Les Cahiers du GERAD ISSN: 0711–2440

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F. Cabo, G. Martín-Herrán,
M. P. Martínez-García

G–2019–34

May 2019

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2019
– Bibliothèque et Archives Canada, 2019

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

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GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Present bias and the inefficiency of the centralized economy. The role of the elasticity of intertemporal substitution

Francisco Cabo [a,b]
Guiomar Martín-Herrán [a,b,c]
María Pilar Martínez-García [d]

[a] *IMUVa, Universidad de Valladolid, 47011 Valladolid, Spain*

[b] *Department Economía Aplicada (Matemáticas). Facultad de Ciencias Económicas y Empresariales, Universidad de Valladolid, 47011 Valladolid, Spain*

[c] *GERAD, Montréal (Québec), Canada, H3T 2A7*

[d] *Universidad de Murcia, 30071 Murcia, Spain*

guiomar@eco.uva.es

May 2019
Les Cahiers du GERAD
G–2019–34

Abstract: We analyze an endogenous growth model with non-constant discounting and a negative externality of growth on utility. With a decreasing rate of impatience, time-consistent agents anticipate the behavior of their future selves and play a game against them. The strategic interaction between subsequent central planners implies slower growth than the market solution, where the externality is not internalized. Indeed, growth can be excessively slow from a social welfare standpoint. Contrary to exponential discounting, under non-constant discounting we prove that the market equilibrium is Pareto-improving provided that the negative externality is sufficiently small. Thus, policy interventions would only be meaningful when the externality surpasses a given threshold. For a specific family of non-constant discount functions we observe that the range for the intensity of the externality compatible with a Pareto-improving market solution widens with the elasticity of intertemporal substitution in consumption. Similarly, this range also widens the more different from constant discounting time preferences are, due either to a wider range of variation for the instantaneous discount rates or because these decay more slowly.

Keywords: Non-constant discounting, elasticity of intertemporal substitution, endogenous growth, social welfare, time-consistent solutions

Acknowledgments: The authors thank Eustasio del Barrio for his invaluable collaboration to prove the existence of a BGP. This study was funded by the Spanish Government (research project ECO2017–82227–P) and by the Regional Government of Castilla y León (research projects VA024P17 and VA105G18), co-financed by FEDER funds.

1 Introduction

Since from Barro (1999), several authors have analyzed economic growth models assuming individuals with non-constant discounting. Recent examples considering a continuous time setting are Farzin and Wendner (2014) and Cabo et al. (2016) for neoclassical growth models, or Strulik (2015) and Cabo et al. (2015) for endogenous growth models. When departing from geometric or exponential discounting, the assumption of a decreasing level of impatience with the time-distance from the present implies that future individuals will be more impatient as anticipated today. Thus, current decisions will not typically be optimal if recomputed later. The consumer's inability to commit to future actions generates time-inconsistent optimal decisions, as future individuals will be inclined to deviate from current planned actions (this problem was raised very early by authors like Ramsey 1928, Strotz 1956 and Pollak 1968). Therefore, the current generation has an interest in manipulating the behavior of future decision-makers. The conflict of interest is solved if current individuals play a game against their future selves, and as a result, optimal decisions become time consistent. This type of behavior can give rise to situations in which the market economy provides higher social welfare than a centralized economy. Krusell et al. (2002) obtain this result for a Neoclassical growth model. When players try to manipulate their future selves they perceive how their current decisions influence future actions differently. The central planner is aware that his investment decisions affect the returns to future savings, while investors in a market economy take the rate of return as given. Under the assumption of decreasing marginal returns, the planner who perceives the decay of future returns would be less inclined to invest than agents in a market economy who perceive future returns as constant. In consequence the market economy grows at a higher rate and provides greater welfare.[1]

This outcome of a Pareto-improving market solution under non-constant discounting is also found in Cabo et al. (2017) for an endogenous growth model. In this approach, the existence of environmental externalities guarantees the result. Utility and production are negatively affected by a pollution externality. If the externality on utility is sufficiently large with respect to the externality on production, the market economy grows faster than the centralized economy where this externality is internalized.[2] Present-biased individuals strongly discount the near future, although impatience decreases with the time distance from the present. Thus, assuming an identical overall impatience, these individuals give a relatively stronger value to the long run than individuals with constant discounting. For these long-run oriented individuals, the faster growth associated with the market economy facilitates greater social welfare. Moreover, a Pareto-improving market solution is more likely when the decay of the instantaneous discount rate is faster. This result has been obtained taking into account a log-utility function (as in Krusell et al. 2002 and Hiraguchi 2014). Although this is a common simplifying hypothesis in this literature, it implies that the long-run growth rates of centralized and market economies mimic those under exponential discounting.

The main objective of the present paper is to analyze to what extent the result of a Pareto-improving market solution relies on the particular hypothesis of a log-utility specification. Thus, we deviate from equal income and substitution effects and additive separability. We analyze whether the result is still valid when considering an elasticity of intertemporal substitution in consumption (EIS) other than one. Under this specification, non-constant discounting does have an impact on the long-run growth rates. To maintain an analytically tractable model for an EIS different from one, the AK-endogenous growth model in Cabo et al. (2017) is simplified by considering a single externality on only utility, which is compatible with a faster growing market economy. Thus, we consider an isoelastic utility dependent on consumption but also on a negative side-effect from production. While for the ease of presentation we focus on pollution, one can also think of other negative externalities on utility generated as a by-product from output growth, like congestion problems or an increment in social problems associated with urban growth.

[1] Hiraguchi (2014) extends this result for a more general non-constant discount function, although in a later work he shows that the result is not valid for a two-sector endogenous growth model (Hiraguchi 2016).

[2] This result is not due to non-constant discounting. An identical result is found in Smulders and Gradus (1996) under exponential discounting.

Under the assumption of an isoelastic utility with an EIS different from one and hence non-separable, the marginal utility of consumption becomes dependent on the negative externality. We study the more common assumption of an EIS lower than one, as well as the more analytically demanding case of an EIS above one. The EIS measures individuals' willingness to transfer present consumption to the future. Moreover, given that the utility is non-separable, the relationship between consumption and pollution crucially depends on whether the EIS is above or below one. If the EIS is lower than one the marginal utility of consumption increases with pollution, showing a compensation effect of pollution on consumption. As pollution grows together with production, the decrease in the marginal utility of consumption (with increasing levels of consumption) is less pronounced. Conversely, if the EIS is greater than one, the marginal utility of consumption decreases with pollution, showing a distaste effect of pollution on consumption. Since both these are substitutes, the marginal utility of consumption decays sharply with the economic growth due to the rise in pollution.

Our research question can be framed within the current debate on the actual value of the EIS and, in particular, on whether it lies above or below one. As already commented, the log-utility is a frequent simplifying assumption in theoretical models of economic growth, specifically when they present complexities like non-constant time preferences. When considering a more realistic isoelastic utility with an EIS different from one, there is no consensus in the literature about its actual value. However, many authors agree that its value differs across countries and even across time. Starting from the well-known paper by Hall (1988), which prescribes a value below 0.1, most of the literature considers EIS below one. Indeed the typical assumption in macroeconomic models is to assume values between 0.5 and 1. An exhaustive review of the estimates given in the literature can be found in Havranek et al. (2015). They collect a great number of estimations within the literature, typically running between 0.1 and 2. This variability is also commented in Thimme (2017), who also highlights that the discussion has moved from zero or positive in the 80s, to below or above one today. A recent example of an EIS equal to 2, based on the standard life-cycle model, is Gruber (2013). Other examples of EIS above 1 can be found in the references provided in both Havranek et al. (2015) and Thimme (2017).

Many of the recent analyses on this topic agree that there is a country-level heterogeneity in EIS. Thus, the survey in Havranek et al. (2015) states that the greater the wealth and the stock market participation, the greater the EIS. Similarly, Thimme (2017) also adds education as a third factor implying higher EIS. The hypothesis that the EIS is likely to be greater than one for countries with an income above a given threshold is also supported by Ben-Gad (2012).

A Pareto-improving market solution fundamentally relies on the assumption of non-constant discounting. Time preference of agents is characterized by a general quasi-hyperbolic discount function in continuous time. Impatience decreases with the time distance from the present. Under these preferences time consistency is ensured if individuals play a game against their future selves. We analyze the unique symmetric Markov perfect equilibrium in linear strategies for this game (see, for example, Karp 2007, Marín-Solano and Navas 2009 or Cabo et al. 2017). This equilibrium characterizes a balanced growth path along which pollution grows with output, although at a lower rate. Given that we assume no externality on production, pollution does not jeopardize economic growth. Hence, we focus on weak sustainability (see, for example Smulders 2000). A balanced growth path exists and is characterized for the centralized as well as for the market economy. The growth rate in each scenario is defined by an implicit equation in which the role played by the constant discount rate (under exponential discounting) is replaced by what is known as the effective discount rate (see, Barro 1999). This rate is built as a weighted mean of all future decreasing instantaneous discount rates. These weights depend on the growth rate and they are defined as the (discounted) relative effect that an increment in the current capital stock has on the utility at every future point in time. A higher EIS comes together with higher weights for very distant instantaneous (and small) discount rates, implying a lower effective discount rate. The same applies for a stronger externality, but only if the EIS lies below one. Conversely, when the substitution effect exceeds the income effect, a greater externality puts more weight on the short run, hence increasing the effective discount rate.

We demonstrate the existence and uniqueness of a balanced growth path equilibrium in both the centralized and the market economy. Even when the EIS is above one, implying exploding future utility gains with current capital increments, a unique equilibrium exists as long as discounting guarantees convergence. Due to the negative effect of the externality on utility, a central planner who internalizes the pollution externality will cut savings with respect to the market economy, hence resulting in a lower long-run growth rate. This result is likewise obtained under constant discounting (see, for example, Smulders and Gradus 1996).

As expected, a higher EIS induces a rise in the growth rate of the economy, in both the centralized and decentralized scenarios. Likewise, a stronger pollution externality reduces growth in the centralized economy. However, its effect on the market economy depends on the EIS. When the EIS is above one, a stronger pollution externality smooths the exploding future utility gains, the incentive to invest shrinks and growth decays. Conversely, in the case of an EIS below one, the effect of current capital increments on ongoing utility decays less sharply, which encourages investment and speeds up economic growth.

