Les Cahiers du GERAD ISSN: 0711–2440

Multi-product production routing under decoupled planning periods

M. Chitsaz,
J.-F. Cordeau,
R. Jans

G–2020–35

June 2020

Citation suggérée : M. Chitsaz, J.-F. Cordeau, R. Jans (Juin 2020). Multi-product production routing under decoupled planning periods, Rapport technique, Les Cahiers du GERAD G–2020–35, GERAD, HEC Montréal, Canada.

Avant de citer ce rapport technique, veuillez visiter notre site Web (https://www.gerad.ca/fr/papers/G-2020-35) afin de mettre à jour vos données de référence, s'il a été publié dans une revue scientifique.

Suggested citation: M. Chitsaz, J.-F. Cordeau, R. Jans (June 2020). Multi-product production routing under decoupled planning periods, Technical report, Les Cahiers du GERAD G–2020–35, GERAD, HEC Montréal, Canada.

Before citing this technical report, please visit our website (https://www.gerad.ca/en/papers/G-2020-35) to update your reference data, if it has been published in a scientific journal.

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2020
– Bibliothèque et Archives Canada, 2020

Legal deposit – Bibliothèque et Archives nationales du Québec, 2020
– Library and Archives Canada, 2020

GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Multi-product production routing under decoupled planning periods

Masoud Chitsaz
Jean-François Cordeau
Raf Jans

GERAD & HEC Montréal, Montréal (Québec), Canada, H3T 2A7

masoud.chitsaz@hec.ca
jean-francois.cordeau@hec.ca
raf.jans@hec.ca

June 2020
Les Cahiers du GERAD
G–2020–35

Abstract: We consider an integrated optimization problem including the production, inventory, and outbound transportation decisions where a central plant fulfills the demand for several final products at its customers. More specifically, we investigate cases where the production planning and routing period lengths are not the same, e.g., days vs. shifts. Thus, we consider the fact that two different discretizations of the planning horizon exist in the decision-making process. This practical feature is a major source of complication for supply chain planners. With respect to the production planning aspect, we consider both big-bucket and small-bucket lot-sizing models. We mathematically formulate the problem under different practical scenarios for the production and route planning period lengths. An exact solution method, as well as heuristic algorithms, are proposed to efficiently solve large problem instances with this feature. To assess the effectiveness of our approach, we generate many test instances and perform an extensive computational study.

Keywords: Production, inventory, distribution, routing, supply chain integration

Acknowledgments: This work was partly supported by the Canadian Natural Sciences and Engineering Research Council under grants 2014–03849 and 2014–04959, and the Professorship in Operations Planning at HEC Montréal. This support is gratefully acknowledged.

1 Introduction

A major task in the supply chain planning process is the coordination of the production plan with the distribution and delivery plans. This entails integrating production scheduling with other important functions of the supply chain such as inventory management, shipment planning, and vehicle routing. Many studies in the literature, including Blumenfeld et al. (1987), Chandra and Fisher (1994), Chen and Vairaktarakis (2005) and Archetti and Speranza (2016), among others, report a significant cost saving potential by coordinating these activities. The problem that arises from the integration of the production and route planning processes is referred to in the literature as the production routing problem (PRP) (Adulyasak et al. 2015).

We investigate in this paper a generalized PRP which takes into account the fact that the production planning and the route planning period lengths are not necessarily identical. The overall planning horizon may, as a consequence, contain a different number of production and route planning periods. For the lot-sizing part of the formulation, we will consider both big-bucket and small-bucket problems. Furthermore, we consider several different products. A single plant coordinates the production scheduling for these multiple products as well as the routing decisions and shipment quantities to the customers. The customers have a time-varying and predetermined demand for each product. The aim is to minimize the total costs of production, inventories and distribution routing subject to the limitations of the problem. The plant has a limited capacity for the production. No backlogging or stockouts are allowed at the plant or at the customers. Both the plant and the customers can carry inventory from one period to the next. The plant, as well as the customers, each have a global storage capacity. The plant manages a limited fleet of capacitated vehicles to handle the shipment of products to the customers and split deliveries are not allowed.

The mathematical models used to solve real-life cases can be different due to the practical conditions which vary from one company to another. One such practical issue, in particular, is the difference in the planning period *lengths* for the production planning and the distribution routing. In such cases, the capacity of the production and routing may be expressed in a different time dimension, which creates the need to have a decoupled discretization of the time horizon. In practice, in some cases, multiple periods of distribution and transportation exist within one production planning period, e.g., the production planning period is one week whereas the routing is done on a daily basis. Conversely, in some other cases, the distribution planning is done using daily truck dispatches, but the production planning is performed on a shift-basis, where one day contains multiple shifts. Consequently, an important aspect of these multi-period problems is to deal with the different period lengths while properly representing the available capacity.

The current literature on the PRP and its variants only considers identical production planning and routing period lengths. This is in many cases an abstraction of the problem in the real world. We investigate the problem of coordinating the production and the routing decisions in a decoupled planning horizon. To the best of our knowledge, this is the first paper looking at this problem in this generality. This is the first contribution of this paper. Next, we present mathematical programming formulations for the problem. Third, we present a unified reformulation for which we develop cutting planes to improve the linear programming relaxation of the original formulation. Fourth, we show how to extend and enhance a state-of-the-art heuristic for the single-product PRP (Chitsaz et al. 2019) to the multi-product PRP (MP-PRP). Based on these advancements, we present an exact solution algorithm to solve MP-PRP. Finally, we show the significant impact of our cutting planes through extensive computational experiments.

The remainder of the paper is organized as follows. We present a review of the related literature in Section 2 in order to position our study with respect to the existing literature. Then, we formally define the problem and express it mathematically in Section 3. We present a reformulation for the problem in Section 4, which we use to prove new valid inequalities in Section 5. In Section 6, we describe the adaptation of a state-of-the-art heuristic to obtain good quality upper bounds for the problem, and further, we show how to enhance the method. The generation of the test instances and computational experiments are presented in Section 7. Finally, Section 8 concludes the paper.

2 Review of the related literature

Adulyasak et al. (2015) provide a comprehensive survey on the PRP including a review of different formulation schemes, various solution techniques, and algorithmic and computational issues. The literature reveals that the PRP has received a rapidly growing interest in the operations research and management community. The majority of the studies focus on the development of heuristic algorithms for this complex problem. Absi et al. (2014), Solyalı and Süral (2017) and Chitsaz et al. (2019) develop multi-phase mixed integer linear programming (MILP)-based heuristics for the single-product PRP.

We focus in this literature review on the related issues of the presence of multiple products and the length of the planning period. In the literature on the lot-sizing problem (LSP) (Pochet and Wolsey 2006), several different assumptions are made with respect to the length of the planning periods for multi-product problems. Typically, a distinction is made between small- and big-bucket models. In the basic big-bucket model, it is assumed that several types of products can be made on a shared resource within one time period, and no sequencing of products is done within a time period. The production of a product in a given period requires a specific setup. All products made in a specific time period can be used to satisfy demand at the end of the same time period. Big-bucket models typically have time periods in the order of a day to a week or even a month. The small-bucket models, on the other hand, assume that at most one type of product can be produced within one time period. A start-up occurs when a machine is set up for a new product which was not produced in the previous period. Typically, the small-bucket models include short production periods of a shift or a day. Within the small-bucket models, a further distinction is made between the Discrete Lot-sizing and Scheduling Problem and the Continuous Setup Lot-sizing Problem. In the former, one imposes that if there is production in a period, it must be at full capacity, whereas in the latter the production quantity can take any value up to the capacity limit.

In the following, we give examples from the literature on the application of big-bucket models in production and distribution planning. Glover et al. (1979) develop a computer-based integrated model for the production, distribution, and inventory planning at Agrico Chemical Company with a 12-month planning horizon and monthly time periods. Martin et al. (1993) optimize production, inventory, and distribution in a multi-plant system for the Flat Glass Products group of Libbey-Owens-Ford over 12 one-month planning periods. De Matta and Guignard (1994b) describe a big-bucket model with a planning horizon consisting of 52 one-week periods. They study the effects of production loss during setup in dynamic production scheduling for process industries producing several products on non-identical flexible processors. Hahn et al. (2000) present the coordinated production planning and scheduling activities among supply chain members of the Hyundai Motor Company at Ulsan, Korea. The company prepares a master production schedule with monthly time periods on a six-month rolling horizon basis. Next, they develop daily production and distribution schedules for each month to make the deliveries possible in one week and not more than 15 days as promised. Brown et al. (2001) study the cost minimization of integrated production, inventory, and distribution plans for the cereal and convenience foods business of Kellogg with weekly periods in a 30-week planning horizon. Çetinkaya et al. (2009) develop a cost-minimization model for integrated production and shipment planning for the Frito-Lay North American plant in Irving, Texas in a finite planning horizon of 12 weeks each representing one period. Neves-Moreira et al. (2019) propose an optimization framework to minimize the total production, inventory and transportation costs in a European meat processing center that produces and distributes multiple meat products among its store chain within working shifts of 8 hours and a break of 1 hour between shifts.

Similarly, some studies from the literature employed small-bucket planning periods for the production planning and scheduling. De Matta and Guignard (1994a) consider the manufacturing operations of a tile company with several production lines. The planning horizon spans over six months and up to the entire year with planning periods of one week for the bottleneck stage. Jans and Degraeve (2004) study the production planning problem at the Solideal group which is one of the major manufacturers and distributors of industrial tires worldwide. The authors report that the production start-ups only take place at the beginning of the morning shifts due to the limited availability of the qualified personnel and adequate supervision throughout the day. The planning period used is one day within a planning horizon of up to 30 days. Silva and Magalhaes (2006) study a production planning problem to minimize the number of tool changeovers while meeting the required due dates at an acrylic fibers production firm in the textile industry. In this study, the planning horizon is divided into four or five weeks with days as planning periods. Marinelli et al. (2007) consider a rolling horizon of one week consisting of five working days (periods) followed by two days off for a capacitated lot-sizing and scheduling problem with parallel machines and shared buffers in a packaging company producing yogurt.

Almost all of the literature on the MP-PRP focuses on the big-bucket LSP as the underlying production model. Chandra and Fisher (1994) were the first to study the effect of the coordination between the production planning and the vehicle routing to minimize the total costs of production, inventories, and transportation. Fumero and Vercellis (1999) study an MP-PRP variant in which split delivery to the customers is allowed. They propose a Lagrangian relaxation approach to solve the problem. Armentano et al. (2011) propose a tabu search with path relinking approach for the problem. Belo-Filho et al. (2015) investigate the coordinated production and distribution of perishable goods. They propose an adaptive large neighborhood search (ALNS) algorithm for the problem. Brahimi and Aouam (2016) study the problem with the possibility of backordering. They develop a solution procedure consisting of a relax-and-fix heuristic and a local search algorithm. Motivated by the industrial gas supply chains, Zhang et al. (2017) introduce an MP-PRP with multiple production capacity levels (modes) in a continuous production environment. They propose an iterative MILP-based heuristic that works with a restricted set of candidate routes at each iteration. The method dynamically updates the set of candidate routes for the next iteration. Miranda et al. (2018) study a rich MP-PRP arising in the context of a Brazilian furniture manufacturer. They consider many practical problem limitations such as sequence-dependent setup times, a heterogeneous fleet of vehicles, and customer time windows and deadlines. They propose a two-phase MILP-based iterative heuristic for the problem. There is only one recent study by Qiu et al. (2018) on the integration of the small-bucket LSP and the vehicle routing problem (VRP). They assume that the production period and routing period have equal lengths. The authors present a MILP to model the problem and provide valid inequalities to tighten the linear programming (LP) relaxation of the proposed model. They further use these inequalities in a branch-and-cut (BC) algorithm.

3 Problem definition and mathematical formulation

We first present common problem assumptions and definitions in Section 3.1. Next, we mathematically define the variables and constraints of the problem in Section 3.2. Finally, we describe specific big- and small-bucket model constraints in Sections 3.3 and 3.4, respectively.

3.1 Common assumptions and definitions

We consider a one-to-many production system where a central plant, denoted by node 0, provides several products for different customers, represented by the set $\mathcal{N}$. We let $\mathcal{N}^+ = \mathcal{N} \cup \{0\}$ represent the set of all nodes including the customers and the central plant. Let $\mathcal{E} = \{(i,j) : i,j \in \mathcal{N}^+, i < j\}$ be the set of all edges connecting the plant and the customers together. We represent by $\mathcal{K}$ the set of all products. In the classical production routing problem, the planning horizon comprises a finite number of discretized time planning periods with an equal length for the production and routing periods (Figure 1).

As indicated in the introduction, we will consider integrated planning problems where the production and routing periods do not necessarily have the same length. We assume that the production and

Figure 1: Planning horizon with equal period lengths

the route planning period lengths can be written as an integer multiple of the *micro period* length, which is defined as the smaller planning period length between the production and the route planning periods. We denote by $\pi \in \mathbb{N}$ and $\rho \in \mathbb{N}$ the integer multiples of the micro period length for the production and the route planning period lengths, respectively. According to the definition, either the production or the route planning period length is equal to the micro period length. Consequently, when the planning period lengths are different, either π or ρ is equal to 1 and the other is strictly greater than 1. In case both planning period lengths are identical, then $\pi = \rho = 1$. Let $\mathcal{T} = \{1, ..., |\mathcal{T}|\}$ be the set of micro periods. We assume that $|\mathcal{T}|$ is divisible by π and ρ. We denote the set of production planning periods by $\mathcal{T}^\pi = \{1, ..., |\mathcal{T}|/\pi\}$. Likewise, we represent the set of route planning periods by $\mathcal{T}^\rho = \{1, ..., |\mathcal{T}|/\rho\}$. Figure 2 shows the situation where the production planning period length is larger than the routing period length, whereas Figure 3 represents the inverse situation.

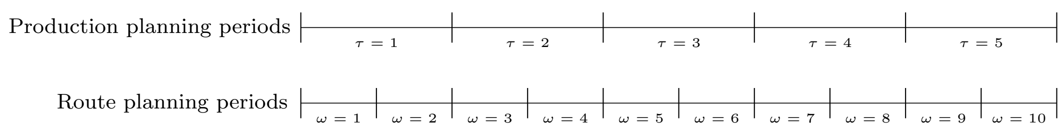

Figure 2: Longer production planning period lengths ($|\mathcal{T}| = 10, \pi = 2, \rho = 1, \tau \in \mathcal{T}^\pi, \omega \in \mathcal{T}^\rho$)

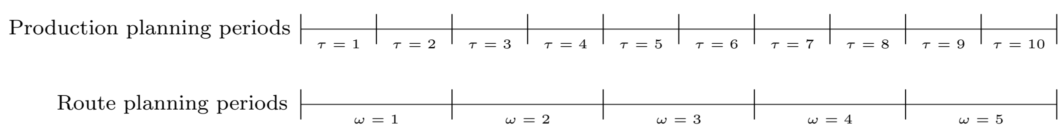

Figure 3: Longer route planning period lengths ($|\mathcal{T}| = 10, \pi = 1, \rho = 2, \tau \in \mathcal{T}^\pi, \omega \in \mathcal{T}^\rho$)

Product availability for shipment. In most production environments and for practical limitations, the production in each period is typically only available for shipment in the next period. This is because the shipments in the same period are already fixed, trucks and drivers are determined and planned to be dispatched. This situation is illustrated in Figure 4 for the case of equal production and route planning periods each equivalent to one day of operation. We index the route planning periods with one period shift/lag. Then, we consider the case that the production in each period is available for shipment in the next routing period which is indexed the same as the current production period. Figure 5 presents the case with longer production period. In this case, when we ship in period $\omega = 3$ or $\omega = 4$, products made in $\tau = 1, 2$ are available for shipment. Figure 6 presents the case with longer routing period in which the shipment in period $\omega = 2$ can include products made in production periods $\tau = 1, 2, 3, 4$.

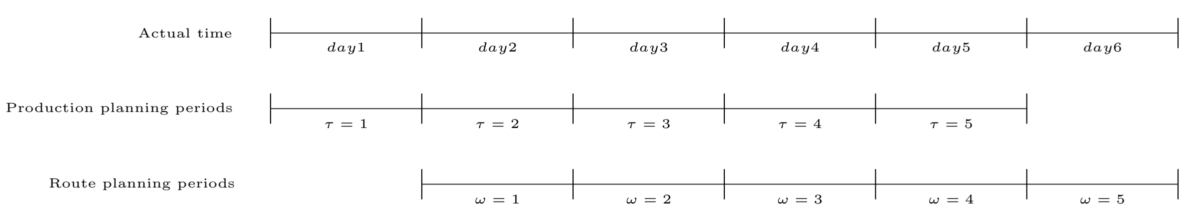

Figure 4: Product availability for shipment with equal period lengths ($|\mathcal{T}| = 5, \pi = 1, \rho = 1, \tau \in \mathcal{T}^\pi, \omega \in \mathcal{T}^\rho$)

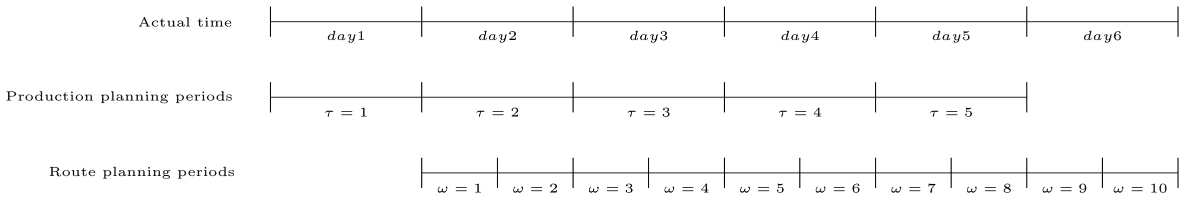

Figure 5: Product availability for shipment with longer production planning period lengths ($|\mathcal{T}| = 10, \pi = 2, \rho = 1, \tau \in \mathcal{T}^{\pi}, \omega \in \mathcal{T}^{\rho}$**)**

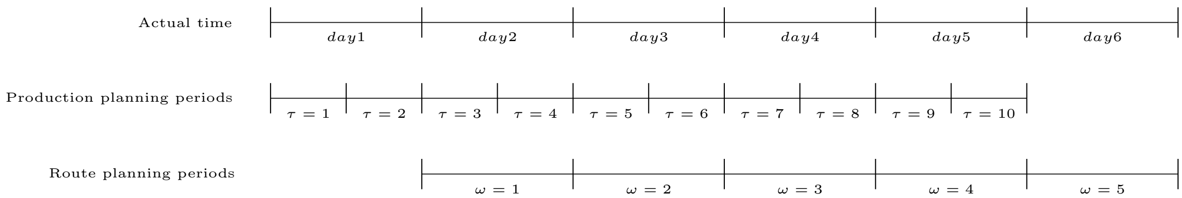

Figure 6: Product availability for shipment with longer route planning period lengths ($|\mathcal{T}| = 10, \pi = 1, \rho = 2, \tau \in \mathcal{T}^{\pi}, \omega \in \mathcal{T}^{\rho}$**)**

A one-period backward graphical shift in the routing period, makes Figures 4 to 6 equivalent to Figures 1 to 3, respectively. Therefore, without loss of generality, the entire production in any period is available for distribution in the period with the same index if the period lengths are equal. If the production period length is larger, the production in any period τ is available for distribution period $\omega = \pi\tau - 1$. If the routing period length is larger, the production in any period τ is available for distribution period $\omega = \lfloor \tau/\rho \rfloor + 1$. This choice of planning period indexing makes it possible to present formulations similar to those in many studies in the literature of the production routing problem (Archetti et al. 2011, Absi et al. 2014, Adulyasak et al. 2014).

