beta}^u - \sum_{s\in S} b_{rs}\bar{\alpha}_s^u$. That is to say, a constraint (12) in the DSP is active at w if and only if the corresponding constraint (61) in the MDSP(u) is active at v. In addition, since $\alpha_s^v \geq 0$, all constraints in the DSP that are active at u are also active at w, thus, $n(R^w) = n(R^u) + n(R^v)$. From this last statement and since (i) $|S^w| = |S^u| + |S^v|$, (ii) $|S^u| = n(R^u)$, (iii) $|S^v| = n(R^v)$, then $|S^w| = n(R^u) + n(R^v) = n(R^w)$ and the point w corresponds to an extreme ray of $\boldsymbol{\Delta}$. □

Proposition 19 implies that any MIS cut and any cut generated by solving the DSP or the BDSPb can be lifted and the resulting cut necessarily corresponds to a stronger extreme ray of $\mathbf{\Delta}$, if one exists, with $\alpha_s = \alpha_s^u, \forall\, s \in S^u$. However, if the procedure is applied to a cut which is not an extreme ray of $\mathbf{\Delta}$, the resulting cut corresponds to a stronger ray of $\mathbf{\Delta}$, if one exists, but not necessarily to an extreme ray of $\mathbf{\Delta}$.

6 Computational Experiments

In this section, we present computational experiments that were performed on instances based on data provided by two major airlines. We first provide a description of these instances, followed by a summary of our computational experiments. These experiments were done to compare the different dual subproblem bounding strategies in a Benders decomposition method, and to evaluate the two proposed strengthening methods. We did not compare all families of feasibility cuts. In fact, from Proposition 18, it is clear that the Benders cuts dominate the PSP cuts (51). Proposition 14 show that all MIS cuts (47) are included in the Benders cuts and that some of the stronger Benders cuts do not correspond to MIS cuts. Furthermore, the generation of an MIS cut implies the solution of an optimization problem in addition to the regular aircraft routing problem. We think that although this family of cuts can be interesting since it leaves out the weakest of the feasibility cuts, the generation procedure is too demanding and would offset the benefits. As a matter of fact, the authors themselves did not implement the proposed cuts (see Cohn and Barnhart (2003)).

6.1 Description of data sets

The test instances come from daily fleet assignment solutions provided by the airlines. The characteristics of the different instances are summarized in Table 1. This table indicates the number of daily legs and the number of short connections (SC).

Table 1: Characteristics of test instances

	B757C	A320D	D9SA	D9SB	B767S	MD80L
Legs	184	258	523	508	510	707
SC	114	183	502	659	370	1183

The aircraft routing problem is solved to determine the sequence of flight legs to be flown by each individual aircraft so as to cover each leg exactly once while ensuring appropriate aircraft maintenance with the available number of aircraft. Recall that the ECP formulation includes all *unique and maximal maintenance-feasible short connection sets* (UM), i.e., columns containing all the information needed from the complete solutions to the aircraft routing problem. Prior to solving ECP, one thus has to solve a series of aircraft routing problems with additional constraints and a modified objective function to generate all UM columns.

The crew pairing problem is solved to determine a minimum-cost set of pairings so that every flight leg is assigned a qualified crew and each pairing follows a set of applicable work rules related to a large number of factors such as flight time, rest time, connection time, etc. One may observe that, to compare the feasibility cuts studied in this paper, the crew pairing part of the integrated problem can be modeled in any way. It is, in fact, independent of the feasibility cut procedure. For the experiments reported in this section, the feasibility constraints are modeled through the use of resources and are handled directly by dynamic programming within a column generation framework (see, *e.g.*, Desaulniers et al. (1998)). Details on the crew cost function and on the work rules considered can be found in Mercier et al. (2005).

6.2 Summary of computational experiments

The integrated aircraft routing and crew pairing problem has been solved with Benders decomposition, using alternately the BDSPa and the BDSPb, with either the basic solution process (*basic*), an auxiliary Pareto-optimal subproblem (*P-O*) or the proposed lifting procedure (*Li*). The different combinations have also been used jointly with two refinements: method *RA* removes, in a preprocessing step, all arcs in the crew networks corresponding to IISC (individually infeasible short connections) in order to eliminate the need for the zero right-hand-side cuts, and method *RC* removes, at every iteration, the linking constraints corresponding to the short connections not chosen by the current crew pairing solution in order to make the subproblem easier to solve. Method RC forces the dual variables corresponding to the removed constraints to take the value zero in an optimal solution. Hence, as for cuts generated from predefined families, the connections not currently used, at a given iteration, are left out of the cut generated with method RC. One can easily observe that the Benders cuts generated would likely be weaker, but still valid. Since the branch-and-bound methodology used for the integer crew pairing problem is heuristic (due to the size and difficulty of the problem) and the purpose of the experiments is to compare the different cut generation procedures, only the LP relaxation of the different instances have been solved. We first report results on the ECP formulation, and then on a different model with a more explicit formulation. Our algorithms were coded in C++ and all experiments were performed on a Sun UltraSPARC-II computer with a 480MHz processor.