Our main result corroborates that the market solution is Pareto-improving considering an isoelastic utility function with an EIS different from one if the pollution externality is sufficiently small. To get an insight about this result, note that the externality on utility leads the central planner to reduce growth, while competitive individuals either reduce growth to a lesser extent or even increase growth. In either case, a gap is opened between the centralized economy and the faster-growing market economy. The slower growth in the centralized solution has two opposite effects on social welfare when agents discount the future at a decreasing rate. First, the standard effect under constant discounting: a less fast growth induces lower damages from pollution at all future times and hence greater social welfare. Second, when impatience decreases with the time distance from the present, agents strongly discount the near future but less so the distant future. These agents value relatively less the short run and relatively more the long run than standard exponential discounting agents (controlling for identical overall impatience). Because these agents are more long-run oriented, a slower growth induces lower welfare. Our result reveals that when the pollution externality is small, the second negative effect is stronger and the centralized economy grows too slowly, so the market equilibrium provides a higher social welfare. Conversely, if the externality is large the first effect prevails and social welfare is greater in the centralized solution. According to this result, policy interventions would only be meaningful when the pollution externality surpasses a given threshold.

Finally, we carry out a sensitivity analysis for a specific family of non-constant discount functions, suggested by Tsoukis et al. (2017). First, we observe that the range of values for the intensity of the pollution externality compatible with a Pareto-improving market solution widens with the EIS. Second, the threshold is also greater the more differently from constant-discounting the agents behave. Thus, as already commented, for wealthy countries with educated citizens who are familiar with financial markets, the EIS is likely to be large, and policy intervention would only be needed for highly damaging pollution problems. Conversely, intervention should be faster in less developed countries.

The paper is organized as follows. The next section briefly presents the model for the centralized and the market economies and proves the existence and uniqueness of a balanced path equilibrium. In Section 3 we carry out a sensitivity analysis of the equilibrium growth rates to changes in the EIS, the pollution externality and the parameters describing the discount function. Section 4 compares social welfare under both scenarios and illustrates how the possibility of a Pareto-improving market solution is affected by the main parameters. The conclusions are summarized in Section 5. All proofs are collected in the Appendix.

2 Endogenous growth with a pollution externality

This section presents an endogenous growth model where an economy produces a final output using linear technology, and generates pollution as a by-product:

$$Y(K) = AK, \quad P(K) = K^{\lambda} \text{ with } \lambda > 0. \tag{1}$$

Capital, K, is the only input, and P represents pollution. Assuming constant population normalized to one, all variables are in per capita terms. Pollution negatively enters the utility function, which also positively depends on consumption, C:

$$U(C,P) = \begin{cases} \dfrac{\left(CP^{-\phi}\right)^{1-1/\sigma} - 1}{1-1/\sigma} & \text{if} \quad \sigma \neq 1, \\ \ln C - \phi \ln P & \text{if} \quad \sigma = 1. \end{cases} \tag{2}$$

Utility is assumed to be isoelastic with a constant elasticity of intertemporal substitution in consumption, $\sigma \in (0,\infty)$. Parameter $\phi > 0$ measures how strongly pollution affects utility. Moreover, taking into account the pollution elasticity of capital in (1), the utility can be written in terms of consumption and capital, with the term $\phi\lambda$, henceforth denoted φ, defining the pollution externality in this economy. Hereinafter we will assume that this externality satisfies the condition[3] $\varphi < 1$. Under this condition, along a balanced path, the positive effect on utility of a constant growth in consumption is stronger than the negative effect of a sustained growth in pollution (associated with the growth in the capital stock).

The cross-effect of consumption and pollution on utility crucially depends on the EIS. When $\sigma < 1$, there exists a "compensation effect" of pollution on consumption, $\partial^2 U/(\partial c\, \partial P) > 0$. The growth in pollution partially counterbalances the fall in the marginal utility of consumption with increasing levels of consumption. This offsetting effect is weaker the greater the EIS, actually turning into a fostering effect for $\sigma > 1$, when there exists a "distaste effect" of pollution on consumption: $\partial^2 U/(\partial c\, \partial P) < 0$. In the usual assumption of log-utility, utility is separable in consumption and pollution and no distaste or compensation effect exists.

We assume agents with a decreasing level of impatience, who discount the future at a rate which decays with the time distance from the present, j. Thus, the instantaneous discount rate $\rho(j)$ satisfies $\rho(j) > 0,\ \dot{\rho}(j) < 0, \forall j \geq 0$, with $\lim_{t\to+\infty} \rho(j) = \rho_\infty > 0$. In consequence, the discount function $\theta(j)$ must satisfy $\dot{\theta}(j)/\theta(j) = -\rho(j)$ with $\theta(0) = 1$. Therefore, this function reads:

$$\theta(j) = e^{-\int_0^j \rho(z)dz}. \tag{3}$$

Having a non-constant instantaneous discount rate, the overall impatience is measured as

$$\hat{\rho} = \left[\int_0^\infty \theta(j)\, dj\right]^{-1}. \tag{4}$$

Note that $\hat{\rho}$ would be the constant discount rate with an equivalent overall level of impatience. In the standard AK-type model with exponential discounting and a pollution externality, the growth rate along the balanced growth path would read $\sigma[A - \hat{\rho}/(1-\varphi)]$. Here, and henceforth, we assume that a balanced growth path with a positive growth rate would be feasible if agents discounted the future at a constant rate. This requires a not excessively large externality and a sufficiently large marginal productivity of capital with respect to the discount rate:

$$A(1-\varphi) > \hat{\rho}. \tag{5}$$

Under the setting of non-constant preferences, we assume that agents acknowledge that the preferences of their future selves will change. As the current time evolves, they will become more impatient. Hence, we consider sophisticated agents who play a game against all their future selves. This hypothesis is considered both for individual agents in a decentralized economy and for the central authority in a centralized economy, who shares the preferences of the current selves. This hypothesis guarantees time consistency- the optimal solution computed at the current time remains valid also if recomputed

[3] Along the balanced path, where consumption, capital and output grow at the same constant rate, this condition guarantees that the maximized Hamiltonian satisfies the Arrow sufficient optimality conditions.

by future generations. Time-consistent solutions are characterized in both the centralized economy, where the central planner internalizes the pollution externality, and in the market economy.

In the centralized economy, at current date t, the central planner maximizes the lifetime utility of the representative consumer (t-agent), taking into account that the decisions to raise capital also enhance production and, consequently, pollution:

$$\max_{C_t(\tau)} \int_t^\infty U\left(C_t(\tau), K_t^\lambda(\tau)\right) \theta(\tau - t)\, d\tau, \tag{6}$$

$$\text{s.t.:} \qquad \dot{K}_t(\tau) = AK_t(\tau) - C_t(\tau), \quad K_t(t) = K_t, \tag{7}$$

where the utility function is given by (2), and $\tau - t$ denotes the time distance from the present.

In a market economy, the representative household maximizes its utility from current time t on, taking pollution as given and subject to its budget constraint:

$$\max_{c_t(\tau)} \int_t^\infty U\left(c_t(\tau), P_t(\tau)\right) \theta(\tau - t)\, d\tau, \tag{8}$$

$$\text{s.t.:} \qquad \dot{k}_t(\tau) = rk_t(\tau) - c_t(\tau), \quad k_t(t) = k_t, \tag{9}$$

where $k_t(\tau)$ denotes household assets and r the market interest rate.[4] Contrary to the central planner, the households in the market solution disregard how an increment in the capital stock will induce higher future pollution.

Following standard modelization in AK-type models, we assume a large number of identical firms that produce following a linear technology, $Y_i = AK_i$. Considering profit maximizing firms, the rental price to capital equals the capital productivity, $r = A$. Finally, assuming identical firms of measure one, $K = K_i$ and consequently, $P = P_i = K^\lambda$, and the aggregate production in (1) follows.

Considering an EIS different from one,[5] in the centralized economy, along a balanced growth path consumption and capital grow at the same constant rate, g, and the propensity to consume out of wealth (capital) remains constant, denoted by ξ. Correspondingly, by (1) pollution grows at rate λg. Thus, the accumulated flow of discounted utility along a balanced path can be written as,

$$\int_t^\infty U\left(\xi K_t(\tau), K_t^\lambda(\tau)\right) \theta(\tau - t)\, d\tau = -\frac{\sigma}{(\sigma-1)\hat{\rho}} + \sigma \frac{\xi^{1-\frac{1}{\sigma}}}{\sigma - 1} K_t^{-\eta} \Omega(g, \sigma, \varphi), \tag{10}$$

with $K_t(\tau)$ the capital stock in the economy along a balanced path and

$$\Omega(g, \sigma, \varphi) = \int_0^\infty \theta(j) e^{-\eta g j} dj, \quad \eta = \frac{1-\sigma}{\sigma}(1-\varphi). \tag{11}$$

In (11) the time distance from the present is denoted by $j = \tau - t$.

In the market economy along a balanced growth path, household consumption and assets grow at the same rate, g, which also determines the growth rate of the total capital in the economy. The propensity to consume out of capital is described as the consumption carried out by each household divided by this household's assets. Thus, the accumulated flow of discounted utility in the market economy reads:

$$\int_t^\infty U\left(\xi k_t(\tau), K_t^\lambda(\tau)\right) \theta(\tau - t)\, d\tau = -\frac{\sigma}{(\sigma-1)\hat{\rho}} + \sigma \frac{\xi^{1-\frac{1}{\sigma}}}{\sigma - 1} k_t^{-\eta_1} K_t^{-\eta_2} \Omega(g, \sigma, \varphi), \tag{12}$$

with $K_t(\tau)$ and $k_t(\tau)$ the capital stock in the economy and the households assets along a balanced path and

$$\eta_1 = \frac{1-\sigma}{\sigma}, \quad \eta_2 = -\frac{1-\sigma}{\sigma}\varphi.$$

[4] Note that without labor, consumers wealth is exclusively determined by the capital stock they own and lend to firms.

[5] The analysis of the particular case $\sigma = 1$ is postponed until Remark 1.

Note that $\eta_1 + \eta_2 = \eta$. The sign of η depends on whether the EIS is greater or lower than one. The assumption of an income effect stronger than or equal to the substitution effect is the most commonly accepted assumption in the literature. Under this assumption $\eta > 0$ and the improper integral $\Omega(g, \sigma, \varphi)$, which defines discounted utility, is convergent. In contrast, under the alternative assumption of $\sigma > 1$, $\eta < 0$ and one needs to establish a ceiling for the elasticity of intertemporal substitution in consumption to guarantee convergence.

Proposition 1 *When considering an infinite time horizon, the improper integral which collects the present value of the stream of discounted utility is convergent if and only if the constant growth rate of the economy satisfies*

$$\eta g + \rho_\infty > 0. \tag{13}$$

This inequality is satisfied for any $\sigma \leq 1$, as well as if $\sigma > 1$ and $g < -\rho_\infty/\eta$.

Under condition (13) the growth rates of the centralized and the decentralized economies are characterized in the next proposition.