Demand. We consider that the demand period length is equal to the route planning period length. Each customer $i \in \mathcal{N}$ has a predetermined demand $d_{ik\omega}$ for each product $k \in \mathcal{K}$ in each period $\omega \in \mathcal{T}^{\rho}$.

Production. The production system has to satisfy the demand for all products at every customer in each demand period without stockouts while respecting the plant's production capacity, which is given by C. We denote by θ_k the necessary capacity consumption to produce one unit of product $k \in \mathcal{K}$. The production of every product $k \in \mathcal{K}$ at the plant in a certain period imposes a fixed setup cost f_k.

Distribution. We consider b_k as the unit size of product $k \in \mathcal{K}$. A limited number of homogeneous vehicles, m, each with a capacity of Q, is available to perform shipments from the plant to the customers using routes that start and end at the plant. When a vehicle travels from location $i \in \mathcal{N}^+$ to $j \in \mathcal{N}^+$ a period-independent routing cost of c_{ij} is incurred.

Inventory bookkeeping. We consider the inventory bookkeeping at the plant to be aligned with the micro periods. When the production planning period length is smaller, this assumption is intuitive (Figure 3). For the case where the routing period length is smaller (Figure 7), during any production period, we have multiple route planning periods and thus it is possible to ship products from the plant within each routing period. Therefore, the level of the products' inventory at the plant may change during the production planning periods. Consequently, when the routing periods are smaller, micro period inventory level tracking enables a precise calculation of the inventory cost at the plant. We let I_{0k0} and I_{ik0} denote the initial inventory of product k at the plant and at the customer i, respectively. The cost at the plant of carrying one unit of product k over to the next micro period is h_{0k}. The cost at customer i to keep one unit of product k in the inventory in one route planning period is h_{ik}. Each customer $i \in \mathcal{N}$ has a global storage capacity L_i. The plant provides a shared storage with the capacity L_0 for all products.

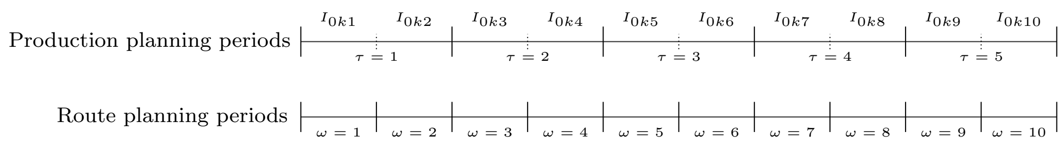

Figure 7: Inventory bookkeeping periods for the longer production planning period lengths ($|\mathcal{T}| = 10, \pi = 2, \rho = 1, \tau \in \mathcal{T}^\pi, \omega \in \mathcal{T}^\rho$)

3.2 Common variables and constraints

For each period $\tau \in \mathcal{T}^\pi$, we let the binary variable $y_{k\tau}$ take value 1 if and only if product $k \in \mathcal{K}$ is produced at the plant and we let $p_{k\tau}$ denote the production quantity. Let I_{0kt} and $I_{ik\omega}$ represent the inventory of product $k \in \mathcal{K}$ at the end of period $t \in \mathcal{T}$ at the plant, and at the end of period $\omega \in \mathcal{T}^\rho$ at the customer $i \in \mathcal{N}$, respectively. Let $q_{ik\omega}$ indicate the shipment quantity of product $k \in \mathcal{K}$ from the plant to the customer i in period $\omega \in \mathcal{T}^\rho$. The variable $x_{ij\omega}$ represents the number of times a vehicle traverses the edge $(i, j) \in \mathcal{E}$ in period $\omega \in \mathcal{T}^\rho$. The binary variable $z_{i\omega}$ takes value 1 if and only if a customer $i \in \mathcal{N}$ is visited in period $\omega \in \mathcal{T}^\rho$. The integer variable $z_{0\omega}$ indicates the number of vehicles dispatched from the plant in period $\omega \in \mathcal{T}^\rho$. The domain of the variables is imposed by constraints (1)–(6):

$$p_{k\tau} \geq 0, y_{k\tau} \in \{0, 1\} \qquad \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}^\pi, \tag{1}$$

$$I_{0kt} \geq 0 \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, \tag{2}$$

$$I_{ik\omega} \geq 0, q_{ik\omega} \geq 0 \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall \omega \in \mathcal{T}^\rho, \tag{3}$$

$$z_{0\omega} \in \mathbb{Z} \qquad \forall \omega \in \mathcal{T}^\rho, \tag{4}$$

$$z_{i\omega} \in \{0, 1\}, x_{0i\omega} \in \{0, 1, 2\} \qquad \forall i \in \mathcal{N}, \forall \omega \in \mathcal{T}^\rho, \tag{5}$$

$$x_{ij\omega} \in \{0, 1\} \qquad \forall (i, j) \in \mathcal{E} : i \neq 0, \forall \omega \in \mathcal{T}^\rho. \tag{6}$$

Constraints (7)–(9) provide the inventory flow balance at the plant. The production and the shipment variables are simultaneously present only during specific micro periods as presented in constraints (7). The cases are *(i)* the first micro period ($t \bmod \pi = 1$) of each large production period ($\pi > 1$ and $\rho = 1$), and *(ii)* the last micro period ($t \bmod \rho = 0$) of each large routing period ($\pi = 1$ and $\rho > 1$). Note that in case we have equal lengths for the production and routing periods, these are the only constraints needed. In the rest of the micro periods of the large production periods ($t \bmod \pi \neq 1, \pi > 1$ and $\rho = 1$), it is only necessary to balance the product inventory and the shipments as in constraints (8). Moreover, no shipment will be possible until the last micro period of the large routing periods ($t \bmod \rho \neq 0, \pi = 1$ and $\rho > 1$). Thus, constraints (9) keep track of the inventory at the plant for such cases:

$$I_{0k,t-1} + p_{k\tau} = \sum_{i \in \mathcal{N}} q_{ik\omega} + I_{0kt}$$

$$\forall k \in \mathcal{K}, \forall t \in \mathcal{T}, (t \bmod \pi = 1, \rho = 1) \vee (\pi = 1, t \bmod \rho = 0), \tau = (t-1)/\pi + 1, \omega = t/\rho \tag{7}$$

$$I_{0k,t-1} = \sum_{i \in \mathcal{N}} q_{ik\omega} + I_{0kt} \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, t \bmod \pi \neq 1, \rho = 1, \omega = t \tag{8}$$

$$I_{0k,t-1} + p_{k\tau} = I_{0kt} \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, \pi = 1, t \bmod \rho \neq 0, \tau = t. \tag{9}$$

The inventory balance constraints at the customers can be written as

$$I_{ik,\omega-1} + q_{ik\omega} = d_{ik\omega} + I_{ik\omega} \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall \omega \in \mathcal{T}^\rho. \tag{10}$$

Constraints (11) set the fleet size for each routing period. Constraints (12) enforce a vehicle to visit a node in case of a shipment to that node. The storage capacity at the plant and at the customers is imposed by constraints (13) and (14), respectively:

$$z_{0\omega} \leq m \qquad \forall \omega \in \mathcal{T}^\rho \tag{11}$$

$$\sum_{k \in \mathcal{K}} b_k q_{ik\omega} \leq Q z_{i\omega} \qquad \forall i \in \mathcal{N}, \forall \omega \in \mathcal{T}^\rho \tag{12}$$

$$\sum_{k \in \mathcal{K}} b_k I_{0kt} \leq L_0 \qquad \forall t \in \mathcal{T} \tag{13}$$

$$\sum_{k \in \mathcal{K}} b_k I_{ik\omega} \leq L_i \qquad \forall i \in \mathcal{N}, \forall \omega \in \mathcal{T}^\rho. \tag{14}$$

Let $\mathcal{E}(\mathcal{A})$ be the set of edges $(i,j) \in \mathcal{E}$ such that $i, j \in \mathcal{A}$, where $\mathcal{A} \subseteq \mathcal{N}$ is a given subset of nodes. Consider $\delta(\mathcal{A})$ as the set of edges incident to a node set $\mathcal{A}$, $\delta(\mathcal{A}) = \{(i,j) \in \mathcal{E} : i \in \mathcal{A}, j \notin \mathcal{A}$ or $i \notin \mathcal{A}, j \in \mathcal{A}\}$. The routing constraints include the node degree requirements (15) and the generalized vehicle routing capacity cuts (16) to eliminate the subtours and to impose the vehicle capacity. We refer to the latter set of constraints as the generalized fractional subtour elimination constraints (GFSEC) (Adulyasak et al. 2014):

$$\sum_{(j,j') \in \delta(i)} x_{jj'\omega} = 2 z_{i\omega} \qquad \forall i \in \mathcal{N}^+, \forall \omega \in \mathcal{T}^\rho \tag{15}$$

$$Q \sum_{(i,j) \in \mathcal{E}(\mathcal{A})} x_{ij\omega} \leq \sum_{i \in \mathcal{A}} (Q z_{i\omega} - \sum_{k \in \mathcal{K}} b_k q_{ik\omega}) \qquad \forall \mathcal{A} \subseteq \mathcal{N}, |\mathcal{A}| \geq 2, \forall \omega \in \mathcal{T}^\rho. \tag{16}$$

3.3 MP-PRP with big-bucket lot-sizing and scheduling

The big-bucket LSP assumes the possibility of producing several products in the same period on one shared resource with limited capacity (Trigeiro et al. 1989). Constraints (17) impose the global production capacity for each production period. The setup for each product is triggered by constraints (18) when its production takes place in any production period:

$$\sum_{k \in \mathcal{K}} \theta_k p_{k\tau} \leq C \qquad \forall \tau \in \mathcal{T}^\pi \tag{17}$$

$$\theta_k p_{k\tau} \leq C y_{k\tau} \qquad \forall k \in \mathcal{K}, \forall \tau \in \mathcal{T}^\pi. \tag{18}$$

The objective is to minimize the total cost of setups, inventory (at the plant and at the customers), and transportation as follows:

$$\min \sum_{k \in \mathcal{K}} (\sum_{\tau \in \mathcal{T}^\pi} f_k y_{k\tau} + \sum_{t \in \mathcal{T}} h_{0k} I_{0kt} + \sum_{\omega \in \mathcal{T}^\rho} \sum_{i \in \mathcal{N}} h_{ik} I_{ik\omega}) + \sum_{\omega \in \mathcal{T}^\rho} \sum_{(i,j) \in \mathcal{E}} c_{ij} x_{ij\omega}. \tag{19}$$

The mixed integer linear program for the PRP with a big-bucket lot-sizing structure, $\mathcal{M}^B_{MP-PRP}$, is to minimize the objective function (19), subject to constraints (1)–(18).

3.4 MP-PRP with small-bucket lot-sizing and scheduling

The small-bucket (continuous) LSP assumes that only one product can be made in every production period (Loparic et al. 2003). We let the binary variable $w_{k\tau}$ be the start-up variable for product k in period τ with an associated start-up cost, g_k. We consider the start-up for product k when it is not produced in period $\tau - 1$, and the machine is set up to produce it in period τ (Pochet and Wolsey 2006):

$$w_{k\tau} \in \{0,1\} \qquad \forall k \in K, \forall \tau \in \mathcal{T}^\pi \tag{20}$$

The start-up variables are modeled in constraints (21). Constraints (22) enforce the requirement that we can only produce one product in any production period. Constraints (23) impose the initial values for the setup variables:

$$w_{k\tau} \geq y_{k\tau} - y_{k,\tau-1} \qquad \forall k \in K, \forall \tau \in \mathcal{T}^\pi \tag{21}$$

$$\sum_{k \in K} y_{k\tau} \leq 1 \qquad \forall \tau \in \mathcal{T}^\pi \tag{22}$$

$$y_{k0} = 0 \qquad \forall k \in K. \tag{23}$$

The objective is to minimize the total cost of start-ups, inventory and transportation as follows:

$$\min \sum_{k\in\mathcal{K}} \Big(\sum_{\tau\in\mathcal{T}^\pi} g_k w_{k\tau} + \sum_{t\in\mathcal{T}} h_{0k} I_{0kt} + \sum_{\omega\in\mathcal{T}^\rho} \sum_{i\in\mathcal{N}} h_{ik} I_{ik\omega} \Big) + \sum_{\omega\in\mathcal{T}^\rho} \sum_{(i,j)\in\mathcal{E}} c_{ij} x_{ij\omega}. \tag{24}$$

The MP-PRP with a small-bucket (continuous) lot-sizing structure, $\mathcal{M}^S_{MP-PRP}$, minimizes the objective function (24), subject to constraints (1)–(16), (18), (20)–(23).

4 A reformulation

Constraints (7)–(9) impose the assumptions on the product flow at the plant level. However, it is not straightforward to strengthen the formulation and derive valid inequalities based on these constraints. We employ some modeling techniques to present these sets of constraints in a unified manner. The general idea is to reformulate the problem using only the micro periods which result in a formulation with the same number of periods at each level. We define π dummy micro periods for every large production planning period ($\pi \geq 1$). We consider ρ dummy micro periods for every large routing period ($\rho \geq 1$). First, we redefine the product demand and the holding cost (problem parameters) at the customers, $\mathbf{d}$ and $\mathbf{h}$, respectively, on the micro periods (equations (25)–(26)):

$$\mathbf{d}_{ikt} = d_{ik\omega}, \qquad \mathbf{h}_{ikt} = h_{ik} \qquad \forall i\in\mathcal{N}, \forall k\in\mathcal{K}, \forall t\in\mathcal{T}, t \bmod \rho = 0, \omega = t/\rho \tag{25}$$

$$\mathbf{d}_{ikt} = 0, \qquad \mathbf{h}_{ikt} = 0 \qquad \forall i\in\mathcal{N}, \forall k\in\mathcal{K}, \forall t\in\mathcal{T}, t \bmod \rho \neq 0. \tag{26}$$

Figure 8 shows an example of how the redefinition works for $\mathcal{T} = 10$ and $\rho = 2$. For all $i\in\mathcal{N}$ and $k\in\mathcal{K}$, we let $\mathbf{d}_{ikt} = 0$ for all $t\in\mathcal{T}$ such that $t \bmod \rho \neq 0$, and we let $\mathbf{d}_{ikt} = d_{ik(t/\rho)}$ for all $t\in\mathcal{T}$ such that $t \bmod \rho = 0$. In addition, we define $\mathbf{d}_{ikt_1t_2}$ as the demand for product $k\in\mathcal{K}$ at customer $i\in\mathcal{N}$ from period t_1 to period t_2 (inclusive), $t_1, t_2 \in \mathcal{T}, t_1 \leq t_2$.

