Some Approaches for Solving the Discretely-Constrained Mixed Complementarity Problem (DC-MCP)

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Outline

Focus on Two Papers

Concentration on two of my co-authored works in this presentation.

- S.A. Gabriel, 2017. "Solving Discretely Constrained Mixed Complementarity Problems Using a Median Function," *Optimization and Engineering*, 18(3), 631-658. [4]
  - Re-express the DC-MCP as a particular (usually binary-constrained) mixed-integer nonlinear program (MINLP), theoretical and numerical results
- S.A. Gabriel, M. Leal, M. Schmidt, 2021. "Solving Binary-Constrained Mixed Complementarity Problems Using Continuous Reformulations," *Computers and Operations Research*, Vol. 131, Issue C. [7]
  - Continuous-variable reformulation of the binary-constrained MCP, theoretical and numerical results

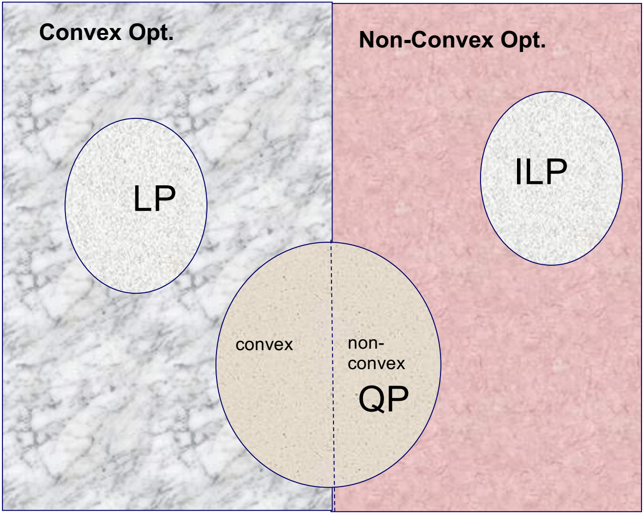
Other related co-authored works in DC-MCP shown in the References.

Some Motivations for Studying the Class of Discretely Constrained Equilibrium Problems

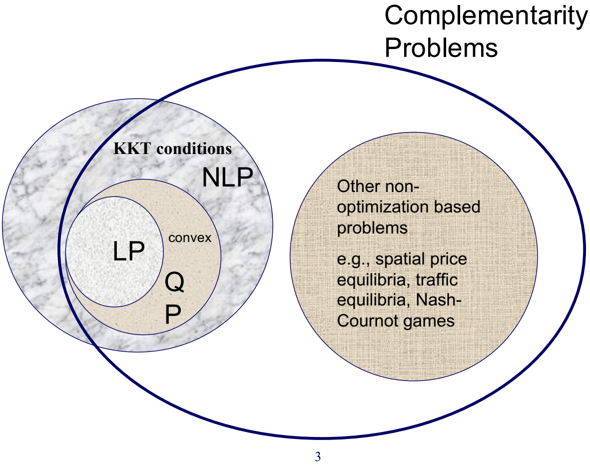
- These problems can apply integer/binary restrictions to equilibriums problem for more realism, richer applications (e.g., game theory plus go-no go decisions, if-then thresholds)
  - Allowing for autonomy in infrastructure networks/supply chains (e.g., energy) that need some additional logic constraints and/or distributional or other equity enforcement (e.g., discrete design variables), or multi-sector coupling (combined with logic variables)
  - Allowing for mixed equilibrium systems that involve volumes as well as discrete units (e.g., road traffic volume but accounting for a discrete number of emergency vehicles)
  - Equilibrium problems combined with combinatorial optimization
  - Relaxation of multi-agent, non-cooperative markets with discrete restrictions (e.g., unit commitment in power), for example [10]