Proposition 2 *The growth rate along a balanced path equilibrium is characterized by*

$$g^* = \sigma \frac{(1-\varphi)A - \Delta(g^*, \sigma, \varphi)}{1-\varphi} \equiv h(g^*) \quad \textit{[Centralized Economy]} \tag{14}$$

$$\tilde{g}^* = \sigma \frac{A - \Delta(\tilde{g}^*, \sigma, \varphi)}{1 - \varphi + \sigma\varphi} \equiv \tilde{h}(\tilde{g}^*) \quad \textit{[Market Economy]}\ ^{6} \tag{15}$$

with

$$\Delta(g, \sigma, \varphi) = \int_0^\infty \rho(j)\omega(j)\, dj, \quad \omega(j) = \frac{\theta(j)e^{-\eta g j}}{\Omega(g, \sigma, \varphi)}. \tag{16}$$

Remark 1 *In the particular case $\sigma = 1$, the accumulated flow of discounted utility along a balanced path (10) in the centralized economy, and (12) in the market economy translates into*

$$\int_t^\infty [\ln(\xi K_t(\tau)) - \phi \ln(K_t^\lambda(\tau))]\theta(\tau - t)\, d\tau = \frac{1}{\hat{\rho}}\left[(1-\varphi)\ln(K_t) + g(1-\varphi)\bar{J}^1 + \ln\xi\right],$$
$$\int_t^\infty [\ln(\xi k_t(\tau)) - \phi \ln(K_t^\lambda(\tau))]\theta(\tau - t)\, d\tau = \frac{1}{\hat{\rho}}\left[\ln(k_t) - \varphi\ln(K_t) + g(1-\varphi)\bar{J}^1 + \ln\xi\right],$$

where $\bar{J}^1 = \hat{\rho}\int_0^\infty j\theta(j)dj$.
Note also that $\Omega(g, 1, \varphi) = 1/\hat{\rho}$ and $\Delta(g, 1, \varphi) = \hat{\rho}$. In consequence, expressions (14) and (15) explicitly define the equilibrium growth rates in the centralized and the market economies:

$$g^{1*} = \frac{(1-\varphi)A - \hat{\rho}}{(1-\varphi)}, \qquad \tilde{g}^{1*} = A - \hat{\rho}. \tag{17}$$

Corollary 1 *Condition for convergence in (13) at the equilibrium reduces to*

$$\sigma < \hat{\sigma} \equiv 1 + \frac{\rho_\infty}{(1-\varphi)A - \rho_\infty}. \tag{18}$$

Note that under condition (5) and taking into account that $\hat{\rho} > \rho_\infty$ then $\hat{\sigma} > 1$, and hence convergence with positive growth rates can also be guaranteed for an elasticity of intertemporal substitution greater than one.

In what follows, we prove, following a three-step procedure, that Equations (14) and (15) implicitly define the growth rate in the centralized and the decentralized economies along the balanced path equilibrium. First we give an intuition on how to construct the term $\Delta(g, \sigma, \varphi)$ which, following

[6] Here, and henceforth, a tilde refers to the market economy.

Barro (1999), can be interpreted as the effective rate of time preference. To this aim we rewrite the first-order optimality conditions in (26) and (32) with the help of expressions (28) and (33) as

$$\frac{\partial U(C_t, K_t^\lambda)}{\partial C_t} = [(1-\varphi)\Omega(g^*,\sigma,\varphi)]^{\frac{1}{\sigma}}\, K_t^{-(1+\eta)}, \qquad \frac{\partial U(c_t, K_t^\lambda)}{\partial c_t} = \Omega(\tilde{g}^*,\sigma,\varphi)^{\frac{1}{\sigma}} k_t^{-(1+\eta_1)} K_t^{-\eta_2},$$

for the centralized and the market solutions, respectively. These equations balance current marginal utility from consumption to the present value of all ongoing effects on utility of a marginal increment in current capital.

To give an interpretation to expression $\Omega(g,\sigma,\varphi)$, note that from (2) it follows that along a balanced path, the utility from a current time t on evolves according to the term $e^{-\eta g j}$. Therefore, this term determines the marginal effect that changes in current capital will have on the utility j periods from t along a balanced path. If $\sigma < 1$, the income effect exceeds the substitution effect, and marginal future effects shrink with the time distance from the present. In contrast, if $\sigma > 1$, current changes induce exploding marginal future effects. Finally, the assumption of log-utility is consistent with an invariant effect of a current change on all future utilities. The effects for different values of σ are depicted in Figure 1 (left).

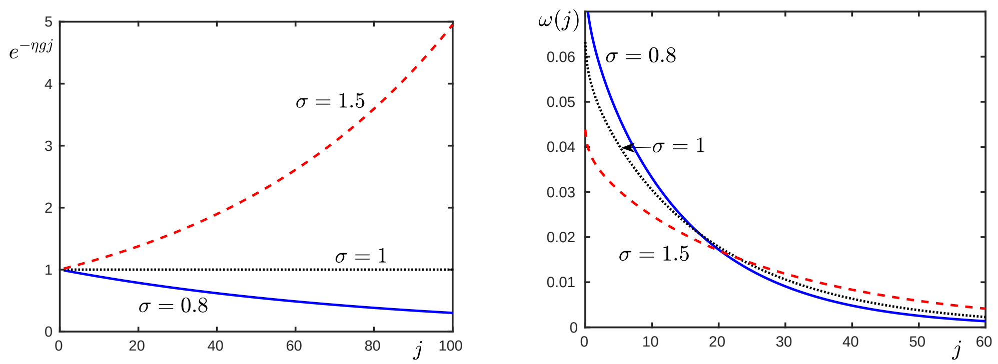

Figure 1: Future marginal effects in utility (left); and weights $\omega(j)$ (right).

A current $t-$agent discounts all these future effects according to the discount function, $\theta(j)$. Therefore, $\Omega(g,\sigma,\varphi)$ collects the total effect of a marginal increment in current capital on all future utility as the aggregated sum of the discounted value of ongoing future marginal effects on utility. This total effect can be decomposed in the effect j periods from now for every $j \geq 0$, which in relative terms is defined by $\omega(j)$ in (16), and depicted in Figure 1 (right). These relative future effects can also be interpreted as the weights defining the effective rate of time preference, $\Delta(g,\sigma,\varphi)$, as a weighted mean of the instantaneous discount rates from the present time on, $\rho(j)$, with weights satisfying $\int_0^\infty \omega(j)dj = 1$.

Second, we explain how a higher growth rate differently impacts the effective discount rate depending on whether the EIS is above or below one. For $\sigma < 1$ (i.e. $\eta > 0$) the decay in the marginal effect of current changes on future utility, $e^{-\eta g j}$, is more pronounced the greater the growth rate of the economy. Thus, a higher growth rate raises the weights given to the instantaneous discount rates closer to the present (which take higher values). In consequence, it also raises the effective discount rate (i.e. individuals are more short-run oriented). In contrast, for $\sigma > 1$ (i.e. $\eta < 0$), the greater the growth rate of the economy is, the faster the marginal future effects, $e^{-\eta g j}$, explode. Hence, a higher growth rate implies more long-run oriented agents. And agents who value more the far distance discount rates (with lower values) present a lower effective discount rate. Finally, when the income and the substitution effect exactly cancels out (i.e. $\sigma = 1$), growth has no effect on the distribution of weights.[7]

These results are summarized in the next proposition.

[7] Alternatively, we could refer to the effect of g on $\Omega(g,\sigma,\varphi)$. A faster growth reduces the total effect of a marginal increment of present capital on ongoing utility if $\sigma < 1$, increasing it if $\sigma > 1$, or leaving it unchanged if $\sigma = 1$.

Proposition 3 *The effective discount rate increases (decreases) with the growth rate of the economy if the EIS is lower (greater) than one, and it is unaffected for a log-utility.*

Lastly, we prove the existence and uniqueness of the balanced path equilibrium since $h(g)$ and $\tilde{h}(g)$ in (14) and (15) cross just once with g. Moreover, the equilibrium growth rate is larger in the market economy.

From Proposition 3 and the definitions in (14) and (15) one can conclude that $h'(g), \tilde{h}'(g)$ decrease (increase) with the growth rate of the economy if σ is lower (greater) than one. As proved in Proposition 4 below, the effective discount rate in the case of zero growth is equal to the constant discount rate having identical overall impatience, $\hat{\rho}$. Thus, $h(0), \tilde{h}(0)$ are clearly positive under condition (5). This proposition also proves that $h(A), \tilde{h}(A)$ are lower than A. Because curves $h(g), \tilde{h}(g)$ have no inflection points (see Lemma 1 in the Appendix), a balanced growth path equilibrium exists in the centralized and in the decentralized economy, as shown in Figure 2.[8]

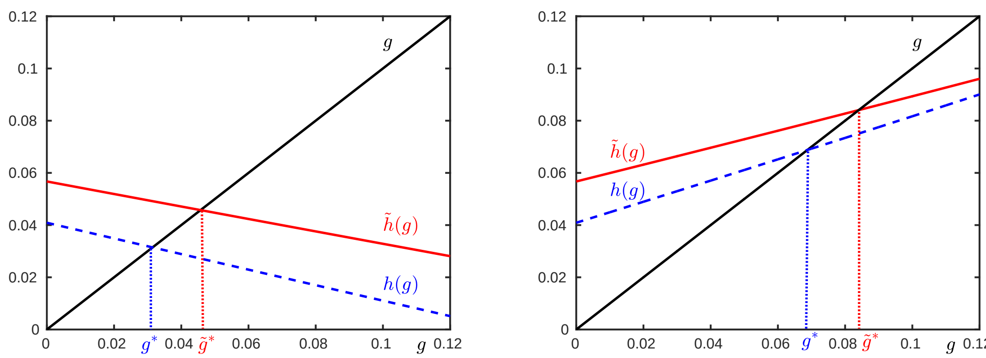

Figure 2: Equilibrium growth rates, g^* and $\tilde{g}^*$, for $\sigma < 1$ (left) $\sigma > 1$ (right).

Proposition 4 *Under conditions (5) and (18) there exists a unique positive solution, g^* and $\tilde{g}^*$, for Equations (14) and (15), respectively. Moreover, $\tilde{g}^* > g^*$.*

3 Sensitivity of the equilibria

In this section we analyze the sensitivity of the equilibrium growth rates under both scenarios to changes in the main parameters of the model: first, the EIS and the pollution externality and, second, the way individuals discount the future.

Proposition 5 *The equilibrium growth rates in the centralized and the market economy are greater the higher the elasticity of intertemporal substitution in consumption.*

As shown in the proof of Proposition 5 in the Appendix, the EIS has a twofold effect on growth. The direct positive effect on growth is the standard effect under constant discounting in a Ramsey economy with pollution. This stems from the fact that a higher σ softens the decay in the marginal utility of consumption with higher consumption levels. Along the balanced path pollution also increases with economic growth, which partially counterbalances the fall in the marginal utility of consumption when $\sigma < 1$, and there exists a "compensation effect" of pollution on consumption. In contrast, for $\sigma > 1$ the rise in pollution exacerbates the fall in the marginal utility of consumption due to a "distaste effect" of pollution on consumption. All in all, a larger EIS has a direct positive effect on growth which is less sharp in the market solution and does not anticipate the effect of consumption decisions on future pollution.