6.2.1 ECP formulation

Tables 2 and 3 present a comparison of the number of Benders cuts and the CPU time needed to solve the LP relaxation when the arcs in the crew networks corresponding to IISC are either kept or removed (RA) in a preprocessing step. One can observe that when the IISC are kept and the lifting procedure is used, the MDSP(u) can be unbounded since some dual variables included in the objective function may not appear in any constraint. The results using the lifting procedure are thus only given when it is used jointly with method RA.

One can clearly see from Table 2 that when the IISC arcs are kept in the crew networks, the number of feasibility cuts needed is always smaller when model (a) is used. Not only the number of cuts generated is smaller, but the first ones are also stronger. The latter statement is illustrated in Figure 1 from Appendix B, where the crew costs are plotted as a function of the number of iterations. One can see from the figure that the lower bounds given by model (a) are always higher than the ones given by model (b). Instances 320D and MD80L were chosen as an illustration, but the same is true for all instances. This is due to the IISC cuts generated. Indeed, when a zero right-hand-side cut was generated at a given iteration in the tests, all IISC in the set taken by the current crew solution had a positive coefficient in the cut when model (a) was used whereas only one of them was included with model (b) (as Propositions 12 and 13 indicated). One can also see from the table that generating Pareto-optimal cuts (P-O) helps in reducing the number of iterations needed.

Table 2: Crew arcs corresponding to IISC are kept†

	B757C	A320D	D9SA	D9SB	B767S	MD80L
Model (a)						
Cuts - Basic	9	20	7	15	12	20
CPU	0.34	1.32	4.83	7.18	5.63	46.28
Cuts - P-O	**7**	**19**	**2**	**6**	**10**	**14**
CPU	0.38	1.61	4.17	5.21	5.40	28.42
Cuts - Basic RC	13	28	19	24	14	51
CPU	0.47	2.00	10.08	10.25	6.64	87.23
Cuts - P-O RC	16	48	10	10	16	66
CPU	0.66	3.25	6.88	6.92	7.88	98.98
Model (b)						
Cuts - Basic	**23**	52	101	146	**56**	216
CPU	0.67	3.04	56.03	83.14	20.86	376.51
Cuts - P-O	24	50	95	**124**	**56**	**196**
CPU	0.86	4.13	66.58	62.02	22.81	373.70
Cuts - Basic RC	24	49	93	139	61	198
CPU	0.75	3.52	51.30	59.93	20.53	395.31
Cuts - P-O RC	26	**44**	**91**	135	**56**	210
CPU	1.05	3.83	63.43	74.59	21.40	394.40

† All CPU times are in minutes.

When model (a) is chosen, Table 2 also shows that the number of cuts needed is greater when RC is used, but that it is not always true with model (b). However, Figure 1 from Appendix B shows that even in the cases where the number of iterations needed is not greater with RC, the generated cuts are generally weaker. In fact, the dual variables corresponding to the removed constraints are not always null. Keeping them thus strengthens the cuts and, except for instance D9SB, the CPU times found in the table show that it is not compensated by the solution time saved by a reduced number of constraints in the Benders primal subproblem. Since method RC did not yield better results when jointly used with method RA, the results for RC are left out of Table 3.

The comparison of Table 3 with Table 2 confirms that the number of iterations needed is reduced with RA. In addition, Table 3 and Figure 2 from Appendix B show that, with

Table 3: Crew arcs corresponding to IISC are removed (RA)†

	B757C	A320D	D9SA	D9SB	B767S	MD80L
Model (a)						
Cuts - Basic	7	17	2	29	**7**	11
CPU	0.31	1.02	3.41	8.01	4.20	17.30
Cuts - P-O	**5**	**15**	2	**9**	8	16
CPU	0.34	1.23	3.32	4.36	4.55	26.97
Cuts - Li	6	**15**	2	20	**7**	**9**
CPU	0.31	1.02	3.24	7.01	3.87	17.19
Model (b)						
Cuts - Basic	12	13	2	10	7	13
CPU	0.39	0.94	3.27	5.33	3.74	20.60
Cuts - P-O	15	**10**	2	**8**	10	14
CPU	0.59	0.93	3.29	4.17	4.92	23.27
Cuts - Li	**7**	14	2	10	**5**	**8**
CPU	0.33	0.97	3.16	4.96	3.49	13.06

† All CPU times are in minutes.

method RA, model (b) is generally better than model (a). When no zero right-hand-side cut has to be generated, extreme rays indeed yield stronger cuts than combinations of extreme rays. Table 3 also shows that the Pareto-optimal cuts, as well as the lifted cuts (Li), help in reducing the number of iterations needed when method RA is used. In fact, the Pareto-optimal cuts are generally outperformed by the proposed lifting procedure in terms of speed of generation, and, in the case of model (b), also in terms of the number of cuts generated. In addition, since the lifting procedure takes less time than solving the Pareto-optimal auxiliary problem, one can see in Figure 3 from Appendix C that even when the number of iterations with Li is larger than with P-O (instances A320B and D9SB), the lower bound given by the crew costs increases as fast with Li.