Outline

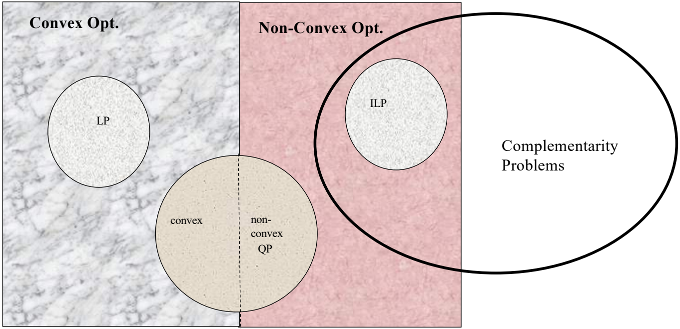
The Big Picture



The Bigger Picture Including the Mixed Complementarity Problem (MCP)



The Bigger Picture with Discrete/Integer Restrictions



LP=linear program
ILP=integer linear program
QP=quadratic program

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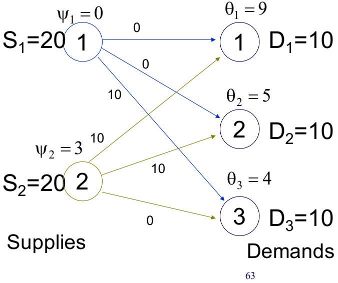
Spatial Price Equilibrium Problem (example of an MCP)

- Consider the classical Transportation Problem with x_{ij} the (primal) flow variables, ψ_i, θ_j the dual variables (Lagrange multipliers) at respectively, supply node i and demand node j
- Not so realistic since ψ_i and θ_j should be elastic not fixed
- Want to have the dual variables a function of the primal variables (x_{ij}) but this can't be done with a linear program ('Catch 22') situation)
- KKT optimality conditions will include: $\psi_i + c_{ij} \geq \theta_j$, $\forall i, j$



Spatial Price Equilibrium Problem (example of an MCP)

- x_{ij}:= flow from supply i to demand j, $S_i := \sum_j x_{ij}, D_j := \sum_i x_{ij}$
- $\psi_i(S_i)$:= inverse supply function
- $\theta_j(D_j)$:= inverse demand function
- c_{ij}:= marginal transport cost

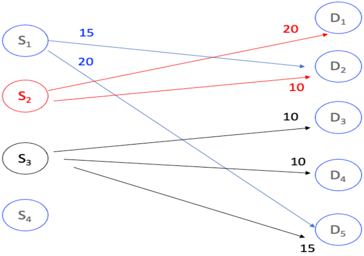
Overall MCP in terms of (nonnegative) flows x_{ij} is thus:

$$0 \leq \psi_i\left(\sum_j x_{ij}\right) + c_{ij} - \theta_j\left(\sum_i x_{ij}\right) \perp x_{ij} \geq 0 \tag{1}$$

- $x_{ij} > 0 \Rightarrow \psi_i\left(\sum_j x_{ij}\right) + c_{ij} = \theta_j\left(\sum_i x_{ij}\right)$ or marginal cost = marginal benefit
- $\psi_i\left(\sum_j x_{ij}\right) + c_{ij} - \theta_j\left(\sum_i x_{ij}\right) > 0 \Rightarrow x_{ij} = 0$ or no flow when marginal cost is higher than marginal benefit

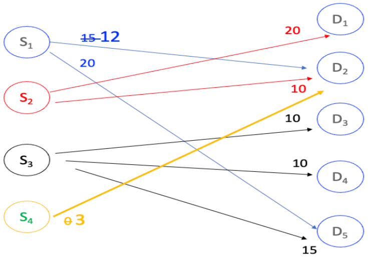
SPE Example with 4 Supply Nodes, 5 Demand Nodes [4]

- Consider a solution of the SPE with numbers on arcs referring to an equilibrium flow (color-coded by supply node).
- **Problem: Supply node 4 is inactive!** This might be a warehouse or depot so costly to keep running.
- Want to keep the equilibrium notion but reroute flows somehow? How to do this?



SPE Example with 4 Supply Nodes, 5 Demand Nodes, Equity-Enforcing Constraints [4]

- If $\sum_j x_{ij} < \delta_i$ then $\sum_j x_{ij} \geq 0.25 \sum_i \sum_j x_{ij}$, re-routing equilibrium flows.
- With δ_i a minimum flow threshold (contractual?) and 0.25 the minimum guranteed flow % (of total flows) for supply node i.
- See the new solution below with $\delta_i = 3, \forall i$. Want to generalize this MCP example, this leads to the DC-MCP.



