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Characteristic functions for cooperative interval games

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Abstract : We extend the α and β characteristic functions (CFs) to cooperative interval games, which constitute an interesting class of games to account for uncertainty and vagueness in decision-making processes. Both characteristic functions are based on a solution of zero-sum interval games when a coalition is the maximizer player and the anti coalition is the minimizer player. We propose an algorithm to define the interval values of a cooperative game in the form of α and β CFs and demonstrate them on an example of a three-person game. Further, we discuss some properties of cooperative interval games and calculate an interval Shapley value. Numerical examples show that the cooperative solution depends on the way we construct a CF to define a cooperative interval game.

Keywords: interval uncertainty, noncooperative interval games, cooperative interval games, characteristic function.

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1 Introduction

In a normal-form game, real number payoff is associated to each strategy profile. As the name suggests, the class of interval games assumes that this payoff is instead defined as an interval. One can interpret the lower and upper values of the interval as pessimistic and optimistic rewards, respectively. Interval games can be seen as one possible approach to deal with some of the inherent uncertainty and vagueness in decision-making processes. Other approaches to deal with stochastic outcomes in a cooperative game are chance-constrained games (Charnes and Granot, 1973), cooperative games in stochastic characteristic function form (Granot, 1977; Suijs et al., 1999), games with random payoffs (Timmer et al., 2005), and fuzzy uncertainty in the values of characteristic functions (Mareš and Vlach, 2004; Nishizaki and Sakawa, 2000).

To the best of our knowledge, the first paper to develop the theory of cooperative interval game is Branzei et al. (2003), where a bankruptcy problem is considered. The theory was further developed in a series of papers by Alparslan Gök et al. (2009a,b) and Branzei et al. (2010a,b), with one objective being the adaptation of classical solution concepts, e.g., the core and the Shapley value, to cooperative interval games (see also Kimms and Drechsel (2009)). Some extensions of the basic theory followed. For instance, Weber et al. (2010) consider ellipsoidal uncertainty, which is a generalization of cooperative interval games because it allows to represent the players mutual dependencies, similarities, and possible affinities to cooperate. Cooperative interval games have been applied to analyze lot sizing problems with uncertain demands (Drechsel and Kimms, 2011).

The class of noncooperative interval games, more specifically interval bimatrix games, was introduced in a series of papers by Shashikhin (2004), Collins and Hu (2005), and Hladík (2010). The two branches have evolved so far in parallel without any connections between the cooperative and noncooperative outcomes.

Our objective is to establish a link by considering cooperative interval games whose primitive are noncooperative games given in normal form. The tool for making this connection is the characteristic function (CF). More specifically, we extend the definition of the classical α and β CFs, which, as all other CFs, measure the strategic strength of all possible coalitions in a cooperative game. Therefore, these values rely on a noncooperative, in fact highly antagonistic, mode of play to compute the set of outcomes (imputations) in a cooperative game. Further, we recall some properties that are of interest in a noninterval (classical) cooperative game and illustrate, using a three-player example, how one can check their satisfaction.

The remaining part of the paper is organized as follows. In Section 2, we recall basic definitions and notions used in cooperative and noncooperative interval games. In Section 3, we define interval α and β CFs by extending (noninterval) α, β CFs to the interval setting, provide an algorithm to compute their values. In Section 4, we provide a numerical example and briefly conclude in Section 5.

2 Interval games

In this section, we recall key definitions and some useful results from noncooperative interval games (NCIGs) mainly from Bhurjee (2016) and Li et al. (2012), and cooperative interval games (CIGs) from Alparslan Gök et al. (2011). Additionally, we provide relevant basic operations from interval calculus that are used in NCIGs and CIGs.

Denote by $I(\mathbb{R})$ the set of all closed intervals in $\mathbb{R}$, by $I(\mathbb{R}_+)$ the set of all closed intervals in $\mathbb{R}_+$, and by $I^n(\mathbb{R}_+)$ the set of all n-dimensional vectors with components in $I(\mathbb{R}_+)$. Let $I, J \in I(\mathbb{R})$ with $I = \left[\underline{I}, \overline{I}\right]$, $J = \left[\underline{J}, \overline{J}\right]$, $|I| = \overline{I} - \underline{I}$, and let β be a strictly positive scalar. We define the following operations on intervals $I, J \in I(\mathbb{R}_+)$:

Addition: $I + J = \left[\underline{I} + \underline{J}, \overline{I} + \overline{J}\right]$;

Multiplication by a positive scalar: $\beta I = \left[\beta\underline{I}, \beta\overline{I}\right]$;
Substraction: $I - J = \left[\underline{I} - \underline{J}, \overline{I} - \overline{J}\right]$ is defined, only if $|I| \geq |J|$;
Multiplication: $I \cdot J = [\underline{I} \cdot \underline{J}, \overline{I} \cdot \overline{J}]$;
Division: $\frac{I}{J} = [\frac{\underline{I}}{\underline{J}}, \frac{\overline{I}}{\overline{J}}]$ is defined, only if $\underline{I} \cdot \overline{J} \leq \underline{J} \cdot \overline{I}$ and $\underline{J}, \overline{J} \neq 0$.

For any two intervals $I = [\underline{I}, \bar{I}]$ and $J = [\underline{J}, \bar{J}]$, we say that I is *weakly better than* J and denote it by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\bar{I} \geq \bar{J}$. If $I \succcurlyeq J$, then for each $x \in J$ there exists $y \in I$ such that $x \leq y$, and for each $y \in I$ there exists $x \in J$ such that $x \leq y$. We say that I is *better than* J and denote it by $I \succ J$, if and only if $I \succcurlyeq J$ and $I \neq J$. We use the notation $I \preccurlyeq J$ to say that I is *weakly less than* J, which is defined if and only if $\underline{I} \leq \underline{J}$ and $\bar{I} \leq \bar{J}$, and $I \prec J$ for I *less than* J, which is defined if and only if $I \preccurlyeq J$ and $I \neq J$.