Our experiments show that the cuts from model (a) exhibit coefficients with equal values and the cuts from both model (a) and model (b) do not include all incompatible short connections when they are not part of the set taken by the current crew solution (as Propositions 6 and 8 suggested). Our computational experiments also show that the proposed lifting procedure has the ability to overcome both weaknesses. The following example shows the first cut generated with model (a) for the fleet D9SA, and the corresponding lifted cut:

$$\begin{aligned} \text{Regular cut}: \quad & x_1 + x_2 + x_3 + x_4 && \leq 3 \\ \text{Lifted cut}: \quad & x_1 + x_2 + x_3 + x_4 + 2x_5 + x_6 && \leq 3. \end{aligned}$$

One can notice that a variable has a coefficient larger than one in the lifted cut. In our computational experiments, when they had a right-hand-side larger than zero, the lifted cuts and the cuts generated from model (b) often showed coefficients with unequal values. Using a short connection can indeed prevent the use of more than one other short connection (even up to four, in our experiments).

In summary, when the ECP formulation is used, model (b) is preferred to model (a) since finding the set of IISC is an easy task in that case and the feasibility cuts generated from extreme rays of the dual subproblem polyhedron are stronger. In addition, the lifting procedure is useful to strengthen the cuts and reduce the total computing time.

It is worth noting that some tests where done with the integrality constraints. A heuristic branch-and-bound method not only influences the computing time, but also has an influence on the number of iterations of the Benders decomposition method. Nevertheless, we believe that the conclusions reached in our experiments can be extended to the integer formulation. The results for instance 757C are shown as an example in Table 4. One will notice that the number of cuts generated is higher with model (b) when RA is used. Although this seems to go against the conclusions drawn in the previous sections, one can observe that it was also the case for this particular instance (and only this one) when the LP relaxation was solved (see Table 3).

Table 4: Instance 757C - Results for the integer formulation†

	Basic (a)	*P-O (a)*	*Li (a)*	*Basic (b)*	*P-O (b)*	*Li (b)*
All arcs						
Cuts	9	**7**	-	26	27	-
CPU	4.39	**4.32**	-	6.72	5.01	-
Cost IP	50 498	50 417	-	50 377	50 397	-
RA						
Cuts	7	6	**5**	13	15	7
CPU	0.59	1.49	2.99	2.32	0.99	**0.53**
Cost IP	50 425	50 275	50 425	50 418	50 353	50 417

† All CPU times are in minutes.

6.2.2 A more explicit formulation

Finding all complete aircraft routing solutions is a difficult task (see Mercier et al. (2005)), and for the larger problems, it is more efficient to solve an integrated model which includes the aircraft routing formulation (*IMARF*) instead of complete solutions. Although IMARF is harder to solve, the increased solution time is well compensated by the time saved by not having to generate the UM columns as in ECP. This section will show that the results on the ECP formulation still hold for the IMARF formulation.

Although the aircraft maintenance constraints are modeled through the use of resources and are handled directly by dynamic programming in a column generation framework, the IMARF formulation includes additional constraints to ensure that all legs are covered by an aircraft and to limit the number of available aircraft. The Benders aircraft dual subproblem of the IMARF formulation thus contains more variables compared to the ECP formulation. The proposed lifting procedure is still valid in this case, but its efficiency is reduced since it only lifts the dual variables associated with the short connection linking constraints. We therefore did not include results for the lifting procedure with the IMARF formulation.

Table 5 presents a comparison of the number of Benders cuts and the CPU time needed to solve the LP relaxation of IMARF when the IISC arcs are kept in the crew networks, while Table 6 gives the time needed to find all IISC (*preprocessing*). As it was the case with ECP, method RC did not prove to be useful and we therefore did not include the corresponding results in the tables.

Table 5: When the IISC arcs are kept - IMARF†

	B757C	A320D	D9SA	D9SB	B767S	MD80L
Model (a)						
Cuts - Basic	22	18	18	50	14	46
CPU	1.21	2.17	16.17	76.66	17.83	215.27
Cuts - P-O	**15**	**12**	**14**	**21**	**11**	**1**
CPU	1.92	3.40	21.98	100.16	42.60	68.33
Model (b)						
Cuts - Basic	32	45	85	**61**	36	85
CPU	1.62	4.87	53.15	47.96	29.24	241.10
Cuts - P-O	**25**	**38**	**72**	**61**	**33**	**82**
CPU	2.58	5.92	69.96	80.25	61.85	390.92

† All CPU times are in minutes.

