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Embedded in Bonds**

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# Pricing Call and Put Options Embedded in Bonds

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### **Abstract**

Call and put options embedded in bonds are of American-style, and cannot be priced in a closed-form. In this paper, we formulate the problem of pricing these options in a stochastic Dynamic Programming (DP) framework. We let the short-term risk-free interest rate move as in Vasicek (1977). We approximate the bond value by a piecewise linear interpolation at each step of the DP procedure, and solve the DP equation in closed-form. Then, we use the DP formulation to establish the basic properties of bonds, price their embedded call and put options, and determine their optimal exercise strategies. Numerical investigations show stability, consistency, and efficiency.

### **Résumé**

Les options de rachat et de remboursement sont de type américain et, de ce fait, ne peuvent être évaluées sous forme fermée. On formule cette question d'évaluation comme un problème de programmation dynamique. On suppose que le sous-jacent, ici le taux d'intérêt instantané, évolue selon la dynamique de Vasicek (1977). À chaque étape de la procédure de programmation dynamique, on estime la valeur de l'obligation par une interpolation linéaire par morceaux et on solutionne l'équation de Bellman sous forme fermée. Cette méthodologie permet de valoriser les obligations avec clauses optionnelles implicites et d'identifier leurs stratégies optimales d'exercice. On établit aussi les propriétés théoriques des titres rachetables et remboursables. Une investigation numérique montre que cette procédure est stable, consistante et efficace.

## 1 Introduction

A *bond* is a contract which pays to its holder a known amount, called the *face value*, at a known future date, called the *maturity*. A bond may also pay periodically to its holder fixed cash dividends called *coupons*. Otherwise, it is called a *zero-coupon bond*. A bond can be interpreted as a loan with a principal equal to the face value and interest payments equal to the coupons (if any). The borrower is the *issuer* of the bond and the lender, i.e., the holder of the bond, is the *investor*. For a description of fixed income securities in general and of bonds in particular, we refer to Fabozzi (1997).

Several bonds contain one or several options coming in various flavors. Firstly, the issuer of the bond may have the right to purchase back its debt for a known amount, called the *call price*, during a specified period within the bond's life. This is the *call option*. Several government bonds contain a call feature [see Bliss and Romn (1995) for the history of callable U.S. Treasury bonds from 1917]. Secondly, the investor may have the right to return the bond to the issuer for a known amount, called the *put price*, during a specified period within the bond's life. This is the *put option*. See for example Brennan and Schwartz (1977). In this way, a *savings bond* can be redeemed at any time before its maturity and a *retractable bond* can be redeemed only at a specified date before its maturity. Similarly, the maturity of an *extensible bond* can be extended to a longer period [Ananthanarayanan and Schwartz (1980) and Longstaff (1990)].

An investor may also have the right to exchange the bond for a given number of an underlying asset during a specified period within the bond's life. In general, this asset is the stock of the bond issuer (a firm in this case). This is called a *conversion option* [Brennan and Schwartz (1980)]. Corporate bonds often contain several embedded options, e.g., the Liquid Yield Option Note (LYON), a product developed by Merrill Lynch Capital Markets in 1985, is a zero-coupon bond that is callable, puttable, and convertible.

All the options described above are an integral part of a bond, and cannot be traded alone as is the case for call and put options on stocks (for example). They are said to be *embedded* in the bond. In general, they are of the American-type, so that the bond with its embedded options can be interpreted as an American-style financial derivative with (possibly) a protection period against early exercising.

Other options embedded in bonds do exist. We give here some examples from the Future contract on long-maturity U.S. Treasury bonds traded on the Chicago Board of Trade (CBOT). This Future contract is an agreement to sell or to buy some U.S. Treasury bonds with a total face value of \$100,000 at a certain date in the future, called the *delivery date*, for a certain price, called the *delivery price*. This contract contains several embedded options discussed by Boyle (1989) and Cohen (1991) among others. The first, called the *timing option*, gives the seller the right to deliver the underlying Treasury bonds at any time during a specified delivery month. The second, called the *quality option*, gives the

seller the right to deliver any U.S. Treasury bond with at least 15 years to the earliest call date or to maturity at the first delivery date. The third, called the *wild card option*, comes from the time difference between the closing hours of the cash and the Future markets. Thus, from 2:00 PM (Chicago time), when the CBOT closes, the seller has until 8:00 PM to decide whether to deliver or not. Bond markets remain open until 4:00 PM.

There are no analytical formulas for valuing American options, even under very simplified assumptions. Numerical methods, essentially trees and finite-differences (FD), are usually used for pricing using a backward induction framework. Recall that trees are particular discrete-time models and FD are numerical solution methods for Partial Differential Equations (PDE). As an alternative approach, the pricing of Bermudan American financial derivatives can be formulated as a Markov Decision process, i.e., a stochastic *Dynamic Programming* (DP) problem as pointed out by Barraquand and Martineau (1995). Here, the DP function, i.e., the value of the bond with its embedded options, is a function of the current time and of the current interest rate, namely the *state variables*. The set of all the (possible) realizations of the state variables defines the *state space*. This value function verifies a DP recurrence (known as the DP or the *Bellman* equation) via the *risk-neutral* principle of asset pricing. Indeed, the DP equation relates the holding value of the bond at the current time as an expectation, under the so-called *risk-neutral* probability measure, of its future value discounted at the risk-free interest rate. The key point with DP is to solve efficiently the DP equation which yields both the bond value and the optimal exercise strategies of its embedded options. For an overview of stochastic DP, we refer to Bertsekas (1987), and for risk-neutral evaluation, we refer to Karatzas and Shreve (1998).

The problem of pricing options embedded in bonds is related to the term structure of interest rates. In this context, the short term risk-free interest rate is very often used as a Markov process in arbitrage-free markets, but zero-coupon bonds and forward rates are also used. Several approaches are suggested in the literature depending on the nature of the underlying asset(s), the dimension of the state space, and the frequency with which the state variables are observed. For example, a one-factor model may be used for the interest rate, a two-factor model may be used for the interest rate and its random volatility, or for the interest rate and a risky asset (say a stock), and so on. The underlying asset(s) may be observed discretely or continuously.