A second effect comes from the assumption of non-constant discounting. The higher the EIS, the stronger the relative effect of current changes on far distant utilities, which leads to a more skewed

[8] In order to be able to plot these graphs we have considered the particular discount function in (19), which will be commented on in Section 3.

distribution of weights toward the long run, as shown in Figure 1 (right). Since the instantaneous discount rate decreases monotonously, the effective discount rate is built by giving higher weights to the lowest discount rates. A lower effective discount rate does indeed induce faster growth.

The direct and the indirect effect are both positive, implying a total positive effect, which can be observed in Figure 3 for the centralized and the market economy.

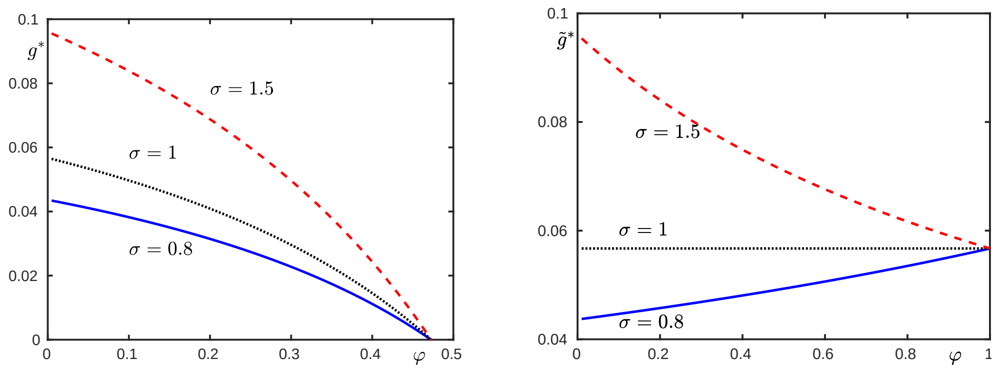

Figure 3: Sensitivity of g^* (left) and $\tilde{g}^*$ (right) w.r.t. σ and φ.

The next proposition highlights that a stronger incidence of pollution on utility, or a higher pollution elasticity of capital, leads to a less fast growth in the centralized economy. However, a stronger incidence of pollution on utility can increase or reduce growth in the market economy depending on whether the EIS is greater or lower than one.

Proposition 6 *The equilibrium growth rate in the centralized economy decreases with the pollution externality.*
The equilibrium growth rate in the market economy increases (decreases) with the pollution externality if the EIS is lower (greater) than one, and is not influenced by the externality for a log-utility.

Proposition 6 analyzes the effect of the size of the pollution externality on the growth rate of the economy. The pollution externality, φ, represents either the impact of pollution on utility, ϕ, or the elasticity of pollution with respect to capital, λ. To analyze its effect we can again distinguish a direct and an indirect effect.

The direct effect mimics the standard effect under constant discounting and can also be decomposed into two separated effects. On the one hand, a greater externality exacerbates the damage caused by pollution along a balanced growth path. Hence, it discourages savings and growth in the centralized solution, which internalizes the losses associated with pollution. On the other hand, as already commented, the decay in the marginal utility from consumption along the balanced path is partially counterbalaced by the increment in pollution under the compensation effect, when $\sigma < 1$. In the market economy, where the pollution externality is not internalized, this second effect will lead to faster growth when $\sigma < 1$. Conversely, when $\sigma > 1$ the distaste effect reinforces the decay in the marginal utility of consumption. The growth rate decreases with the pollution externality, specially in the centralized solution.

A greater externality also indirectly affects growth through its effect on the effective discount rate, $\Delta(g, \sigma, \varphi)$. This indirect effect crucially depends on whether the income effect exceeds the substitution effect or vice versa. Assuming a rise in the pollution externality, the marginal effect of current changes on future utilities, $e^{-\eta g j}$, shrinks more smoothly (for $\sigma < 1$), or expands less rapidly (for $\sigma > 1$). Hence, the relative weights, $\omega(j)$, given to the far distant future increase for $\sigma < 1$ and decrease for $\sigma > 1$. In consequence, individuals with a low willingness to substitute future for present consumption become less short-run oriented, their effective discount rate decreases, inducing a rise in the growth rate. Conversely, if individuals are willing to substitute intertemporally between consumption, they become less long-run oriented, face a higher discount rate and wish for lower growth.

For the centralized solution the direct effect of the pollution externality on growth is negative, while the indirect effect can be positive or negative depending on whether the EIS lies below or above 1. For $\sigma > 1$ both effects lead to lower growth. By contrast, for $\sigma < 1$ the indirect effect partially counterbalances the negative direct effect, although as proved in the proof of Proposition 6, the negative effect prevails. For the market solution, the direct and the indirect effects always have the same sign. For a low EIS a greater externality induces faster growth, both directly and due to a lower effective discount rate, and the opposite is true for a large EIS.

Next we analyze the impact of non-constant discounting on the growth rates in the centralized and the market economies. This effect cannot be characterized for a general discount function as in (3). Therefore, we rely on a particular family of discount functions presented in Tsoukis et al. (2017). These functions can be embodied as

$$\theta(j) = e^{-\rho_\infty j}(1+\delta j)^{-\gamma/\delta}, \tag{19}$$

with $\rho_\infty > 0, \gamma \in (0,1)$, $\delta > 0$ and $\gamma/\delta < 1$. In consequence, the instantaneous discount rate associated with this expression is

$$\rho(j) = \rho_\infty + \gamma/(1+\delta j).$$

Therefore, $\rho(j)$ decreases from $\rho_\infty + \gamma$ to ρ_∞ as the time distance from the present tends to infinity.

Computing the integral $\int_0^\infty \theta(j)dj$, considering the Laplace transform, and taking into account the assumption of identical overall level of impatience in (4) one gets

$$\hat{\rho} = \frac{e^{-\rho_\infty/\delta}\delta^{\gamma/\delta}\rho_\infty^{1-\gamma/\delta}}{\Gamma\left(1-\frac{\gamma}{\delta}, \frac{\rho_\infty}{\delta}\right)}, \tag{20}$$

where $\Gamma(a,b) = \int_b^\infty x^{a-1}e^{-x}dx$ is the incomplete gamma function.[9]

For this particular discount function we can numerically compute[10] the growth rates of the economy for the market and the centralized economies from (14) and (15). This enables us to study how the parameters defining the discount function affect growth in either scenario, as well as the gap between the two growth rates. Function (19) is described by three parameters. However, we aim to characterize how growth is affected by the way individuals discount future utility, but maintaining the overall level of impatience unchanged. Hence, we perform a controlled experiment, and analyze the effect of γ and ρ_∞, when δ adjusts so that condition (20) remains unaltered. An increment in γ (controlled by δ) implies that the range for the instantaneous discount rate, $[\rho_\infty + \gamma, \rho_\infty)$, widens. To maintain overall impatience the short run is strongly discounted and the mid and long run are more weakly discounted (see Figure 4 top). On the other hand, a higher ρ_∞ does not modify the width of this range, but the instantaneous discount rate decreases more sharply, as shown in Figure 5 top. As a consequence, under the same overall impatience, the short run is more strongly valued, and less so the long run.

Individuals whose preferences show higher γ are more distant from non-constant discounting, and, hence, more long-run oriented. When the EIS is lower than 1 ($\sigma = 0.8$) the marginal effect of changes in current capital on future utility decreases with the time distance from the present. Hence, the more strongly the long run is valued, the lower the aggregate value of the ongoing effects, $\Omega(g,\sigma,\varphi)$.[11] This reduces the incentive on current savings, so slowing down economic growth. Figure 4 bottom-left shows that a rise in γ, from 0.2 to 0.9, reduces the growth rate of the economy, and the reduction is stronger the smaller the pollution externality. If conversely, the EIS is greater than 1 ($\sigma = 1.5$), the marginal effect of changes in current capital on future utility increases with the time distance from the present.

[9] This follows the expression proposed by Tsoukis et al. (2017). We have corrected for a mistake by changing the standard Gamma function for an incomplete Gamma function.

[10] In all figures we consider as benchmark the following parameters values: $\gamma = 0.5$, $\delta = 10$, $\rho_\infty = 0.05$, $\varphi = 0.2$, $A = 0.12$. These parameters have been calibrated to obtain an overall impatience $\hat{\rho} = 0.0633$ and growth rates ranging between 0.04 and 0.1 in the different scenarios.

[11] Note that a reduction in $\Omega(g,\sigma,\varphi)$ is equivalent to an increment in the effective discount rate, from equation (30), and the consequent decay in the growth rate of the economy.

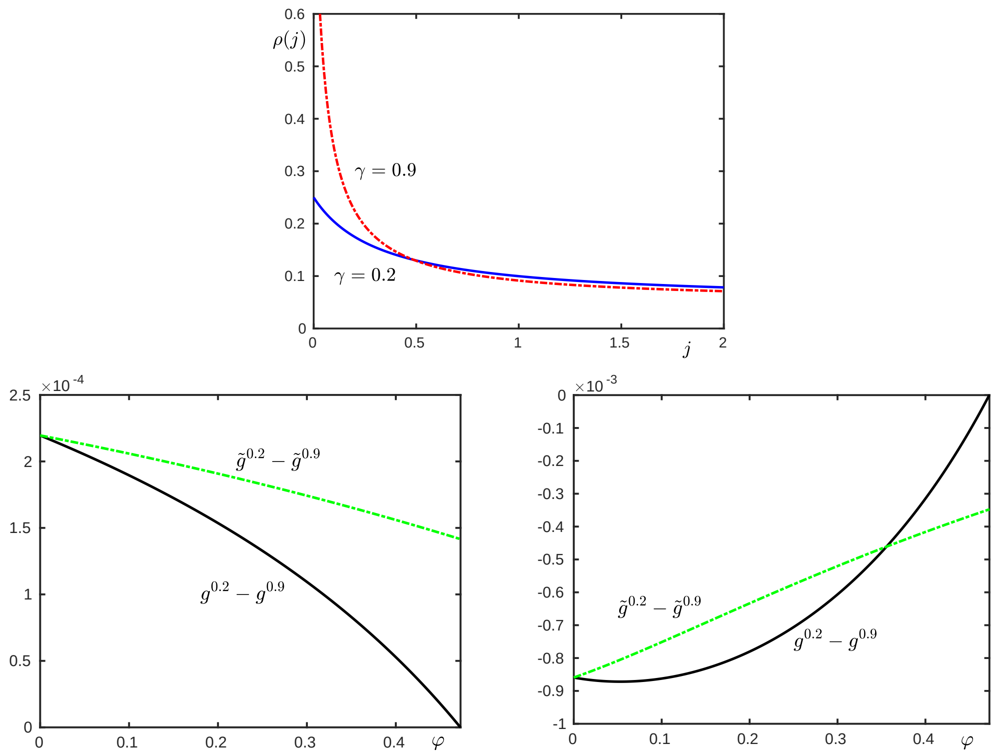

Figure 4: Effect of γ. Bottom left $\sigma = 0.8$; bottom right $\sigma = 1.5$.