2.1 Noncooperative interval games

A n-player non-zero-sum interval game in normal form is defined by

$$G = \{N, X_1, \dots, X_n, u_i, \dots, u_n\}, \tag{1}$$

where $N = \{1, \dots, n\}$ is the set of players, X_i the finite set of strategies of Player $i \in N$, and $u_i : \prod_{i \in N} X_i \to I(\mathbb{R})$ her payoff function.

In particular, a two-player noncooperative interval game, usually called interval bimatrix game, is represented by two matrices $(\tilde{A}, \tilde{B})$, where each entry is a pair of intervals $(\tilde{a}_{ij}, \tilde{b}_{ij})$, $i = 1, \dots, m$, $j = 1, \dots, \ell$. The entries $\tilde{a}_{ij} = [a_{ijL}, a_{ijR}]$ and $\tilde{b}_{ij} = [b_{ijL}, b_{ijR}]$ are the payoffs (or utilities) of players 1 and 2, respectively, when they choose their ith and jth pure strategies, respectively.

The set of mixed strategies $\bar{X}_i$ of Player $i \in N$ is defined by

$$\bar{X}_i = \left\{ \xi_i = \left(\xi_i^1, \dots, \xi_i^{|X_i|}\right) \in \mathbb{R}^{|X_i|} : \sum_{j=1}^{|X_i|} \xi_i^j = 1,\, \xi_i^j \geq 0,\, \forall j = 1, \dots, |X_i| \right\},$$

where ξ_i^j is the probability that Player i chooses strategy $x_i^j \in X_i$. We denote by $\xi_i(x_j)$ the probability of Player i choosing strategy x_j when using a mixed strategy ξ_i. The expected payoff of Player i in an interval game with mixed set of strategies is defined as follows:

$$H_i(\xi_1, \dots, \xi_n) = \sum_{x_1 \in X_1} \dots \sum_{x_n \in X_n} u_i(x_1, \dots, x_n)\xi_1(x_1) \cdot \ldots \cdot \xi_n(x_n),$$

where $u_i(x_1, \dots, x_n)$ is an interval or an element of $I(\mathbb{R})$ for any $(\xi_1, \dots, \xi_n) \in \bar{X}_1 \times \ldots \times \bar{X}_n$.

Remark 1. In the paper, we consider n-player NCIGs where the payoff function of any player takes values in the cone $I(\mathbb{R})$ endowed with the partial order $\succcurlyeq$.

2.2 Cooperative interval games

Denote by S a coalition, that is, a subset of the set of players N. We recall that a characteristic function $v(\cdot)$ is defined by $v : 2^N \to \mathbb{R}$, $v(\varnothing) = 0$. Von Neumann and Morgenstern (1944) interpreted the value $v(S)$ as the sum of the gains that coalition $S \subseteq N$ can guarantee its members. In cooperative games without externalities, i.e., when the payoff of coalition S is independent of the actions of the players in $N \backslash S$, the value $v(S)$ is obtained by optimizing the (possibly weighted) sum of the coalition members' payoffs. In games with externalities, the outcome of S depends on the behavior of the players in $N \backslash S$, which leads to different approaches to computing the values of v. The reader is referred to Parilina et al. (2022) for a brief survey on cooperative games, characteristic functions, and solution concepts in noninterval games in the meaning of classical games.

A cooperative interval game in coalitional form is a pair $< N, w >$, where $N = \{1, 2, \dots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\varnothing) = [0, 0]$ (Alparslan Gök et al., 2009b). For each coalition $S \subseteq N$, the value $w(S)$ is given by the interval $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$, $\overline{w}(S)$ are the lower and upper bounds of $w(S)$, respectively. We denote the *length* game by $< N, |w| >$, where $|w|(S) = \overline{w}(S) - \underline{w}(S)$ for any $S \in 2^N$. Note that $\overline{w} = \underline{w} + |w|$.

Denote by IG^N the family of all cooperative interval games with the player set N. Note that if all the values are degenerate intervals, i.e., $\underline{w}(S) = \overline{w}(S) = w(S)$, then we have a classical cooperative game with scalar characteristic function values. This implies that classical cooperative games is embedded in the class of CIGs.

Now, we recall some properties of CIGs (see Alparslan Gök et al. (2009b)):

- For any $w_1, w_2 \in IG^N$, we say that $w_1 \preccurlyeq w_2$ if $w_1(S) \preccurlyeq w_2(S)$ for any $S \subseteq N$.
- For any $w_1, w_2 \in IG^N$, and $\lambda \in \mathbb{R}_+$, we define interval cooperative games $< N, w_1 + w_2 >$ and $< N, \lambda w >$ by introducing corresponding characteristic functions $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\lambda w)(S) = \lambda w(S)$ for any coalition $S \subseteq N$. So, we say that IG^N endowed with $\preccurlyeq$ is a *partially ordered set* and has a *cone structure* with respect to addition and multiplication by a nonnegative scalar.
- A game $< N, w >$ is *supermodular* if
$$w(S) + w(T) \preccurlyeq w(S \cup T) + w(S \cap T), \quad \forall S, T \subseteq N. \tag{2}$$
From formula (2) it follows that a game $< N, w >$ is supermodular if and only if its *border games* $< N, \underline{w} >$ and $< N, \overline{w} >$ are convex.
- A game $w \in IG^N$ is called *convex* if $< N, w >$ is supermodular and its length game $< N, |w| >$ is also supermodular.
- A game $< N, w >$ is called *size monotonic* if the length game is monotonic, i.e., $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. The class of size monotonic interval games on set N is denoted by $SMIG^N$.

The interval marginal operators and the interval Shapley value were defined on $SMIG^N$ in Alparslan Gök et al. (2009b) as follows.