Outline

Complementarity Problems, Mixed Complementarity Problems (MCP) [5]

(Mixed) Nonlinear Complementarity Problem MNCP

Having a function $F: R^n \to R^n$, find an $x \in R^{n_1}$, $y \in R^{n_2}$ such that

$$F_i(x,y) \geq 0, x_i \geq 0, F_i(x,y) * x_i = 0 \; for \; i = 1,\ldots,n_1$$

$$F_i(x,y) = 0, y_i \text{ free}, \; for \; i = n_1+1,\ldots,n$$

Example

$$F(x_1,x_2,y_1) = \begin{pmatrix} F_1(x_1,x_2,y_1) \\ F_2(x_1,x_2,y_1) \\ F_3(x_1,x_2,y_1) \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - y_1 \\ x_1 + x_2 + y_1 - 2 \end{pmatrix}$$ so we want to find x_1, x_2, y_1 s.t.

$$x_1 + x_2 \geq 0 \qquad x_1 \geq 0 \qquad (x_1 + x_2) * x_1 = 0$$

$$x_1 - y_1 \geq 0 \qquad x_2 \geq 0 \qquad (x_1 - y_1) * x_2 = 0$$

$$x_1 + x_2 + y_1 - 2 = 0 \qquad y_1 \text{ free}$$

One solution: $(x_1,x_2,y_1) = (0,2,0)$, why? Any others?

If all functions (linear) affine, we get the linear complementarity problem (LCP)

Sources for Complementarity Problems: Linear Programming

Consider a (primal) linear program in the variables $x \in R^n$:

$$\begin{aligned} min_x \quad & c^T x && && \text{(2a)} \\ s.t. \quad & Ax \geq b && (y) && \text{(2b)} \\ & x \geq 0 && && \text{(2c)} \end{aligned}$$

and corresponding dual linear program in the variables $y \in R^m$

$$\begin{aligned} max_y \quad & b^T y && && \text{(3a)} \\ s.t. \quad & A^T y \leq c && (x) && \text{(3b)} \\ & y \geq 0 && && \text{(3c)} \end{aligned}$$

and complementary slackness for both primal and dual problems:

$$(Ax - b)^T y = 0, (c - A^T y)^T x = 0 \tag{4}$$

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Sources for Complementarity Problems: LP Primal and Dual Feasibility, Complementary Slackness

We can rewrite things a bit to get the following equivalent form. Find $x \in R^n, y \in R^m$ such that:

$$0 \leq c - A^T y \perp x \geq 0 \tag{5a}$$

$$0 \leq Ax - b \perp y \geq 0 \tag{5b}$$

This is exactly the (monotone) linear complementarity problem (LCP) in nonnegative variables (x, y) and is exactly the KKT optimality conditions as applied to the primal LP. Here

$$F(x,y) = \begin{pmatrix} F_x(x,y) \\ F_y(x,y) \end{pmatrix} = \begin{pmatrix} c - A^T y \\ Ax - b \end{pmatrix} \texttt{ or} \tag{6}$$

$$F(x,y) = \begin{pmatrix} c \\ -b \end{pmatrix} + \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{7}$$

Sources for Complementarity Problems: KKT Optimality Conditions for Nonlinear Programs

Consider a nonlinear program of the following form where $f : R^n \to R, g_i : R^n \to R, i = 1, \ldots, m, h_j : R^n \to R, j = 1, \ldots, p$:

$$\begin{aligned} min_x \quad & f(x) && && \text{(8a)} \\ s.t. \quad & g_i(x) \leq 0 && i = 1, \ldots, m \quad (\lambda_i) && \text{(8b)} \\ & h_j(x) = 0 && j = 1, \ldots, p \quad (\gamma_j) && \text{(8c)} \end{aligned}$$