In consequence, long-run oriented individuals will obtain greater ongoing utility gains from current capital increments, $\Omega(g, \sigma, \varphi)$. Thus, they will save more the greater the value of γ, fostering growth. This is shown in Figure 4 bottom-right, which also shows that the gap between the two scenarios widens with the intensity of the externality. Identical reasoning applies for a lower ρ_∞, as shown in Figures 5 bottom.

4 Social welfare comparison

In the previous section, considering a pollution externality in the utility of consumption, we have seen that the market economy grows faster than the centralized economy. Faster growth might lead to higher welfare in the decentralized scenario when individuals discount the future at a decreasing rate. This result is proved in the next proposition, provided that the pollution externality is small enough. By that we imply that either the incidence of pollution on utility, ϕ, or the pollution elasticity of capital, λ, is small.

Proposition 7 *Under conditions (5) and (18) the social welfare in the market economy surpasses the social welfare in the centralized economy provided that the pollution externality is sufficiently small.*

Assuming identical households of measure one, $k_t = K_t$, and then the gap in the optimal social welfare between the market and the centralized economy can be computed when $\sigma \neq 1$, taking into account expressions (27), (28) and (33) as

$$\widetilde{W}(K_t, K_t) - W(K_t) = K_t^{-\eta} H(\varphi), \tag{21}$$

where

$$H(\varphi) = -\frac{\sigma}{1-\sigma}\left[\Omega(\tilde{g}^*, \sigma, \varphi)^{\frac{1}{\sigma}} - (1-\varphi)^{\frac{1-\sigma}{\sigma}}\Omega(g^*, \sigma, \varphi)^{\frac{1}{\sigma}}\right]. \tag{22}$$

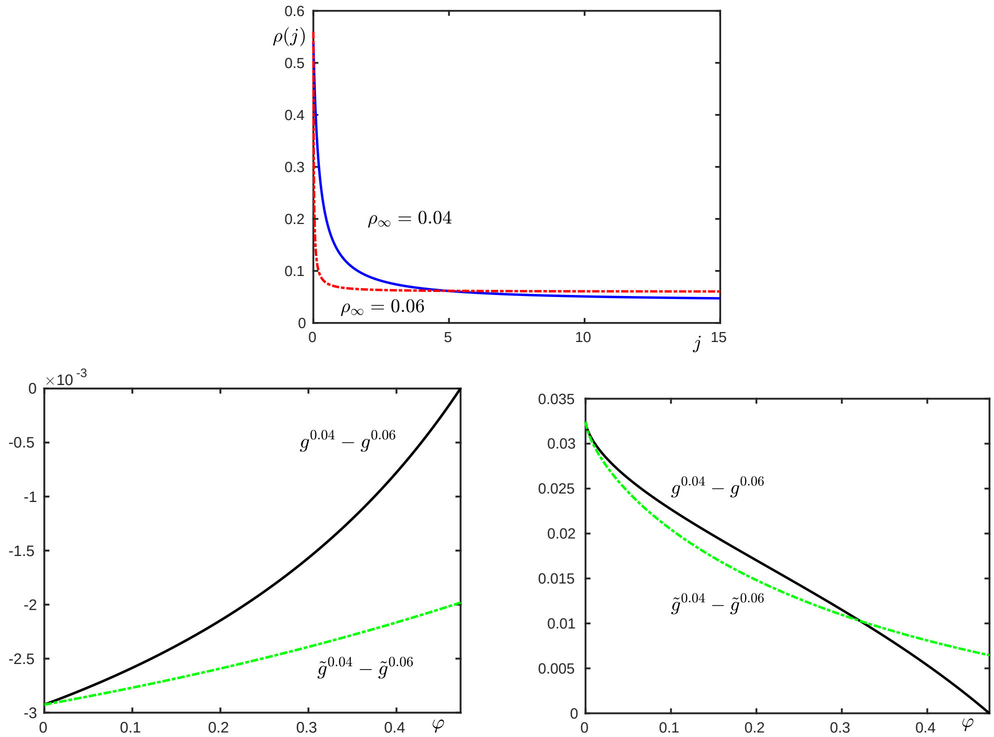

Figure 5: Effect of ρ_∞. Bottom left $\sigma = 0.8$; bottom right $\sigma = 1.5$.

Likewise, for the particular case $\sigma = 1$, from (17), (35) and (36), this gap reads:

$$\widetilde{W}^1(K_t, K_t) - W^1(K_t) = H^1(\varphi) = \varphi \bar{J}^1 + \frac{\ln(1-\varphi)}{\hat{\rho}}, \tag{23}$$

with $\bar{J}^1$ defined in Remark 1.

Note first that, in the general case, the capital stock at current time t determines the magnitude of this gap in (21), but not its sign. Neither does it influence this gap when $\sigma = 1$. From expressions (21) and (23) it follows that the social welfare in the market economy surpasses the social welfare in the centralized economy whenever the intensity of the pollution externality, φ, is such that function $H(\varphi)$ or $H^1(\varphi)$, defined in (22) and (23) is positive.[12] For the extreme case of zero externality, $\varphi = 0$, the centralized and the market solutions coincide and no difference in welfare exists, $H(0) = H^1(0) = 0$. Proposition 7 analytically proves that these functions always take a positive value for a sufficiently small φ. Moreover, we have numerically found that this result typically reverses for a large pollution externality, as is illustrated in Figure 6. As this figure shows, there exists a threshold value, $\hat{\varphi} > 0$, above which the centralized solution provides higher social welfare and hence policy interventions might improve social welfare.

Next, we describe how the window within which the market economy is Pareto-superior, $(0, \hat{\varphi})$, is affected by the parameters describing preferences. To this aim we distinguish between the effect of preferences through the EIS, and the effect of temporal discounting. A higher EIS does not monotonously imply a wider gap $\widetilde{W}(K_t, K_t) - W(K_t)$. It does, however, widen the range of values for the pollution externality, $\hat{\varphi}$, for which the market solution is welfare improving. This result is presented for three different values of σ in Figure 6. Moreover, it can be generalized for any σ leading to a positive growth rate of the economy. Increasing σ till the upper bound $\hat{\sigma}$ (above which convergence ceases), $\hat{\varphi}$ monotonously increases.

To analyze the effect of temporal discounting, as in the previous section, we focus on two parameters to describe the way households discount the future, γ and ρ_∞, controlled by the corresponding

[12] If functions $H(\varphi)$ or $H^1(\varphi)$ are computed in the case of constant discounting, they take negative values for any φ, consistently with the standard result of the Pareto-efficiency of the centralized equilibrium.

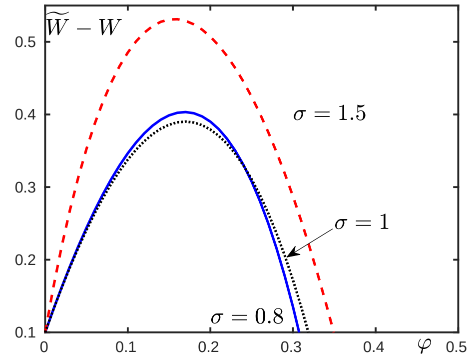

Figure 6: $\widetilde{W}(K_t, K_t) - W(K_t)$ **with respecto to** φ**.**

variation in δ so as to maintain overall impatience unchanged. A higher γ represents a wider range of variation in instantaneous discount rates, and hence, a time preference more dissimilar from constant discounting. Conversely, a higher ρ_∞ corresponds with a faster decay in instantaneous discount rates, and a discount function closer to exponential discounting. Figure 7 (left) depicts the threshold of the pollution externality below which the market provides higher social welfare, $\hat{\varphi}$, as a function of γ. It shows that the wider the range in instantaneous discount rates, the wider the interval of the pollution externality within which policy intervention would not be needed. A similar analysis, but with respect to ρ_∞ is presented in Figure 7 (right). It shows that this interval narrows with the speed of decay in instantaneous discount rates.[13] Thus, both charts in Figure 7 highlight that the more different from constant-discounting individuals' behaviour is, the greater the pollution externality for a policy intervention to be necessary needs to be.

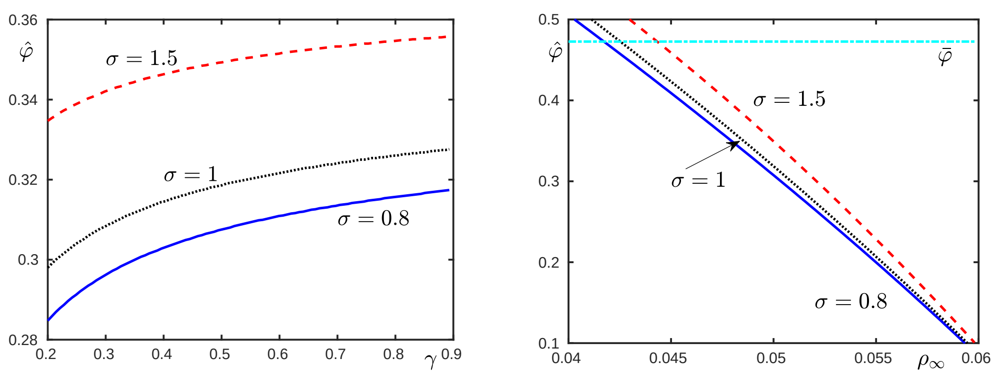

Figure 7: Effect of γ **(left) and** ρ_∞ **(right) on** $\hat{\varphi}$**.**

5 Concluding remarks

This paper analyzes a simple AK-endogenous growth model under the following main assumptions: i) a negative externality (e.g. pollution) enters the consumers' utility; ii) time preferences are characterized by a decreasing level of impatience with the time-distance from the present; and iii) current individuals acknowledge that the discount rate of future cohorts will not be as low as anticipated, and hence try to manipulate their future selves, playing a game against them all. Two main consequences stem from these assumptions. First, the Markov perfect equilibrium in linear strategies for this game is characterized by a slower growth when played by current against future central planners than when played by competitive agents, who ignore how investment on production capacity increases pollution. Second, agents with a decaying rate of temporal discount are more long-run oriented than standard

[13] In this chart $\bar{\varphi}$ represents the upper bound for φ that guarantees positive growth. A similar bound applies in the left chart, not plotted as it lies above the vertical span of the graph.

individuals with a constant discount rate and the same overall impatience. This can be explained because present-biased individuals strongly discount the near future and the distant future much less. Tying these two consequences together we have proved the main finding of our paper: if the pollution externality on utility is sufficiently small, then the strategic interaction between subsequent central planners could lead to a too slow growth rate, so the social welfare of long-run oriented individuals ends up being smaller than in the market economy.