Denote by $\Pi(N)$ the set of permutations $\sigma : N \to N$ on N. The interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})^N$ corresponding to σ, associates with each $w \in SMIG^N$ the interval marginal vector $m^\sigma(w)$ with respect to σ defined by $m_i^\sigma(w) = w\left(P^\sigma(i) \cup \{i\}\right) - w\left(P^\sigma(i)\right)$ for each $i \in N$, where $P^\sigma(i) := \left\{r \in N \mid \sigma^{-1}(r) < \sigma^{-1}(i)\right\}$, and $\sigma^{-1}(i)$ denotes the entrance number of player i.

The interval Shapley value $\Phi : SMIG^N \to I^n(\mathbb{R})$ is then defined by

$$\Phi(w) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N.$$

3 Defining interval α and β characteristic functions

In this section, we first provide definitions of α and β characteristic functions (CFs) in a noninterval setting. Then, we introduce the definitions of α and β CFs for an interval setting.

3.1 α and β CFs for noninterval games

Consider a noninterval game in normal form defined in (1).

Definition 1. The α characteristic function (α CF) is given by

$$\nu^\alpha(S) = \max_{x_S} \min_{x_{N \setminus S}} \sum_{s \in S} u_s(x_S, x_{N \setminus S}),$$

where x_S is a strategy of coalition S and $x_{N \setminus S}$ is a strategy of coalition $N \setminus S$.

The α CF was introduced in Von Neumann and Morgenstern (1944). The value $\nu^\alpha(S)$ represents the maximum payoff that coalition S can guarantee for itself irrespective of the strategies used by the left-out players, that is, the players in $N \setminus S$. In this concept, coalition S moves first, choosing a joint strategy x_S that maximizes its payoff, before the joint strategy $x_{N\setminus S}$ of the players in $N \setminus S$ is chosen.

Definition 2. The β characteristic function is given by

$$\nu^\beta(S) = \min_{x_{N\setminus S}} \max_{x_S} \sum_{s\in S} u_s(x_S, x_{N\setminus S})$$

The value $\nu^\beta(S)$ is the maximum payoff that coalition S cannot be prevented from getting by players in $N \setminus S$. In this case, the players in coalition S choose their joint strategy x_S after the left-out players in $N \setminus S$ have chosen their joint strategy $x_{N\setminus S}$. For more details about α and β characteristic functions, see Aumann and Peleg (1960).

3.2 Maxmin and minmax operators in interval matrix games

To compute maxmin and minmax in interval matrix games we need to specify a procedure of interval ordering, on which maximization and minimization operations are based. Following Li (2011) and Nayak and Pal (2009), we define the analogs of maxmin and minmax operators for an interval matrix game given by the matrix with entries that are intervals, that is, $\tilde{A} = \{\tilde{a}_{ij}\}_{\substack{i=1,\dots,m\\ j=1,\dots,\ell}}$, where $\tilde{a}_{ij} = [a_{ijL}, a_{ijR}]$. We define the mid-point and half-width of interval $\tilde{a}$ as $\tilde{a} = [a_L, a_R]$ as $m(\tilde{a}) = (a_L + a_R)/2$ and $h(\tilde{a}) = (a_R - a_L)/2$, respectively.

Let $\tilde{a}, \tilde{b} \in I(\mathbb{R})$ be two intervals. For any $\tilde{a}, \tilde{b} \in I(\mathbb{R})$ such that $m(\tilde{a}) \le m(\tilde{b})$ and $h(\tilde{a}) + h(\tilde{b}) \neq 0$, an *acceptability index* to the premise $\tilde{a} \lesssim \tilde{b}$ is defined as follows (see Nayak and Pal (2009)):

$$\psi(\tilde{a} \lesssim \tilde{b}) = \frac{m(\tilde{b}) - m(\tilde{a})}{h(\tilde{b}) + h(\tilde{a})},$$

which is interpreted as "the value judgment or satisfaction degree of the Player that the interval $\tilde{a}$ is not superior to the interval $\tilde{b}$" (or $\tilde{b}$ is not inferior to $\tilde{a}$) in terms of value. In the interval value case, "not inferior to" and "not superior to" are analogous to "not less than" and "not greater than" in the real number set, respectively. We define the ordering of the interval based on acceptability index $\psi(\tilde{a} \lesssim \tilde{b})$ in the following way. Let operator $\bigvee$ correspond to the max operator and $\bigwedge$ to min operator. They are defined as follows:

$$\tilde{a} \bigvee \tilde{b} = \begin{cases} \tilde{b}, & \text{if } \psi(\tilde{a} \lesssim \tilde{b}) > 0, \\ \tilde{a}, & \text{if } \psi(\tilde{a} \lesssim \tilde{b}) = 0 \text{ and} \\ & h(\tilde{a}) < h(\tilde{b}) \text{ and Player is pessimistic}, \\ \tilde{b}, & \text{if } \psi(\tilde{a} \lesssim \tilde{b}) = 0 \text{ and} \\ & h(\tilde{a}) < h(\tilde{b}) \text{ and Player is optimistic}, \end{cases} \tag{3}$$

$$\tilde{a} \bigwedge \tilde{b} = \begin{cases} \tilde{b}, & \text{if } \psi(\tilde{b} \lesssim \tilde{a}) > 0, \\ \tilde{a}, & \text{if } \psi(\tilde{b} \lesssim \tilde{a}) = 0 \text{ and} \\ & h(\tilde{a}) > h(\tilde{b}) \text{ and Player is pessimistic}, \\ \tilde{b}, & \text{if } \psi(\tilde{b} \lesssim \tilde{a}) = 0 \text{ and} \\ & h(\tilde{a}) > h(\tilde{b}) \text{ and Player is optimistic}, \end{cases} \tag{4}$$