The KKT conditions are to find primal variables $x \in R^n$ and Lagrange multipliers $\lambda \in R^m_+$, $\gamma \in R^p$ such that:

$$\begin{aligned} 0 =& \nabla f(x) + \sum_i \nabla g_i(x)\lambda_i + \sum_j \nabla h_j(x)\gamma_j, x \texttt{ free} && \text{(9a)} \\ 0 \leq& - g(x) \perp \lambda \geq 0 && \text{(9b)} \\ 0 =& h(x), \quad \gamma \text{ free} && \text{(9c)} \end{aligned}$$

Selected Sources for Complementarity Problems: KKT Optimality Conditions for Nonlinear Programs

Putting all these conditions together, we get a mixed complementarity problem of the following form. Find vectors $\lambda \in R^m_+$, $x \in R^n, \gamma \in R^p$ such that:

$$F(x, \lambda, \gamma) = \begin{pmatrix} \nabla f(x) + \sum_i \nabla g_i(x)\lambda_i + \sum_j \nabla h_j(x)\gamma_j \\ -g(x) \\ h(x) \end{pmatrix}$$

with

$$\begin{aligned} 0 =& F_x(x, \lambda, \gamma) & x \text{ free} \quad (10a) \\ 0 \leq& F_\lambda(x, \lambda, \gamma) & \perp \lambda \geq 0 \quad (10b) \\ 0 =& F_\gamma(x, \lambda, \gamma) & \gamma \text{ free} \quad (10c) \end{aligned}$$

$\Rightarrow$ Connection to game theory problems

Outline

The Bounded MCP (generalizes MCP [4, 5])

For the following two definitions, $F : R^n \rightarrow R^n$ is a given function and the vectors $l, u \in R^n \cup \{-\infty, +\infty\}$ with $l \leq u$.

- **Definition:** The bounded mixed complementarity problem is to find a vector $x \in R^n$ so that

$$\begin{array}{ll} F_i(x) \geq 0 & x_i = l_i \\ F_i(x) = 0 & l_i < x_i < u_i \\ F_i(x) \leq 0 & x_i = u_i \end{array} \qquad (11)$$

  This generalizes the earlier MCP that had $(l_i, u_i) = (0, \infty)$ (nonnegative variables) or $(l_i, u_i) = (-\infty, \infty)$ (free variables).
- **Definition:** Let $S \subseteq R^n$ be a given subset of the variables x in a bounded MCP as given above. The DC-MCP is to find a solution x to (11) that also satisfies $x_i \in Z, \forall i \in S$.

DC-MCP, Encoding Integer Values

- In most cases, the set S refers to integer-valued variables but could be more generally discrete levels.
- If a variable $x_i, i \in S$ is to be binary, then $F_i(x) = x_i(1 - x_i) = 0$, with $l_i = -\infty, u_i = +\infty$.
- More generally for integer values $\{-m_1, -m_1 + 1, \ldots, 0, 1, \ldots, m_2\}$, we have $F_i(x) = (-m_1 - x_i) \times \ldots (x_i) \times \ldots (m_2 - x_i) = 0$, with $l_i = -\infty, u_i = +\infty$.
- Can add additional constraints on the continuous and integer-valued variables in a designated set X; see for example, the next slides on the median function.
- Can also just add auxiliary (e.g., binary) variables for example to the MINLP formulation shown in the next slides.

Outline

H Function

As described in [9] and [3], the bounded MCP(F) can be equivalently recast as finding the zero of the function $H : R^n \to R^n$ defined by

$$H_i(x) = x_i - mid(l_i, u_i, x_i - F_i(x)), \forall i \tag{12}$$

where $mid(a, b, c)$ represents the median of the three scalars a, b, c.
Consider an example where $H : R^2 \to R^2$ with

$$F_1(x, y) = x + y, \qquad (l_1, u_1) = (0, +\infty) \tag{13a}$$

$$F_2(x, y) = y, \qquad (l_2, u_2) = (-\infty, +\infty) \tag{13b}$$

So that,

$$\begin{pmatrix} H_1(x, y) \\ H_2(x, y) \end{pmatrix} = \begin{pmatrix} x - mid(0, +\infty, x - (x + y)) \\ y - mid(-\infty, +\infty, y - (y)) \end{pmatrix} = \begin{pmatrix} \begin{cases} x + y & y \leq 0 \\ x & y > 0 \end{cases} \\ y \end{pmatrix} \tag{13c}$$