For discrete-time models, the discrete interest rate is the most widely used, through a binomial tree in the spirit of Cox, Ross, and Rubinstein (1979). These include models by Ho and Lee (1986), Black, Derman, and Toy (1990), and Kalotay, Williams, and Fabozzi (1993). All these models are calibrated to exactly mimic the initial yield curve.

For continuous-time models, the short-term risk-free interest rate is very often modeled as a diffusion process. As pointed out by Chan et al. (1992), most of the alternative dynamics for the interest rate are described by the general stochastic differential equation (SDE)

$$dr_t = (\alpha + \beta r_t) dt + \sigma r_t^\gamma dB_t, \quad \text{for } 0 \leq t \leq T, \quad (1)$$

where  $\alpha$ ,  $\beta$ ,  $\sigma$ , and  $\gamma$  are real parameters and  $B_t$ , for  $0 \leq t \leq T$ , is a standard Brownian motion. Table 1 presents various versions of equation (1) used in the literature.

Table 1: Models for the Short-Term Interest Rate

| Model                        | $\alpha$ | $\beta$ | $\sigma$ | $\gamma$ |
|------------------------------|----------|---------|----------|----------|
| 1. Merton (1973)             |          | 0       |          | 0        |
| 2. Vasicek (1977)            |          |         |          | 0        |
| 3. Brennan-Schwartz (1977)   | 0        | 0       |          | 1        |
| 4. Brennan-Schwartz (1980)   |          |         |          | 1        |
| 5. Marsh-Rosenfeld (1983)    | 0        |         |          |          |
| 6. Cox-Ingersoll-Ross (1985) |          |         |          | 1/2      |

The model by Merton (1973) is simply a standard Brownian motion with a drift. Vasicek (1977) uses a mean-reverting Ornstein-Uhlenbeck process. This model gives nice distributional results and ensures explicit formulas for zero-coupon bonds and for several European-style interest rate derivatives, but it has the undesirable property of allowing negative interest rates (though with very low probabilities). Several authors take advantage of its properties to price various interest rate derivatives (often in closed-form), e.g., Jamshidian (1989) and Rabinovitch (1989). Brennan and Schwartz (1977, 1980) are pioneers on the modeling of options embedded in bonds. They use the PDE approach and FD algorithms. They let the interest rate move as a geometric Brownian motion (GBM) without a drift to price the call and the put options (Model 3) and as a mean-reverting proportional process to price the conversion option (Model 4). Model 3 has also been used by Dothan (1978) to price bonds in closed-form, and Model 4 by Courtadon (1982) to price several European as well as American options on bonds. Notice that Model 3 is a special case of Model 4 and that the latter includes the GBM process of Black and Scholes (1973). Marsh and Rosenfeld (1983) use the constant elasticity of variance (CEV) process (among others), introduced previously for options pricing [see Cox (1996)]. Cox, Ingersoll, and Ross (CIR) (1985) use the mean-reverting square-root process (MRSR) to handle the interest rate movements. Their basic one-factor formulation is based on an equilibrium concept, known to preclude against arbitrage opportunities. It is extendible to several factors, ensures strictly positive interest rates, and gives explicit formulas for zero-coupon bonds and for some European-style interest rate derivatives. Other authors use the MRSR equation to price various interest rate derivatives, e.g., Richard (1978), Ananthanarayanan and Schwartz (1980), and Schaefer and Schwartz (1984). Other dynamics for interest rates can also be found, e.g., Brennan and Schwartz (1979) and Clewlow and Strickland (1997).

The underlying assets may also be zero-coupon bonds, e.g., Briys, Crouhy, and Schöbel (1991), or forward rates, e.g., Heath, Jarrow, and Morton (HJM) (1992). The HJM model is one of the most flexible as it integrates several systematic sources of risk and random coefficients.

Continuous-time models are parsimonious, so that matching all theoretical bond values with their market counterparts gives much more equations than the number of parameters to estimate. A remedy, proposed by Hull and White (1990a), is to augment the model, i.e., add parameters until a calibration becomes possible. This leads to the extended Vasicek model and to the extended CIR model. See the note by Carverhill (1995) and the response by Hull and White (1995) for a discussion about the performance of their “extended” models. Hull and White (1990b, 1993, 1994a, 1994b, 1996) interpret the FD method as a trinomial tree and price several interest rate derivatives within their “extended” models.

In this paper, we formulate the problem of pricing options embedded in bonds as a stochastic DP model, the focus being on the solution of the DP equation. Precisely, we consider call and put options. It is well known that the call option tends to decrease the price of the bond, while the put option has the opposite effect. It is also well known that the call option is more likely exercised by the bond issuer when the interest rates are low, while the put option is more likely exercised by the investor when the interest rates are high. We use the DP formulation to establish these basic intuitions, to evaluate the impact of these options on the bond price, and to determine their optimal exercise strategies.

We adopt the Vasicek (1977) model for the short-term risk-free interest rate. By the *finite-elements* technique, we approximate the DP function by a piecewise linear interpolation, and, subsequently, we solve the DP equation in closed-form. The PDE approach and FD algorithms could but are not used in this context [Wilmott, Dewynne, and Howison (1997)]. Indeed, the DP formulation and the finite elements technique is a viable alternative to the PDE approach and FD algorithms if the joint distribution of the state variables is known explicitly and the number of exercise opportunities is limited [see Chapter 2].

In Section 2, we present the Vasicek formulation and its distributional properties. In Section 3, we give the DP formulation. In Sections 4 and 5, we derive some theoretical properties of the bond value and solve the DP equation. In Section 6, we give some results. In Section 7, we conclude.

## 2 The Model

Vasicek (1977) introduced a continuous-time, finite horizon, and frictionless market for the short-term risk-free interest rate in which *risk-neutral* evaluation is possible. We present this model directly under the risk-neutral probability measure, denoted by  $Q$ , whose existence is guaranteed by the no-arbitrage property.