Following Li (2011) and Nayak and Pal (2009), the strategy profile (k, r) in pure strategies of the interval matrix game with payoff matrix $\tilde{A}$ is defined as a saddle point of the interval matrix game if $\bigvee_{1\le i\le m} \bigwedge_{1\le j\le \ell} \tilde{a}_{ij}$ being an analog of maxmin and $\bigwedge_{1\le j\le \ell} \bigvee_{1\le i\le m} \tilde{a}_{ij}$ as an analog of minmax exist and they are equal, i.e.

$$\tilde{a}_{kr} = \bigvee_{1\le i\le m} \bigwedge_{1\le j\le \ell} \tilde{a}_{ij} = \bigwedge_{1\le j\le \ell} \bigvee_{1\le i\le m} \tilde{a}_{ij}.$$

In the next section, we use maxmin and minmax for interval matrix games determined as follows:

$$\max_{1\le i\le m}\min_{1\le j\le \ell}\tilde{a}_{ij} := \bigvee_{1\le i\le m}\bigwedge_{1\le j\le \ell}\tilde{a}_{ij}, \tag{5}$$

$$\min_{1\le j\le \ell}\max_{1\le i\le m}\tilde{a}_{ij} := \bigwedge_{1\le j\le \ell}\bigvee_{1\le i\le m}\tilde{a}_{ij}. \tag{6}$$

3.3 α and β CFs for interval games

We extend the definitions of α and β characteristic functions to interval games defined by (1) and propose a method for constructing them.

Definition 3. The interval α CF is given by

$$\begin{aligned} w^{\alpha} &= \max_{x_S\in X_S}\min_{x_{N\setminus S}\in X_{N\setminus S}}\sum_{s\in S}u_s(x_S,x_{N\setminus S}) \\ &= \max_{x_S\in X_S}\min_{x_{N\setminus S}\in X_{N\setminus S}}\sum_{s\in S}[\underline{u}_s(x_S,x_{N\setminus S}),\bar{u}_s(x_S,x_{N\setminus S})], \end{aligned}$$

where $\underline{u}_s$, $\bar{u}_s$ are the lower and upper utility values, respectively.

When coalitions S and $N\setminus S$ select strategies x_S and $x_{N\setminus S}$, their interval payoffs are given by $u_S(x_S,x_{N\setminus S})$ and $u_{N\setminus S}(x_S,x_{N\setminus S})$, respectively.

Definition 4. The interval β CF is given by

$$\begin{aligned} w^{\beta} &= \min_{x_{N\setminus S}\in X_{N\setminus S}}\max_{x_S\in X_S}\sum_{s\in S}u_s(x_S,x_{N\setminus S}) \\ &= \min_{x_{N\setminus S}\in X_{N\setminus S}}\max_{x_S\in X_S}\sum_{s\in S}[\underline{u}_s(x_S,x_{N\setminus S}),\bar{u}_s(x_S,x_{N\setminus S})]. \end{aligned}$$

If the interval values of maxmin and minmax in pure strategies coincide, then the interval matrix game has a saddle point (which is also an interval) and the values of α and β CFs coincide. Otherwise, the game has no saddle point and these values are different.

When each player $i\in N$ chooses the strategies from the set of mixed strategies $\bar{X}_i$, we compute the lower and upper bounds of the matrix game in the mixed strategies (Li (2011)):

$$\max_{x_S\in \bar{X}_S}\min_{x_{N\setminus S}\in \bar{X}_{N\setminus S}}\sum_{s\in S}[\underline{u}_s(x_S,x_{N\setminus S}),\bar{u}_s(x_S,x_{N\setminus S})], \tag{7}$$

$$\min_{x_{N\setminus S}\in \bar{X}_{N\setminus S}}\max_{x_S\in \bar{X}_S}\sum_{s\in S}[\underline{u}_s(x_S,x_{N\setminus S}),\bar{u}_s(x_S,x_{N\setminus S})]. \tag{8}$$

Li (2011) proved that

$$\begin{aligned} &\max_{x_S\in \bar{X}_S}\min_{x_{N\setminus S}\in \bar{X}_{N\setminus S}}\sum_{s\in S}[\underline{u}_s(x_S,x_{N\setminus S}),\bar{u}_s(x_S,x_{N\setminus S})] \\ \lesssim\ &\min_{x_{N\setminus S}\in \bar{X}_{N\setminus S}}\max_{x_S\in \bar{X}_S}\sum_{s\in S}[\underline{u}_s(x_S,x_{N\setminus S}),\bar{u}_s(x_S,x_{N\setminus S})]. \end{aligned} \tag{9}$$

We give the comments on interval CFs given by Definitions 3 and 4:

1. Inequality (9) holds for interval matrix games, and this result is different from the one obtained for noninterval matrix games. In the latter case, the maxmin and minmax with mixed strategies always coincide and are equal to the value of the matrix game. Consequently, the α and β CF values are equal for noninterval normal-form games played in mixed strategies. This is not the case for the class of interval games. This is a first important difference between these two classes of games that is worth highlighting.

2. We note that both maxmin and minmax (in terms of (5) and (6)) in pure strategies exist since the set of pure strategies X_i is finite for any player $i \in N$ and when for any two intervals $\tilde{a}$ and $\tilde{b}$ to be compared the condition that $h(\tilde{a}) + h(\tilde{b}) \neq 0$ holds true. We note that both maxmin and minmax in mixed strategies exist (see Li (2011)).
3. If the interval matrix game of coalition S against coalition $N \setminus S$ has a saddle point in pure strategies, then the value of the game coincides with the values of α and β CF calculated for S. Otherwise, we can find maxmin and minmax in pure strategies to set the values of α and β CF, respectively. Another way to define these values is to calculate maxmin and minmax in mixed strategies by solving corresponding linear programming problems (Li (2011)).
4. The values of CFs depend on how we define max and min operations on intervals. In our paper, we show how CFs can be constructed based on definitions of maxmin and minmax given by (5) and (6) using acceptability index. One can differently order intervals using "weakly better" (denoted by $\succcurlyeq$) or "better" (denoted by $\succ$) approaches described at the end of Section 2. We do not discuss existence of maxmin and minmax values (in pure or mixed strategies) in interval matrix games in such a case of ordering. Another approach to compare intervals and to define the value of a matrix game is a lexicographic method proposed in Li (2011). The discussion of different approaches on interval ordering is given in Sengupta et al. (2001).