H is in General Non-smooth, Example for $F : R^2 \to R^2$

Thus,

$$\|H(x,y)\|_1 = \begin{cases} |x+y| + |y| & y \leq 0 \\ |x| + |y| & y > 0 \end{cases} \quad (14a)$$

$$= \begin{cases} |x+y| - y & y \leq 0 \\ |x| + y & y > 0 \end{cases} \quad (14b)$$

$$= \begin{cases} |y| - y = -2y & y \leq 0, \text{ for } x = 0 \\ y & y > 0, \text{ for } x = 0 \end{cases} \quad (14c)$$

which for $x = 0$ fixed, is continuous in y but kinked, hence non-differentiable in the sense of Fréchet at the point $y = 0$ as shown.

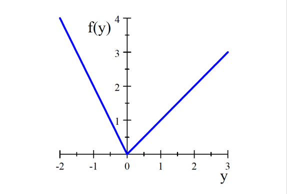

Figure 1: Example of the mid function being non-smooth.

DC-MCP Definition

Definition: *A vector pair (x,y) is a relaxed DC-MCP solution if it solves (15).*

$$\min_{x \in X} \|H(x,y)\| \tag{15a}$$

$$s.t.\ x_i \in R_+, i \in I_x \backslash D_x \tag{15b}$$

$$x_i \in Z_+, i \in D_x \tag{15c}$$

$$y_j \in R\ , j \in I_y \backslash D_y \tag{15d}$$

$$y_j \in Z, j \in D_y \tag{15e}$$

- Where H is given in (12), $\|\cdot\|$ is any vector norm, I_x, I_y are the index sets, respectively for x, y and D_x, D_y are respective subsets.
- If for a solution to this optimization problem: (x^*, y^*), $\|H(x^*, y^*)\| = 0$, then (x^*, y^*) is also a DC-MCP solution.
- Can add any integer if-then type constraints (or other constraints) like for the equity-enforcing SPE example via the set X.

MINLP Problem for the Bounded MCP

$$\min_{x,y,z^+,z^-,w^+,w^-,b,\widetilde{b}} f = \sum_{i \in I_x} \left(z_i^+ + z_i^-\right) + \sum_{j \in I_y}^{n} \left(w_j^+ + w_j^-\right) \quad (16a)$$

$$s.t. - M b_i \leq x_i - F_i(x,y) - l_i \leq M(1 - b_i), \forall i \in I_x \quad (16b)$$

$$-M\widetilde{b}_i \leq x_i - F_i(x,y) - u_i \leq M(1 - \widetilde{b}_i), \forall i \in I_x \quad (16c)$$

$$-M\left(2 - b_i - \widetilde{b}_i\right) \leq z_i^+ - z_i^- - x_i + l_i \leq M\left(2 - b_i - \widetilde{b}_i\right) \quad (16d)$$

$$-M\left(1 + b_i - \widetilde{b}_i\right) \leq z_i^+ - z_i^- - F_i\left(x,y\right) \leq M\left(1 + b_i - \widetilde{b}_i\right) \quad (16e)$$

$$-M\left(b_i + \widetilde{b}_i\right) \leq z_i^+ - z_i^- - x_i + u_i \leq M\left(b_i + \widetilde{b}_i\right) \quad (16f)$$

$$w_j^+ - w_j^- = F_j(x,y), \forall j \in I_y \quad (16g)$$

$$x_i \in R_+, i \in I_x \backslash D_x, x_i \in Z_+, i \in D_x, b_i, \widetilde{b}_i \in \{0,1\}, \forall i \in I_x \quad (16h)$$

$$y_j \in R\ , j \in I_y \backslash D_y, y_j \in Z, j \in D_y \quad (16i)$$

$$z_i^+, z_i^- \geq 0, \forall i \in I_x, w_j^+, w_j^- \geq 0, \forall j \in I_y \quad (16j)$$