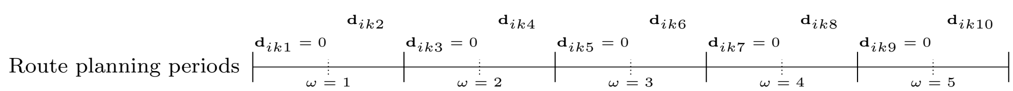

Figure 8: Dummy micro periods in the case of longer route planning period lengths ($|\mathcal{T}| = 10, \pi = 1, \rho = 2, \omega \in \mathcal{T}^\rho$)

Next, for each micro period $t\in\mathcal{T}$, we define variables $\mathbf{y}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{z}$ and $\mathbf{x}$ similar to y, p, q, z and x, respectively. Furthermore, we define new inventory variables, $\mathbf{I}_{ikt}$ on the micro periods $t\in\mathcal{T}$ only for the customers $i\in\mathcal{N}$ and for all $k\in\mathcal{K}$. Note that the inventory variables of the original formulation (Section 3) for the plant, I_{0kt}, are already defined on the micro periods $t\in\mathcal{T}$. The reformulation for the big-bucket MP-PRP can be written as the following $\mathcal{R}^B_{MP-PRP}$ model:

$$(\mathcal{R}^B_{MP-PRP}) \quad \min \sum_{t\in\mathcal{T}} \Big\{ \sum_{k\in\mathcal{K}} \big(f_k \mathbf{y}_{kt} + h_{0k} I_{0kt} + \sum_{i\in\mathcal{N}} \mathbf{h}_{ikt} \mathbf{I}_{ikt} \big) + \sum_{(i,j)\in\mathcal{E}} c_{ij} \mathbf{x}_{ijt} \Big\} \tag{27}$$

s.t. (2),(13), and

$$I_{0k,t-1} + \mathbf{p}_{kt} = \sum_{i\in\mathcal{N}} \mathbf{q}_{ikt} + I_{0kt} \qquad \forall k\in\mathcal{K}, \forall t\in\mathcal{T} \tag{28}$$

$$\mathbf{I}_{ik,t-1} + \mathbf{q}_{ikt} = \mathbf{d}_{ikt} + \mathbf{I}_{ikt} \qquad \forall i\in\mathcal{N}, \forall k\in\mathcal{K}, \forall t\in\mathcal{T} \tag{29}$$

$$\sum_{k\in\mathcal{K}} \theta_k \mathbf{p}_{kt} \leq C \qquad \forall t\in\mathcal{T} \tag{30}$$

$$\theta_k \mathbf{p}_{kt} \leq C \mathbf{y}_{kt} \qquad \forall k\in\mathcal{K}, \forall t\in\mathcal{T} \tag{31}$$

$$\mathbf{z}_{0t} \leq m \qquad \forall t\in\mathcal{T} \tag{32}$$

$$\sum_{k\in\mathcal{K}} b_k \mathbf{q}_{ikt} \leq Q \mathbf{z}_{it} \qquad \forall i\in\mathcal{N}, \forall t\in\mathcal{T} \tag{33}$$

$$\sum_{k\in\mathcal{K}} b_k \mathbf{I}_{ikt} \le L_i \qquad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{34}$$

$$\sum_{(j,j')\in\delta(i)} \mathbf{x}_{jj't} = 2\mathbf{z}_{it} \qquad \forall i \in \mathcal{N}^+, \forall t \in \mathcal{T} \tag{35}$$

$$Q \sum_{(i,j)\in\mathcal{E}(\mathcal{A})} \mathbf{x}_{ijt} \le \sum_{i\in\mathcal{A}} (Q\mathbf{z}_{it} - \sum_{k\in\mathcal{K}} b_k \mathbf{q}_{ikt}) \qquad \forall \mathcal{A} \subseteq \mathcal{N}, |\mathcal{A}| \ge 2, \forall t \in \mathcal{T} \tag{36}$$

$$\mathbf{y}_{kt} = 0 \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, t \bmod \pi \ne 1, \rho = 1 \tag{37}$$

$$\mathbf{z}_{it} = 0 \qquad \forall i \in \mathcal{N}, \forall t \in \mathcal{T}, \pi = 1, t \bmod \rho \ne 0 \tag{38}$$

$$\mathbf{z}_{0t} = 0 \qquad \forall t \in \mathcal{T}, \pi = 1, t \bmod \rho \ne 0 \tag{39}$$

$$\mathbf{p}_{kt} \ge 0, \mathbf{y}_{kt} \in \{0,1\} \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{40}$$

$$\mathbf{I}_{ikt} \ge 0, \mathbf{q}_{ikt} \ge 0 \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{41}$$

$$\mathbf{z}_{0t} \in \mathbb{Z} \qquad \forall t \in \mathcal{T} \tag{42}$$

$$\mathbf{z}_{it} \in \{0,1\}, \mathbf{x}_{0it} \in \{0,1,2\} \qquad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{43}$$

$$\mathbf{x}_{ijt} \in \{0,1\} \qquad \forall (i,j) \in \mathcal{E} : i \ne 0, \forall t \in \mathcal{T}. \tag{44}$$

The objective function (27) minimizes the total production, inventory, and transportation costs over the micro periods. Constraints (28) and (29) impose the product flow balance at the plant and at the customers, respectively. Constraints (30) and (31) are production capacity constraints. Constraints (32)–(34) enforce the fleet size, shipment capacity, and storage capacity at the customers. Constraints (35)–(36) are the node degree and subtour elimination constraints for the micro periods. Constraints (37) prevent setups in the micro periods where no production is possible. Constraints (38)–(39) forbid node visits and vehicle dispatches in the micro periods where no shipment is available. Constraints (40)–(44) define the domain for the reformulation variables.

Next, for each micro period $t \in \mathcal{T}$, we define variables $\mathbf{w}$ similar to w. The reformulation for the small-bucket MP-PRP, $\mathcal{R}^S_{MP-PRP}$, can be written as follows:

$$(\mathcal{R}^S_{MP-PRP}) \quad \min \sum_{t\in\mathcal{T}} \Big\{ \sum_{k\in\mathcal{K}} \big(g_k \mathbf{w}_{kt} + h_{0k} I_{0kt} + \sum_{i\in\mathcal{N}} \mathbf{h}_{ikt} \mathbf{I}_{ikt}\big) + \sum_{(i,j)\in\mathcal{E}} c_{ij} \mathbf{x}_{ijt} \Big\}, \tag{45}$$

s.t. (2), (13), (28)–(29), (31)–(44), and

$$\mathbf{w}_{kt} \ge \mathbf{y}_{kt} - \mathbf{y}_{k,t-\pi} \qquad \forall k \in K, \forall t \in \mathcal{T}, t \bmod \pi = 1, \rho = 1, \tag{46}$$

$$\sum_{k\in\mathcal{K}} \mathbf{y}_{kt} \le 1 \qquad \forall t \in \mathcal{T}, \tag{47}$$

$$\mathbf{w}_{kt} = 0 \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}, t \bmod \pi \ne 1, \rho = 1, \tag{48}$$

$$\mathbf{y}_{k0} = 0 \qquad \forall k \in \mathcal{K}, \tag{49}$$

$$\mathbf{w}_{kt} \ge 0 \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{T}. \tag{50}$$

Constraints (46)–(47) (together with (31)) impose the small-bucket LSP assumptions on the setup and start-up variables. Note that in constraints (46), the setup variables in each period t depend on the setup variables in periods t and $t-\pi$. Constraints (48) prevent start-ups in the micro periods where no production is possible. Constraints (49) force the initial values for the setup variables. Constraints (50) define the domain for the start-up variables.

Theorem 1 *$\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$ are valid reformulations for $\mathcal{M}^B_{MP-PRP}$ and $\mathcal{M}^S_{MP-PRP}$, respectively.*

Proof. See Appendix A.

5 Valid inequalities

We develop several valid inequalities to improve the LP relaxation bound of $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$. The inequalities in this section are inspired by prior work on similar problems: Archetti et al. (2007) for the IRP; Archetti et al. (2011) and Adulyasak et al. (2014) for the single product PRP; Chitsaz et al. (2020) for the assembly routing problem (ARP) which considers an assembly production structure; and Atamtürk and Küçükyavuz (2005) for the lot-sizing with inventory bounds and fixed costs. First, we present (l, S)-type and cut-set-type inequalities for the the lot-sizing structures of the models. Then, we provide inequalities concerning the distribution and routing structure of the models. The proofs of the propositions are provided in Appendix A.

5.1 Inequalities for the production and inventory flow structures

The *(l,S)* inequalities were introduced in Barany et al. (1984) where l refers to a period ($l \leq |T|$), and S is a subset of periods $\{1, ..., l\}$ not necessarily contiguous. Their cardinality is exponential and they are known to provide the convex hull for the single-item uncapacitated LSP. Pochet and Wolsey (1994) showed that when the sum of unit production and inventory costs in every period is larger than or equal to the unit production cost in the next period, it is sufficient to consider only a polynomial subset of these inequalities to describe the convex hull. These inequalities improve the linear relaxation bound of the lot-sizing structure (28)–(29) and (31). Because these two sets of constraints are present in both models, inequalities (51) are valid for $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$.

Proposition 1

$$\sum_{e=t_1}^{t_2} \mathbf{p}_{ke} \leq I_{0kt_2} + \sum_{i\in\mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{e=t_1}^{t_2} \big(\sum_{i\in\mathcal{N}} \mathbf{d}_{iket_2}\big)\mathbf{y}_{ke} \quad \forall k \in \mathcal{K}, \forall t_1, t_2 \in \mathcal{T}, t_1 \leq t_2 \tag{51}$$

are valid for $\mathcal{R}^B_{MP-PRP}$, $\mathcal{R}^S_{MP-PRP}$.

Next, we present lower bounds for the total number of required production setups ($\mathbf{y}_{kt}$) from period $e = 1$ to $t \in \mathcal{T}$ and for each product $k \in \mathcal{K}$.

Proposition 2 *Inequalities*

$$\left\lceil \frac{\max\left\{0, \sum_{i\in\mathcal{N}} \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} - I_{0k0}\right\}}{C/\theta_k} \right\rceil \leq \sum_{e=1}^{t} \mathbf{y}_{ke} \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{52}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ *and* $\mathcal{R}^S_{MP-PRP}$.

5.2 Inequalities for the distribution and inventory flow structures

Constraints (29) and (33) form a structure similar to those of constraints (28) and (31). Therefore, we present new (l, S)-type inequalities in Proposition 3.

Proposition 3 *Inequalities*

$$\sum_{e=t_1}^{t_2} \mathbf{q}_{ike} \leq \mathbf{I}_{ik,t_2} + \sum_{e=t_1}^{t_2} \mathbf{d}_{iket_2}\mathbf{z}_{ie} \quad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall t_1, t_2 \in \mathcal{T}, t_1 \leq t_2 \tag{53}$$

are valid for $\mathcal{R}^B_{MP-PRP}$, $\mathcal{R}^S_{MP-PRP}$.

In Propositions 4 and 5, we present lower bounds for the total number of required vehicle dispatches ($\mathbf{z}_{0t}$), and node visits ($\mathbf{z}_{it}$), respectively, from period $e = 1$ to $t \in \mathcal{T}$.

Proposition 4 *Inequalities*

$$\left\lceil \frac{1}{Q} \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} \right\rceil \leq \sum_{e=1}^{t} \mathbf{z}_{0e} \quad \forall t \in \mathcal{T} \tag{54}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ *and* $\mathcal{R}^S_{MP-PRP}$.

Proposition 5 *Inequalities*

$$\left\lceil \frac{\sum_{k \in \mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\}}{\min\left\{Q, L_i + \max_{1 \leq \theta \leq t}\{\sum_{k \in \mathcal{K}} b_k \mathbf{d}_{ik\theta}\}\right\}} \right\rceil \leq \sum_{e=1}^{t} \mathbf{z}_{ie} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{55}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ *and* $\mathcal{R}^S_{MP-PRP}$.

One observes that the LHS of inequalities (52) and (54)–(55) includes only problem parameters and hence returns integer values. In addition, we add two more sets of inequalities to improve the routing structure of both models. Inequalities (56) require a vehicle dispatch in case a node has to be visited in a certain period. The other set of inequalities, (57), is the adaptation of the Dantzig-Fulkerson-Johnson (DFJ) constraints to eliminate infeasible paths and maintain connectivity on the vehicle routes. They were first proposed by Dantzig et al. (1954) for the travelling salesman problem (TSP). These inequalities require that, in an integral solution, the number of edges in any subset of visited nodes is smaller than the cardinality of the set:

$$\mathbf{z}_{it} \leq \mathbf{z}_{0t} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{56}$$

$$\sum_{(i,j) \in E(\mathcal{A})} \mathbf{x}_{ijt} \leq \sum_{i \in \mathcal{A}} \mathbf{z}_{it} - \mathbf{z}_{\alpha t} \quad \forall \mathcal{A} \subseteq \mathcal{N}, |\mathcal{A}| \geq 2, \forall \alpha \in \mathcal{A}, \forall t \in \mathcal{T}. \tag{57}$$

The cardinality of these inequalities is exponential and thus they cannot be added a priori to the model in practical applications. These inequalities do not impose the vehicle capacity.

6 An upper bound heuristic

To obtain high-quality feasible solutions for the MP-PRP instances, we adapt the unified matheuristic proposed by Chitsaz et al. (2019). The authors applied this algorithm (CCJ-DH) to an assembly routing problem (ARP) where each supplier provides a distinct component. In addition, they applied CCJ-DH on the classic PRP and IRP instances where the plant/depot distributes only one type of product among many customers. In both studies, the authors report small optimality gaps for the solutions obtained by this heuristic especially on the large-scale instances of these problems. Therefore, to obtain high-quality feasible solutions for the MP-PRP instances, we adapt the unified matheuristic proposed in Chitsaz et al. (2019). We pass the solution obtained by this heuristic as cutoff values to our branch-and-cut algorithm.

The underlying idea in this algorithm is to heuristically solve the complex routing part and efficiently communicate the obtained routing costs in the objective function and with the rest of the model. This matheuristic works by decomposing the model into three independent subproblems and solving them iteratively. The first subproblem ($\mathcal{M}_y$) is a special LSP. This subproblem returns a setup schedule using an approximation of the total transportation cost in the objective function based on the number of dispatched vehicles. Using this given setup schedule, the second subproblem ($\mathcal{M}_z$) determines node visits and shipment quantities. In this subproblem, another approximation of the total transportation cost is considered in the objective function: the node visit transportation cost. Finally, the third subproblem considers a separate VRP for each period t. When the routing subproblems are solved, the algorithm updates the node visit cost approximation in the $\mathcal{M}_z$ model for the next iteration. This procedure is repeated to reach a local optimum. Then, the algorithm adds a diversification constraint (Fischetti et al. 2004) to the $\mathcal{M}_y$ model to change the setup schedule to explore other parts of the feasible solution space. The algorithm uses similar diversification constraints to generate new node visit patterns using the $\mathcal{M}_z$ model. The method terminates when a stopping condition is met.

Since we consider the multi-product variant of the PRP, we take this extension into account, compared to CCJ-DH implementation of Chitsaz et al. (2019), in the calculation of product inventories and inventory costs at the customers as well as the total shipment amount from the plant to each customer in all subproblems. However, the existence of multiple products as well as longer planning periods results in much larger subproblems which slow down the solution of the $\mathcal{M}_y$ and $\mathcal{M}_z$ models in this implementation. Efficiently solving these subproblems is a crucial step in the adaptation of CCJ-DH to obtain quality solutions for the MP-PRP variants. To overcome this challenge and to obtain a more efficient algorithm, we enhance the performance of CCJ-DH by adding relevant inequalities from Section 5. We add inequalities (51)–(52) and (54) to the $\mathcal{M}_y$ subproblem. Moreover, we incorporate inequalities (53) and (55) in the $\mathcal{M}_z$ subproblem.

7 Computational experiments

The computational experiments were performed on the Calcul Québec computing infrastructure with Intel Xeon X5650 @ 2.67 GHz processors and a memory limit of 25 GB. The BC procedure is implemented in C++ using the CPLEX 12.7 callable library. All experiments were performed in sequential form using one thread. We consider the best-bound node selection strategy for the BB search tree. We do not change any other CPLEX parameter. The algorithm applies the valid inequalities at the root node and adds GFSECs (36) and DFJ (57) at each node of the search tree as cutting planes whenever they are violated by more than 0.1 unit. To separate GFSECs, we use algorithm $\mathcal{A}1$ which is presented in Chitsaz et al. (2020). When a violated GFSEC (36) is found, the BC method also adds the corresponding DFJ (57). In our experiments, we set a time limit of one hour both for the BC method and for CCJ-DH.

7.1 MP-PRP test bed

Although some studies were conducted on the MP-PRP, there is no standard data set available for this problem. Therefore, we have developed the data sets for each of the extensions of the MP-PRP. The test instances were generated on the basis of the following data:

- micro period planning horizon $|\mathcal{T}|$: 12, 18, 24, 30; number of products $|\mathcal{K}|$: 4, 6, 8;
- number of customers $|\mathcal{N}|$ (increasing by steps of 5 for all $|\mathcal{T}|$ values): 5 to 35 for $|\mathcal{T}| = 12$, 5 to 30 for $|\mathcal{T}| = 18$, 5 to 25 for $|\mathcal{T}| = 24$, 5 to 20 for $|\mathcal{T}| = 30$;
- demand at customer i for product k in period t: constant over time, and random integer in the set $\{0, 1, 2\}$;
- storage capacity L_0 at the plant: uniformly distributed random integer (UDRI) in the interval $\big[|\mathcal{T}||\mathcal{K}||\mathcal{N}|/4, |\mathcal{T}||\mathcal{K}||\mathcal{N}|/3\big]$
- storage capacity L_i at customer i: UDRI in the interval $\big[|\mathcal{T}||\mathcal{K}|/4, |\mathcal{T}||\mathcal{K}|/3\big]$;
- production capacity C: UDRI in the interval $\big[|\mathcal{T}||\mathcal{K}||\mathcal{N}|/5, |\mathcal{T}||\mathcal{K}||\mathcal{N}|/4\big]$;
- production resource consumption θ_k for product k: random integer in the set $\{1, 2\}$;
- unit size b_k of product k: random integer in the set $\{1, 2\}$;
- truck capacity Q: $\big[10|\mathcal{K}|, 20|\mathcal{K}|$; number of trucks m: $|\mathcal{N}|$;
- initial inventory I_{0k0} of product k at the plant: UDRI in the interval $[0, 3|\mathcal{K}||\mathcal{N}|/2]$, initial inventory I_{ik0} of product k at customer i: UDRI in the interval $[0, 3|\mathcal{K}|/2]$;
- fixed setup/start-up cost f_k and g_k for product k: UDRI in the interval $[5000, 6000]$;
- holding cost h_{0k} of product k in each micro period at the plant: random integer in the set $\{1, 2\}$, holding cost h_{ik} of product k in each micro period at customer i: random integer in the set $\{3, 4\}$;

- longitude and latitude coordinates of the nodes (plant and the customers): UDRI in the interval $[0, 1500]$, transportation cost c_{ij}: Euclidean distance between nodes (rounded up to the nearest integer).