Although it is not as blatant as with the ECP formulation, one can see from Table 5 that model (a) again dominates model (b) when the IISC arcs are kept in the networks. The table also shows that the number of Pareto-optimal cuts (*P-O*) generated is smaller than the number of regular Benders cuts. One can notice that the total computing time is not necessarily reduced when Pareto-optimal cuts are generated (solving the Pareto-optimal auxiliary problem is time-consuming), but systematically reducing the number of cuts might be preferred when the integer problem is solved. Table 6 shows that the preprocessing step needed for method RA is more time consuming than the whole solution process when the arcs are kept (even for the IP). Hence, when the IMARF formulation is used, finding the set of IISC is a hard task and it is better to keep all arcs in the networks. In that case, model (a) is preferred to model (b).

Table 6: Identifying the IISC - IMARF

	B757C	A320D	D9SA	D9SB	B767S	MD80L
Preprocessing (min.)	1.90	4.91	76.91	179.53	130.67	1612.80

One can observe that, with this formulation, a procedure using the MIS feasibility cuts (47) could not generate the stronger zero right-hand-side cuts when the IISC arcs are kept in the crew networks since, from Proposition 14, all MIS cuts correspond to extreme rays of $\boldsymbol{\Delta}$.

7 Conclusion

When the integrated aircraft routing and crew pairing problem is solved either with Benders decomposition or another cut generation procedure, one has to verify at each iteration, in a subproblem, if the set of short connections used by the current crew solution is feasible for the aircraft routing problem. This paper has shown that there is no simple form for the extreme rays of the dual subproblem polyhedron. The dual variables can indeed take different values in an extreme ray. In addition, feasibility cuts generated from predefined families (forcing each coefficient to take the value one) are all included in the set of Benders feasibility cuts, and may also be dominated by the latter.

To avoid having to identify extreme rays, one can make the Benders dual subproblem bounded. To achieve this, artificial variables must be introduced in the aircraft routing primal subproblem. This paper has compared different bounding methods and pointed out that the choice of artificial variables has an impact on the dual subproblem polyhedron, and thus, on the feasibility cuts generated. It was shown that cuts generated when a unique artificial variable is used correspond to extreme rays while it is not the case for cuts generated when a different artificial variable is added in each short connection linking constraint. The latter strategy may indeed favor cuts with equal coefficients even when they are weaker. In addition, the different types of cuts do not necessarily include short connections that are not taken by the current crew pairing solution, at a given iteration, even if they are incompatible with those included. For this reason, and to allow the dual variables to take different values, a lifting procedure was proposed which can be used on all types of feasibility cuts. It was shown that when the procedure is applied on a cut corresponding to an extreme ray, the resulting cut also corresponds to an extreme ray. On test instances containing up to 700 daily legs, it was observed that the proposed lifting procedure is useful to strengthen the cuts and to reduce the total computing time.

The paper has also shown that for cuts preventing the use of a set of individually infeasible short connections for the aircraft routing problem (IISC), some linear combinations of extreme rays give stronger cuts than extreme rays. This counter-intuitive result could potentially be applied to situations where some other blocks of dual constraints are independent, or only weakly linked by the variables (IISC cuts are generated when some dual variables are only present in the non-negativity constraints). The numerical results have pointed out that the bounding method generating cuts corresponding to extreme rays is preferred only when the IISC are identified in a preprocessing step.

Appendix A

Proof of Lemma 1. Recall that a ray t of a polyhedral cone $C \subseteq R^n$ is an extreme ray if and only if there are $n-1$ linearly independent constraints that are active at t (see *e.g.* Bertsimas and Tsitsiklis (1997)). In the Benders DSP, the total number of variables is $|S|+1$. For q to be an extreme ray of $\boldsymbol{\Delta}$, the maximum number of linearly independent constraints satisfied at equality by q thus has to be equal to $|S|$. Because of constraints (13), the variables taking the value 0 in q correspond to active non-negativity constraints. Therefore, only the number of positive variables α_s, i.e., $|S^q|$, has to be considered and compared to the maximum number of linearly independent constraints (12) which are active at q, i.e., $n(R^q)$.

Proof of Lemma 2. Recall that a point p of a polyhedron P of dimension n is an extreme point if and only if there are at least n linearly independent active constraints of P at p. In the Benders BDSPa, the total number of variables is $|S|+1$ (with variable β). For p to be an extreme point of $\boldsymbol{\Delta}^{B_a}$, the maximum number of linearly independent constraints satisfied at equality by p thus has to be greater than or equal to $|S|+1$. Since a non-negativity constraint is active for every null variable α, one does not need to consider constraints (35) and variables $\alpha_s, s \in S \backslash S^p$. Therefore, the point p is an extreme point of $\boldsymbol{\Delta}^{B_a}$ if and only if $n(H^p) \geq |S^p|+1$.