Outline

Example of Continuous-Variable Formulation (see [7] for other examples)

Here is an example of a continuous reformulation of the DC-MCP noticing that we can reformulate it as the following system of inequalities (x now is the set of nonnegative and free variables combined):

$$(x_i - l_i)F_i(x) \le 0 \quad i \in \{1, \dots, n\} \tag{17a}$$

$$(u_i - x_i)F_i(x) \ge 0, \quad i \in \{1, \dots, n\} \tag{17b}$$

$$x_i \in [l_i, u_i], \quad i \in \{1, \dots, n\} \tag{17c}$$

assuming $l_i < u_i$ for each i. Now the binary-constrained MCP (BC-MCP) which we consider for the next few slides, in addition has:

$$x_i \in \{0, 1\}, \quad i \in S \subseteq \{1, \dots, n\} \tag{18}$$

Example of Continuous-Variable Formulation: Complementarity-Constrained Formulation (CCF)

Note that $x_i \in \{0, 1\}$ is equivalent to

$$x_i \in [0, 1] \subseteq R, \quad x_i(1 - x_i) = 0.$$

Thus, we can replace the BC-MCP equivalently by

$$(x_i - l_i)F_i(x) \le 0 \quad i \in \{1, \ldots, n\} \tag{19a}$$
$$(u_i - x_i)F_i(x) \ge 0, \quad i \in \{1, \ldots, n\} \tag{19b}$$
$$x_i \in [l_i, u_i] \quad i \in \{1, \ldots n\} \tag{19c}$$
$$0 \le (1 - x_i) \perp x_i \ge 0 \quad i \in S \tag{19d}$$

The MPEC-like condition (19d) is again a complementarity problem.

Relaxed Complementarity-Constrained Problem (RCCF-ϵ)

- For computational reasons, better to not solve the above directly as an NLP.
- For example, can try the regularization scheme of Scholtes, which relaxes the MPEC constraint (19d) to get the following formulation:

$$(x_i - l_i)F_i(x) \leq 0, \quad i \in \{1, \dots, n\} \tag{20a}$$

$$(u_i - x_i)F_i(x) \geq 0, \quad i \in \{1, \dots, n\} \tag{20b}$$

$$x_i \in [l_i, u_i], \quad i \in \{1, \dots, n\} \tag{20c}$$

$$x_i(1 - x_i) \leq \epsilon, \quad i \in S \tag{20d}$$

$$x_i \in [0, 1], \quad i \in S \tag{20e}$$

where $\epsilon > 0$ is a given regularization parameter that is iteratively decreased. See [7] for details on theoretical and numerical results and other continuous-variable formulations.

Selected Theoretical Results: Median Function Formulation

The following result shows the correspondence between the above problem (16) and solving the DC-MCP expressed as (15). For each $i \in I_x$, assume that $l_i < u_i$.

Theorem

1. *Consider any feasible solution* $\left(x, y, z^+, z^-, w^+, w^-, b, \widetilde{b}\right)$ *to (16). Then, for* $z_i \triangleq z_i^+ - z_i^-, w_j \triangleq w_j^+ - w_j^-, z_i = H_i(x, y), \forall i \in I_x, w_j = H_j(x, y), \forall j \in I_y$
2. *Consider any optimal solution* $\left(x^*, y^*, {z^+}^*, {z^-}^*, {w^+}^*, {w^-}^*, b^*, \widetilde{b^*}\right)$ *to (16). Then at most one of* $\left(z_i^{+*}, z_i^{-*}\right)$ *is nonzero and at most one of* $\left(w_i^+, w_i^-\right)$ *is nonzero.*
3. *Consider any optimal solution* $\left(x^*, y^*, {z^+}^*, {z^-}^*, {w^+}^*, {w^-}^*, b^*, \widetilde{b^*}\right)$ *to (16) with corresponding optimal objective function value* f^*. *Then,*

$$\|H(x^*, y^*)\|_1 = f^* = 0 \Leftrightarrow (x^*, y^*) \text{ solve the DC-MCP (11).}$$