For each combination of the number of planning periods and customers we randomly generated 5 instances. As a result, the test bed includes medium ($|\mathcal{T}| = 12, |\mathcal{K}| = 4, |\mathcal{N}| = 5$) to very large size ($|\mathcal{T}| = 30, |\mathcal{K}| = 8, |\mathcal{N}| = 20$ or $|\mathcal{T}| = 12, |\mathcal{K}| = 8, |\mathcal{N}| = 35$) instances. Overall, instances are generated with 22 combinations of the planning horizons and numbers of customers, three numbers of product sizes and 5 instances per category. We apply the $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$ models for each instance. We consider $\pi = \{1, 2, 3\}, \rho = 1$ for the $\mathcal{R}^B_{MP-PRP}$ model, and $\pi = 1, \rho = \{1, 2, 3\}$ for the $\mathcal{R}^S_{MP-PRP}$ model. Note that the case where $\pi = \rho = 1$ corresponds to the case with equal period lengths at the production and routing levels and can be applied for both $\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$ models. Considering 6 combinations of the π and ρ parameters for both models, our test bed includes 1980 instances (990 instances for each model).

7.2 Performance of the heuristic

We report in Table 1 the performance of CCJ-DH with and without the addition of the valid inequalities. The results are presented for both small- and big-bucket models for $\rho = \pi = 1$. Each row in this table corresponds to a combination of the number of planning periods, number of products, and number of customers. In these tables, columns 4 to 12 and 13 to 21 include the results for the small-bucket and big-bucket MP-PRP instances, respectively. Columns four and five show the number of executed CCJ-DH iterations in the time limit for CCJ-DH without applying valid inequalities (None), and for the case where CCJ-DH is equipped with the valid inequalities (All), respectively. Column six presents the percent change in the number of iterations between these two implementations. Columns seven and eight show the average solution time in seconds for CCJ-DH with and without the inequalities, respectively. Column nine presents the percent change in the solution times. Columns 10 and 11 show the average solution values obtained by CCJ-DH without and with applying the valid inequalities, respectively. Column 12 presents the percent change in the average solution values. The same information is provided in columns 13 to 21 for the big-bucket model.

By adding the valid inequalities we were expecting to obtain better solution times. In addition, we also obtained better solution values due to the fact that on average the algorithm is able to perform more iterations in the one-hour time limit. On the small-bucket MP-PRP instances, the average number of iterations is increased by more than 29% and the average computing time is decreased by 34.2%. Moreover, on average, the solution values are improved by 0.4%. On the big-bucket MP-PRP instances, the improvement in the average solution values is 4.0%. This is obtained by a 26.7% increase in the number of iterations while the solution time is decreased by more than 38%. This is a significant improvement in the performance of CCJ-DH which is obtained by incorporating the valid inequalities.

7.3 Performance of valid inequalities

We further compare the effect of the valid inequalities on the performance of the BC method. In Tables 2–7, we report a summary of the results on the performance of the BC when we apply no inequality (None) or we employ inequalities (51)–(57) (All). These tables present CPU times, the average lower bound values as a percentage of the upper bound obtained by the BC without applying CCJ-DH cutoffs (%UB) and as a percentage of the best upper bound (%BUB) for each BC setting. To calculate the best upper bound (BUB) for each BC setting, we considered the upper bounds obtained by either that BC setting or CCJ-DH. When we do not consider the valid inequalities in the BC method (None), we do not apply them in CCJ-DH either. For the case where we include all inequalities in the BC method (All), we apply them in the heuristic cutoff procedure as well. In these tables, a zero value under %UB columns means that no feasible solution (UB) is found by the BC method. The results indicate that the BC performs better, in terms of the average solution time and optimality gap, when all inequalities are applied. Furthermore, in all cases for the planning period length scenarios and the bucket size models, valid inequalities create a significant improvement in the final results (%BUB). More specifically, on the big-bucket MP-PRP instances with four products ($k = 4$), employing the valid inequalities improves %BUB on average from 74.9% to 91.3%, 82.2% to 92.9%, and 86.5% to 95.0%,

Table 1: Performance of enhanced CCJ-DH with valid inequalities

			Small-Bucket LSP ($\rho = 1$)									Big-Bucket LSP ($\pi = 1$)								
			Iterations			CPU (s)			Avg Solution Value			Iterations			CPU (s)			Avg Solution Value		
l	k	n	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)
12	4	5	200	200	-0.2	248	196	-21.0	34183	34190	0.0	200	200	0.0	80	60	-25.6	34230	34301	0.2
		10	200	200	0.0	574	212	-63.1	40150	40011	-0.3	200	200	0.2	331	92	-72.1	39880	39962	0.2
		15	200	200	0.0	912	267	-70.7	50775	50810	0.1	201	200	-0.4	494	151	-69.3	50331	50132	-0.4
		20	200	200	0.0	1430	294	-79.4	56281	56240	-0.1	200	200	0.0	970	173	-82.1	55796	55658	-0.2
		25	200	200	-0.2	2670	429	-83.9	63327	63562	0.4	200	200	0.0	1265	230	-81.8	62888	62911	0.0
		30	192	200	4.4	3244	515	-84.1	69228	69054	-0.3	200	200	-0.2	2597	397	-84.7	67852	67820	0.0
		35	177	200	13.3	3365	622	-81.5	78314	78079	-0.3	191	200	4.7	3040	653	-78.5	77693	77652	-0.1
	6	5	200	200	0.0	365	241	-34.0	43093	43082	0.0	200	200	0.0	202	167	-17.0	42723	42707	0.0
		10	200	200	0.2	1478	329	-77.7	48757	48891	0.3	200	200	0.0	631	225	-64.4	48410	48525	0.2
		15	200	200	0.0	1718	980	-42.9	56683	56640	-0.1	200	200	0.0	1856	835	-55.0	55924	56452	0.9
		20	190	194	1.9	3170	1611	-49.2	61770	61672	-0.2	197	200	1.4	2814	460	-83.6	61090	60595	-0.8
		25	188	197	4.5	3211	1679	-47.7	67404	67317	-0.1	186	200	7.5	3302	751	-77.3	66360	66248	-0.2
		30	158	188	18.7	3582	2174	-39.3	72819	72740	-0.1	166	200	20.8	3559	1398	-60.7	71737	70892	-1.2
		35	133	186	39.6	3597	2221	-38.3	77567	77207	-0.5	170	200	17.6	3579	1085	-69.7	75884	75480	-0.5
	8	5	200	200	0.0	1450	240	-83.5	53041	52767	-0.5	200	200	0.2	794	215	-72.9	53173	53073	-0.2
		10	184	200	8.9	3562	455	-87.2	63032	62842	-0.3	185	200	7.9	3508	299	-91.5	62413	61482	-1.5
		15	167	200	20.1	3526	861	-75.6	68983	68863	-0.2	149	200	34.3	3589	780	-78.3	68040	67274	-1.1
		20	140	200	43.1	3584	1600	-55.3	80499	79842	-0.8	143	200	40.1	3583	952	-73.4	80092	76515	-4.5
		25	149	200	34.6	3393	1264	-62.7	82656	81318	-1.6	166	200	20.8	3584	662	-81.5	80274	78480	-2.2
		30	144	200	38.7	3081	1701	-44.8	90853	89879	-1.1	132	200	51.5	3588	1178	-67.2	89828	85252	-5.1
		35	123	200	62.9	3597	2118	-41.1	95675	95265	-0.4	127	200	58.3	3596	1602	-55.5	93896	89776	-4.4
18	4	5	193	200	3.4	1529	1081	-29.3	44005	44193	0.4	195	195	0.0	1381	1163	-15.8	47542	47354	-0.4
		10	184	200	8.7	3062	1228	-59.9	50401	50341	-0.1	193	200	3.5	2986	1350	-54.8	50394	50390	0.0
		15	165	200	21.1	3581	2304	-35.7	65459	65661	0.3	173	200	16.0	3588	2542	-29.2	67004	66833	-0.3
		20	168	200	19.2	3598	3093	-14.0	83147	83209	0.1	175	195	11.8	3367	3219	-4.4	84535	84373	-0.2
		25	153	195	27.4	3371	3139	-6.9	91539	90713	-0.9	164	195	18.9	3280	3268	-0.4	91764	91674	-0.1
		30	155	187	20.4	3583	3063	-14.5	105551	105052	-0.5	159	179	12.3	3597	3582	-0.4	107328	106134	-1.1
	6	5	200	200	0.2	2162	1159	-46.4	48619	48686	0.1	200	200	0.0	1184	360	-69.6	47936	47955	0.0
		10	133	189	42.0	3582	1433	-60.0	62216	61944	-0.4	163	188	15.3	3586	1449	-59.6	61406	61210	-0.3
		15	133	190	43.0	3589	2054	-42.8	73024	72573	-0.6	171	200	17.2	3021	1071	-64.5	72569	70633	-2.7
		20	115	200	74.5	3598	2243	-37.7	84906	84728	-0.2	127	186	46.6	3589	1469	-59.1	85001	82481	-3.0
		25	108	182	69.5	3589	2882	-19.7	96786	96242	-0.6	130	200	54.3	3579	1131	-68.4	94421	92337	-2.2
		30	109	185	70.0	3597	3050	-15.2	102082	101396	-0.7	110	200	82.1	3578	1561	-56.4	101295	98367	-2.9
	8	5	186	200	7.5	2764	754	-72.7	61821	61516	-0.5	200	200	0.0	2508	603	-75.9	60828	60521	-0.5
		10	118	200	70.4	3579	927	-74.1	75394	75052	-0.5	129	200	55.0	3594	778	-78.3	74838	73190	-2.2
		15	104	200	92.7	3567	1786	-49.9	86876	86392	-0.6	105	200	91.2	3596	1913	-46.8	87568	83270	-4.9
		20	99	200	101.2	3581	2668	-25.5	99424	98795	-0.6	100	200	98.8	3588	2519	-29.8	101322	95169	-6.1
		25	93	200	114.1	3583	3043	-15.1	111463	110933	-0.5	96	182	89.6	3596	3464	-3.6	114865	105700	-8.0
		30	94	200	113.7	3573	3035	-15.1	126134	124048	-1.7	94	188	100.0	3583	3361	-6.2	126910	119132	-6.1
24	4	5	186	200	7.5	2105	1427	-32.2	45357	45304	-0.1	191	187	-2.1	1836	1617	-11.9	48114	48210	0.2
		10	163	192	17.4	3586	2572	-28.3	70579	70707	0.2	171	195	14.4	3237	2835	-12.4	77565	77060	-0.7
		15	150	178	19.1	3582	3549	-0.9	89644	89110	-0.6	141	170	20.6	3571	3541	-0.8	92714	91905	-0.9
		20	153	187	22.5	3394	3402	0.2	104324	104300	0.0	153	186	21.3	3572	3462	-3.1	109874	109908	0.0
		25	148	182	22.7	3575	3588	0.4	128542	128189	-0.3	164	182	10.9	3556	3569	0.4	132887	131388	-1.1
	6	5	183	200	9.5	3052	1056	-65.4	58642	58960	0.5	200	200	0.0	1368	775	-43.3	58364	58643	0.5
		10	107	186	73.8	3592	2053	-42.9	74421	74210	-0.3	130	181	38.9	3590	2318	-35.4	74195	72214	-2.7
		15	101	172	70.7	3582	2765	-22.8	98203	96800	-1.4	116	165	42.4	3589	2591	-27.8	102216	97108	-5.0
		20	97	164	69.6	3587	3376	-5.9	111720	112166	0.4	109	198	82.0	3579	2355	-34.2	118017	106025	-10.2
		25	85	185	118.6	3004	3243	8.0	129831	130443	0.5	125	200	59.5	3305	2228	-32.6	133399	125202	-6.1
	8	5	126	194	53.2	3560	1706	-52.1	70954	70295	-0.9	152	200	31.6	3042	1993	-34.5	69948	69255	-1.0
		10	101	191	89.9	3589	2012	-43.9	89034	87965	-1.2	107	200	86.6	3589	1683	-53.1	90867	86005	-5.4
		15	112	161	44.4	3348	2838	-15.2	108971	109180	0.2	91	198	116.6	3570	2910	-18.5	114113	103271	-9.5
		20	90	145	60.3	3588	2895	-19.3	129890	129134	-0.6	92	191	107.6	3576	3303	-7.6	132371	121384	-8.3
		25	88	145	63.9	3592	3088	-14.0	149381	149307	0.0	91	187	105.5	3590	3286	-8.5	155235	139698	-10.0
30	4	5	148	169	14.8	3593	3082	-14.2	56334	56214	-0.2	168	172	2.1	3439	3381	-1.7	59314	59069	-0.4
		10	146	180	23.0	3588	3314	-7.6	86405	87148	0.9	154	162	5.3	3580	3452	-3.6	96627	94970	-1.7
		15	138	181	30.9	3584	3578	-0.2	107240	107804	0.5	147	158	7.2	3576	3594	0.5	113827	112225	-1.4
		20	133	168	26.0	3588	3578	-0.3	131404	131804	0.3	141	169	20.1	3585	3585	0.0	139196	137726	-1.1
	6	5	113	179	58.5	3578	2537	-29.1	64532	64005	-0.8	147	189	28.2	3508	2074	-40.9	63748	62687	-1.7
		10	99	166	67.5	3598	3307	-8.1	97895	96699	-1.2	126	186	48.4	3253	2292	-29.5	99542	94685	-4.9
		15	95	176	84.9	3609	2762	-23.5	113507	114587	1.0	95	200	110.1	3583	2200	-38.6	123983	112317	-9.4
		20	109	167	52.8	3590	3587	-0.1	150344	146634	-2.5	119	200	67.5	3300	3097	-6.2	154903	145043	-6.4
	8	5	106	198	87.9	3599	2772	-23.0	82451	82577	0.2	121	198	63.4	3588	2461	-31.4	85282	80352	-5.8
		10	92	169	84.1	3589	3288	-8.4	110313	109752	-0.5	94	181	93.8	3580	2758	-23.0	122004	104552	-14.3
		15	89	92	2.7	3588	2219	-38.1	144806	145270	0.3	91	168	83.6	3575	2240	-37.3	154682	134254	-13.2
		20	89	94	5.9	3587	2495	-30.5	170344	168063	-1.3	89	161	80.5	3585	2687	-25.1	179248	155739	-13.1
Average			144	187	29.6	3107	2046	-34.2	84833	84521	-0.4	152	192	26.7	2953	1810	-38.7	86641	83164	-4.0

Table 2: Performance of branch-and-cut algorithm on the big-bucket LSP ($k = 4$)

		$\pi = 1$						$\pi = 2$						$\pi = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	1059	99.8	99.8	383	100.0	100.0	235	100.0	100.0	184	100.0	100.0	170	100.0	100.0	68	100.0	100.0
	10	3584	93.8	94.6	3587	96.2	96.5	3582	95.2	97.3	3587	97.1	97.1	3581	95.5	95.5	2934	98.3	98.3
	15	3587	28.3	91.1	3589	15.9	93.4	3585	47.6	93.0	3589	15.6	94.3	3585	67.4	94.9	3588	61.3	95.9
	20	3588	25.2	83.1	3589	13.5	92.2	3587	28.8	89.0	3589	10.6	93.3	3586	30.1	89.6	3589	13.7	95.7
	25	3588	15.9	72.7	3589	8.5	89.7	3589	18.3	83.1	3590	11.5	91.5	3589	16.0	86.0	3589	8.5	93.3
	30	3590	7.9	67.2	3589	6.6	89.6	3590	9.6	82.6	3588	9.5	90.9	3590	10.2	85.3	3590	10.4	94.6
	35	3590	3.9	62.1	3590	7.9	88.3	3590	4.8	67.4	3590	8.8	90.4	3591	7.1	72.1	3590	9.3	93.7
18	5	3584	90.2	90.5	2604	98.0	98.0	3212	98.4	98.4	2224	98.9	98.9	2437	98.6	98.6	2187	99.5	99.5
	10	3588	30.8	86.8	3588	17.1	92.6	3586	46.1	92.6	3589	16.8	94.3	3587	50.4	94.7	3589	62.6	94.9
	15	3590	13.7	76.2	3590	11.9	91.2	3590	18.9	88.7	3590	12.0	93.2	3589	23.1	93.6	3590	28.6	96.5
	20	3589	11.7	70.0	3590	5.8	88.7	3589	13.6	78.5	3590	7.8	91.2	3589	15.3	87.9	3590	10.4	93.8
	25	3590	5.4	64.1	3590	7.2	88.8	3590	8.7	74.6	3590	10.0	91.0	3590	7.0	85.3	3590	8.5	94.2
	30	3590	3.9	58.3	3590	5.1	89.0	3590	5.2	67.5	3590	7.5	91.5	3591	6.0	73.3	3590	8.0	94.8
24	5	3587	90.5	94.2	3216	97.9	97.9	3537	97.4	98.0	3187	98.8	98.8	3584	97.4	97.7	3297	99.1	99.1
	10	3589	22.4	81.9	3590	10.0	91.7	3589	27.8	91.7	3590	43.7	93.0	3588	34.1	94.3	3590	15.3	96.4
	15	3590	6.7	66.3	3590	8.9	88.3	3590	10.3	74.4	3590	11.4	91.0	3590	13.2	84.0	3590	12.0	93.7
	20	3590	5.3	61.0	3590	11.1	88.0	3590	6.8	66.5	3590	11.1	89.9	3590	8.3	75.7	3590	12.2	92.1
	25	3590	5.4	53.2	3590	7.6	85.9	3591	7.3	60.7	3590	8.3	88.0	3591	8.9	69.6	3589	9.3	90.6
30	5	3588	60.4	83.8	3590	81.4	96.8	3587	84.6	93.3	3589	96.6	97.7	3588	82.6	94.2	3432	74.3	98.3
	10	3590	12.8	74.1	3590	6.8	88.8	3590	21.0	84.9	3590	9.7	90.8	3589	30.7	90.0	3590	8.3	93.3
	15	3590	5.6	60.8	3590	9.3	87.4	3590	7.7	65.0	3590	9.3	88.4	3590	9.4	71.5	3590	11.2	90.6
	20	3590	5.7	55.8	3589	7.3	86.8	3590	8.1	62.3	3590	7.9	88.5	3590	9.3	69.5	3590	11.5	90.1
Avg		3474	29.3	74.9	3382	28.8	91.3	3417	34.8	82.2	3354	31.9	92.9	3381	37.3	86.5	3316	35.1	95.0