Next, we provide an algorithm to construct the α and β CFs by Definition 3 and 4, respectively. As we mentioned in Item 2, the maxmin and minmax values in pure strategies exist and we provide the algorithm where they are found in pure strategies, but the same algorithm can be reformulated for the class of mixed strategies for which existence of maxmin and minmax is proved in Li (2011). The procedure of finding maxmin and minmax values in mixed strategies in interval matrix games is also described in Li (2011).

3.4 Constructing interval characteristic functions

To determine the interval α CF, we implement the following steps:

1. Choose a coalition $S \subseteq N$.
2. Define a zero-sum interval game with coalition S being the maximizer player and coalition $N \setminus S$ the minimizer player:
   (a) The set of strategies of coalition S is $X_S = \prod_{i \in S} X_i$, where X_i is the set of pure strategies of Player i in game (1). The set of strategies of coalition $N \setminus S$ is $X_{N\setminus S} = \prod_{i \in N\setminus S} X_i$.
   (b) The payoff function of coalition S is given by
$$u_S(x_S, x_{N\setminus S}) = [\underline{u}_S(x_S, x_{N\setminus S}), \overline{u}_S(x_S, x_{N\setminus S})],$$
   where $x_S \in X_S, x_{N\setminus S} \in X_{N\setminus S}$, $\underline{u}_S(x_S, x_{N\setminus S}) = \sum_{s\in S} \underline{u}_s(x_S, x_{N\setminus S})$, $\overline{u}_S(x_S, x_{N\setminus S}) = \sum_{s\in S} \overline{u}_s(x_S, x_{N\setminus S})$.
   Similarly, for coalition $N \setminus S$, we have
$$u_{N\setminus S}(x_S, x_{N\setminus S}) = [\underline{u}_{N\setminus S}(x_S, x_{N\setminus S}), \overline{u}_{N\setminus S}(x_S, x_{N\setminus S})],$$
   where
$$\underline{u}_{N\setminus S}(x_S, x_{N\setminus S}) = \sum_{i\in N\setminus S} \underline{u}_i(x_S, x_{N\setminus S}),$$
$$\overline{u}_{N\setminus S}(x_S, x_{N\setminus S}) = \sum_{i\in N\setminus S} \overline{u}_i(x_S, x_{N\setminus S}).$$

   Hence, a zero-sum interval game is defined as follows:
$$\Gamma_S = \big\{\{S, N \setminus S\}, \{X_S, X_{N\setminus S}\}, \{u_S, u_{N\setminus S}\}\big\}.$$

3. Find a maxmin value in zero-sum interval game Γ_S. The procedure of finding maxmin value is described in Section 3.2. It consists in determining the gain floor of the maximizer player (coalition S) and the loss ceiling of the minimizer player (coalition $N \backslash S$) when choosing strategies $x_S \in X_S$ and $x_{N\backslash S} \in X_{N\backslash S}$, respectively. That is, we have

$$\hat{v}^* = \max_{x_S \in X_S} \min_{x_{N\backslash S} \in X_{N\backslash S}} u_S(x_S, x_{N\backslash S}),$$

which corresponds to coalition S's gain-floor, and

$$\hat{\omega}^* = \min_{x_{N\backslash S} \in X_{N\backslash S}} \max_{x_S \in X_S} u_S(x_S, x_{N\backslash S}),$$

which is the loss ceiling of coalition $N \setminus S$. Obviously, both quantities are intervals.
Set the value of α characteristic function for coalition S as the maxmin value:

$$w^\alpha(S) = \max_{x_S \in X_S} \min_{x_{N\backslash S} \in X_{N\backslash S}} u_S(x_S, x_{N\backslash S}).$$

4. Repeat steps 1-3 for all coalitions $S \subseteq N$.

The algorithm can also be used for computing the interval β CF by changing a maxmin value in Step 3 by a minmax value.

Next, we give the following result.

Proposition 1. Both interval α and interval β CFs, defined in Section 3.3, always exist. Moreover, $w^\alpha(N) = w^\beta(N)$, and for any $S \subset N$, we have $w^\alpha(S) \lesssim w^\beta(S)$.

Proof. The existence of interval α and β CFs follows from the existence of maxmin and minmax values in interval zero-sum games, which is proved in Li (2011).

Now, we prove that for any $S \subset N$, $w^\alpha(S) \lesssim w^\beta(S)$. For any coalition S and any strategy $x_S \in \bar{X}_S$, we have

$$\min_{x_{N\backslash S}} \left\{ \sum_{i \in S} u_i(x_S, x_{N\backslash S}) \right\} \lesssim \sum_{i \in S} u_i(x_S, x_{N\backslash S}).$$

Similarly, for any coalition $N \setminus S$ and strategy $x_{N\backslash S}$, it holds that

$$\max_{x_S} \left\{ \sum_{i \in S} u_i(x_S, x_{N\backslash S}) \right\} \gtrsim \sum_{i \in S} u_i(x_S, x_{N\backslash S}).$$

Thus, for any coalitions S and $N \setminus S$, we obtain

$$\min_{x_{N\backslash S}} \left\{ \sum_{i \in S} u_i(x_S, x_{N\backslash S}) \right\} \lesssim \max_{x_S} \left\{ \sum_{i \in S} u_i(x_S, x_{N\backslash S}) \right\}.$$

Therefore,

$$\min_{x_{N\backslash S}} \left\{ \sum_{i \in S} u_i(x_S, x_{N\backslash S}) \right\} \lesssim \min_{x_{N\backslash S}} \max_{x_S} \left\{ \sum_{i \in S} u_i(x_S, x_{N\backslash S}) \right\},$$

and finally, we have

$$\max_{x_S} \min_{x_{N\backslash S}} \left\{ \sum_{i \in S} u_i(x_S, x_{N\backslash S}) \right\} \lesssim \min_{x_{N\backslash S}} \max_{x_S} \left\{ \sum_{i \in S} u_i(x_S, x_{N\backslash S}) \right\},$$

that is,

$$w^\alpha(S) \lesssim w^\beta(S).$$

For $S = N$, we have

$$w^\alpha(N) = \max_{x_N} \min_{\varnothing} \sum_{i \in N} u_i(x_N) = \min_{\varnothing} \max_{x_N} \sum_{i \in N} u_i(x_N) = w^\beta(N).$$ □

Proposition 1 states that for any interval game α CF assigns smaller or equal value to any coalition S in comparison with β CF. This result also holds true for noninterval games.