We have analyzed how the centralized and the market equilibria respond to three fundamental aspects: the EIS (the most important for our analysis), the intensity of the pollution externality and the discount function. It is important to recall that we do not present a doomsday model, so, in both the centralized and in the market economy a unique balanced growth path exists, and the continuous growth of pollution does not reduce utility as it is more than compensated by a faster growth in consumption. The growth rate of the economy in both scenarios depends on the three aspects above. The value of the EIS has an immediately clear consequence, given the assumption of a non-separable utility function in consumption and pollution. If the EIS is lower than one, as is commonly assumed in macroeconomic models, a compensation effect of pollution on consumption fosters economic growth. In contrast, an EIS above one would imply a distaste effect of pollution on consumption, so discouraging growth. Regardless of the existence of a compensation or a distaste effect, a higher EIS always leads to faster growth.

As for the pollution externality, its effect on growth depends fundamentally on the EIS. The externality directly induces growth reductions in the centralized equilibrium. However, for the market equilibrium, if the EIS is lower than one, given the compensation effect, the pollution externality fosters growth. Conversely, when the EIS is greater than one, the distaste effect implies a reduction in economic growth with the pollution externality.

Time preferences also affect growth differently depending on whether the EIS is above or below one. To understand this dichotomy, we highlight first how this value influences optimal consumption/saving decisions. Current time optimizing agents must take into account how a rise in current capital will affect current utility, as well as the utility of all future selves (against whom current agents are playing). Under the assumption of Markov perfect linear strategies, the propensity to consume out of wealth is constant and, therefore, the marginal effect on ongoing utility can be computed and crucially depends on whether the EIS is above or below one. It decreases (if the EIS is lower than one), increases (if the EIS is greater than one), or remains constant (if the EIS is equal to one). When the instantaneous discount rate decreases, these preferences are more dissimilar from exponential discounting the wider the range of variation for the instantaneous discount rate, or the smoother their decay with the time distance from the present. Considering identical overall impatience, the more dissimilar from exponential discounting, the more strongly the agents discount the nearby future with respect to the far distant future, and the more long-run oriented these individuals are. When the EIS is lower than one, current savings have decreasing future marginal effects on utility. Therefore, the more long-run oriented individuals are, the lower are the aggregate returns to current savings, and a lower incentive to save slows down growth. Conversely, if the EIS is greater than one, the marginal effect of current savings increases with the time distance from the present, and being more long-run oriented leads individuals to enhance savings and growth.

The relationship between the elasticity of intertemporal substitution in consumption and the way agents discount the future plays a fundamental role in the result of a Pareto-improving market solution. The sequence of impacts that current changes have on the payoffs of all future selves, discounted at a non-constant rate, defines the relative value that the current generation assigns to all future instants of time (i.e. to all future selves). This sequence depends on the elasticity of intertemporal substitution in consumption. The higher the EIS the softer the effect on nearby utilities and the stronger the effect on the utility of far-distant generations. In consequence, individuals with a higher EIS can be regarded as more long-run oriented. The strategic interaction between subsequent central planners, which leads to slow growth, can provide a lower social welfare than the market solution because individuals with non-constant discounting strongly value the long-run and, hence, higher growth rates. The more long-run oriented these agents are, the more likely a Pareto-improving market solution is. We numerically observe that a rise in the EIS widens the range of values for the pollution externality compatible with a Pareto-improving market solution.

Similarly, the more dissimilar from exponential discounting the time preferences are, the more long-run oriented the individuals are. Therefore, we also observe that a discount function more distant from constant discounting (although keeping overall impatience unchanged) also corresponds to a wider range of values for the pollution externality compatible with a Pareto-improving market solution.

Assuming that agents' degree of impatience decreases with the time distance from the present, our analysis comes together with a policy implication. If pollution stemming from the productive process partially reduces the utility from consumption and it has no influence on the production capacity. If, moreover, this externality is not too large, then the market equilibrium provides a higher welfare than the strategic interaction between subsequent central planners. Under these circumstances no policy actions would be needed. In contrast, these policy actions might be necessary when the externality surpasses a given threshold. This threshold is larger the more dissimilar from exponential discounting the agents are and the greater the EIS. Thus, policy actions would be less necessary in wealthy countries with a well-educated population familiar with financial markets, where according to the empirical evidence the EIS is larger (see, for example, Ben-Gad 2012, Havranek et al. 2015 and Thimme 2017).

The analysis has been carried out for a toy AK-model. Under this simple specification we have been able to prove existence and uniqueness of the centralized and decentralized balanced growth path equilibria. Moreover, the reaction of the equilibrium growth rates to changes in the EIS and the pollution externality can also be derived. Finally, the result of a Pareto-improving market solution for a sufficiently small pollution externality can also be analytically proved for any plausible EIS. Numerical results could be explored for more complex models.

A Appendix.

Proof of Proposition 1. From (3), and taking into account that $\rho(j)$ is strictly decreasing towards $\rho_\infty > 0$, then

$$\theta(j) = e^{-\int_0^j \rho(z)dz} < e^{-\rho_\infty j}.$$

In consequence, $\Omega(g,\sigma,\varphi)$ is upper bounded:

$$\Omega(g,\sigma,\varphi) < \int_0^\infty e^{-(\eta g+\rho_\infty)j} dj.$$

Therefore, $\eta g + \rho_\infty > 0$ guarantees the convergence of the integral $\Omega(g,\sigma,\varphi)$.

In contrast, if condition (13) does not hold, then $-\eta g > \rho_\infty$, $\sigma > 1$ and $\eta < 0$. Therefore, since $\rho(j)$ is strictly decreasing towards $\rho_\infty > 0$, two possible cases arise:

1. If $\rho(0) > -\eta g$, then there exists a $\check{j}$ such that $\rho(\check{j}) = -\eta g$ and $\rho(z) \in (\rho_\infty, -\eta g)$ for all $z > \check{j}$. Thus,
$$\Omega(g,\sigma,\varphi) = \int_0^{\check{j}} e^{-\eta g j} e^{-\int_0^j \rho(z)dz} dj + \int_{\check{j}}^\infty e^{-\eta g j} e^{-\int_0^j \rho(z)dz} dj. \tag{24}$$
The second integral is lower bounded:
$$e^{-\int_0^{\check{j}} \rho(z)dz} \int_{\check{j}}^\infty e^{-\eta g j} e^{-\int_{\check{j}}^j \rho(z)dz} dj > e^{-\int_0^{\check{j}} \rho(z)dz} \int_{\check{j}}^\infty e^{-\eta g j} e^{\int_{\check{j}}^j \eta g dz} dj \equiv e^{-\int_0^{\check{j}} \rho(z)dz} \int_{\check{j}}^\infty e^{-\eta g \check{j}} dj.$$
Because this last integral is clearly divergent, this is also true for $\Omega(g,\sigma,\varphi)$.
2. If $\rho(0) \leq -\eta g$, then, on replacing $\check{j}$ by zero in the second integral in (24), and following the same reasoning, it immediately follows that $\Omega(g,\sigma,\varphi) > \int_0^\infty dj$, and, hence, divergent.

□

Proof of Proposition 2. Case $\sigma \neq 1$.

Centralized economy Following Karp (2007), an optimal stationary solution of problem (6)–(7) needs to satisfy the following Bellman equation:

$$\int_t^\infty U\left(C_t^*(\tau),(K_t^*)^\lambda(\tau)\right)\,\theta(\tau-t)\,\rho(\tau-t)\,d\tau=\max_{C_t}\left\{U(C_t,K_t^\lambda)+W'(K_t)[AK_t-C_t]\right\}, \tag{25}$$

where $C_t^*(\tau)$ is the consumption and $(K_t^*)^\lambda(\tau)$ the pollution along the optimal path for the t-agent, and $W(K_t)=\int_t^\infty U\left(C_t^*(\tau),(K_t^*)^\lambda(\tau)\right)\,\theta(\tau-t)\,d\tau$ denotes the value function. The argument for variables in the RHS of (25) is the current time t (omitted when no confusion can arise), at which the t-agent solves his dynamic problem.

First-order condition for optimality reads:

$$\frac{\partial U(C_t,K_t^\lambda)}{\partial C_t}=W'(K_t). \tag{26}$$

Along a balanced growth path, from equation (10) the value function can be written as $W(K_t)=W_0+W_1K_t^{-\eta}$, with

$$W_0=\frac{\sigma}{(1-\sigma)\hat{\rho}},\qquad W_1=\sigma\frac{(\xi^*)^{1-\frac{1}{\sigma}}}{\sigma-1}\Omega(g^*,\sigma,\varphi), \tag{27}$$

where ξ^* represents the optimal propensity to consume out of capital, to be determined. Plugging $C_t^*(\tau)=\xi^*K_t(\tau)$ into the first-order condition in (26), and taking into account the expression for W_1, one gets

$$\xi^*=\frac{1}{(1-\varphi)\Omega(g^*,\sigma,\varphi)},\qquad W_1=-\frac{1}{\eta}\left[(1-\varphi)\Omega(g^*,\sigma,\varphi)\right]^{\frac{1}{\sigma}}. \tag{28}$$

Taking into account ξ^* and the dynamics for the capital stock in (7), the optimal growth rate along the balanced path satisfies

$$g^*=A-\xi^*=A-\frac{1}{(1-\varphi)\Omega(g^*,\sigma,\varphi)}. \tag{29}$$

Assuming convergence under (13) and integrating $\Omega(g,\sigma,\varphi)$ by parts, one gets

$$\Omega(g,\sigma,\varphi)=\frac{1}{\eta g}-\frac{1}{\eta g}\int_0^\infty e^{-\eta g j}\theta(j)\rho(j)\,dj\Leftrightarrow\Delta(g,\sigma,\varphi)=\frac{1}{\Omega(g,\sigma,\varphi)}-\eta g, \tag{30}$$

with $\Delta(g,\sigma,\varphi)$ defined in (16). Thus, from (29) the alternative way of characterizing the optimal growth rate in (14) and (16) follows.