Table 3: Performance of branch-and-cut algorithm on the big-bucket LSP ($k = 6$)

		$\pi = 1$						$\pi = 2$						$\pi = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	686	100.0	100.0	56	100.0	100.0	99	100.0	100.0	21	100.0	100.0	78	100.0	100.0	25	100.0	100.0
	10	3587	96.4	97.2	1315	99.8	99.8	3009	97.3	97.5	1069	99.7	99.7	2733	99.9	99.9	864	99.9	99.9
	15	3588	17.8	79.9	3590	65.0	96.1	3588	53.3	92.8	3590	97.1	97.3	3588	56.6	96.9	3050	82.8	98.1
	20	3589	10.6	77.5	3590	26.1	95.6	3588	14.0	84.8	3590	30.9	96.2	3590	18.2	95.2	3590	47.1	97.1
	25	3590	4.6	52.8	3591	4.4	94.2	3590	7.6	64.8	3590	8.0	94.8	3590	9.4	83.3	3591	8.3	95.3
	30	3590	0.8	49.1	3590	1.6	94.0	3591	5.1	60.0	3590	5.6	94.8	3590	3.3	73.4	3590	6.3	94.9
	35	3591	0.7	42.4	3590	2.6	92.7	3590	2.9	54.0	3590	8.1	93.9	3590	0.0	58.6	3590	3.9	93.7
18	5	3021	92.8	92.8	297	100.0	100.0	1391	99.8	99.8	145	100.0	100.0	1365	100.0	100.0	49	100.0	100.0
	10	3589	24.1	87.0	3590	67.2	97.9	3589	33.8	95.6	3558	97.0	98.4	3588	39.4	95.8	3590	98.7	98.9
	15	3589	7.7	70.2	3590	3.2	95.5	3590	11.1	88.6	3590	5.6	96.0	3590	16.3	93.0	3590	12.8	98.6
	20	3590	5.1	59.4	3590	1.4	92.1	3590	7.6	71.6	3590	1.9	94.1	3590	10.3	84.8	3590	6.7	98.2
	25	3590	3.5	48.8	3590	2.6	91.9	3590	2.1	55.0	3590	4.8	92.3	3590	5.2	61.4	3590	8.0	97.0
	30	3590	0.6	43.6	3590	7.4	91.3	3591	0.8	55.1	3590	3.3	92.4	3591	0.0	59.0	3590	7.3	96.7
24	5	3588	86.1	86.7	2948	99.4	99.4	3588	96.4	96.4	1934	99.9	99.9	3588	96.6	97.1	455	100.0	100.0
	10	3589	12.8	72.0	3590	66.0	96.9	3589	18.7	91.3	3590	96.5	98.3	3589	23.5	93.1	3590	80.8	98.4
	15	3590	3.9	57.3	3590	0.0	91.5	3590	6.7	67.5	3590	2.1	95.3	3590	10.3	77.0	3590	7.5	95.5
	20	3590	3.7	48.1	3590	2.5	92.8	3590	5.7	57.5	3590	5.2	93.5	3591	7.0	62.2	3590	3.9	94.4
	25	3590	2.8	44.5	3590	3.8	92.0	3591	1.8	50.8	3590	5.9	92.0	3591	1.2	55.0	3590	9.8	97.6
30	5	3589	63.1	76.6	1733	100.0	100.0	3589	67.0	94.0	973	100.0	100.0	3589	85.9	97.3	1191	99.6	99.9
	10	3590	6.8	64.3	3590	2.4	93.7	3590	10.6	74.4	3590	7.4	94.5	3590	14.6	81.7	3590	7.1	95.1
	15	3590	3.2	47.2	3589	1.3	93.9	3590	6.2	56.8	3589	2.2	94.3	3590	8.9	63.2	3589	4.5	96.8
	20	3590	3.6	46.4	3589	3.9	90.8	3590	5.1	51.4	3589	3.3	91.5	3590	6.4	57.0	3589	9.2	94.6
Avg		3432	25.0	65.6	3063	34.6	95.1	3305	29.7	75.4	2961	40.2	95.9	3290	32.4	81.1	2867	41.1	97.3

respectively for $\pi = 1$, $\pi = 2$ and $\pi = 3$ (Table 2). On the same LSP type MP-PRP instances with six products ($k = 6$), the addition of the valid inequalities increases %BUB on average from 65.6% to 95.1%, 75.4% to 95.9%, and 81.1% to 97.3%, respectively for $\pi = 1$, $\pi = 2$ and $\pi = 3$ (Table 3). On the big-bucket MP-PRP instances with eight products ($k = 8$) which consist of the highest number of products, the implementation of the valid inequalities increases %BUB on average from 56.6% to 96.8%, 67.2% to 97.3%, and 76.1% to 97.7%, respectively for $\pi = 1$, $\pi = 2$ and $\pi = 3$ (Table 4). This indicates the substantial impact of applying the valid inequalities.

Similarly, on the small-bucket MP-PRP instances with four products ($k = 4$), employing the valid inequalities improves %BUB on average from 63.8% to 84.9%, 77.3% to 89.7%, and 86.5% to 92.1%, respectively for $\rho = 1$, $\rho = 2$ and $\rho = 3$ (Table 5). On the small-bucket instances with six products ($k = 6$), the addition of the valid inequalities increases %BUB on average from 49.7% to 86.4%, 65.8% to 90.4%, and 76.5% to 94.3%, respectively for $\rho = 1$, $\rho = 2$ and $\rho = 3$ (Table 6). On the big-bucket instances with eight products ($k = 8$) with the largest number of products, the addition of the valid inequalities increases %BUB on average from 44.3% to 86.1%, 59.4% to 90.7%, and 67.7% to 93.8%, respectively for $\rho = 1$, $\rho = 2$ and $\rho = 3$ (Table 7). This indicates the substantial impact of applying the

Table 4: Performance of branch-and-cut algorithm on the big-bucket LSP ($k = 8$)

		$\pi = 1$						$\pi = 2$						$\pi = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	618	100.0	100.0	7	100.0	100.0	1586	96.7	96.8	5	100.0	100.0	87	100.0	100.0	4	100.0	100.0
	10	3587	33.0	59.9	170	100.0	100.0	3588	49.7	70.1	139	100.0	100.0	3588	62.4	89.3	123	100.0	100.0
	15	3589	34.7	63.0	2365	99.8	99.8	3589	51.1	74.7	947	100.0	100.0	3589	51.6	88.0	1464	100.0	100.0
	20	3590	7.9	51.2	3590	97.1	97.2	3587	9.0	63.5	3590	96.6	97.0	3588	11.5	72.7	3589	98.9	99.0
	25	3588	3.5	48.4	3590	21.5	95.3	3587	7.3	59.0	3590	38.9	95.6	3588	8.2	65.3	3590	95.2	97.4
	30	3588	2.2	42.3	3590	1.4	94.2	3588	4.9	50.3	3590	0.0	94.7	3588	7.1	62.0	3590	18.4	95.4
	35	3587	0.0	38.0	3590	0.0	93.0	3588	0.9	48.8	3590	1.5	93.2	3588	2.6	58.2	3590	1.8	93.9
18	5	3585	82.8	89.4	34	100.0	100.0	1968	100.0	100.0	11	100.0	100.0	2268	99.7	99.7	15	100.0	100.0
	10	3586	17.4	68.8	2702	99.4	99.4	3587	25.4	85.5	1990	99.7	99.7	3586	30.2	93.9	1364	100.0	100.0
	15	3587	5.6	53.5	3589	40.4	97.6	3587	9.0	71.9	3590	48.8	98.3	3587	11.9	84.4	3147	78.5	98.9
	20	3587	1.8	53.0	3589	1.3	94.9	3588	5.8	63.8	3590	0.0	95.5	3589	9.9	78.6	3589	2.6	95.4
	25	3588	3.6	45.2	3589	2.8	94.9	3588	3.2	51.8	3590	1.9	95.2	3587	7.1	61.1	3589	2.3	95.1
	30	3588	0.8	36.8	3590	1.3	93.0	3588	0.0	46.8	3590	3.3	93.8	3587	0.0	57.0	3590	1.8	93.8
24	5	3586	66.5	84.9	383	100.0	100.0	3585	79.2	92.7	158	100.0	100.0	3585	83.8	89.5	142	100.0	100.0
	10	3586	9.6	62.4	3576	78.9	97.9	3589	13.9	75.8	3437	83.1	99.0	3587	15.9	90.6	3471	99.6	99.6
	15	3589	4.3	50.5	3581	0.0	96.6	3590	5.8	64.0	3586	1.9	96.9	3590	8.5	77.3	3589	7.1	97.9
	20	3590	3.9	47.9	3589	0.0	94.9	3589	4.3	54.8	3590	0.0	95.8	3590	6.1	64.5	3590	0.0	98.1
	25	3589	3.4	43.5	3590	1.4	94.7	3589	5.0	50.2	3590	5.0	95.9	3589	6.4	55.4	3590	2.3	97.7
30	5	3588	30.1	70.3	763	100.0	100.0	3588	73.5	87.7	434	100.0	100.0	3589	86.4	90.7	553	100.0	100.0
	10	3589	5.6	52.3	3590	2.2	96.6	3590	7.8	68.1	3590	3.4	96.8	3589	12.6	77.4	3590	4.8	96.2
	15	3589	4.0	44.0	3590	0.0	96.2	3589	5.6	57.1	3590	1.8	97.3	3589	8.1	64.7	3589	8.3	93.7
	20	3589	3.1	40.1	3590	0.0	94.4	3589	4.6	46.2	3590	1.8	95.9	3589	6.3	53.6	3590	4.6	97.2
	Avg	3453	19.3	56.6	2738	43.1	96.8	3423	25.6	67.2	2608	44.9	97.3	3369	28.9	76.1	2589	51.2	97.7

Table 5: Performance of branch-and-cut algorithm on the small-bucket LSP ($k = 4$)

		$\rho = 1$						$\rho = 2$						$\rho = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	1540	99.7	99.7	1003	99.8	99.8	380	100.0	100.0	74	100.0	100.0	25	100.0	100.0	14	100.0	100.0
	10	3586	85.6	94.0	3588	95.1	95.7	3581	98.2	98.2	3079	98.2	98.2	2418	99.5	99.5	2028	100.0	100.0
	15	3589	40.3	70.1	3590	0.0	92.2	3584	51.2	91.3	3587	52.7	93.5	3581	0.0	93.8	3583	38.3	95.3
	20	3589	0.0	64.1	3590	12.6	91.4	3587	27.0	85.5	3588	15.1	92.7	3584	13.2	93.6	3585	0.0	94.5
	25	3590	0.0	59.5	3591	0.0	88.1	3588	0.0	71.4	3589	0.0	91.1	3586	0.0	91.7	3587	12.5	92.8
	30	3591	0.0	56.0	3590	0.0	85.5	3589	0.0	70.2	3590	3.8	91.1	3588	0.0	83.2	3589	0.0	91.9
	35	3590	0.0	46.6	3590	0.0	84.3	3590	0.0	68.9	3591	0.0	89.8	3589	0.0	79.0	3590	0.0	91.4
18	5	3587	91.6	92.4	3589	97.6	97.6	2752	98.7	98.7	1788	99.7	99.7	1252	99.8	99.8	1256	100.0	100.0
	10	3589	48.5	70.7	3589	16.4	90.9	3585	28.9	88.3	3587	70.1	93.7	3583	73.0	94.1	3584	52.1	94.1
	15	3590	8.1	60.6	3590	2.1	88.1	3587	12.0	77.2	3588	31.6	90.6	3586	28.8	90.6	3586	33.4	92.7
	20	3590	0.0	59.5	3590	0.0	81.0	3589	0.0	70.1	3590	0.0	89.9	3587	14.2	87.1	3588	0.0	90.5
	25	3591	0.0	51.7	3590	0.0	81.5	3590	0.0	68.0	3590	0.0	89.7	3589	0.0	81.2	3589	0.0	91.0
	30	3590	0.0	46.5	3590	0.0	78.3	3591	0.0	65.0	3590	0.0	84.8	3590	0.0	71.9	3590	0.0	89.8
24	5	3588	77.0	81.3	3589	96.3	96.3	3368	97.3	98.0	3001	99.1	99.1	1789	100.0	100.0	1356	99.9	99.9
	10	3589	28.5	70.9	3590	18.9	89.4	3587	62.5	78.5	3589	41.2	91.7	3585	27.2	89.3	3587	26.6	92.9
	15	3590	7.4	57.2	3590	0.0	77.0	3588	21.0	67.4	3590	0.0	86.7	3587	12.7	80.1	3588	28.9	88.2
	20	3590	0.0	47.5	3590	0.0	75.0	3590	0.0	64.2	3591	0.0	84.1	3589	0.0	75.2	3590	0.0	88.9
	25	3590	0.0	44.1	3590	0.0	69.8	3590	0.0	60.0	3590	0.0	74.0	3590	0.0	74.4	3590	0.0	83.3
30	5	3589	13.6	74.6	3590	54.4	92.5	3586	75.6	87.8	3079	77.3	95.7	3456	91.0	94.6	2940	76.2	96.0
	10	3590	7.1	58.9	3591	0.0	73.8	3588	8.9	69.0	3590	4.4	85.6	3587	39.3	82.1	3589	13.0	87.3
	15	3591	0.0	50.7	3590	0.0	72.1	3589	0.0	62.9	3590	4.0	80.5	3588	0.0	73.6	3590	0.0	85.4
	20	3590	0.0	46.3	3591	0.0	67.7	3590	0.0	60.0	3590	0.0	71.4	3590	0.0	68.7	3590	4.9	81.0
	Avg	3496	23.1	63.8	3472	22.4	84.9	3394	31.0	77.3	3275	31.7	89.7	3178	31.8	86.5	3118	31.2	92.1

valid inequalities. In Appendix B, we report the lower bound improvements obtained by incorporating the valid inequalities in the small- and big-bucket models. Generally, when the instances are harder to solve (smaller ρ and π), the impact of the inequalities on the lower bound improvement is bigger.

7.4 Analysis of the cost shares

Finally, we analyze the cost component shares on different MP-PRP instances. Tables 8 and 9 present the different cost component values and proportions for $\rho = \{1, 2, 3\}$ and $\pi = \{1, 2, 3\}$, respectively for the small- and big-bucket LSP instances. In Table 8, columns three, 10, and 17 show the total cost values. Columns four to nine present the production, inventory, and the transportation costs and shares (in percent), respectively for $\rho = 1$. Columns 11 to 16 and 18 to 23 do the same for the cases where $\rho = 2$ and $\rho = 3$, respectively. Table 9 reports the same information for big-bucket LSP instances. In all cases, the share of the production setup cost decreases when for the same number of periods, the number of customers increases. In most situations, for any number of periods and π (or ρ) combination, the share of the inventory cost, and the share of the transportation cost increase when

Table 6: Performance of branch-and-cut algorithm on the small-bucket LSP ($k = 6$)

		$\rho = 1$						$\rho = 2$						$\rho = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	3095	98.4	98.4	1247	99.9	99.9	986	100.0	100.0	109	100.0	100.0	127	100.0	100.0	14	100.0	100.0
	10	3583	65.8	68.2	3581	98.2	98.2	3578	88.5	88.7	2360	99.7	99.7	3570	99.4	99.4	272	100.0	100.0
	15	3584	27.2	54.2	3581	50.9	96.5	3580	14.6	77.2	3577	93.2	97.5	3573	68.9	90.0	3577	78.3	98.8
	20	3581	0.0	49.9	3583	0.0	94.2	3578	13.3	70.4	3580	38.8	96.6	3579	47.9	88.9	2981	58.3	97.3
	25	3587	5.3	40.1	3585	0.0	91.6	3582	0.0	62.2	3585	0.0	95.4	3581	25.0	83.2	3581	19.5	96.2
	30	3587	0.0	33.4	3590	0.0	88.3	3580	0.0	60.0	3586	0.0	95.1	3577	0.0	78.5	3588	0.0	97.0
	35	3581	0.0	29.3	3590	0.0	84.4	3580	0.0	57.1	3585	0.0	92.5	3578	0.0	72.9	3583	0.0	95.4
18	5	3577	69.7	72.7	3184	99.5	99.5	3573	91.1	91.5	2049	99.8	99.8	3351	97.5	97.5	168	100.0	100.0
	10	3579	38.4	60.4	3584	56.9	96.2	3576	37.5	70.8	3581	78.0	97.2	3573	41.6	80.3	3463	97.8	98.2
	15	3586	0.0	50.8	3584	0.0	89.2	3583	32.5	71.0	3584	47.0	95.0	3575	49.5	77.1	3579	57.6	96.6
	20	3586	0.0	42.5	3584	0.0	83.9	3583	0.0	60.7	3582	14.3	92.8	3582	10.5	71.4	3580	19.5	95.9
	25	3586	0.0	39.3	3584	0.0	81.0	3581	0.0	56.6	3584	0.0	88.0	3584	0.0	67.7	3582	0.0	94.3
	30	3586	0.0	30.7	3577	0.0	81.5	3591	0.0	52.9	3577	0.0	82.2	3585	0.0	66.3	3576	0.0	93.6
24	5	3580	67.7	71.1	3404	97.3	97.3	3575	84.4	84.4	3193	98.0	98.2	3578	94.3	94.3	1405	99.8	99.8
	10	3584	16.5	52.0	3582	0.0	88.0	3581	39.8	61.3	3581	83.4	94.5	3580	38.0	69.7	3571	96.8	98.0
	15	3585	0.0	44.5	3576	0.0	79.9	3583	18.9	57.2	3578	0.0	87.4	3581	37.3	63.4	3581	17.4	91.6
	20	3584	0.0	42.3	3577	0.0	76.1	3584	0.0	53.9	3577	0.0	79.8	3581	0.0	63.6	3581	13.8	88.3
	25	3585	0.0	35.6	3583	0.0	71.0	3584	0.0	47.2	3583	0.0	74.9	3584	0.0	60.5	3578	0.0	85.5
30	5	3582	35.6	55.4	3579	92.1	93.9	3581	64.8	69.1	3585	94.4	96.3	3579	78.6	81.4	2236	97.8	97.8
	10	3583	0.0	43.9	3587	11.4	73.6	3586	43.5	55.7	3586	0.0	83.7	3586	27.7	60.7	3581	17.9	87.7
	15	3590	0.0	39.3	3582	0.0	71.2	3587	7.3	52.1	3583	0.0	75.6	3580	25.2	58.8	3582	27.8	84.8
	20	3583	0.0	38.5	3582	0.0	64.5	3582	0.0	46.7	3582	0.0	67.6	3580	0.0	57.0	3582	0.0	77.9
Avg		3562	19.3	49.7	3450	27.6	86.4	3463	28.9	65.8	3281	38.5	90.4	3412	38.2	76.5	2920	45.6	94.3