4 An example

To illustrate the construction of interval α and β CFs given by a game $< N, w^j >$, $j \in \{\alpha, \beta\}$, we provide an example with three players. Each player has two strategies, that is, $X = \{x_1, x_2\}$ for Player 1, $Y = \{y_1, y_2\}$ for Player 2, and $Z = \{z_1, z_2\}$ for Player 3.

We represent players' utilities by matrices, where Player 1 chooses a row, Player 2 a column, and Player 3 a matrix:

Player 3: z_1

	y_1	y_2
x_1	$([1,2],[6,8],[1,3])$	$([3,5],[0,1],[4,6])$
x_2	$([4,6],[8,9],[2,5])$	$([7,10],[4,5],[2,4])$

Player 3: z_2

	y_1	y_2
x_1	$([0,2],[1,3],[2,4])$	$([5,7],[3,5],[0,3])$
x_2	$([3,6],[4,6],[5,6])$	$([7,9],[6,8],[0,2])$

Step 1. We start with coalition $S = \{1\}$, therefore Players 2 and 3 form a coalition $\{2,3\}$ that plays against Player 1. While we are calculating the value of $w(\{1\})$, we can also calculate the value $w(\{2,3\})$. So, we consider the similar steps for both $S = \{1\}$ and $S = \{2,3\}$.

Step 2. The sets of strategies of coalition $\{1\}$ is $X_{\{1\}} = \{x_1, x_2\}$ and of coalition $\{2,3\}$ is $X_{\{2,3\}} = \{(y_1,z_1),(y_2,z_1),(y_1,z_2),(y_2,z_2)\}$. We can write two corresponding zero-sum interval games:

		Coalition $\{2,3\}$			
		(y_1,z_1)	(y_2,z_1)	(y_1,z_2)	(y_2,z_2)
Coalition $\{1\}$	x_1	$[1,2]$	$[3,5]$	$[0,2]$	$[5,7]$
	x_2	$[4,6]$	$[7,10]$	$[3,6]$	$[7,9]$

		Coalition $\{1\}$	
		x_1	x_2
Coalition $\{2,3\}$	(y_1,z_1)	$[7,11]$	$[10,14]$
	(y_2,z_1)	$[4,7]$	$[6,9]$
	(y_1,z_2)	$[3,7]$	$[9,12]$
	(y_2,z_2)	$[3,8]$	$[6,10]$

Step 3. We find maxmin/minmax values in zero-sum interval game $\Gamma_{\{1\}}$:

$$\max_{x_{\{1\}}} \min_{x_{\{2,3\}}} u_1(x_{\{1\}}, x_{\{2,3\}}) = [3,6],$$
$$\min_{x_{\{2,3\}}} \max_{x_{\{1\}}} u_1(x_{\{1\}}, x_{\{2,3\}}) = [3,6].$$

Therefore, we set the values of α and β CFs for coalition $S = \{1\}$ as follows:

$$w^\alpha(\{1\}) = w^\beta(\{1\}) = [3,6].$$

We make similar calculations for coalition $\{2,3\}$ and find maxmin/minmax values in $\Gamma_{\{2,3\}}$:

$$\max_{x_{\{2,3\}}} \min_{x_{\{1\}}} \sum_{i \in \{2,3\}} u_i(x_{\{1\}}, x_{\{2,3\}}) = [7,11],$$
$$\min_{x_{\{1\}}} \max_{x_{\{2,3\}}} \sum_{i \in \{2,3\}} u_i(x_{\{1\}}, x_{\{2,3\}}) = [7,11],$$

and set the values of α and β CFs for coalition $S = \{2, 3\}$ as follows:

$$w^{\alpha}(\{2,3\}) = w^{\beta}(\{2,3\}) = [7, 11].$$

We proceed by considering other coalitions $\{2\}$ and $\{1, 3\}$, $\{3\}$ and $\{1, 2\}$ in a similar way. For the grand coalition $\{1, 2, 3\}$, the value of α and β CFs are defined as follows:

$$\begin{aligned} w^{\alpha}(\{1,2,3\}) = w^{\beta}(\{1,2,3\}) &= \max_{x\in X, y\in Y, z\in Z} \sum_{i\in N} u_i(x, y, z) \\ &= \max\{[8, 13], \ldots, [14, 20], [13, 19]\} = [14, 20]. \end{aligned}$$

The values of α and β CFs are given in Table 1.

Table 1: The values of α and β CFs

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
w^{α}	[3,6]	[1,3]	[1,3]	[11,15]	[7,11]	[7,11]	[14,20]
w^{β}	[3,6]	[3,5]	[2,4]	[12,15]	[8,12]	[7,11]	[14,20]

We notice that the values of α and β CFs are different for coalitions $\{2\}$, $\{3\}$, $\{1, 2\}$, and $\{1, 3\}$, which highlights that the choice of a CF is in itself an issue in a cooperative game.