Market economy The Bellman equation in the market economy for the optimization problem (8)–(9) reads:

$$\begin{aligned}&\int_t^\infty U\left(c_t^*(\tau),(K_t^*)^\lambda(\tau)\right)\,\theta(\tau-t)\,\rho(\tau-t)\,d\tau=\\&\max_{c_t}\left\{U(c_t,K_t^\lambda)+\widetilde{W}'_{k_t}(k_t,K_t)[rk_t-c_t]+\widetilde{W}'_{K_t}(k_t,K_t)\dot{K}_t\right\},\end{aligned} \tag{31}$$

where $c_t^*(\tau)$ is household consumption, $(K_t^*)^\lambda$ the pollution along the optimal path for the t-agent, and $\widetilde{W}(k_t,K_t)=\int_t^\infty U\left(c_t^*(\tau),(K_t^*)^\lambda(\tau)\right)\,\theta(\tau-t)\,d\tau$, denotes the value function. First-order condition for optimality reads:

$$\frac{\partial U(c_t,K_t^\lambda)}{\partial c_t}=\frac{\partial\widetilde{W}(k_t,K_t)}{\partial k_t}. \tag{32}$$

Along a balanced growth path, from equation (12) the value function can be written as $\widetilde{W}(k_t,K_t)=\widetilde{W}_0+\widetilde{W}_1k_t^{-\eta_1}K_t^{-\eta_2}$, with

$$\widetilde{W}_0=W_0,\qquad\widetilde{W}_1=\sigma\frac{\left(\tilde{\xi}^*\right)^{1-\frac{1}{\sigma}}}{\sigma-1}\Omega(\tilde{g}^*,\sigma,\varphi),$$

where $\tilde{\xi}^*$ represents the optimal propensity to consume out of capital in the market economy.

Plugging $c_t^*(\tau) = \tilde{\xi}^* k_t(\tau)$ into the first-order condition in (32), and taking into account the expression for $\widetilde{W}_1$, one gets

$$\tilde{\xi}^* = \frac{1}{\Omega(\tilde{g}^*, \sigma, \varphi)}, \qquad \widetilde{W}_1 = \frac{\sigma}{\sigma - 1}\Omega(\tilde{g}^*, \sigma, \varphi)^{\frac{1}{\sigma}}. \tag{33}$$

Therefore, from (9), the optimal growth rate along the balanced path satisfies

$$\tilde{g}^* = r - \tilde{\xi}^* = A - \frac{1}{\Omega(\tilde{g}^*, \sigma, \varphi)}. \tag{34}$$

Finally, from (30) the expression in (15) follows.

Case $\sigma = 1$.

Following the same methodology as above, one can obtain the value functions for the centralized economy, $W^1(K_t) = W_0^1 + W_1^1 \ln K_t$:

$$W_0^1 = \frac{g^{1*}(1-\varphi)\bar{J}^1 + \ln\left(\frac{\hat{\rho}}{1-\varphi}\right)}{\hat{\rho}}, \qquad W_1^1 = \frac{1-\varphi}{\hat{\rho}}, \tag{35}$$

as well as for the market economy, $\widetilde{W}^1(k_t, K_t) = \widetilde{W}_0^1 + \widetilde{W}_1^1 \ln k_t + \widetilde{W}_2^1 \ln K_t$:

$$\widetilde{W}_0^1 = \frac{\tilde{g}^{1*}(1-\varphi)\bar{J}^1 + \ln \hat{\rho}}{\hat{\rho}}, \quad \widetilde{W}_1^1 = \frac{1}{\hat{\rho}}, \quad \widetilde{W}_2^1 = -\frac{\varphi}{\hat{\rho}}. \tag{36}$$

Likewise, the growth rates g^{1*} and $\tilde{g}^{1*}$ in (17) follow. □

Proof of Corollary 1. In the centralized economy the growth rate is implicitly defined in (14). Hence, when condition (13) is evaluated at the optimum, it depends on σ directly, but also indirectly through its effect on g^*. As σ tends to the value at which $\eta g^* + \rho_\infty$ vanishes, so $\Omega(g^*, \sigma, \varphi)$ tends to infinity. Therefore, from (29) g^* converges towards A. In consequence, condition (13) becomes (18). The same reasoning would apply in the market economy taking into account expression (34), and leading to the same value of $\hat{\sigma}$. □

Proof of Proposition 3. The derivative of $\omega(j)$ with respect to g, reads:

$$\frac{\partial \omega(j)}{\partial g} = \frac{\theta(j)e^{-\eta g j}}{\Omega(g, \sigma, \varphi)}\eta\left[\bar{J}(g, \sigma, \varphi) - j\right], \quad \text{with} \quad \bar{J}(g, \sigma, \varphi) = \int_0^\infty j\omega(j)\, dj.$$

From this expression it is straightforward to compute the derivative of $\Delta(g, \sigma, \varphi)$ with respect to g:

$$\frac{\partial \Delta(g, \sigma, \varphi)}{\partial g} = -\eta\left(1 - \frac{\bar{J}(g, \sigma, \varphi)}{\Omega(g, \sigma, \varphi)}\right) = -\eta Cov(g, \sigma, \varphi) \gtreqless 0 \text{ if } \sigma \lesseqgtr 1. \tag{37}$$

Note that the term in round brackets above can be proved to be the covariance between j and $\rho(j)$ (denoted as $Cov(g, \sigma, \varphi)$), which takes a negative value (see, Cabo et al. (2017)). □

Lemma 1 *Functions $h(g)$ and $\tilde{h}(g)$ are strictly convex for $\sigma \neq 1$ and linear for $\sigma = 1$.*

Proof. $h(g)$ and $\tilde{h}(g)$ are strictly convex if and only if $\Delta(g, \sigma, \varphi)$ is strictly concave as a function of g. To compute the second derivative of $\Delta(g, \sigma, \varphi)$ with respect to g, we take into account the first derivative in (37) and the following expressions:

$$\frac{\partial \bar{J}(g, \sigma, \varphi)}{\partial g} = -\eta Var(g, \sigma, \varphi), \quad \frac{\partial \Omega(g, \sigma, \varphi)}{\partial g} = -\eta \Omega(g, \sigma, \varphi)\bar{J}(g, \sigma, \varphi),$$

where $Var(g,\sigma,\varphi)$ would be the variance when considering $\omega(j)$ as a density function. Thus,

$$\frac{\partial^2\Delta(g,\sigma,\varphi)}{\partial g^2} = \frac{\eta^2}{\Omega(g,\sigma,\varphi)}\left(\bar{J}^2(g,\sigma,\varphi) - Var(g,\sigma,\varphi)\right).$$

To conclude, we need to prove that $Var(g,\sigma,\varphi) > \bar{J}^2(g,\sigma,\varphi)$.

The logarithm of $\omega(j)$ reads:

$$f(j) = \ln(\omega(j)) = B - \eta g j - \int_0^j \rho(z)dz, \quad \text{where} \quad B = -\ln(\Omega(g,\sigma,\varphi)).$$

Because $\dot{\rho}(z) < 0$, then $f''(j) > 0$ and hence $\omega(j)$ is log-convex.

This proof makes use of some of the ideas implemented in Karlin *et al.* (1961) for log-concave density functions but here applied for a log-convex function. Following these authors, we define

$$g(j) = \int_0^\infty w(t+j)n(t)dt = \int_j^\infty w(u)n(u-j)du,$$

where

$$n(u) = \begin{cases} u^2 & \text{if} \quad u > 0, \\ 0 & \text{if} \quad u \le 0. \end{cases}$$

Under this specification, provided that $w(j)$ is log-convex, then $g(j)$ is also log-convex (see Boyd and Vandenberghe, 2004, pages 105–106) and, therefore,

$$g''(j)g(j) - \left(g'(j)\right)^2 \ge 0. \tag{38}$$

These derivatives can be computed as

$$g'(j) = \left(\int_j^\infty w(u)(u-j)^2du\right)' = -2\int_j^\infty w(u)(u-j)du,$$
$$g''(j) = 2\int_j^\infty w(u)du.$$

Particularizing for $j = 0$, we obtain the first two raw moments: $g(0) = \mu_2$, $g'(0) = -2\mu_1$, $g''(0) = 2$. Thus, inequality (38) reads $\mu_2 - 2\mu_1^2 \ge 0$. Or, equivalently $Var(g,\sigma,\varphi) - \bar{J}^2(g,\sigma,\varphi) \ge 0$. Indeed, the inequality is strict unless $f(j)$ refers to the exponential density function, which is both log-concave and log-convex. □

Proof of Proposition 4. From (16) it follows that

$$\Delta(0,\sigma,\varphi) = \int_0^\infty \rho(j)\frac{\theta(j)}{\int_0^\infty \theta(i)di} = \hat{\rho}\int_0^\infty -\dot{\theta}(j)dj = \hat{\rho}.$$

Therefore,

$$h(0) = \left(A - \frac{\hat{\rho}}{1-\varphi}\right)\sigma, \quad \tilde{h}(0) = \frac{A-\hat{\rho}}{1-\varphi+\sigma\varphi}\sigma, \tag{39}$$

which are positive from condition (5).

For $\sigma \le 1$, $h'(g), \tilde{h}'(g) \le 0$ for all $g \in [0, A]$, therefore, an equilibrium, g^*, $\tilde{g}^*$, always exists and is unique.

For $\sigma > 1$, $h'(g), \tilde{h}'(g) > 0$ for all $g \in [0, A]$, and existence requires $h(A), \tilde{h}(A) < A$. Under condition (5) both inequalities are satisfied if and only if

$$\sigma < 1 + \frac{\Delta(A, \sigma, \varphi)}{A(1-\varphi) - \Delta(A, \sigma, \varphi)}.$$

Note that the denominator in this expression is positive under condition (5) because $\Delta(g, \sigma, \varphi)$ decreases with g when the EIS is larger than one. From the definition in (16) is follows that $\Delta(g, \sigma, \varphi) \geq \rho_\infty$ for all $g \in [0, A]$. Moreover, function $x/(A(1-\varphi) - x)$ increases with x and, therefore,

$$\sigma < \hat{\sigma} < 1 + \frac{\Delta(A, \sigma, \varphi)}{A(1-\varphi) - \Delta(A, \sigma, \varphi)},$$

which proves that condition (5) together with (18) imply existence.

Likewise, it can be proved that $\tilde{h}(g) > h(g)$ for all $g \in [0, A]$ if and only if

$$\sigma < 1 + \frac{\Delta(g, \sigma, \varphi)}{A(1-\varphi) - \Delta(g, \sigma, \varphi)},$$

which holds true under the same reasoning.