Lemma 20 *If p is an extreme point of $\boldsymbol{\Delta}^{B_a}$, then $n(S^p) \leq |S^p|$, i.e., the maximum number of linearly independent active constraints (33) at p is bounded by the number of positive variables α at the point.*

Proof. There are a total of $|S^p|+1$ positive variables contributing to constraints (33). The maximum number of linearly independent active constraints (33) at p, $n(S^p)$, is thus bounded by $|S^p|+1$. Furthermore, since the variable β is present, with the same coefficient, in all constraints (33), the latter bound can be improved, and $n(S^p) \leq |S^p|$. □

Proof of Lemma 4. If $p \in P_{\boldsymbol{\Delta}^{B_a}}$, then $n(H^p) \geq |S^p|+1$ in the BDSPa (see Lemma 2). Since there cannot be more than $|S^p|$ linearly independent active constraints (33) (see Lemma 20), or more than $|S^p|$ active constraints (34), at least one constraint of each type is satisfied at equality at p, i.e., $\alpha_s = \bar{c}$ for at least one short connection $s \in S^p$, and $\beta^p = \max_{r \in R} \sum_{s \in S^p} b_{rs}\alpha_s^p$.

Proof of Lemma 5. If, for a given set F^t, $p \in \bar{P}^{F^t}_{\boldsymbol{\Delta}^{B_a}}$, then there is at least one constraint (33) which is active at p (see Lemma 4). Hence, $R^p \neq 0$ and $\sum_{s \in S} b_{rs}\alpha_s^p = \beta^p$, $\forall\, r \in R^p$. Recall that, for a given set F^t, the value of the objective function (32) of the BDSPa at the point p is $v_a(p, F^t) = \sum_{s \in F^t} \alpha_s^p - \beta^p$. Let $r_1 \in R^p$ be any active constraint (33) at p. Therefore, $\beta^p = \sum_{s \in S} b_{r_1 s}\alpha_s^p$ and $v_a(p, F^t) = \sum_{s \in F^t} \alpha_s^p - \sum_{s \in S} b_{r_1 s}\alpha_s^p$. One can observe that the variables from $S \backslash F^t$ can never positively contribute to the objective function value. If $\sum_{s \in F^t} b_{rs}\alpha_s^p < \beta^p$, $\forall\, r \in R^p$, i.e., all active constraints (33) include contributions from variables α in $S \backslash F^t$, then one can construct a point $p' = (\boldsymbol{\alpha}^{p'}, \beta^{p'})$, where (i) $\alpha_s^{p'} = \alpha_s^p$, $\forall\, s \in F^t$,

(ii) $\alpha_s^{p'} = 0, \forall\, s \in S\backslash F^t$, and (iii) $\beta_{p'} = \beta_p - \min_{r\in R^{p'}} \sum_{s\in S\backslash F^t} b_{rs}\alpha_s^p$. One can observe that p' is feasible since it satisfies all constraints (33)–(35) from the BDSPa. The objective value at p' is $v_a(p', F^t) = \sum_{s\in F^t} \alpha_s^{p'} - \beta^{p'} = \sum_{s\in F^t} \alpha_s^p - \beta_p + \min_{r\in R^{p'}} \sum_{s\in S\backslash F^t} b_{rs}\alpha_s^p = v_a(p, F^t) + \min_{r\in R^{p'}} \sum_{s\in S\backslash F^t} b_{rs}\alpha_s^p$. Since $\sum_{s\in F^t} b_{rs}\alpha_s^p < \beta^p, \forall\, r \in R^p$, $\min_{r\in R^{p'}} \sum_{s\in S\backslash F^t} b_{rs}\alpha_s^p > 0$, and thus $v_a(p', F^t) > v_a(p, F^t)$. This is impossible since $p \in \bar{P}^{F^t}_{\mathbf{\Delta}^{B_a}}$.

Proof of Lemma 7. If $\bar{c} = 1$, then $\alpha_s^{p_2} = 1, \forall\, s \in (S^p \cap F^t)$. Let $r_1 \in R^{p_1}$ be one active constraint at p_1 and $S_1^p = \{s \in S^p \,|\, b_{r_1 s} = 1\}$ be the set of short connections from S^p included in r_1 ($|R^{p_1}| \geq 1$ from Lemma 4). Let r_2 and S_2^p correspond to p_2 as r_1 and S_1^p correspond to p_1. With the active constraint r_1, one can calculate $\beta^{p_1} = \sum_{s\in S_1^p} \alpha_s^{p_1}$ and obtain $v_a(p_1, F^t) = \sum_{s\in F^t} \alpha_s^{p_1} - \beta^{p_1} = \sum_{s\in F^t} \alpha_s^{p_1} - \sum_{s\in S_1^p} \alpha_s^{p_1}$. The same is true for $v_a(p_2, F^t)$. Although $S^{p_1} = S^{p_2}$, it is easy to see that R^{p_2} can differ from R^{p_1} since there exists at least one short connection $s_i \in (S^p \cap F^t)$ with $\alpha_{s_i}^{p_1} \neq \alpha_{s_i}^{p_2}$. Only two possible situations can occur: either there is an active constraint $r_i \in R^{p_1}$ that is also active at p_2, or there is none.