Remark 2. We can easily notice that if use the ordering like $\succcurlyeq$ such that for any two intervals $I = [\underline{I}, \bar{I}]$ and $J = [\underline{J}, \bar{J}]$, $I \succcurlyeq J$ if and only if $\underline{I} \geq \underline{J}$ and $\bar{I} \geq \bar{J}$, then the maxmin (α CF) and minmax (β CF) values exist and they are equal to the values given in Table 1.

Based on the above results, we can make the following observations:

- Both interval w^{α} and w^{β} are not superadditive. Indeed, it is easy to verify that $|w^{\alpha}|(\{1, 3\}) = 4 \ngeq 5 = (|w^{\alpha}|(\{1\}) + |w^{\alpha}|(\{3\}))$ and $|w^{\beta}|(\{N\}) = 6 \ngeq 7 = (|w^{\alpha}|(\{2, 3\}) + |w^{\alpha}|(\{1\}))$. Recalling that α CF for noninterval games is superadditive, we conclude that this result cannot be transposed to interval games.
- Both games $< N, w^i >, i \in \{\alpha, \beta\}$ are size monotonic, i.e., $|w^i|(S) \leq |w^i|(T)$ for all $S, T \in 2^N$ with $S \subset T$.

Indeed, we have

α CF	β CF
$\|w^{\alpha}\|(\{1\}) = 3 < 4 = \|w^{\alpha}\|(\{1,2\})$	$\|w^{\beta}\|(\{1\}) = 3 = 3 = \|w^{\beta}\|(\{1,2\})$
$\|w^{\alpha}\|(\{1\}) = 3 < 4 = \|w^{\alpha}\|(\{1,3\})$	$\|w^{\beta}\|(\{1\}) = 3 < 4 = \|w^{\beta}\|(\{1,3\})$
$\|w^{\alpha}\|(\{2\}) = 2 = 2 = \|w^{\alpha}\|(\{1,2\})$	$\|w^{\beta}\|(\{2\}) = 2 < 4 = \|w^{\beta}\|(\{2,3\})$
$\|w^{\alpha}\|(\{2\}) = 2 < 4 = \|w^{\alpha}\|(\{2,3\})$	$\|w^{\beta}\|(\{2\}) = 2 = 2 = \|w^{\beta}\|(\{1,2\})$
$\|w^{\alpha}\|(\{3\}) = 2 < 4 = \|w^{\alpha}\|(\{1,3\})$	$\|w^{\beta}\|(\{3\}) = 2 < 4 = \|w^{\beta}\|(\{1,3\})$
$\|w^{\alpha}\|(\{3\}) = 2 < 4 = \|w^{\alpha}\|(\{2,3\})$	$\|w^{\beta}\|(\{3\}) = 2 < 4 = \|w^{\beta}\|(\{2,3\})$
$\|w^{\alpha}\|(\{1,2\}) = 4 < 6 = \|w^{\alpha}\|(\{1,2,3\})$	$\|w^{\beta}\|(\{1,2\}) = 3 < 6 = \|w^{\beta}\|(\{1,2,3\})$
$\|w^{\alpha}\|(\{1,3\}) = 3 < 6 = \|w^{\alpha}\|(\{1,2,3\})$	$\|w^{\beta}\|(\{1,3\}) = 3 < 6 = \|w^{\beta}\|(\{1,2,3\})$
$\|w^{\alpha}\|(\{2,3\}) = 4 < 6 = \|w^{\alpha}\|(\{1,2,3\})$	$\|w^{\beta}\|(\{2,3\}) = 4 < 6 = \|w^{\beta}\|(\{1,2,3\})$

- Both interval w^{α} and w^{β} are not supermodular because $w^i(S) + w^i(T) \preccurlyeq w^i(S \cup T) + w^i(S \cap T)$, $\forall S, T \in 2^N$ does not hold. It is easily seen that for $S = \{1, 2\}$ and $T = \{2, 3\}$ that

$$w^{\alpha}(\{1,2\}) + w^{\alpha}(\{2,3\}) = [18, 26] \succcurlyeq [15, 23] = w^{\alpha}(\{N\}) + w^{\alpha}(\{2\}).$$

Similarly, for $S = \{1, 2\}$ and $T = \{1, 3\}$ we have

$$w^{\beta}(\{1,2\}) + w^{\beta}(\{1,3\}) = [20, 27] \succcurlyeq [17, 26] = w^{\beta}(\{N\}) + w^{\beta}(\{1\}).$$

- Both interval w^α and w^β are nonconvex because their length games $< N, \left|w^i\right| >, i \in \{\alpha, \beta\}$ are not supermodular. Indeed, we have

$$|w^\alpha|(\{1\}) + |w^\alpha|(\{2\}) = 5 \nleq 4 = |w^\alpha|(\{1,2\}),$$
$$|w^\beta|(\{1\}) + |w^\beta|(\{2\}) = 5 \nleq 3 = |w^\beta|(\{1,2\}).$$

Let us now compute the interval Shapley value for α and β CFs using the following formulas:

$$\Phi_i(w^j) = \sum_{S: i \notin S} \frac{|S|!(n-1-|S|)!}{n!}(w^j(S \cup \{i\}) - w^j(S)), \quad i = 1,2,3,$$

where $j \in \{\alpha, \beta\}$.

We obtain

w \ Φ	Φ_1	Φ_2	Φ_3
w^α	$\left[\frac{36}{6}, \frac{50}{6}\right]$	$\left[\frac{30}{6}, \frac{41}{6}\right]$	$\left[\frac{18}{6}, \frac{29}{6}\right]$
w^β	$\left[\frac{35}{6}, \frac{48}{6}\right]$	$\left[\frac{32}{6}, \frac{42}{6}\right]$	$\left[\frac{17}{6}, \frac{30}{6}\right]$

We clearly see that the CFs do not yield the same Shapley values. Here, Player 1 prefers the α CF and Player 2 prefers the β CF, while Player 3 cannot prefer one of the interval values with ordering $\preccurlyeq$ (the intervals $\left[\frac{18}{6}, \frac{29}{6}\right]$ and $\left[\frac{17}{6}, \frac{30}{6}\right]$ are incomparable with $\preccurlyeq$). If we consider the interval ordering defined by (3) and (4), then again Player 1 prefers the α CF, Player 2 — β CF, while Player 3 prefers α CF if she is pessimistic and β CF if she is optimistic.