Selected Numerical Results: Median Function Formulation

Problem #	n_x	n_y	There is a continuous solution that is integer?	Type of Problem
1a	10	10	yes, by construction	Small Illustrative
1b	10	10	no	Small Illustrative
1c	10	10	no	Small Illustrative
1d	1000	1000	yes, by construction	Large random
1e	1000	1000	yes, but not known in advance	Large random
2a	4	0	yes	Energy duopoly
2b	4	0	no	Energy duopoly
2c1	4	0	no	Energy duopoly
2c2	4	0	no	Energy duopoly
3a	12	0	yes, but not known in advance	Spatial Price Equilibrium
3b	12	0	yes, but not known in advance	Spatial Price Equilibrium
3c	12	0	yes, but not known in advance	Spatial Price Equilibrium

Table 1: Summary of numerical results.

Main Conclusions

- DC-MCP approach can apply integer restrictions to an equilibrium problem setting for more realism, richer applications (e.g., SPE)
  - Allowing for autonomy in infrastructure networks/supply chains (e.g., energy) that need some additional logic constraints and/or distributional or other equity enforcement (e.g., discrete design variables), or multi-sector coupling (combined with logic variables)
  - Allowing for mixed equilibrium systems that involve volumes as well as discrete units (e.g., road traffic volume but accounting for a discrete number of emergency vehicles)
  - Equilibrium problems combined with combinatorial optimization (generalizing the SPE example)
  - Relaxation of multi-agent, non-cooperative markets with discrete restrictions (e.g., unit commitment in power), for example [10]
- There are many approaches to solving DC-MCPs involving complementarity and optimization modeling, some mentioned here, others in the References and other literature
- Lots of directions to explore both theoretically and numerically

Relevant Publications

[1] F. D. Fomeni, S.A. Gabriel, M. J. Anjos, 2019."Applications of Logic Constrained Equilibria to Traffic Networks and to Power Systems with Storage, *Journal of the Operational Research Society*, 70(2), 310-325.

[2] F. D. Fomeni, S.A. Gabriel, M. J. Anjos, 2019. "An RLT Approach for Solving the Binary-Constrained Mixed Linear Complementarity, *Computers and Operations Research*,110, 48-59.

[3] S. A. Gabriel, 1998. "An NE/SQP Method for the Bounded Nonlinear Complementarity Problem," *Journal of Optimization Theory and Applications*, 97(2), 493-506.

[4] S.A. Gabriel, 2017. "Solving Discretely Constrained Mixed Complementarity Problems Using a Median Function," *Optimization and Engineering*, 18(3), 631-658.

[5] S.A. Gabriel, A.J. Conejo, J.D. Fuller, B.F. Hobbs, B.F. and C. Ruiz, 2012. *Complementarity modeling in energy markets* (Vol. 180). Springer Science & Business Media.

[6] S.A. Gabriel, A.J. Conejo, C. Ruiz, Sauleh Siddiqui , 2013. "Solving Discretely-Constrained, Mixed Linear Complementarity Problems with Applications in Energy, " *Computers and Operations Research*, 40(5), 1339-1350.

[7] S.A. Gabriel, M. Leal, M. Schmidt, 2021. "Solving Binary-Constrained Mixed Complementarity Problems Using Continuous Reformulations," *Computers and Operations Research*, Vol. 131, Issue C.

[8] S.A. Gabriel, S. Siddiqui, A.J. Conejo, C. Ruiz, 2013, "Discretely-Constrained, Nash-Cournot Games in Energy," *Networks and Spatial Economics*, 13(3), 307-326.

[9] S. A. Gabriel and J. J. More, 1997. "Smoothing of Mixed Complementarity Problems," 105-116 in *Complementarity and Variational Problems State of the Art*, SIAM, Proceedings of the International Conference on Complementarity Problems, The Johns Hopkins University, Baltimore, Maryland.

[10] R. Weinhold and S.A. Gabriel, 2020. "Discretely Constrained Mixed Complementary Problems: Application and Analysis of a Stylised Electricity Market," *Journal of the Operational Research Society*, 71(2), 237-249.