Table 7: Performance of branch-and-cut algorithm on the small-bucket LSP ($k = 8$)

		$\rho = 1$						$\rho = 2$						$\rho = 3$					
		None			All			None			All			None			All		
l	n	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB	CPU	%UB	%BUB
12	5	3586	80.7	81.2	2276	99.8	99.8	3581	89.5	89.5	1261	100.0	100.0	2823	99.6	99.6	41	100.0	100.0
	10	3588	0.0	59.6	3589	99.1	99.1	3585	17.1	78.1	1372	99.9	99.9	3583	63.1	77.2	180	100.0	100.0
	15	3589	0.0	47.8	3590	97.7	97.7	3587	56.3	69.8	3588	98.6	98.7	3584	29.1	72.9	1529	100.0	100.0
	20	3590	0.0	43.9	3590	37.1	92.2	3588	0.0	65.4	3589	96.6	96.6	3586	0.0	70.5	3375	98.8	98.8
	25	3590	0.0	36.7	3590	0.0	90.8	3589	0.0	59.6	3590	71.4	94.9	3587	40.9	69.4	3588	98.1	98.1
	30	3590	0.0	37.4	3590	17.5	89.1	3590	0.0	52.4	3590	0.0	90.7	3589	13.3	65.4	3589	72.7	94.7
	35	3590	0.0	27.4	3590	0.0	84.9	3590	0.0	52.0	3590	15.2	91.0	3590	0.0	65.6	3589	38.3	95.0
18	5	3588	44.3	61.1	2708	99.9	99.9	3586	73.0	74.1	1208	99.9	99.9	3584	84.7	84.7	305	100.0	100.0
	10	3589	0.0	48.6	3590	57.8	95.6	3587	12.8	65.5	3588	98.3	98.3	3586	47.6	69.5	3106	99.6	99.6
	15	3590	0.0	43.2	3590	0.0	86.9	3588	15.5	57.2	3589	91.9	94.4	3587	36.8	66.7	3588	98.1	98.1
	20	3590	0.0	40.3	3590	0.0	85.5	3589	0.0	60.2	3590	51.3	93.4	3588	11.3	66.5	3588	57.5	96.0
	25	3590	0.0	40.1	3590	0.0	83.6	3590	0.0	53.4	3590	0.0	88.0	3589	0.0	65.1	3590	35.8	94.1
	30	3590	0.0	27.7	3590	0.0	81.3	3590	0.0	48.4	3590	0.0	83.8	3590	0.0	64.0	3590	0.0	91.8
24	5	3588	30.9	54.5	3590	98.1	98.1	3586	41.3	65.8	3153	99.2	99.2	3585	72.1	72.2	1899	99.9	99.9
	10	3590	7.0	44.8	3590	16.1	86.2	3587	9.1	56.1	3590	95.5	95.7	3586	51.7	63.3	3589	97.2	97.2
	15	3590	0.0	39.3	3590	0.0	79.8	3589	8.3	52.9	3590	36.0	89.7	3588	37.0	59.4	3589	71.2	90.7
	20	3590	0.0	38.6	3590	0.0	79.3	3590	0.0	51.9	3590	0.0	80.2	3589	0.0	60.7	3590	17.9	86.1
	25	3590	0.0	39.2	3590	0.0	70.0	3590	0.0	46.5	3590	0.0	77.2	3589	0.0	59.1	3591	0.0	85.0
30	5	3589	34.2	54.1	3590	65.8	92.1	3587	36.4	64.5	3059	99.3	99.3	3587	73.8	73.9	1810	99.6	99.6
	10	3590	0.0	40.0	3590	0.0	75.9	3589	16.9	52.4	3590	66.1	83.1	3588	44.2	56.6	3590	83.5	85.7
	15	3589	0.0	34.0	3590	0.0	65.9	3588	0.0	47.8	3590	0.0	71.2	3587	7.7	54.2	3590	45.3	79.0
	20	3589	0.0	35.5	3590	0.0	60.7	3589	0.0	43.3	3590	0.0	70.1	3588	0.0	53.7	3590	0.0	74.9
Avg		3589	9.0	44.3	3490	31.3	86.1	3588	17.1	59.4	3231	55.4	90.7	3552	32.4	67.7	2841	68.8	93.8

the number of customers increases. Note that the production costs that are taken into account in the model are the fixed production costs. The variable production costs are not included, since the total demand for all customers needs to be satisfied and hence the total variable production cost represents a fixed amount that is left out of the objective function.

Figures (9) and (10) present a comparison of the cost component share (in percentage) for different numbers of customers and periods $l = 12$ and 30 when small- and big-bucket LSP instances are considered, respectively. These figures show that by increasing the number of planning periods it is possible to schedule the production in such a way that the share of the production setups decreases. Similar tendencies are observed for instances with periods $l = 18$ and 24. The challenge for the practitioners is in designing and developing efficient methods to both obtain feasible solutions and proving the quality of those solutions.

Table 8: Cost component values and proportions for small-bucket LSP

		$\rho = 1$							$\rho = 2$							$\rho = 3$						
l	n	Total	Production		Inventory		Transport		Total	Production		Inventory		Transport		Total	Production		Inventory		Transport	
12	5	43346	32273	73.5%	3646	8.1%	7428	18.4%	43669	32273	72.9%	3745	8.2%	7652	18.9%	44212	32273	72.2%	3594	7.9%	8345	20.0%
	10	50581	32780	63.9%	7555	14.3%	10246	21.8%	50879	32780	63.5%	7925	14.9%	10174	21.6%	52214	32780	62.0%	7216	13.2%	12219	24.7%
	15	58771	33160	55.4%	10861	18.1%	14750	26.4%	59427	33160	54.8%	11685	19.2%	14582	25.9%	60249	33160	54.1%	10972	17.9%	16117	28.0%
	20	65918	32788	49.0%	14936	22.0%	18194	29.0%	67142	32788	48.2%	15349	22.2%	19005	29.6%	67604	32788	47.9%	15443	21.9%	19374	30.2%
	25	70732	33066	46.1%	16897	23.4%	20769	30.5%	72047	33066	45.3%	17580	24.0%	21400	30.8%	72604	33066	44.9%	17753	23.8%	21784	31.3%
	30	77224	33166	42.3%	20703	26.4%	23355	31.3%	78886	33166	41.4%	21388	26.7%	24332	31.9%	79423	33166	41.1%	21186	26.0%	25072	32.9%
	35	83517	32816	38.9%	24080	28.3%	26621	32.8%	84989	32816	38.2%	24059	27.8%	28114	33.9%	85562	32816	37.9%	24026	27.5%	28720	34.6%
18	5	51465	32888	63.0%	7781	14.7%	10796	22.2%	51956	32936	62.5%	8072	15.1%	10947	22.4%	53110	33261	61.9%	7323	13.4%	12526	24.7%
	10	62446	32897	51.8%	14587	22.8%	14962	25.5%	62935	32897	51.4%	14442	22.3%	15595	26.3%	63142	32897	51.2%	14843	22.8%	15401	26.0%
	15	74875	32854	43.3%	21881	28.6%	20140	28.1%	75578	32854	42.9%	21930	28.4%	20794	28.7%	75353	32854	43.0%	21110	27.4%	21389	29.6%
	20	88911	32689	36.4%	28301	31.4%	27920	32.2%	89445	32689	36.2%	28200	31.1%	28556	32.8%	89075	32689	36.3%	27124	30.0%	29262	33.6%
	25	99296	33013	32.8%	34436	34.1%	31847	33.0%	100226	33013	32.5%	35083	34.4%	32130	33.0%	99719	33013	32.7%	33676	33.2%	33029	34.1%
	30	110165	32990	29.6%	40596	36.3%	36713	34.1%	111469	33377	29.7%	40757	36.1%	37335	34.3%	111398	33377	29.7%	40129	35.4%	37891	34.9%
24	5	58186	33016	55.8%	11711	19.6%	13460	24.6%	58582	33016	55.4%	12429	20.6%	13137	24.0%	60098	33351	54.4%	11362	18.2%	15586	27.5%
	10	77628	33284	42.3%	22990	29.4%	21421	28.3%	78191	33284	42.0%	23174	29.3%	21759	28.7%	78071	33284	42.1%	22939	29.1%	21848	28.8%
	15	98363	33036	33.1%	35823	36.0%	29505	30.9%	99483	33375	33.1%	36821	36.5%	29286	30.4%	99498	33728	33.3%	35001	34.6%	30902	32.1%
	20	115200	34125	29.1%	44300	37.9%	36909	33.0%	114614	33747	29.0%	43630	37.5%	37237	33.5%	114753	33726	28.9%	42816	36.5%	38544	34.6%
	25	135980	34282	25.0%	57031	41.5%	44667	33.5%	135970	34944	25.4%	56176	40.7%	44917	33.8%	135521	33919	24.7%	56620	41.3%	44981	34.0%
30	5	67599	33073	48.3%	18422	26.6%	16170	25.1%	68019	33073	48.0%	18928	27.2%	16085	24.8%	69663	33791	47.2%	19066	26.0%	18272	26.8%
	10	97866	34012	34.3%	36414	36.6%	27707	29.2%	98690	35412	35.4%	35474	35.2%	27937	29.4%	97996	35057	35.3%	34994	34.8%	28345	29.9%
	15	122554	34973	28.1%	52127	41.4%	36055	30.5%	121843	34349	27.8%	51825	41.7%	35802	30.6%	121253	34638	28.1%	51142	41.1%	35739	30.8%
	20	148833	37500	25.0%	64408	42.5%	46993	32.4%	148201	40102	26.6%	62499	41.5%	45866	31.9%	147073	38239	25.7%	62440	41.8%	46527	32.5%

Table 9: Cost component values and proportions for big-bucket LSP

		$\pi = 1$							$\pi = 2$							$\pi = 3$						
l	n	Total	Production		Inventory		Transport		Total	Production		Inventory		Transport		Total	Production		Inventory		Transport	
12	5	43360	32273	73.5%	3335	7.4%	7753	19.1%	43294	32273	73.6%	3331	7.4%	7691	19.0%	43167	32273	73.8%	3475	7.9%	7420	18.3%
	10	49989	32780	64.5%	6619	12.9%	10657	22.7%	50009	32780	64.5%	6637	12.9%	10658	22.6%	50003	32780	64.5%	6759	13.1%	10531	22.5%
	15	57953	33160	56.1%	9727	16.6%	15133	27.3%	57961	33160	56.1%	9728	16.6%	15140	27.3%	57941	33160	56.2%	9823	16.8%	14958	27.0%
	20	64256	32788	50.3%	13111	20.0%	18357	29.7%	64336	32788	50.2%	13134	20.1%	18414	29.7%	64231	32788	50.3%	13123	20.1%	18320	29.6%
	25	69213	33066	47.2%	15006	21.4%	21141	31.5%	69176	33066	47.2%	15151	21.6%	20959	31.3%	69177	33066	47.2%	15137	21.6%	20973	31.2%
	30	74655	33166	43.8%	17920	23.7%	23569	32.5%	74742	33166	43.7%	17837	23.6%	23739	32.7%	74640	33166	43.8%	17768	23.5%	23707	32.7%
	35	80969	32816	40.2%	20296	24.9%	27857	35.0%	80893	32816	40.2%	20239	24.8%	27838	35.0%	80913	32816	40.2%	20176	24.7%	27921	35.0%
18	5	51943	34055	64.4%	6903	13.1%	11185	22.5%	51802	34055	64.7%	7016	13.4%	10865	21.9%	52154	34055	64.3%	7229	13.7%	10964	22.0%
	10	61597	32897	52.5%	13577	21.6%	15122	25.9%	61414	32897	52.6%	13312	21.3%	15205	26.1%	61548	32897	52.5%	13341	21.2%	15309	26.2%
	15	73579	33529	45.1%	19502	26.2%	20547	28.7%	73451	33529	45.2%	19920	26.7%	20003	28.1%	73520	33529	45.1%	19483	26.2%	20508	28.7%
	20	87341	33066	37.6%	26673	30.2%	27602	32.1%	86730	33066	37.9%	26528	30.4%	27136	31.7%	86865	33066	37.9%	25938	29.7%	27861	32.5%
	25	96570	33013	33.9%	32145	33.0%	31413	33.1%	96471	33013	33.9%	32398	33.3%	31127	32.8%	96495	33013	33.9%	32331	33.2%	31152	32.8%
	30	107878	33362	30.7%	38345	35.2%	36171	34.0%	107441	33362	30.9%	38078	35.2%	36001	33.9%	107456	33362	30.9%	37883	35.0%	36211	34.1%
24	5	58703	33752	56.6%	11442	19.2%	13523	24.2%	58601	33752	56.7%	11440	19.2%	13450	24.0%	58655	33752	56.7%	11691	19.6%	13213	23.7%
	10	78427	35936	45.6%	21351	27.3%	21140	27.1%	78426	35936	45.6%	21564	27.5%	20926	26.9%	78464	35936	45.5%	20909	26.7%	21619	27.7%
	15	97428	34063	34.7%	34803	35.4%	28562	29.9%	96433	33723	34.6%	34286	35.3%	28424	30.0%	96250	33723	34.7%	33975	35.1%	28552	30.2%
	20	112439	34923	30.9%	41918	37.1%	35598	32.0%	112490	34923	30.9%	41730	36.9%	35837	32.3%	112607	34923	30.8%	41635	36.8%	36048	32.4%
	25	132096	35711	27.0%	53485	40.4%	42900	32.7%	132198	35711	27.0%	53564	40.4%	42923	32.6%	132138	35711	27.0%	53369	40.3%	43058	32.8%
30	5	67369	34090	50.2%	17681	26.0%	15598	23.9%	67339	34090	50.1%	17443	25.6%	15873	24.2%	67323	34090	50.2%	17373	25.6%	15861	24.2%
	10	98069	37351	37.9%	33407	33.9%	27311	28.1%	98206	37351	37.9%	33483	34.0%	27371	28.2%	98355	37351	37.8%	33042	33.5%	27962	28.7%
	15	119599	36496	30.3%	48146	39.8%	34957	29.8%	120592	36933	30.4%	48075	39.4%	35584	30.1%	119876	36496	30.3%	48332	39.9%	35047	29.8%
	20	146169	43719	29.9%	57433	38.9%	45017	31.2%	146122	43719	30.0%	57715	39.1%	44688	31.0%	146593	43719	29.9%	57793	39.0%	45080	31.1%

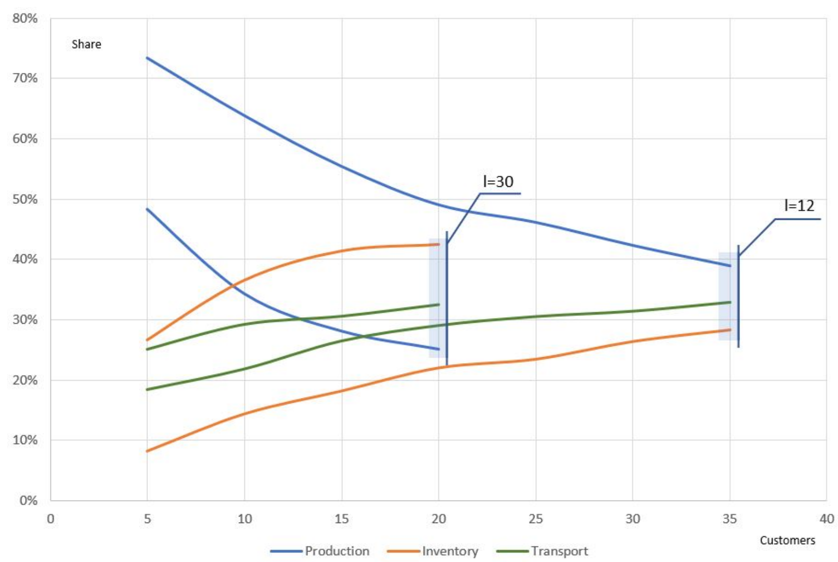

Figure 9: Cost share (%) comparison for different number of customers and periods in small-bucket LSP with $\rho = 1$

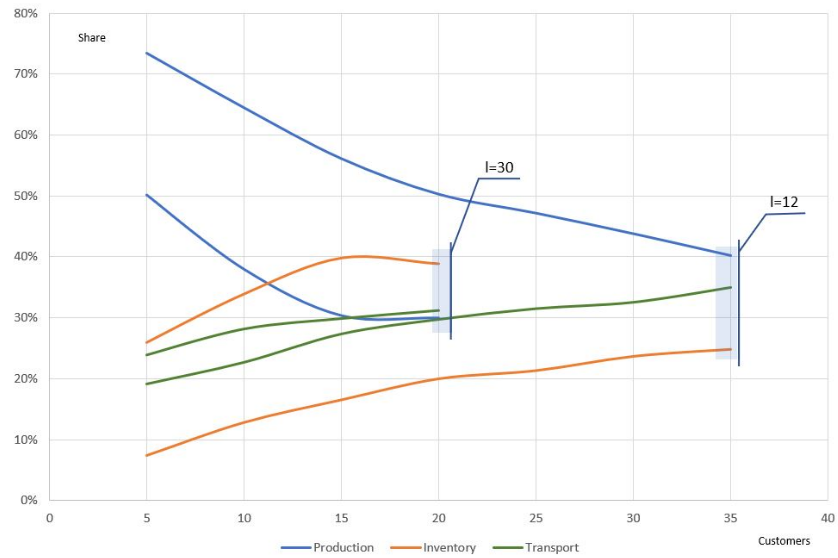

Figure 10: Cost share (%) comparison for different number of customers and periods in big-bucket LSP with $\pi = 1$

8 Summary

While classical production routing problems have received considerable attention from the research community, all studies on this problem and its variants consider identical production and route planning period lengths. In this paper, we have presented formulations for a multi-product production routing problem with the possibility of incorporating different production and route planning period lengths. This is the first attempt in the literature to consider such a practical limitation. We model both big-bucket and small-bucket lot-sizing problems at the production level. Next, we have adapted a state-of-the-art matheuristic to obtain quality solutions for instances of this problem with different numbers of products, planning periods, and customers. We have developed many sets of valid inequalities that exploit the structure of the problem. The effectiveness of the derived valid inequalities within our branch-and-cut algorithm was tested through an extensive set of computational experiments. The availability of an exact algorithm has allowed us to measure the quality of the upper bounding heuristic. We have shown that by including the relevant valid inequalities in the heuristic, significant improvements in terms of the number of iterations, the solution time and quality can be achieved. We observe that for the same numbers of micro periods, customers and products, the problem can be solved more efficiently when the number of production planning periods or routing periods decreases. One explanation is that in these cases the number of decision variables will quickly decrease in our proposed reformulation model.