Assume that $r_1 \in (R^{p_1} \cap R^{p_2})$. In this case, $\beta^{p_2} = \beta^{p_1} + \sum_{s\in(S_1^p\cap F^t)}(1 - \alpha_s^{p_1})$. Therefore, $v_a(p_2, F^t) = v_a(p_1, F^t) + \sum_{s\in F^t}(1-\alpha_s^{p_1}) - \sum_{s\in(S_1^p\cap F^t)}(1-\alpha_s^{p_1})$. Since (i) $\alpha_s^{p_1} \leq 1, \forall\, s \in S^{p_1}$, (ii) $\exists\, s_i \in F^t \,|\, \alpha_{s_i}^{p_1} < 1$ and (iii) $(S_1^p \cap F^t) \subset F^t$, then $v_a(p_2, F^t) \geq v_a(p_1, F^t)$. If, in addition, no aircraft routings $r \in R^{p_1}$ contain all short connections $s_i \in F^t$ such that $\alpha_{s_i}^{p_1} < 1$, then $\sum_{s\in F^t}(1 - \alpha_s^{p_1}) > \sum_{s\in(S_1^p\cap F^t)}(1 - \alpha_s^{p_1})$ and $v_a(p_2, F^t) > v_a(p_1, F^t)$.

Assume that $|R^{p_1} \cap R^{p_2}| = 0$. In that case, $\beta^{p_2} = \beta^{p_1} + \sum_{s\in(F^t\cap S_1^p\cap S_2^p)}(1 - \alpha_s^{p_1}) - \sum_{s\in(S_1^p\backslash S_2^p)} \alpha_s^{p_1} + \sum_{s\in(S_2^p\backslash S_1^p)} \alpha_s^{p_2}$. Therefore, $v_a(p_2, F^t) = v_a(p_1, F^t) + \sum_{s\in F^t} \alpha_s^{p_2} - \sum_{s\in F^t} \alpha_s^{p_1} - \sum_{s\in S_2^p} \alpha_s^{p_2} + \beta^{p_1}$. Since r_2 is not active at p_1, $\sum_{s\in S_2^p} \alpha_s^{p_1} < \beta^{p_1}$. Hence, $\sum_{s\in F^t} \alpha_s^{p_1} - \sum_{s\in(F^t\backslash S_2^p)} \alpha_s^{p_1} + \sum_{s\in(S_2^p\backslash F^t)} \alpha_s^{p_1} < \beta^{p_1}$. Seeing that (i) $\sum_{s\in(F^t\backslash S_2^p)} \alpha_s^{p_2} > \sum_{s\in(F^t\backslash S_2^p)} \alpha_s^{p_1}$, (ii) $\sum_{s\in(F^t\backslash S_2^p)} \alpha_s^{p_2} = \sum_{s\in F^t} \alpha_s^{p_2} - \sum_{s\in(F^t\cap S_2^p)} \alpha_s^{p_2}$, (iii) $\sum_{s\in(S_2^p\backslash F^t)} \alpha_s^{p_1} = \sum_{s\in(S_2^p\backslash F^t)} \alpha_s^{p_2}$ and (iv) $\sum_{s\in(F^t\cap S_2^p)} \alpha_s^{p_2} + \sum_{s\in(S_2^p\backslash F^t)} \alpha_s^{p_2} = \sum_{s\in S_2^p} \alpha_s^{p_2}$, then $\sum_{s\in F^t} \alpha_s^{p_1} - \sum_{s\in F^t} \alpha_s^{p_2} + \sum_{s\in S_2^p} \alpha_s^{p_2} < \beta^{p_1}$ and $v_a(p_2, F^t) > v_a(p_1, F^t)$.

Proof of Lemma 9. If $p \in P_{\mathbf{\Delta}^{B_b}}$, then $n(H^p) \geq |S^p| + 1$ in the BDSPb (see Lemma 2). From Lemma 20, there cannot be more than $|S^p|$ linearly independent active constraints (41). From that, and since there is only one constraint (42), the latter constraint is active and there are exactly $|S^p|$ linearly independent active constraints (41) at p in the BDSPb.

Proof of Lemma 10. If $q \in R_{\mathbf{\Delta}}$, then $n(R^q) = |S^q|$ in the DSP. By definition, $\sum_{s\in S^q} b_{rs}\alpha_s^q = \beta^q, \forall\, r \in R^q$. Since $S^p = S^q$, $\sum_{s\in S^p} b_{rs}\alpha_s^p = \sum_{s\in S^q} b_{rs}(\alpha_s^q / \sum_{s\in S^q} \alpha_s^q) = \beta^q / \sum_{s\in S^q} \alpha_s^q = \beta^p, \forall\, r \in R^q$. Therefore, $R^p = R^q$ and $n(R^p) = |S^p|$ in the BDSPb. In addition, constraint (42) is satisfied as an equality since $\sum_{s\in S} \alpha_s^p = \sum_{s\in S^q}(\alpha_s^q / \sum_{s\in S^q} \alpha_s^q) = 1$. Hence, $n(H^p) = |S^p| + 1$ in the BDSPb and, from Lemma 20, $p \in P_{\mathbf{\Delta}^{B_b}}$.