The risk-free interest rate moves under  $Q$  according to the SDE

$$dr_t = \kappa(\bar{r} - r_t) dt + \sigma dB(t), \quad \text{for } 0 \leq t \leq T, \quad (2)$$

where  $\kappa$ ,  $\bar{r}$ , and  $\sigma$  are real positive constants and  $B(t)$ , for  $0 \leq t \leq T$ , is a standard Brownian motion whose augmented natural filtration is denoted by  $\mathcal{F}(t)$ , for  $0 \leq t \leq T$ . Over time, the interest rate process is pushed towards its *reverting level*  $\bar{r}$  at the *reverting rate*  $\kappa$ , and these random reverting cycles are more or less amplified depending on the *volatility* parameter  $\sigma$ . We show later that the standard error of the future interest rate depends on  $\sigma$ .

In this model, a financial derivative can be priced as an expectation under  $Q$  of its future payoff discounted at the risk-free rate. This is the fundamental risk-neutral principle of asset pricing. Of course, bonds may be priced in that way. For example, for  $0 \leq t' \leq t \leq t'' \leq T$ , the rational price at  $t'$  of an optionless zero-coupon bond paying 1 at  $T$  is

$$\begin{aligned} v_{t'}(r) &= E \left[ e^{-\int_{t'}^{t''} r_t dt} v_{t''}(r_{t''}) \mid \mathcal{F}(t'), r_{t'} = r \right] \\ &= E \left[ e^{-\int_{t'}^{t''} r_t dt} v_{t''}(r_{t''}) \mid r_{t'} = r \right], \end{aligned} \quad (3)$$

where  $v_{t''}$  is the value of the bond at  $t''$ . Equation (3) at  $t' = 0$  and  $t'' = T$  gives

$$v_0(r) = E \left[ e^{-\int_0^T r_t dt} \right],$$

where  $r$  is the interest rate at time 0.

In the Vasicek (1977) model, it is well known that the random variables  $r_{t''}$  and  $\int_{t'}^{t''} r_u du$  are normal [see Elliott and Kopp (1999) for example]. In the following, we extend this result to the random vector  $\left( r_{t''}, \int_{t'}^{t''} r_u du \right)'$ .

**Lemma 1** For  $f$  and  $g$  two real functions continuously differentiable in  $[t', t'']$ , for  $0 \leq t' \leq t'' \leq T$ , and  $W(t)$ , for  $t \in [0, T]$ , a standard Brownian motion, one has

$$\begin{aligned} & \int_{t'}^{t''} \left( \int_{t'}^u f(t) g(u) dW(t) \right) du \\ &= \int_{t'}^{t''} \left( \int_{t'}^{t''} f(t) g(u) I(t \in [t', u]) dW(t) \right) du \\ &= \int_{t'}^{t''} \left( \int_{t'}^{t''} f(t) g(u) I(u \in [t, t'']) du \right) dW(t) \\ &= \int_{t'}^{t''} \left( \int_t^{t''} f(t) g(u) du \right) dW(t). \end{aligned}$$



**Proof.** We use the integration by parts theorem in stochastic calculus [Øksendal (1995)]. We define the function  $h(t) = f(t) \int_t^{t''} g(u) du$  and transform the right hand integral as

$$\begin{aligned}
\int_{t'}^{t''} \left( \int_t^{t''} f(t) g(u) du \right) dW(t) &= \int_{t'}^{t''} h(t) dW(t) \\
&= h(t'') W(t'') - h(t') W(t') - \\
&\quad \int_{t'}^{t''} \frac{\partial h}{\partial t}(t) W(t) dt \\
&= - \int_{t'}^{t''} f(t') g(u) W(t') du \\
&\quad + \int_{t'}^{t''} f(t) g(t) W(t) dt \\
&\quad - \int_{t'}^{t''} \left( \int_t^{t''} \frac{\partial f}{\partial t}(t) g(u) W(t) du \right) dt.
\end{aligned}$$

We now use the same theorem to transform  $\int_{t'}^u f(t) g(u) dW(t)$  and thereafter the left hand integral as

$$\begin{aligned}
\int_{t'}^{t''} \left( \int_{t'}^u f(t) g(u) dW(t) \right) du &= \int_{t'}^{t''} f(u) g(u) W(u) du - \\
&\quad \int_{t'}^{t''} f(t') g(u) W(t') du - \\
&\quad \int_{t'}^{t''} \left( \int_{t'}^u \frac{\partial f}{\partial t}(t) g(u) W(t) dt \right) du.
\end{aligned}$$

The final result comes from the basic properties of multi-dimensional real integrals. ■

**Proposition 2** *Conditioning on the information available at time  $t'$ , that is, for  $r_{t'} = r$ , the random vector*

$$\left( r_{t''}, \int_{t'}^{t''} r_u du \right)', \quad 0 \leq t' \leq t'' \leq T,$$

*is a normal vector with mean*

$$\mu(r) = \begin{bmatrix} \mu_1(r) = \bar{r} + e^{-\kappa \Delta t} (r - \bar{r}) \\ \mu_2(r) = \bar{r} \Delta t + \frac{1 - e^{-\kappa \Delta t}}{\kappa} (r - \bar{r}) \end{bmatrix}$$

and variance

$$\Sigma = \begin{bmatrix} \sigma_1^2 = \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa\Delta t}) & \sigma_{12} = \frac{\sigma^2}{2\kappa^2} (1 - 2e^{-\kappa\Delta t} + e^{-2\kappa\Delta t}) \\ \sigma_{21} = \sigma_{12} & \sigma_2^2 = \frac{\sigma^2}{2\kappa^3} (-3 + 2\kappa\Delta t + 4e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}) \end{bmatrix},$$

where  $\Delta t = t'' - t'$ . It admits the following decomposition

$$\begin{cases} r_{t''} = \mu_1(r) + \sigma_1 Z_1 \\ \int_{t'}^{t''} r_u du = \mu_2(r) + \sigma_2 [\rho Z_1 + \sqrt{1 - \rho^2} Z_2] \end{cases},$$

where  $(Z_1, Z_2)'$  is a standard normal vector and  $\rho = \sigma_{12} / (\sigma_1 \sigma_2)$  is the (conditional) coefficient of correlation between  $r_{t''}$  and  $\int_{t'}^{t''} r_u du$ .