Appendix

A Proofs

Theorem 1 *$\mathcal{R}^B_{MP-PRP}$ and $\mathcal{R}^S_{MP-PRP}$ are valid reformulations for $\mathcal{M}^B_{MP-PRP}$ and $\mathcal{M}^S_{MP-PRP}$, respectively.*

Proof. First we show that for every feasible solution of the $\mathcal{M}^B_{MP-PRP}$ model, there exists a feasible solution to the $\mathcal{R}^B_{MP-PRP}$ model with the same solution value. Suppose that $\bar{y}$, $\bar{p}$, $\bar{I}$, $\bar{q}$, $\bar{z}$ and $\bar{x}$ satisfy the system of (1)–(18) (feasible in $\mathcal{M}^B_{MP-PRP}$).

- For every $\tau \in \mathcal{T}^\pi$ and for every $k \in \mathcal{K}$, we let $\bar{\mathbf{y}}_{kt} = \bar{y}_{k\tau}$ and $\bar{\mathbf{p}}_{kt} = \bar{p}_{k\tau}$ where $t = \pi(\tau - 1) + 1$. Constraints (37) fix the rest of the $\bar{\mathbf{y}}$ and $\bar{\mathbf{p}}$ variables to zero.

- For every $\omega \in \mathcal{T}^\rho$ and for every $i \in \mathcal{N}$, we let $\bar{\mathbf{z}}_{it} = \bar{z}_{i\omega}$ where $t = \omega\rho$. Constraints (38) fix the rest of the $\bar{\mathbf{z}}$ variables to zero.
- For every $\omega \in \mathcal{T}^\rho$, for every $i \in \mathcal{N}$ and for every $k \in \mathcal{K}$, we let $\bar{\mathbf{q}}_{ikt} = \bar{q}_{ik\omega}$ where $t = \omega\rho$. Constraints (38) fix the rest of the $\bar{\mathbf{q}}$ variables to zero.
- For every $\omega \in \mathcal{T}^\rho$, we let $\bar{\mathbf{z}}_{0t} = \bar{z}_{0\omega}$ where $t = \omega\rho$. Constraints (39) fix the rest of the $\bar{\mathbf{z}}_{0t}$ variables to zero.
- For every $\omega \in \mathcal{T}^\rho$ and for every $(i,j) \in \mathcal{E}$, we let $\bar{\mathbf{x}}_{ijt} = \bar{x}_{ij\omega}$ where $t = \omega\rho$. Constraints (35) and (38)–(39) force the rest of the $\bar{\mathbf{x}}$ variables to zero.
- For every $\omega \in \mathcal{T}^\rho$, for every $i \in \mathcal{N}$ and for every $k \in \mathcal{K}$, we let $\bar{\mathbf{I}}_{ikt} = \bar{I}_{ik\omega}$ where $t = \omega\rho$. For the rest of the micro periods $(t \in \mathcal{T}, t \bmod \rho \neq 0)$, we let $\bar{\mathbf{I}}_{ikt} = \bar{I}_{ik,\lfloor \frac{t}{\rho} \rfloor}$.
- The inventory variables (and hence the solutions) at the plant level, $\bar{I}_{0kt}$, are defined on the micro periods and are the same in both formulations.

One observes that the solution $\bar{\mathbf{y}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{I}}$, $\bar{\mathbf{q}}$, $\bar{\mathbf{z}}$ satisfies the system of constraints (2), (13), (28)–(44) and hence is feasible in $\mathcal{R}^B_{MP-PRP}$. Similarly, we can show that for every feasible solution in $\mathcal{R}^B_{MP-PRP}$ there exists a feasible solution in $\mathcal{M}^B_{MP-PRP}$. Thus, $\mathcal{R}^B_{MP-PRP}$ is a valid reformulation of $\mathcal{M}^B_{MP-PRP}$. In the same way, we can show $\mathcal{R}^S_{MP-PRP}$ is a valid reformulation of $\mathcal{M}^S_{MP-PRP}$. □

Proposition 1

$$\sum_{e=t_1}^{t_2} \mathbf{p}_{ke} \leq I_{0kt_2} + \sum_{i \in \mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{e=t_1}^{t_2} \Big(\sum_{i \in \mathcal{N}} \mathbf{d}_{iket_2} \Big) \mathbf{y}_{ke} \forall k \in \mathcal{K}, \forall t_1, t_2 \in \mathcal{T}, t_1 \leq t_2 \tag{51}$$

are valid for $\mathcal{R}^B_{MP-PRP}$, $\mathcal{R}^S_{MP-PRP}$.

Proof. If $\sum_{e=t_1}^{t_2} \mathbf{y}_{ke} = 0$, then no setup will be done during periods t_1 to t_2 and hence no production of product $k \in \mathcal{K}$ is possible during these periods ($\sum_{e=t_1}^{t_2} \mathbf{p}_{ke} = 0$). Then, inequalities (51) are satisfied because the left-hand-side (LHS) will be equal to zero and the inventory variables in the right-hand-side (RHS) are nonnegative. Otherwise, let θ be the first period in which the production setup for product $k \in \mathcal{K}$ is performed, i.e., $\theta = \min_e \{t_1 \leq e \leq t_2 | \mathbf{y}_{ke} = 1\}$. Then,

$$\begin{aligned}
\sum_{e=t_1}^{t_2} \mathbf{p}_{ke} &= \sum_{e=\theta}^{t_2} \mathbf{p}_{ke} \\
&= \sum_{e=\theta}^{t_2} (I_{0ke} - I_{0k,e-1} + \sum_{i \in \mathcal{N}} \mathbf{q}_{ike}) \\
&= \sum_{e=\theta}^{t_2} \Big(I_{0ke} - I_{0k,e-1} + \sum_{i \in \mathcal{N}} (\mathbf{I}_{ike} - \mathbf{I}_{ik,e-1} + \mathbf{d}_{ike}) \Big) \\
&= I_{0kt_2} - I_{0k,\theta-1} + \sum_{i \in \mathcal{N}} (\mathbf{I}_{ikt_2} - \mathbf{I}_{ik,\theta-1} + \mathbf{d}_{ik\theta t_2}) \\
&\leq I_{0kt_2} + \sum_{i \in \mathcal{N}} (\mathbf{I}_{ikt_2} + \mathbf{d}_{ik\theta t_2}) \\
&= I_{0kt_2} + \sum_{i \in \mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{i \in \mathcal{N}} \mathbf{d}_{ik\theta t_2} \mathbf{y}_{k\theta} \\
&\leq I_{0kt_2} + \sum_{i \in \mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{e=\theta}^{t_2} (\sum_{i \in \mathcal{N}} \mathbf{d}_{iket_2}) \mathbf{y}_{ke} \\
&= I_{0kt_2} + \sum_{i \in \mathcal{N}} \mathbf{I}_{ikt_2} + \sum_{e=t_1}^{t_2} (\sum_{i \in \mathcal{N}} \mathbf{d}_{iket_2}) \mathbf{y}_{ke}.
\end{aligned}$$

The first four equations follow from the definition of θ, constraints (28), constraints (29), and the definition of $\mathbf{d}_{ikt_1t_2}$, respectively. The first inequality holds due to the non-negativity of inventory variables. The next equation is valid because $\mathbf{y}_{k\theta} = 1$. The last inequality is valid since the $\mathbf{y}_{ke}$ variables are nonnegative. The last equation holds due to the assumption that there is no setup from period t_1 to θ. □

Proposition 2 *Inequalities*

$$\left\lceil \frac{\max\left\{0, \sum_{i\in\mathcal{N}} \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} - I_{0k0}\right\}}{C/\theta_k} \right\rceil \leq \sum_{e=1}^{t} \mathbf{y}_{ke} \quad \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{52}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ *and* $\mathcal{R}^S_{MP-PRP}$.

Proof. First we show:

$$\begin{aligned}
\sum_{e=1}^{t} \mathbf{p}_{ke} &= \sum_{e=1}^{t} \big(\sum_{i\in\mathcal{N}} \mathbf{q}_{ike} + I_{0ke} - I_{0k,e-1} \big) \\
&= \sum_{e=1}^{t} \Big(\sum_{i\in\mathcal{N}} (\mathbf{d}_{ike} + \mathbf{I}_{ike} - \mathbf{I}_{ik,e-1}) + I_{0ke} - I_{0k,e-1} \Big) \\
&= \sum_{i\in\mathcal{N}} (\mathbf{d}_{ik1t} + \mathbf{I}_{ikt} - \mathbf{I}_{ik0}) + I_{0kt} - I_{0k0} \\
&\geq \sum_{i\in\mathcal{N}} (\mathbf{d}_{ik1t} - \mathbf{I}_{ik0}) - I_{0k0}.
\end{aligned}$$

The first two equation are obtained based on constraints (28) and (29), respectively. The third equation holds due to the definition of $\mathbf{d}_{ikt_1t_2}$. The first inequality follows from the non-negativity of inventory variables. We can write

$$\textstyle\sum_{e=1}^{t} \mathbf{p}_{ke} \geq \max\left\{0, \sum_{i\in\mathcal{N}} \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} - I_{0k0}\right\},$$

because only a strictly positive product shortage triggers the production at the plant. Finally, the validity of the proposition comes from the fact that:

$$\begin{aligned}
\max\left\{0, \sum_{i\in\mathcal{N}} \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} - I_{0k0}\right\} &\leq \sum_{e=1}^{t} \mathbf{p}_{ke} \\
&\leq C/\theta_k \sum_{e=1}^{t} \mathbf{y}_{ke}.
\end{aligned}$$

□

Proposition 3 *Inequalities*

$$\sum_{e=t_1}^{t_2} \mathbf{q}_{ike} \leq \mathbf{I}_{ik,t_2} + \sum_{e=t_1}^{t_2} \mathbf{d}_{iket_2} \mathbf{z}_{ie} \quad \forall i \in \mathcal{N}, \forall k \in \mathcal{K}, \forall t_1, t_2 \in \mathcal{T}, t_1 \leq t_2 \tag{53}$$

are valid for $\mathcal{R}^B_{MP-PRP}$, $\mathcal{R}^S_{MP-PRP}$.

Proof. If $\sum_{e=t_1}^{t_2} \mathbf{z}_{ie} = 0$, then customer $i \in \mathcal{N}$ will not be visited during periods t_1 to t_2. This results in no shipment of product $k \in \mathcal{K}$ to that customer during the associated periods. Then, inequalities (53) are satisfied because the inventory variables in the RHS are nonnegative. Otherwise, let θ be the first period in which customer $i \in \mathcal{N}$ is visited, i.e., $\theta = \min_e\{t_1 \leq e \leq t_2 | \mathbf{z}_{ie} = 1\}$. Then,

$$\begin{aligned}
\sum_{e=t_1}^{t_2} \mathbf{q}_{ike} &= \sum_{e=\theta}^{t_2} \mathbf{q}_{ike} \\
&= \sum_{e=\theta}^{t_2} (\mathbf{I}_{ike} - \mathbf{I}_{ik,e-1} + \mathbf{d}_{ike}) \\
&= \mathbf{I}_{ikt_2} - \mathbf{I}_{ik,\theta-1} + \mathbf{d}_{ik\theta t_2} \\
&\leq \mathbf{I}_{ikt_2} + \mathbf{d}_{ik\theta t_2} \\
&= \mathbf{I}_{ikt_2} + \mathbf{d}_{ik\theta t_2}\mathbf{z}_{i\theta} \\
&\leq \mathbf{I}_{ikt_2} + \sum_{e=\theta}^{t_2} \mathbf{d}_{iket_2}\mathbf{z}_{ie} \\
&= \mathbf{I}_{ikt_2} + \sum_{e=t_1}^{t_2} \mathbf{d}_{iket_2}\mathbf{z}_{ie}.
\end{aligned}$$

The first three equations hold because of the definition of θ, constraints (10) for periods θ to t_1, and the definition of $\mathbf{d}_{ikt_1t_2}$. The first inequality is valid due to the non-negativity of the inventory variables. The fourth equation follows from $\mathbf{z}_{i\theta} = 1$. The last inequality and equation are valid because the $\mathbf{y}_{ke}$ variables are nonnegative. □

Proposition 4 *Inequalities*

$$\left\lceil \frac{1}{Q} \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} \right\rceil \leq \sum_{e=1}^{t} \mathbf{z}_{0e} \quad \forall t \in \mathcal{T} \tag{54}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ *and* $\mathcal{R}^S_{MP-PRP}$.

Proof. We have

$$\begin{aligned}
\sum_{e=1}^{t} Q\mathbf{z}_{0e} &\geq \sum_{e=1}^{t} \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} b_k \mathbf{q}_{ike} \\
&= \sum_{e=1}^{t} \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} b_k (\mathbf{d}_{ike} + \mathbf{I}_{ike} - \mathbf{I}_{ik,e-1}) \\
&= \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} b_k (\mathbf{d}_{ik1e} + \mathbf{I}_{ike} - \mathbf{I}_{ik0}) \\
&\geq \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} b_k (\mathbf{d}_{ik1e} - \mathbf{I}_{ik0}).
\end{aligned}$$

The first inequality is valid since the LHS is the total fleet capacity for period $e = 1$ to t, and the RHS is the total shipment for the same periods. The first equation follows from constraints (29). The second equation is valid due to the definition of $\mathbf{d}_{ikt_1t_2}$. The second inequality holds due to the non-negativity of inventory variables. The proposition is valid because only strictly positive demand shortages necessitate vehicles' dispatch. □

Proposition 5 *Inequalities*

$$\left\lceil \frac{\sum_{k \in \mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\}}{\min\left\{Q, L_i + \max_{1 \leq \theta \leq t}\{\sum_{k \in \mathcal{K}} b_k \mathbf{d}_{ik\theta}\}\right\}} \right\rceil \leq \sum_{e=1}^{t} \mathbf{z}_{ie} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \tag{55}$$

are valid for $\mathcal{R}^B_{MP-PRP}$ *and* $\mathcal{R}^S_{MP-PRP}$.

Proof. Similar to the proof presented in Proposition 4 we have

$$\sum_{k\in\mathcal{K}} b_k(\mathbf{d}_{ik1t} - \mathbf{I}_{ik0}) \leq \sum_{e=1}^{t}\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ike}.$$

Thus,

$$\sum_{k\in\mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} \leq \sum_{e=1}^{t}\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ike},$$

is valid for the reason that only strictly positive product shortage volumes force shipments. The vehicle capacity constraints (33) provide the first upper bound:

$$\sum_{e=1}^{t}\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ike} \leq Q\sum_{e=1}^{t}\mathbf{z}_{ie}.$$

Next, we have

$$\begin{aligned}\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ikt} &= \sum_{k\in\mathcal{K}} b_k(\mathbf{d}_{ikt} + \mathbf{I}_{ikt} - \mathbf{I}_{ik,t-1})\\ &\leq \sum_{k\in\mathcal{K}} b_k(\mathbf{d}_{ikt} + \mathbf{I}_{ikt})\\ &\leq \sum_{k\in\mathcal{K}} b_k\mathbf{d}_{ikt} + L_i,\end{aligned}$$

which gives

$$\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ikt} \leq \big(\sum_{k\in\mathcal{K}} b_k\mathbf{d}_{ikt} + L_i\big)\mathbf{z}_{it}.$$

Therefore, we deduce

$$\sum_{k\in\mathcal{K}} b_k \max\{0, \mathbf{d}_{ik1t} - \mathbf{I}_{ik0}\} \leq \sum_{e=1}^{t}\sum_{k\in\mathcal{K}} b_k\mathbf{q}_{ike} \leq \min\big\{Q, L_i + \max_{1\leq\theta\leq t}\{\sum_{k\in\mathcal{K}} b_k\mathbf{d}_{ik\theta}\}\big\}\sum_{e=1}^{t}\mathbf{z}_{ie}.$$

□

B Impact of inequalities on lower bound improvement

Table 10 reports the improvement of the lower bounds obtained by incorporating the valid inequalities in the small- and big-bucket models. On the small-bucket instances, applying the valid inequalities results in an average increase of the lower bounds by 70.5%, 38.7%, and 25.6%, respectively for $\rho = 1$, $\rho = 2$ and $\rho = 3$. On the big-bucket instances, the lower bound improvements obtained by the addition of the valid inequalities are 48.8%, 32.8%, and 22.7%, respectively for $\pi = 1$, $\pi = 2$ and $\pi = 3$. Notice that in this table, for the cases where the inequalities improve the lower bound more than twice, the percentage increase reported is more than 100%. Overall, the larger (more periods, products, and nodes) and the harder to solve (smaller ρ and π) the instances are, the bigger the improvement is.