5 Concluding remarks

In this paper, we introduce interval α and β characteristic functions. In particular, our analysis shows that these different approaches yield different values for interval games and consequently different cooperative interval solutions. A simple three-person game example highlights the difference between the interval Shapley values calculated with α and β CFs. We find that both games $< N, w^\alpha >$ and $< N, w^\beta >$ may be nonconvex. The similarity with noninterval games is that for any interval game, the α approach to constructing CF values assigns a smaller or equal values to any coalition S in comparison with the β approach.

Two extensions are worth considering. First, attempt to define an algorithm that allows to compute all imputations in the core of an interval game with discrete strategy sets. Second, determining characteristic function values when the strategy sets are continuous and the payoff functions are defined by utility functions.

References

Alparslan Gök, S. Z., Branzei, R., and Tijs, S. (2009a). Convex interval games. Journal of Applied Mathematics and Decision Sciences, art.no. 342089.

Alparslan Gök, S. Z., Miquel, S., and Tijs, S. H. (2009b). Cooperation under interval uncertainty. Mathematical Methods of Operations Research, 69:99–109.

Alparslan Gök, S. Z., Branzei, O., Branzei, R., and Tijs, S. (2011). Set-valued solution concepts using interval-type payoffs for interval games. Journal of Mathematical Economics, 47(4-5):621–626.

Aumann, R. J. and Peleg, B. (1960). Von Neumann-Morgenstern solutions to cooperative games without side payments. Bulletin of the American Mathematical Society, 66(3):173–179.

Bhurjee, A. K. (2016). Existence of equilibrium points for bimatrix game with interval payoffs. International Game Theory Review, 18(01), 1650002.

Branzei, R., Dimitrov, D., and Tijs, S. (2003). Shapley-like values for interval bankruptcy games. Economics Bulletin, 3:1–8.

Branzei, R., Branzei, O., Alparslan Gök, S. Z., and Tijs, S. (2010a). Cooperative interval games: a survey. Central European Journal of Operations Research, 18:397–411.

Branzei, R., Tijs, S., and Alparslan Gök, S. Z. (2010b). How to handle interval solutions for cooperative interval games. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 18(02):123–132.

Charnes, A. and Granot, D. (1973). Prior solutions: Extensions of convex nucleus solutions to chance-constrained games. Center for Cybernetic Studies, University of Texas, 323-332.

Collins, W. D. and Hu, C.Y. (2005). Fuzzily Determined Interval Matrix Games. Proceedings of the BISCSE'05, University of California.

Drechsel, J. and Kimms, A. (2011). Cooperative lot sizing with transshipments and scarce capacities: solutions and fair cost allocations. International Journal of Production Research, 49(9):2643–2668.

Granot, D. (1977). Cooperative games in stochastic characteristic function form. Management Science, 23(6):621–630.

Hladík, Milan. (2010). Interval valued bimatrix games. Kybernetika, 46(3):435–446.

Kimms, A. and Drechsel, J. (2009). Cost sharing under uncertainty: an algorithmic approach to cooperative interval-valued games. Business Research, 2:206–213.

Li, D.-F. (2011). Notes on "linear programming technique to solve two-person matrix games with interval pay-offs". Asia-Pacific Journal of Operational Research, 28(06):705–737.

Li, D.-F., Nan, J.-X., and Zhang, M.-J. (2012). Interval programming models for matrix games with interval payoffs. Optimization Methods and Software, 27(1):1–16.

Mare, M. (2001). Fuzzy cooperative games: cooperation with vague expectations. Physica-Verlag, Heidelberg.

Mareš, M. and Vlach, M. (2004). Fuzzy classes of cooperative games with transferable utility. Scientiae Mathematicae Japonica, 2:269–278.

Nayak, P.K. and Pal, M. (2009). Linear programming technique to solve two person matrix games with interval pay-offs. Asia-Pacific Journal of Operational Research, 26(2):285–305.

Nishizaki, I. and Sakawa, M. (2000). Fuzzy cooperative games arising from linear production programming problems with fuzzy parameters. Fuzzy Sets and Systems, 114(1):11–21.

Parilina, E., Reddy, P. V., and Zaccour, G. (2022). Theory and Applications of Dynamic Games: A Course on Noncooperative and Cooperative Games Played Over Event Trees, (Vol.51). Springer Nature.

Sengupta, A., Pal, T.P., Chakraborty, D. (2001). Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming, Fuzzy Sets and Systems, 119(1):129–138.

Shapley, L. S. (1953). A value for n-person games. In Kuhn, H. W. and Tucker, A. W., editors, Contributions to the Theory of Games II, volume 28 of Annals of Mathematics Studies, pages 307–317. Princeton University Press, Princeton. Springer Nature.

Shashikhin, V. N. (2004). Antagonistic Game with Interval Payoff Functions. Cybernetics and Systems Analysis, 40(4):556–564.

Suijs, J., Borm, P., De Waegenaere, A., and Tijs, S. (1999). Cooperative games with stochastic payoffs. European Journal of Operational Research, 113(1):193–205.

Timmer, J., Borm, P., and Tijs, S. (2005). Convexity in stochastic cooperative situations. International Game Theory Review, 7(01):25–42.

Von Neumann, J. and Morgenstern, O. (1944). The Theory of Games and Economic Behavior, Princeton University Press.

Weber, G., Branzei, R., and Alparslan-Gök, S. (2010). On cooperative ellipsoidal games. In 24th Mini EURO Conference-On Continuous Optimization and Information-Based Technologies in the Financial Sector, MEC EurOPT, 369–372.