**Proof.** From equation (2), one can apply Ito's lemma to the process  $\phi(t, r_t) = e^{\kappa t} r_t$ , for  $t \in [0, T]$ , and show that

$$r_u = \bar{r} + e^{-\kappa(u-t')} (r - \bar{r}) + \sigma \int_{t'}^u e^{-\kappa(u-t)} dB(t),$$

and consequently that

$$\begin{aligned} \int_{t'}^{t''} r_u du &= \bar{r}\Delta t + \frac{1 - e^{-\kappa\Delta t}}{\kappa} (r - \bar{r}) + \sigma \int_{t'}^{t''} \left( \int_{t'}^u e^{-\kappa(u-t)} dB(t) \right) du \\ &= \bar{r}\Delta t + \frac{1 - e^{-\kappa\Delta t}}{\kappa} (r - \bar{r}) + \sigma \int_{t'}^{t''} \left( \int_t^{t''} e^{-\kappa(u-t)} du \right) dB(t). \end{aligned}$$

The last equality comes from the Lemma. Conditioning on the information available at time  $t' \in [0, T]$ , we can decompose each component of the vector  $(r_{t''}, \int_{t'}^{t''} r_u du)'$  into a deterministic part and a random part. The latter part turns out to be a limit of linear combinations of the same standard Brownian motion taken at different points in time. We conclude that the random variables  $r_{t''}$  and  $\int_{t'}^{t''} r_u du$ , conditioned on  $F(t')$ , are jointly normal.

Now, from basic properties of stochastic integrals [Øksendal (1995)], one can derive the conditional mean and the conditional variance of the vector  $(r_{t''}, \int_{t'}^{t''} r_u du)'$ . Its conditional mean is

$$\begin{aligned} &E \left[ \left( r_{t''}, \int_{t'}^{t''} r_u du \right)' \mid F(t') \right] \\ &= \left( \bar{r} + e^{-\kappa\Delta t} (r - \bar{r}), \bar{r}\Delta t + \frac{1 - e^{-\kappa\Delta t}}{\kappa} (r - \bar{r}) \right)', \end{aligned}$$

since the centered random vector

$$\left( \int_{t'}^{t''} e^{-\kappa(u-t)} dB(t), \int_{t'}^{t''} \left( \int_t^{t''} e^{-\kappa(u-t)} du \right) dB(t) \right)',$$

is independent of  $F(t')$ . The conditional variance of  $r_{t''}$  is

$$\begin{aligned} \text{Var} [r_{t''} | F(t')] &= E \left[ \left( \int_{t'}^{t''} \sigma e^{-\kappa(t''-t)} dB(t) \right)^2 | F(t') \right] \\ &= \sigma^2 \int_{t'}^{t''} e^{-2\kappa(t''-t)} dt \\ &= \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa\Delta t}). \end{aligned}$$

The conditional variance of  $\int_{t'}^{t''} r_u du$  is

$$\begin{aligned} &\text{Var} \left[ \int_{t'}^{t''} r_u du | F(t') \right] \\ &= E \left[ \left( \int_{t'}^{t''} \left( \int_t^{t''} \sigma e^{-\kappa(u-t)} du \right) dB(t) \right)^2 | F(t') \right] \\ &= \sigma^2 \int_{t'}^{t''} \left( \int_t^{t''} e^{-\kappa(u-t)} du \right)^2 dt \\ &= \frac{\sigma^2}{2\kappa^3} (-3 + 2\kappa\Delta t + 4e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}). \end{aligned}$$

The conditional covariance between  $r_{t''}$  and  $\int_{t'}^{t''} r_u du$  is

$$\begin{aligned} &\text{Cov} \left[ r_{t''}, \int_{t'}^{t''} r_u du | F(t') \right] \\ &= E \left[ \int_{t'}^{t''} \sigma e^{-\kappa(t''-t)} dB(t) \int_{t'}^{t''} \left( \int_t^{t''} \sigma e^{-\kappa(u-t)} du \right) dB(t) | F(t') \right] \\ &= \sigma^2 \int_{t'}^{t''} e^{-\kappa(t''-t)} \left( \int_t^{t''} e^{-\kappa(u-t)} du \right) dt \\ &= \frac{\sigma^2}{2\kappa^2} (1 - 2e^{-\kappa\Delta t} + e^{-2\kappa\Delta t}). \end{aligned}$$

■

### 3 The DP Formulation

In this section, we present the DP function and the DP equation for a zero-coupon bond with its embedded call and put options. We essentially use the risk-neutral principle of asset pricing to assess the DP formulation.

Let  $t_0, \dots, t_M$  be a sequence of dates such that  $\Delta t = t_{m+1} - t_m$ , for  $m = 0, \dots, M - 1$ , where  $t_0 = 0$  is the origin and  $t_M = T$  is the maturity of the bond. We assume that the exercise opportunities of the embedded options are at  $t_m$ , for  $m = 1, \dots, M$ . In practice, the first increment of time  $t_1 - t_0$  and the last one  $t_M - t_{M-1}$  may be different from  $\Delta t$ .

Let  $c_m$  and  $p_m$  be the call and the put prices at  $t_m$ , respectively. If the issuer calls back the bond at  $t_m$ , he pays a known amount  $c_m$  to the investor, and, at the same date, if the investor puts the bond, he receives a known amount  $p_m$  from the issuer. Assume that the call and put prices verify  $0 < p_m \leq c_m$ , as is usual in practice, and  $c_M = p_M = 1$ . Finally, let  $v_m(r)$  be the value of the bond and  $v_m^h(r)$  its holding value at time  $t_m$ , where  $r$  is the interest rate at that time.