Proof of Lemma 17. PSP cuts correspond to a point in $\mathbf{\Delta}^{B_a}$, where $\bar{\alpha}_s = 1, \forall\, s \in F^t$, and $\bar{\beta} = |S(\bar{r}^t) \cap F^t|$. When $\bar{c} = 1$, the point is feasible and there are exactly $|F^t|$ active constraints (34) at $(\bar{\boldsymbol{\alpha}}, \bar{\beta})$. In addition, at least one constraint (33) is also active at that point since, from the definition of β, there exists at least one routing using exactly β short connections. PSP cuts thus correspond to an extreme point of $\boldsymbol{\Delta}^{B_a}$ since $n(H^p) \geq |S^p|+1$ (see Lemma 2). The objective function (32) chooses a solution that maximizes the difference between the number of positive variables from F^t in the solution and the value of β. One cannot increase the number of positive variables from F^t in the PSP cuts since they are all already included, and a decrease in the number of variables from F^t in the cut would result in a smaller or equal decrease in the value of β. PSP cuts thus correspond to an optimal extreme point of $\boldsymbol{\Delta}^{B_a}$.

Appendix B

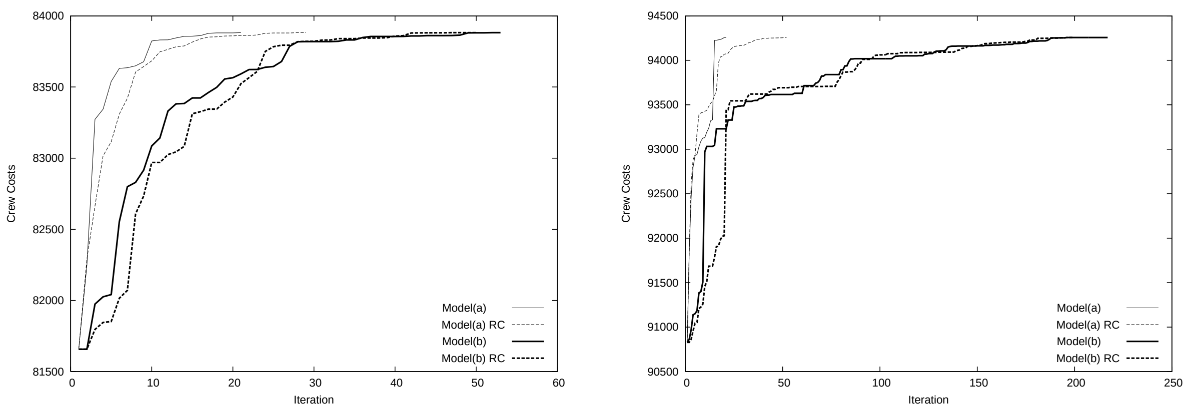

Figure 1: Crew costs as a function of the number of iteration – ECP (320D and MD80LA)

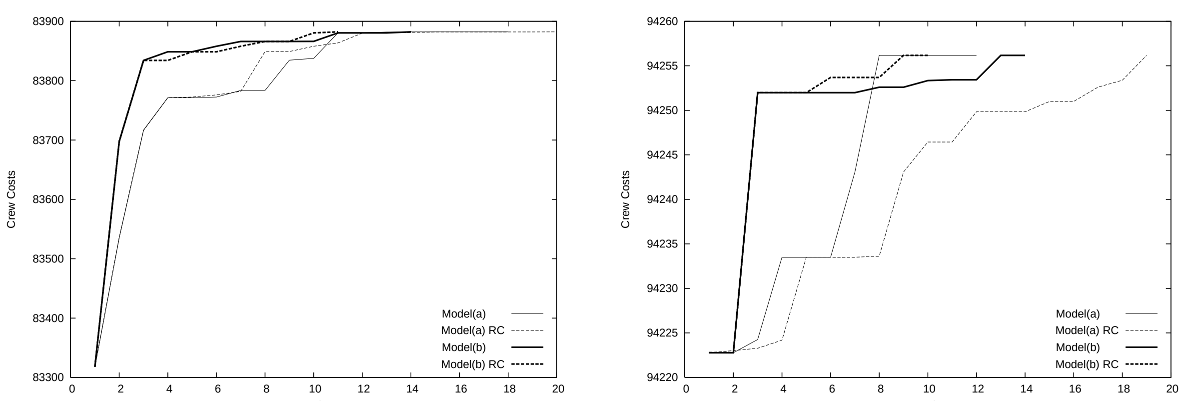

Figure 2: Crew costs as a function of the number of iteration – ECP RA (320D and MD80LA)