Table 10: Lower bound improvement with valid inequalities

			Small-Bucket LSP									Big-Bucket LSP								
			$\rho = 1$			$\rho = 2$			$\rho = 3$			$\pi = 1$			$\pi = 2$			$\pi = 3$		
l	k	n	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)	None	All	(%)
12	4	5	32209	32277	0.2	32722	32722	0.0	33124	33124	0.0	32149	32231	0.3	32296	32296	0.0	32796	32797	0.0
		10	37338	37826	1.3	38464	38663	0.5	38576	38743	0.4	37270	38021	2.0	38165	38841	1.8	40690	41902	3.0
		15	35620	46818	31.4	46793	47836	2.2	47675	47998	0.7	45810	46788	2.1	46775	47339	1.2	47581	48216	1.3
		20	36045	51379	42.5	48405	52614	8.7	52233	52553	0.6	46319	51293	10.7	49448	52076	5.3	50711	54006	6.5
		25	37750	56009	48.4	45767	58209	27.2	57714	58240	0.9	45843	56432	23.1	52459	57442	9.5	54083	58588	8.3
		30	38823	58977	51.9	48663	63216	29.9	56972	63068	10.7	45756	60708	32.7	56186	61856	10.1	58221	64360	10.5
		35	36530	65812	80.2	54449	71021	30.4	61081	70838	16.0	48263	68569	42.1	52506	70189	33.7	56035	72779	29.9
	6	5	41381	41807	1.0	41868	41869	0.0	42431	42431	0.0	41549	41551	0.0	41725	41726	0.0	41633	41634	0.0
		10	33196	47535	43.2	42981	47923	11.5	48248	48491	0.5	46726	47330	1.3	46612	47484	1.9	47389	47435	0.1
		15	30768	54604	77.5	43703	55253	26.4	51466	56256	9.3	44733	54216	21.2	51539	54458	5.7	53907	54722	1.5
		20	30843	58067	88.3	43318	59456	37.3	55455	60735	9.5	47355	57941	22.4	51515	58387	13.3	57946	58790	1.5
		25	27061	61644	127.8	42123	64643	53.5	57162	66127	15.7	35044	62377	78.0	42987	62716	45.9	55226	63144	14.3
		30	24347	64207	163.7	43812	69387	58.4	58190	71632	23.1	35242	66655	89.1	42974	67181	56.3	52642	67172	27.6
		35	22774	65167	186.1	44497	71968	61.7	57718	75366	30.6	32224	69960	117.1	41089	70541	71.7	44369	70382	58.6
	8	5	42840	51861	21.1	47042	52465	11.5	52703	52911	0.4	51640	51646	0.0	50137	51646	3.0	51718	51720	0.0
		10	37550	61433	63.6	49309	62554	26.9	49285	62761	27.3	37317	60607	62.4	43597	60667	39.2	55322	60799	9.9
		15	33033	65826	99.3	49858	68103	36.6	52410	68933	31.5	42816	65689	53.4	50349	65761	30.6	59367	66560	12.1
		20	35314	73324	107.6	54670	77393	41.6	59535	79663	33.8	40900	73662	80.1	49006	74018	51.0	55803	74858	34.1
		25	30298	73816	143.6	50749	78962	55.6	59812	81722	36.6	38864	74784	92.4	46606	75041	61.0	51947	76288	46.9
		30	33969	80129	135.9	49220	85443	73.6	62520	89341	42.9	37900	80278	111.8	43849	80730	84.1	53250	81233	52.6
		35	26186	80766	208.4	51421	89173	73.4	66248	93864	41.7	35488	83457	135.2	44488	83776	88.3	52561	84461	60.7
18	4	5	39762	41065	3.3	41912	42162	0.6	43152	43201	0.1	40764	44644	9.5	44757	45002	0.5	46421	46750	0.7
		10	35596	45740	28.5	44683	46895	5.0	46987	47177	0.4	43648	46648	6.9	46533	47592	2.3	48684	49143	0.9
		15	39802	57618	44.8	50978	59761	17.2	59067	60456	2.4	51158	60741	18.7	59330	62126	4.7	62987	64340	2.1
		20	49660	67317	35.6	58505	74722	27.7	72111	74928	3.9	59455	74832	25.9	66375	76946	15.9	74651	79524	6.5
		25	47834	73942	54.6	61869	81651	32.0	73752	82170	11.4	58963	81305	37.9	69300	83406	20.4	79007	86216	9.1
		30	49062	82540	68.2	69313	89963	29.8	75892	94457	24.5	62571	94454	51.0	72992	97335	33.4	79337	100585	26.8
	6	5	34801	47730	37.2	44269	48296	9.1	47045	48300	2.7	43733	47348	8.3	47474	47534	0.1	48132	48133	0.0
		10	37572	59059	57.2	44256	59903	35.4	49879	60562	21.4	53397	58827	10.2	57935	59162	2.1	59077	60428	2.3
		15	37114	64768	74.5	52130	69758	33.8	56146	70331	25.3	51033	67416	32.1	63308	67732	7.0	68518	70757	3.3
		20	36140	71067	96.6	51488	78906	53.3	59969	80467	34.2	50530	75849	50.1	59250	76265	28.7	71374	79473	11.3
		25	38148	77909	104.2	55098	85132	54.5	64799	90294	39.3	46074	84758	84.0	51369	85181	65.8	58356	89193	52.8
		30	31380	82567	163.1	54260	83962	54.7	67555	95073	40.7	44227	89723	102.9	54882	90230	64.4	59948	94703	58.0
	8	5	37779	60800	60.9	45791	61484	34.3	52629	62055	17.9	53963	59791	10.8	59814	59823	0.0	60239	60329	0.2
		10	36584	71601	95.7	49792	74072	48.8	53239	75366	41.6	51253	71851	40.2	63055	71997	14.2	69282	72735	5.0
		15	37505	75001	100.0	49870	82299	65.0	58549	85056	45.3	46867	81133	73.1	60809	81250	33.6	70824	81722	15.4
		20	40095	84329	110.3	60068	93408	55.5	66786	95822	43.5	53448	90199	68.8	60627	90448	49.2	74627	90514	21.3
		25	44748	92675	107.1	60036	99228	65.3	73512	106397	44.7	51905	100244	93.1	55547	100348	80.7	64873	100563	55.0
		30	34879	100873	189.2	61152	105629	72.7	81373	116921	43.7	46385	110800	138.9	56943	110893	94.7	67506	111051	64.5
24	4	5	36759	42541	15.7	43126	43587	1.1	43972	43972	0.0	43961	45648	3.8	45847	46200	0.8	48018	48345	0.7
		10	50151	62422	24.5	54811	63598	16.0	61985	64155	3.5	63563	70675	11.2	70741	72049	1.8	74116	74980	1.2
		15	51188	68504	33.8	61052	78158	28.0	71569	79344	10.9	61418	81087	32.0	69306	83543	20.5	78508	85904	9.4
		20	49617	78196	57.6	66930	86937	29.9	77359	91181	17.9	67335	96631	43.5	73930	98913	33.8	83805	101629	21.3
		25	56931	89466	57.1	76430	94160	23.2	93332	104852	12.3	70695	112880	59.7	81229	115793	42.6	92936	118924	28.0
	6	5	41723	55935	34.1	48982	56743	15.8	54927	57256	4.2	50097	56736	13.3	55861	57344	2.7	56765	58253	2.6
		10	38617	65411	69.4	45974	70576	53.5	51752	71797	38.7	53410	69747	30.6	66603	70258	5.5	70141	73279	4.5
		15	43692	77271	76.9	56361	85754	52.2	61809	89464	44.7	58591	88695	51.4	65734	89648	36.4	77048	94911	23.2
		20	47217	85307	80.7	60577	88328	45.8	70340	97616	38.8	56619	98396	73.8	62872	99074	57.6	71175	104064	46.2
		25	46044	92487	100.9	62432	97536	56.2	78391	110354	40.8	59196	115140	94.5	65476	115886	77.0	73358	122441	66.9
	8	5	38655	68186	76.4	46545	69136	48.5	51114	70006	37.0	58883	67577	14.8	64180	67725	5.5	62049	68559	10.5
		10	39921	75831	90.0	49937	84312	68.8	56734	86259	52.0	56690	83155	46.7	66275	83691	26.3	78877	85354	8.2
		15	42789	87070	103.5	57925	98548	70.1	65219	100268	53.7	57544	99720	73.3	68433	100247	46.5	82500	102014	23.7
		20	50100	102655	104.9	67126	103666	54.4	78414	113330	44.5	63257	115110	82.0	70139	116232	65.7	82650	119220	44.2
		25	58504	104333	78.3	70501	116094	64.7	88498	128494	45.2	67318	132210	96.4	72592	133255	83.6	81444	136383	67.5
30	4	5	41914	51320	22.4	49256	52661	6.9	52745	53506	1.4	49829	55906	12.2	55029	57370	4.3	58277	59437	2.0
		10	50881	64490	26.7	59218	73973	24.9	70425	75481	7.2	71829	84310	17.4	81387	86160	5.9	86628	88413	2.1
		15	54462	77646	42.6	67796	86645	27.8	78062	91194	16.8	69166	98065	41.8	74267	100129	34.8	81083	102269	26.1
		20	60842	89133	46.5	78042	92806	18.9	87915	104617	19.0	76932	119510	55.3	85681	121507	41.8	96420	124022	28.6
	6	5	35596	59195	66.3	44430	60855	37.0	51872	62278	20.1	48684	61609	26.5	59139	61885	4.6	62722	63888	1.9
		10	42848	71091	65.9	53919	81988	52.1	58050	84645	45.8	63260	88507	39.9	72299	89646	24.0	83197	94600	13.7
		15	44575	81966	83.9	59043	86568	46.6	66065	95959	45.2	58460	105409	80.3	65654	106106	61.6	76228	110982	45.6
		20	57842	94307	63.0	69322	99637	43.7	82484	113483	37.6	71824	131624	83.3	76242	132635	74.0	86641	137613	58.8
	8	5	44742	76233	70.4	52823	80236	51.9	60955	80973	32.8	60149	79002	31.3	72118	79472	10.2	75038	80878	7.8
		10	44031	83196	88.9	57220	92055	60.9	62042	95017	53.1	63638	100983	58.7	75878	103136	35.9	87215	106309	21.9
		15	49287	96033	94.8	68020	102163	50.2	77332	113785	47.1	67947	129109	90.0	81801	132076	61.5	93833	136203	45.2
		20	60258	101687	68.8	72111	116843	62.0	88653	124326	40.2	71759	146924	104.7	75811	149248	96.9	89133	152974	71.6
Average			40781	69512	70.5	53412	74086	38.7	61591	77385	25.6	52131	77562	48.8	59078	78434	32.8	65558	80452	22.7

References

Absi, N., Archetti, C., Dauzère-Pérès, S. and Feillet, D. (2014). A two-phase iterative heuristic approach for the production routing problem, Transportation Science 49(4): 784–795.

Adulyasak, Y., Cordeau, J.-F. and Jans, R. (2014). Formulations and branch-and-cut algorithms for multivehicle production and inventory routing problems, INFORMS Journal on Computing 26(1): 103–120.

Adulyasak, Y., Cordeau, J.-F. and Jans, R. (2015). The production routing problem: A review of formulations and solution algorithms, Computers & Operations Research 55: 141–152.

Archetti, C., Bertazzi, L., Laporte, G. and Speranza, M. G. (2007). A branch-and-cut algorithm for a vendor-managed inventory-routing problem, Transportation Science 41(3): 382–391.

Archetti, C., Bertazzi, L., Paletta, G. and Speranza, M. G. (2011). Analysis of the maximum level policy in a production-distribution system, Computers & Operations Research 38(12): 1731–1746.

Archetti, C. and Speranza, M. G. (2016). The inventory routing problem: the value of integration, International Transactions in Operational Research 23(3): 393–407.

Armentano, V. A., Shiguemoto, A. and Løkketangen, A. (2011). Tabu search with path relinking for an integrated production-distribution problem, Computers & Operations Research 38(8): 1199–1209.

Atamtürk, A. and Küçükyavuz, S. (2005). Lot sizing with inventory bounds and fixed costs: Polyhedral study and computation, Operations Research 53(4): 711–730.

Barany, I., Van Roy, T. and Wolsey, L. A. (1984). Uncapacitated lot-sizing: The convex hull of solutions, in B. Korte and K. Ritter (eds), Mathematical Programming at Oberwolfach II, Springer, pp. 32–43.

Belo-Filho, M., Amorim, P. and Almada-Lobo, B. (2015). An adaptive large neighbourhood search for the operational integrated production and distribution problem of perishable products, International Journal of Production Research 53(20): 6040–6058.

Blumenfeld, D. E., Burns, L. D., Daganzo, C. F., Frick, M. C. and Hall, R. W. (1987). Reducing logistics costs at General Motors, Interfaces 17(1): 26–47.

Brahimi, N. and Aouam, T. (2016). Multi-item production routing problem with backordering: a MILP approach, International Journal of Production Research 54(4): 1076–1093.

Brown, G., Keegan, J., Vigus, B. and Wood, K. (2001). The Kellogg company optimizes production, inventory, and distribution, Interfaces 31(6): 1–15.

Çetinkaya, S., Üster, H., Easwaran, G. and Keskin, B. B. (2009). An integrated outbound logistics model for Frito-Lay: Coordinating aggregate-level production and distribution decisions, Interfaces 39(5): 460–475.

Chandra, P. and Fisher, M. L. (1994). Coordination of production and distribution planning, European Journal of Operational Research 72(3): 503–517.

Chen, Z.-L. and Vairaktarakis, G. L. (2005). Integrated scheduling of production and distribution operations, Management Science 51(4): 614–628.

Chitsaz, M., Cordeau, J.-F. and Jans, R. (2019). A unified decomposition matheuristic for assembly, production and inventory routing, INFORMS Journal on Computing 31(1): 134–152.

Chitsaz, M., Cordeau, J.-F. and Jans, R. (2020). A branch-and-cut algorithm for an assembly routing problem, European Journal of Operational Research 282(3): 896–910.

Dantzig, G. B., Fulkerson, R. and Johnson, S. (1954). Solution of a large-scale traveling-salesman problem, Journal of the Operations Research Society of America 2(4): 393–410.

De Matta, R. and Guignard, M. (1994a). Dynamic production scheduling for a process industry, Operations Research 42(3): 492–503.

De Matta, R. and Guignard, M. (1994b). Studying the effects of production loss due to setup in dynamic production scheduling, European Journal of Operational Research 72(1): 62–73.

Fischetti, M., Polo, C. and Scantamburlo, M. (2004). A local branching heuristic for mixed-integer programs with 2-level variables, with an application to a telecommunication network design problem, Networks 44(2): 61–72.

Fumero, F. and Vercellis, C. (1999). Synchronized development of production, inventory, and distribution schedules, Transportation Science 33(3): 330–340.

Glover, F., Jones, G., Karney, D., Klingman, D. and Mote, J. (1979). An integrated production, distribution, and inventory planning system, Interfaces 9(5): 21–35.

Hahn, C. K., Duplaga, E. A. and Hartley, J. L. (2000). Supply-chain synchronization: lessons from Hyundai Motor Company, Interfaces 30(4): 32–45.

Jans, R. and Degraeve, Z. (2004). An industrial extension of the discrete lot-sizing and scheduling problem, IIE transactions 36(1): 47–58.

Loparic, M., Marchand, H. and Wolsey, L. A. (2003). Dynamic knapsack sets and capacitated lot-sizing, Mathematical Programming 95(1): 53–69.

Marinelli, F., Nenni, M. E. and Sforza, A. (2007). Capacitated lot sizing and scheduling with parallel machines and shared buffers: A case study in a packaging company, Annals of Operations Research 150(1): 177–192.

Martin, C. H., Dent, D. C. and Eckhart, J. C. (1993). Integrated production, distribution, and inventory planning at libbey-owens-ford, Interfaces 23(3): 68–78.

Miranda, P. L., Cordeau, J.-F., Ferreira, D., Jans, R. and Morabito, R. (2018). A decomposition heuristic for a rich production routing problem, Computers & Operations Research 98: 211–230.

Neves-Moreira, F., Almada-Lobo, B., Cordeau, J.-F., Guimaraes, L. and Jans, R. (2019). Solving a large multi-product production-routing problem with delivery time windows, Omega 86: 154–172.

Pochet, Y. and Wolsey, L. A. (1994). Polyhedra for lot-sizing with Wagner-Whitin costs, Mathematical Programming 67(1-3): 297–323.

Pochet, Y. and Wolsey, L. A. (2006). Production Planning by Mixed Integer Programming, Springer Science & Business Media.

Qiu, Y., Wang, L., Xu, X., Fang, X. and Pardalos, P. M. (2018). Formulations and branch-and-cut algorithms for multi-product multi-vehicle production routing problems with startup cost, Expert Systems with Applications 98: 1–10.

Silva, C. and Magalhaes, J. M. (2006). Heuristic lot size scheduling on unrelated parallel machines with applications in the textile industry, Computers & Industrial Engineering 50(1-2): 76–89.

Solyalı, O. and Süral, H. (2017). A multi-phase heuristic for the production routing problem, Computers & Operations Research 87: 114–124.

Trigeiro, W. W., Thomas, L. J. and McClain, J. O. (1989). Capacitated lot sizing with setup times, Management Science 35(3): 353–366.

Zhang, Q., Sundaramoorthy, A., Grossmann, I. E. and Pinto, J. M. (2017). Multiscale production routing in multicommodity supply chains with complex production facilities, Computers & Operations Research 79: 207–222.