At the maturity date  $t_M = T$ , the value of the bond is

$$v_M(r) = 1, \quad \text{for all } r, \quad (4)$$

where 1 is the face value of the bond with its embedded options. Table 2 gives the “payoff” at time  $t_m$  by the issuer to the investor under decision pairs.

Table 2: The bond payoff under decision pairs

|             | Investor |       |
|-------------|----------|-------|
| Issuer      | Put      | Hold  |
| Call        | $p_m$    | $c_m$ |
| Not to Call | $p_m$    | —     |

From Table 2, we can specify the optimal strategies of the two agents at each monitoring date. The issuer has a dominating strategy; he will call the bond if  $v_m^h(r) > c_m$ , otherwise, he is better not to call. In the first case, the investor will put the bond if  $p_m > c_m$  (a possibility that we exclude). In the second case, the investor will put the bond if  $v_m^h(r) < p_m$ . Therefore, for  $m = 1, \dots, M - 1$ , one has

$$v_m(r) = \begin{cases} c_m & \text{if } v_m^h(r) > c_m \\ v_m^h(r) & \text{if } p_m \leq v_m^h(r) \leq c_m \\ p_m & \text{if } v_m^h(r) < p_m \end{cases} . \quad (5)$$

By the risk-neutral principle of asset pricing, we obtain the holding value of the bond  $v_m^h(r)$  at time  $t_m$  from its value  $v_{m+1}$  at time  $t_{m+1}$  by

$$\begin{aligned} v_m^h(r) &= E \left[ e^{-\int_{t_m}^{t_{m+1}} r_t dt} v_{m+1}(r_{t_{m+1}}) \mid \mathcal{F}(t_m), r_{t_m} = r \right] \\ &= E_{m,r} \left[ e^{-\int_{t_m}^{t_{m+1}} r_t dt} v_{m+1}(r_{t_{m+1}}) \right]. \end{aligned} \quad (6)$$

At the origin, the bond value is a function of the observed interest rate

$$v_0(r) = v_0^h(r), \quad \text{for all } r. \quad (7)$$

Equations (4-7) define the stochastic DP formulation. In particular, equations (4), (5), and (7) define respectively the DP function at maturity, at a given step of the DP procedure, and at the origin. Solving the expectation in (6) backwards from the maturity to the origin yields both the initial value of the bond and the optimal exercise strategies of its embedded options. We return to this point later.

## 4 Properties of the Bond's Value

In this section, we study the shape of the value function for a zero-coupon bond with its embedded options. We essentially use the properties of real integrals.

**Proposition 3** *For  $m = 0, \dots, M-1$ , the holding value of the bond,  $v_m^h(r)$ , is a strictly positive, continuous, and strictly decreasing function of  $r$ . The value function,  $v_m(r)$ , verifies*

$$v_m(r) = \begin{cases} c_m, & \text{if } r \leq a_m \\ v_m^h(r), & \text{if } a_m < r < b_m \\ p_m, & \text{if } r \geq b_m \end{cases},$$

where  $a_m$  and  $b_m$  are two thresholds associated with time  $t_m$ .

**Proof.** The proof proceeds by induction on  $m$ . For  $m = M-1$ , equation (6) gives

$$\begin{aligned} v_{M-1}^h(r) &= E_{M-1,r} \left[ e^{-\int_{t_{M-1}}^T r_t dt} \right] \\ &= e^{-\mu_2(r) + \sigma_2^2/2}, \end{aligned}$$

because by conditioning on the information available at time  $t_{M-1}$ , the random variable  $e^{-\int_{t_{M-1}}^T r_t dt}$  is lognormal with parameters  $\mu_2(r)$  and  $\sigma_2$  [see Proposition 6]. Clearly, the holding value of the bond  $v_{M-1}^h(r)$  is a strictly positive, continuous, and strictly decreasing function of  $r$ , since  $\mu_2(r)$  is a continuous and strictly increasing function of  $r$ . The

thresholds  $a_{M-1}$  and  $b_{M-1}$  at step  $M-1$  do exist since  $v_{M-1}^h(r) \rightarrow +\infty$  when  $r \rightarrow -\infty$  and  $v_{M-1}^h(r) \rightarrow 0$  when  $r \rightarrow +\infty$  [see equation (5)].

Now, we show that the properties under interest are verified at step  $m$  once they are verified at step  $m+1$ . From equation (6), one has

$$\begin{aligned} & v_m^h(r) \\ &= E_{m,r} \left[ e^{-\int_{t_m}^{t_{m+1}} r_t dt} v_{m+1}(r_{t_{m+1}}) \right] \\ &= \iint_{\mathbb{R}^2} e^{-\mu_2(r) - \sigma_2(\rho z_1 + \sqrt{1-\rho^2} z_2)} v_{m+1}(\mu_1(r) + \sigma_1 z_1) \phi(z_1) \phi(z_2) dz_1 dz_2 \\ &= e^{-\mu_2(r) + \sigma_2^2(1-\rho^2)/2} \int_{-\infty}^{+\infty} e^{-\sigma_2 \rho z_1} v_{m+1}(\mu_1(r) + \sigma_1 z_1) \phi(z_1) dz_1, \end{aligned}$$

where  $\phi$  is the density function of a standard normal random variable.

The holding value is strictly positive and continuous by the Lebesgue's dominated convergence theorem [Billingsley (1995)]. It is a strictly decreasing function of  $r$  since  $\mu_1(r)$  and  $\mu_2(r)$  are strictly increasing functions of  $r$  and  $v_{m+1}(r)$  is a strictly positive, bounded, bounded away from zero, and non-increasing function of  $r$ . Finally, the thresholds  $a_m$  and  $b_m$  at step  $m$  do exist since  $v_m^h(r) \rightarrow +\infty$  when  $r \rightarrow -\infty$  and  $v_m^h(r) \rightarrow 0$  when  $r \rightarrow +\infty$ . ■

## 5 Solving the DP Equation

In this section, we show how to compute the expectation in (6) for each  $m$ . The idea here is to partition the real axis into a collection of intervals and then to approximate the bond value by a piecewise linear interpolation, so that computation becomes feasible in closed-form.