Appendix C

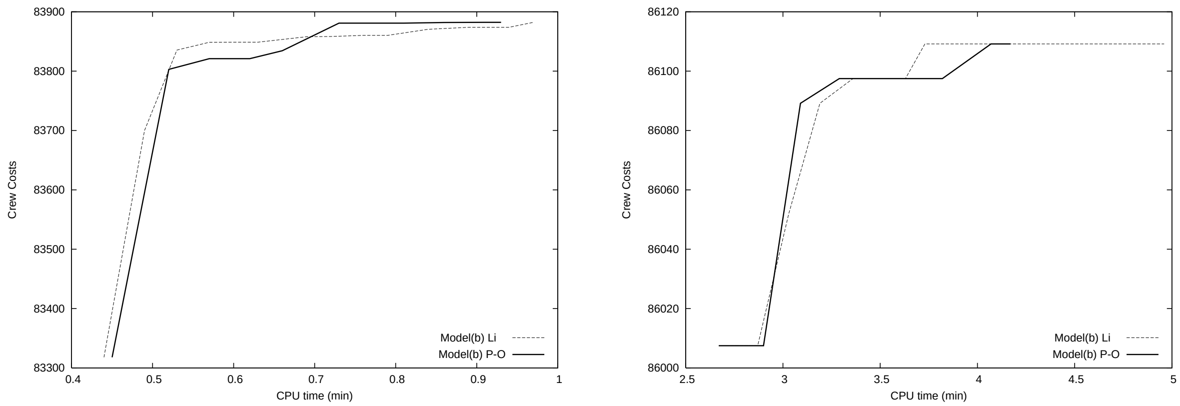

Figure 3: Crew costs as a function of CPU time – ECP RA (320D and D9SB)

References

J.F. Benders. “Partitioning Procedures for Solving Mixed-Variables Programming Problems.” *Numerische Mathematik*, **4**:238–252 (1962).

D. Bertsimas and J.N. Tsitsiklis. *Introduction to Linear Optimization*. Athena Scientific, Nashua, U.S.A., 1997.

A.M. Cohn and C. Barnhart. “Improving Crew Scheduling by Incorporating Key Maintenance Routing Decisions.” *Operations Research*, **51**:387–396 (2003).

J.-F. Cordeau, F. Pasin and M.M Salomon. “An Integrated Model for Logistics Network Design.” *Annals of Operations Research*, **144**:59–82 (2006).

J.-F. Cordeau, F. Soumis and J. Desrosiers. “Simultaneous Assignment of Locomotives and Cars to Passenger Trains.” *Operations Research*, **49**:531–548 (2001a).

J.-F. Cordeau, G. Stojković, F. Soumis and J. Desrosiers. “Benders Decomposition for Simultaneous Aircraft Routing and Crew Scheduling.” *Transportation Science*, **35**:375–388 (2001b).

G. Desaulniers, J. Desrosiers, I. Ioachim, M.M. Solomon, F. Soumis and D. Villeneuve. “A Unified Framework for Deterministic Time Constrained Vehicle Routing and Crew Scheduling Problems.” In T.G. Crainic and G. Laporte, editors, *Fleet Management and Logistics*, pages 57–93. Kluwer, Boston, 1998.

H.-C. Huang, Z. Liang and R. Li. “Solving Integrated Aircraft Routing and Crew Scheduling Problem by Benders Decomposition.” Atlanta. INFORMS Annual Meeting (2003).

D. Klabjan, E.L. Johnson and G.L. Nemhauser. “Airline Crew Scheduling with Time Windows and Plane Count Constraints.” *Transportation Science*, **36**:337–348 (2002).

T.L. Magnanti and R.T. Wong. “Accelerating Benders Decomposition: Algorithmic Enhancement and Model Selection Criteria.” *Operations Research*, **29**:464–484 (1981).

A. Mercier. *Méthodes de décomposition pour la planification intégrée des rotations d'équipages et des itinéraires d'avions*. Ph.D. dissertation, École Polytechnique de Montréal, Canada, 2006.

A. Mercier, J.-F. Cordeau and F. Soumis. “A computational Study of Benders Decomposition for the Integrated Aircraft Routing and Crew Scheduling Problem.” *Computers & Operations Research*, **32**:1451–1476 (2005).

G. L. Nemhauser and L. A. Wolsey. *Integer and Combinatorial Optimization*. Wiley, New York, 1988.

R.V. Rasmussen. *Hybrid IP/CP Methods for Solving Sports Scheduling problems*. Ph.D. dissertation, University of Aarhus, 2006.

T. Santoso, S. Ahmed, Goetschalckx and A. Shapiro. “A stochastic programming approach for supply chain network design under uncertainty.” *European Journal of Operational Research*, **167**:96–115 (2005).

G. Yu, editor. *Operations Research in the Airline Industry*. Kluwer, Boston, 1998.