Let  $a_0 = -\infty < a_1 < \dots < a_p < a_{p+1} = +\infty$  be a set of points and  $R_1, \dots, R_{p+1}$  be a partition of  $\mathbb{R}$  into  $(p+1)$  intervals such that  $R_1 = (-\infty, a_1)$  and

$$R_i = [a_{i-1}, a_i) \quad \text{for } i = 2, \dots, p+1.$$

Given an approximation  $\tilde{v}_m$  of the bond value  $v_m$  at the points  $a_k$  and step  $m$ , we interpolate this function by a piecewise linear interpolation of the form

$$\hat{v}_m(a) = \sum_{i=1}^{p+1} (\alpha_i^m + \beta_i^m a) I_i(a), \quad (8)$$

where

$$I_i(a) = \begin{cases} 1 & \text{if } a \in R_i \\ 0 & \text{elsewhere} \end{cases}.$$

Its local coefficients  $\alpha_i^m$  and  $\beta_i^m$  are obtained by solving the linear equations

$$\tilde{v}_m(a_i) = \widehat{v}_m(a_i), \quad \text{for } i = 2, \dots, p, \quad (9)$$

and, for  $i \in \{1, p+1\}$ , they are identical to those of the adjacent interval. Other piecewise polynomial approximations, such as quadratic and cubic splines, could be used in this context [see de Boor (1978) for a general discussion].

Assume now that  $\widehat{v}_{m+1}$  is known, and so are its local coefficients at step  $m+1$  as in (8). The expectation in (6) at step  $m$  becomes

$$\begin{aligned} \tilde{v}_m^h(a_k) &= E_{m,a_k} \left[ e^{-\int_{t_m}^{t_{m+1}} r_t dt} \widehat{v}_{m+1}(r_{t_{m+1}}) \right] \\ &= \sum_{i=1}^{p+1} \left( \alpha_i^{m+1} E_{m,a_k} \left[ e^{-\int_{t_m}^{t_{m+1}} r_t dt} I_i(r_{t_{m+1}}) \right] + \right. \\ &\quad \left. \beta_i^{m+1} E_{m,a_k} \left[ e^{-\int_{t_m}^{t_{m+1}} r_t dt} r_{t_{m+1}} I_i(r_{t_{m+1}}) \right] \right), \end{aligned} \quad (10)$$

where  $\tilde{v}_m^h$  denotes the approximate holding value of the bond. Now, we use the decomposition of the random vector  $\left( r_{t_{m+1}}, \int_{t_m}^{t_{m+1}} r_t dt \right)'$  in Proposition 2 to compute explicitly the integrals in (10).

Indeed, for  $k = 1, \dots, p$  and  $i = 1, \dots, p+1$ , the first integrals can be expressed as

$$\begin{aligned} A_{k,i} &= E_{m,a_k} \left[ e^{-\int_{t_m}^{t_{m+1}} r_t dt} I_i(r_{t_{m+1}}) \right] \\ &= e^{-\mu_2(a_k) + \sigma_2^2/2} [\Phi(a_{k,i}) - \Phi(a_{k,i-1})], \end{aligned}$$

and the second ones as

$$\begin{aligned} B_{k,i} &= E_{m,a_k} \left[ e^{-\int_{t_m}^{t_{m+1}} r_t dt} r_{t_{m+1}} I_i(r_{t_{m+1}}) \right] \\ &= e^{-\mu_2(a_k) + \sigma_2^2/2} \times \\ &\quad \left[ (\mu_1(a_k) - \sigma_{12}) (\Phi(a_{k,i}) - \Phi(a_{k,i-1})) - \sigma_1 \left( e^{-a_{k,i}^2} - e^{-a_{k,i-1}^2} \right) / \sqrt{2\pi} \right], \end{aligned}$$

where  $\Phi$  is the cumulative density function of a standard normal random variable and

$$a_{k,j} = (a_j - \mu_1(a_k) + \sigma_{12}) / \sigma_1, \quad \text{for } j \in \{i-1, i\}.$$

In this case, the future interest rate may take negative values [see Proposition 1], but with very low probabilities. We take  $a_1 = \mu_1(r) - 6\sigma_1$ ,  $a_2 = \mu_1(r) - 4\sigma_1$ ,  $a_{p-1} = \mu_1(r) + 4\sigma_1$ , and  $a_p = \mu_1(r) + 6\sigma_1$ , where  $r$  is the interest rate at time 0 and  $\Delta t = T$  the maturity

of the bond. The integer  $p$  is a parameter to be specified. Then, we select the points  $a_k$ , for  $k = 2, \dots, p-1$ , to be equally spaced within the interval  $[a_2, a_{p-1}]$ .

Finally, we explain how the DP procedure works:

FOR  $m = M, 0$  by step -1

FOR  $k = 1, p$

    Compute  $\tilde{v}_m^h(a_k)$  for all the points  $a_k$  by (10);

    Compute  $\tilde{v}_m(a_k)$  for all the points  $a_k$  by (5);

NEXT  $k$

    Compute the coefficients of  $\hat{v}_m$  at step  $m$  by (9);

NEXT  $m$

## 6 Numerical Experiments

The value of an embedded option is obtained by making the difference between the prices of the bond with and without the option. In this section, we price by DP a call and a put options embedded in a zero-coupon bond. We price first some optionless zero-coupon bonds by DP and compare the results with the exact solution of Vasicek (1977).

Consider a zero-coupon bond paying 1 at the maturity  $T$  (in years) and let  $r_0 = 4.5\%$ ,  $\kappa = 1$ ,  $\bar{r} = 5\%$ , and  $\sigma = 0.01$ . The number of monitoring dates is denoted by  $M$ . Table 3 gives the values of some optionless zero-coupon bonds computed by DP. Its last column indicates the exact solution of Vasicek (1977). CPU times are given in seconds and they are for the last line (the most expensive). In Table 4, all the parameters values are the same as in Table 3 except for the interest rate at the origin, increased to 5.5%.

Table 3: Optionless Zero-Coupon Bonds ( $r_0 = 4.5\%$ )

|           | $p$    |        |        |         |
|-----------|--------|--------|--------|---------|
| $(T, M)$  | 25     | 50     | 100    | Vasicek |
| (1, 2)    | 0.9543 | 0.9543 | 0.9543 | 0.9543  |
| (2, 4)    | 0.9088 | 0.9088 | 0.9088 | 0.9088  |
| (5, 10)   | 0.7828 | 0.7828 | 0.7828 | 0.7828  |
| (10, 20)  | 0.6097 | 0.6097 | 0.6097 | 0.6098  |
| CPU (sec) | 0.08   | 0.37   | 1.01   |         |



Table 4: Optionless Zero-Coupon Bonds ( $r_0 = 5.5\%$ )

|           | $p$    |        |        |         |
|-----------|--------|--------|--------|---------|
| $(T, M)$  | 25     | 50     | 100    | Vasicek |
| (1, 2)    | 0.9482 | 0.9482 | 0.9482 | 0.9482  |
| (2, 4)    | 0.9010 | 0.9010 | 0.9010 | 0.9010  |
| (5, 10)   | 0.7750 | 0.7750 | 0.7750 | 0.7751  |
| (10, 20)  | 0.6037 | 0.6037 | 0.6037 | 0.6038  |
| CPU (sec) | 0.09   | 0.28   | 1.07   |         |

As expected, the value of a zero-coupon bond decreases when the maturity or the interest rate increases. In view of the results, the DP procedure appears to be stable, consistent, and efficient. CPU times are obtained with an old 100 Mhz Silicon Graphics (and an f77 compiler).

Now, consider the 5 years zero-coupon bond in Table 4 for which we add a call and a put features. The call and put prices are specified in the contract and could be selected arbitrarily. Here, they are determined as in Mason et al. (1995) [see the case “Waste Management, Inc.”].

Table 5: The Call and Put Prices

| $m$ | $t_m$ (in years) | $c_m$   | $p_m$   |
|-----|------------------|---------|---------|
| 1   | 0.5              | 0.83070 | 0.78914 |
| 2   | 1                | 0.84734 | 0.80749 |
| 3   | 1.5              | 0.86452 | 0.83040 |
| 4   | 2                | 0.88223 | 0.85824 |
| 5   | 2.5              | 0.90051 | 0.88039 |
| 6   | 3                | 0.91935 | 0.90311 |
| 7   | 3.5              | 0.92641 | 0.92641 |
| 8   | 4                | 0.95032 | 0.95032 |
| 9   | 4.5              | 0.97484 | 0.97484 |
| 10  | 5                | 1       | 1       |

In this example, the bond under interest will be exercised for sure after 3.5 years (if not exercised before). The last column of Table 6 reports the options values. They are obtained by making the difference between the price of the bond with its embedded option(s) and the price of its optionless counterpart (0.7751).

Table 6: Pricing the Embedded Call and Put Options

|           |          | $p$    |        |        |         |              |
|-----------|----------|--------|--------|--------|---------|--------------|
| $(T, M)$  | Option   | 25     | 50     | 100    | Vasicek | Option Value |
| (5, 10)   | Call     | 0.7726 | 0.7727 | 0.7727 | 0.7751  | 0.0024       |
| (5, 10)   | Put      | 0.7767 | 0.7766 | 0.7766 | 0.7751  | 0.0015       |
| (5, 10)   | Call+Put | 0.7751 | 0.7751 | 0.7751 | 0.7751  | 0            |
| CPU (sec) |          | 0.09   | 0.3    | 1.04   |         |              |

As expected, the call option tends to decrease the bond value and the put option has the opposite effect. In this case, the inclusion of both the call and the put features has negligible effect on the bond value (no effect for the precision given in this example). They tend to compensate each other.

## 7 Conclusion

American-style financial derivatives do not admit closed-form solutions even under very simplified assumptions. In this paper, we address the problem of pricing the embedded call and put options via a stochastic DP formulation, the focus being on the solution of the DP equation. We let the interest rate move as in Vasicek (1977), we approximate the bond value by a piecewise linear interpolation, and solve the DP equation in closed-form. We use this formulation to price zero-coupon bonds and their embedded options. Results show that the bond value decreases when the call option is included, and increases when the put option is included. Numerical investigation shows stability, consistency, and efficiency. Also, we use the DP formulation to establish some theoretical properties of the bond value with its embedded options. It is a strictly positive, continuous, and non-increasing function of the interest rate. It is equal to the call price for “low” interest rates and to the put price for “high” interest rates.

If the interest rate moves according to the GBM, the CEV, or the CIR process, the market remains arbitrage-free and risk-neutral evaluation is preserved. In this context, the interest rate is Markov and the stochastic DP formulation could be applied.

## References

- [1] Ananthanarayanan, A.L. and E.S. Schwartz, “Retractable and Extendible Bonds: The Canadian Experience,” *Journal of Finance*, 35 (1980), 31-47.
- [2] Barraquand, J. and D. Martineau, “Numerical Valuation of High Dimensional Multivariate American Securities,” *Journal of Financial and Quantitative Analysis*, 30 (1995), 383-405.

- [3] Bertsekas, D.P., *Dynamic Programming: Deterministic and Stochastic Models*, Prentice-Hall, New Jersey, 1987.
- [4] Billingsley, P., *Probability and Measure*, Third Edition, John Wiley and Sons, New York, 1995.
- [5] Black, F., E. Derman, and W. Toy, "A One-Factor Model of Interest Rates and its Application to Treasury Bond Options," *Financial Analysts Journal*, 46 (1990), 33-40.
- [6] Black, F. and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (1973), 637-654.
- [7] Bliss, R.R. and E.I. Ronn, "To Call or not to Call? Optimal Call Policies for Callable U.S. Treasury Bonds," *Economic Review*, 80 (1995), 1-15.
- [8] Boyle, P.P., "The Quality Option and the Timing Option in Futures Contracts," *Journal of Finance*, 44 (1989), 101-113.
- [9] Brennan, M.J. and E.S. Schwartz, "Savings Bonds, Retractable Bonds, and Callable Bonds," *Journal of Financial Economics*, 5 (1977), 67-88.
- [10] Brennan, M.J. and E.S. Schwartz, "A Continuous Time Approach to the Pricing of Bonds," *Journal of Banking and Finance*, 3 (1979), 133-155.
- [11] Brennan, M.J. and E.S. Schwartz, "Analyzing Convertible Bonds," *Journal of Financial and Quantitative Analysis*, 15 (1980), 907-929.
- [12] Briys, E., M. Crouhy, and R. Schöbel, "The Pricing of Default-Free Interest Rate Cap, Floor, and Collar Agreements," *Journal of Finance*, 46 (1991), 1879-1892.
- [13] Carverhill, A., "A Note on the Models of Hull and White for Pricing Options on the Term Structure," *Journal of Fixed Income*, 5 (1995), 89-96.
- [14] Chan, K.C., G.A. Karolyi, F.A. Longstaff, and A.B. Sanders, "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate," *Journal of Finance*, 47 (1992), 1209-1227.
- [15] Clewlow, L. and C. Strickland, "Monte Carlo Valuation of Interest Rate Derivatives under Stochastic Volatility," *Journal of Fixed Income*, 7 (1997), 35-45.
- [16] Cohen, H., "Evaluating Embedded Options," *Economic Review*, 76 (1991), 9-17.
- [17] Courtadon, G., "The Pricing of Options on Default-Free Bonds," *Journal of Financial and Quantitative Analysis*, 17 (1982), 75-100.
- [18] Cox, J.C., "The Constant Elasticity of Variance Option Pricing Model," *Journal of Portfolio Management*, 22 (1996), 15-17.
- [19] Cox, J.C., J.E. Ingersoll, and S.A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53 (1985), 385-407.
- [20] Cox, J.C., S.A. Ross, and M. Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7 (1979), 229-263.
- [21] de Boor, C., *A Practical Guide to Splines*, Springer-Verlag, New York, 1978.

- [22] Dothan, L.U., "On the Term Structure of Interest Rates," *Journal of Financial Economics*, 6 (1978), 59-69.
- [23] Elliott, R.J. and P.E. Kopp, *Mathematics of Financial Markets*, Springer-Verlag, New York, 1999.
- [24] Fabozzi, F.J., *The Handbook of Fixed Income Securities*, Fifth Edition, McGraw-Hill, New York, 1997.
- [25] Heath, D., R. Jarrow, and A. Morton, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, 60 (1992), 77-105.
- [26] Ho, T.S.Y. and S.B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41 (1986), 1011-1029.
- [27] Hull, J. and A. White, "Pricing Interest-Rate Derivative Securities," *Review of Financial Studies*, 3 (1990a), 573-592.
- [28] Hull, J. and A. White, "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis*, 25 (1990b), 87-100.
- [29] Hull, J. and A. White, "One Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities," *Journal of Financial and Quantitative Analysis*, 28 (1993), 235-254.
- [30] Hull, J. and A. White, "Numerical Procedures for Implementing Term Structure Models I: Single-Factor Models," *Journal of Derivatives*, 2 (1994a), 7-16.
- [31] Hull, J. and A. White, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, 2 (1994b), 37-48.
- [32] Hull, J. and A. White, "A Note on the Models of Hull and White for Pricing Options on the Term Structure: Response," *Journal of Fixed Income*, 5 (1995), 97-103.
- [33] Hull, J. and A. White, "Using Hull-White Interest Rate Trees," *Journal of Derivatives*, 4 (1996), 26-36.
- [34] Jamshidian, F., "An Exact Bond Option Formula," *Journal of Finance*, 44 (1989), 205-209.
- [35] Kalotay, A.J., G.O. Williams, and F.J. Fabozzi, "A Model for Valuing Bonds and Embedded Options," *Financial Analysts Journal*, 49 (1993), 35-47.
- [36] Karatzas, I. and S.E. Shreve, *Methods of Mathematical Finance*, Springer-Verlag, New York, 1998.
- [37] Longstaff, F.A., "Pricing Options with Extendible Maturities: Analysis and Applications," *Journal of Finance*, 45 (1990), 935-957.
- [38] Marsh, T.A. and E.R. Rosenfeld, "Stochastic Processes for Interest Rates and Equilibrium Bond Prices," *Journal of Finance*, 38 (1983), 635-646.

- [39] Mason, S., R. Merton, A. Perold, and P. Tufano, *Cases in Financial Engineering*, Prentice-Hall, New Jersey, 1995.
- [40] Merton, R.C., "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (1973), 141-183.
- [41] Øksendal, B., *Stochastic Differential Equations, An Introduction with Applications*, Fourth Edition, Springer-Verlag, Germany, 1995.
- [42] Rabinovitch, R., "Pricing Stock and Bond Options when the Default-Free Rate is Stochastic," *Journal of Financial and Quantitative Analysis*, 24 (1989), 447-457.
- [43] Richard, S.F., "An Arbitrage Model of the Term Structure of Interest Rates," *Journal of Financial Economics*, 6 (1978), 33-57.
- [44] Schaefer, S.M. and E.S. Schwartz, "A Two-Factor Model of the Term Structure: An Appropriate Analytical Solution," *Journal of Financial and Quantitative Analysis*, 19 (1984), 413-424.
- [45] Vasicek, O., "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (1977), 177-188.
- [46] Wilmott, P., J. Dewynne, and S. Howison, *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press, United Kingdom, 1993.