Lemma 1 proves that $h(g)$ and $\tilde{h}(g)$ are convex functions of g. Therefore, since $h(0), \tilde{h}(0) > 0$ and $h(A), \tilde{h}(A) < A$ the equilibrium in each scenario is unique. From this uniqueness it follows that $h'(g^*)$, $\tilde{h}'(\tilde{g}^*) < 1$. In consequence, since $\tilde{h}(g) > h(g)$ for all $g \in [0, A]$, then $\tilde{g}^* > g^*$ as shown in Figures 2. □

Proof of Proposition 5. The direct effect of σ on $h(g)$ and $\tilde{h}(g)$ can be computed as

$$\frac{\partial h(g)}{\partial \sigma} = A - \frac{\Delta(g, \sigma, \varphi)}{1-\varphi} > 0,$$
$$\frac{\partial \tilde{h}(g)}{\partial \sigma} = \frac{(A - \Delta(g, \sigma, \varphi))(1-\varphi)}{(1-\varphi(1-\sigma))^2} > 0.$$

To compute the indirect effect of σ, one needs to first compute its effect on $\Delta(g, \sigma, \varphi)$. From (16) and (37) it follows that

$$\frac{\partial \Delta(g, \sigma, \varphi)}{\partial \sigma} = g \frac{1-\varphi}{\sigma^2} Cov(g, \sigma, \varphi) < 0.$$

In consequence, the indirect effect of σ through $\Delta(g, \sigma, \varphi)$ would also be positive:

$$\frac{\partial h(g)}{\partial \Delta} \frac{\partial \Delta(g, \sigma, \varphi)}{\partial \sigma} = -g \frac{1}{\sigma} Cov(g, \sigma, \varphi) > 0,$$
$$\frac{\partial \tilde{h}(g)}{\partial \Delta} \frac{\partial \Delta(g, \sigma, \varphi)}{\partial \sigma} = -g \frac{1-\varphi}{\sigma(1-\varphi(1-\sigma))} Cov(g, \sigma, \varphi) > 0.$$

Adding the direct and indirect effects, curves $h(g)$ and $\tilde{h}(g)$ shift upward with σ. For $\sigma \leq 1$, $h'(g), \tilde{h}'(g) \leq 0$ as shown in Figure 2 (left). Hence higher σ undoubtedly leads to higher equilibrium growth rates. For $\sigma > 1$, $h'(g), \tilde{h}'(g) > 0$ and $h(0), \tilde{h}(0)$ in (39) are strictly positive, as depicted in Figure 2 (right). Uniqueness implies $h'(g^*), \tilde{h}'(\tilde{g}^*) < 1$ and hence also ensures a positive effect of σ on the equilibrium growth rates. □

Proof of Proposition 6. The direct effect of φ on $h(g)$ and $\tilde{h}(g)$ can be computed as

$$\frac{\partial h(g)}{\partial \varphi} = -\sigma \frac{\Delta(g,\sigma,\varphi)}{(1-\varphi)^2} < 0,$$
$$\frac{\partial \tilde{h}(g)}{\partial \varphi} = \sigma \frac{(A - \Delta(g,\sigma,\varphi))(1-\sigma)}{(1-\varphi(1-\sigma))^2} \gtreqless 0 \text{ if } \sigma \lesseqgtr 1. \tag{40}$$

To compute the indirect effect of φ, one needs to first compute its effect on $\Delta(g,\sigma,\varphi)$. From (16) it follows that

$$\frac{\partial \Delta(g,\sigma,\varphi)}{\partial \varphi} = g \frac{1-\sigma}{\sigma} Cov(g,\sigma,\varphi) \lesseqgtr 0 \text{ if } \sigma \lesseqgtr 1. \tag{41}$$

In consequence, the indirect effect of φ through $\Delta(g,\sigma,\varphi)$ reads:

$$\frac{\partial h(g)}{\partial \Delta} \frac{\partial \Delta(g,\sigma,\varphi)}{\partial \varphi} = -g \frac{1-\sigma}{1-\varphi} Cov(g,\sigma,\varphi) \gtreqless 0 \text{ if } \sigma \lesseqgtr 1,$$
$$\frac{\partial \tilde{h}(g)}{\partial \Delta} \frac{\partial \Delta(g,\sigma,\varphi)}{\partial \varphi} = -g \frac{1-\sigma}{1-\varphi(1-\sigma)} Cov(g,\sigma,\varphi) \gtreqless 0 \text{ if } \sigma \lesseqgtr 1. \tag{42}$$

In the market economy, adding togueter direct and indirect effects, curve $\tilde{h}(g)$ shifts up with φ when $\sigma < 1$, and shifts down when $\sigma > 1$. Therefore $\tilde{g}^*$ grows (resp. decreases) with φ when $\sigma < 1$ (resp. $\sigma > 1$). From the expressions in (40) and (42) it is straightforward to conclude that φ does not affect $\tilde{g}^*$ when $\sigma = 1$.

In the centralized economy, the direct effect is always negative. Moreover, the indirect effect is also negative for $\sigma > 1$, and then the pollution externality undoubtedly shifts $h(g)$ downward. Conversely, when $\sigma < 1$ the indirect effect is positive and the total effect can be computed adding toguether direct and indirect effects, and taking into account (30):

$$\frac{dh(g)}{d\varphi} = \frac{(1-\sigma)g}{(1-\varphi)\Omega(g,\sigma,\varphi)} \left(\bar{J}(g,\sigma,\varphi) - \frac{1}{\eta g} \right). \tag{43}$$

While $\bar{J}(g,\sigma,\varphi)$ is the mean value for the $\omega(j)$ density function in (16), the term $1/(\eta g)$ can be regarded as the mean value for an exponential density function, $\eta g e^{-\eta g j}$. At $j = 0$, $\omega(0) = 1/\Omega(g,\sigma,\varphi)$ is greater than ηg (otherwise $\Delta(g,\sigma,\varphi)$ in (30) would be negative). Moreover, since

$$\frac{d\left(\frac{\theta(j)}{\Omega(g,\sigma,\varphi)} - \eta g\right)}{dj} < 0,$$

the two density functions cross only once, the tail in the exponential is thicker, and therefore,

$$\bar{J}(g,\sigma,\varphi) < \frac{1}{\eta g}.$$

The total derivative is negative, and hence $h(g)$ shifts downward also for $\sigma < 1$. In consequence g^* decreases with φ regardless of the value of σ. □

Proof of Proposition 7. Case $\sigma \neq 1$.

The value functions in the centralized and the market scenarios read:

$$W(K_t) = W_0 + W_1 K_t^{-\eta}, \quad \widetilde{W}(k_t, K_t) = \widetilde{W}_0 + \widetilde{W}_1 k_t^{-\eta_1} K_t^{-\eta_2},$$

where $\eta_1 + \eta_2 = \eta$. Taking into account that $k_t = K_t$, $W_0 = \widetilde{W}_0$ and the expressions of W_1 and $\widetilde{W}_1$ in (28) and (33), the difference follows:

$$\widetilde{W}(K_t, K_t) - W(K_t) = K_t^{-\eta} H(\varphi),$$

where

$$H(\varphi) = -\frac{\sigma}{1-\sigma}\left[\Omega(\tilde{g}^*, \sigma, \varphi)^{\frac{1}{\sigma}} - (1-\varphi)^{\frac{1-\sigma}{\sigma}}\Omega(g^*, \sigma, \varphi)^{\frac{1}{\sigma}}\right].$$

From (14) and (15) when $\varphi = 0$ it follows that $g^* = \tilde{g}^*$ and, hence, $H(0) = 0$. In what follows we prove that $H'(0) > 0$, which implies the existence of a non-empty interval for φ within which $H(\varphi) > 0$, implying $\widetilde{W}(K_t, K_t) > W(K_t)$.

To simplify the notation, henceforth we skip the arguments in functions Ω, Δ and $\bar{J}$. Thus, we denote:

$$\Omega = \Omega(g^*, \sigma, \varphi), \quad \Delta = \Delta(g^*, \sigma, \varphi), \quad \bar{J} = \bar{J}(g^*, \sigma, \varphi),$$
$$\tilde{\Omega} = \Omega(\tilde{g}^*, \sigma, \varphi), \quad \tilde{\Delta} = \Delta(\tilde{g}^*, \sigma, \varphi), \quad \tilde{\bar{J}} = \tilde{\bar{J}}(g^*, \sigma, \varphi).$$

The derivative, $H'(\varphi)$ reads:

$$H'(\varphi) = -\frac{1}{1-\sigma}\left[\tilde{\Omega}^{\frac{1-\sigma}{\sigma}}\frac{d\tilde{\Omega}}{d\varphi} + (1-\sigma)(1-\varphi)^{\frac{1-2\sigma}{\sigma}}\Omega^{\frac{1}{\sigma}} - (1-\varphi)^{\frac{1-\sigma}{\sigma}}\Omega^{\frac{1-\sigma}{\sigma}}\frac{d\Omega}{d\varphi}\right]. \tag{44}$$

Taking into account that (14) and (15) implicitly define g^* and $\tilde{g}^*$ as functions of φ, the derivatives above can be written as

$$\frac{d\Omega}{d\varphi} = \frac{\partial\Omega}{\partial g}\frac{dg^*}{d\varphi} + \frac{\partial\Omega}{\partial\varphi}, \quad \frac{d\tilde{\Omega}}{d\varphi} = \frac{\partial\tilde{\Omega}}{\partial g}\frac{d\tilde{g}^*}{d\varphi} + \frac{\partial\tilde{\Omega}}{\partial\varphi}.$$

The partial derivatives of $\tilde{\Omega}$ and Ω w.r.t. φ can be computed taking into account (30), (37) and (41):

$$\frac{\partial\Omega}{\partial g} = -\eta\Omega\bar{J}, \quad \frac{\partial\Omega}{\partial\varphi} = \frac{1-\sigma}{\sigma}g^*\Omega\bar{J}, \quad \frac{\partial\tilde{\Omega}}{\partial g} = -\eta\tilde{\Omega}\tilde{\bar{J}}, \quad \frac{\partial\tilde{\Omega}}{\partial\varphi} = \frac{1-\sigma}{\sigma}\tilde{g}^*\tilde{\Omega}\tilde{\bar{J}}.$$

Implicit differentiation in (14) and (15) leads to

$$\frac{dg^*}{d\varphi} = \frac{\frac{\partial h(g^*)}{\partial\varphi} + \frac{\partial h(g^*)}{\partial\Delta}\frac{\partial\Delta}{\partial\varphi}}{1-h'^*)}, \quad \frac{d\tilde{g}^*}{d\varphi} = \frac{\frac{\partial\tilde{h}(\tilde{g}^*)}{\partial\varphi} + \frac{\partial\tilde{h}(\tilde{g}^*)}{\partial\Delta}\frac{\partial\tilde{\Delta}}{\partial\varphi}}{1-\tilde{h}'(\tilde{g}^*)}.$$

The total effect of φ on $h(g^*)$ is computed in (43). Likewise, taking into account (40), (42), (30) and (34), the total effect of φ on $\tilde{h}(\tilde{g}^*)$, after some computation reads:

$$\frac{d\tilde{h}(\tilde{g}^*)}{d\varphi} = \sigma\frac{1-\sigma}{\tilde{\Omega}(1-\varphi(1-\sigma))^2}\tilde{g}^*\tilde{\bar{J}}(1+\eta).$$

Moreover, from (37) it follows that

$$h'(g^*) = -\frac{\sigma}{1-\varphi}\frac{\partial\Delta}{\partial g} = (1-\sigma)Cov(g^*, \sigma, \varphi), \tag{45}$$
$$\tilde{h}'(\tilde{g}^*) = -\frac{\sigma}{1-\varphi(1-\sigma)}\frac{\partial\tilde{\Delta}}{\partial g} = \frac{\sigma\eta}{1-\varphi(1-\sigma)}Cov(\tilde{g}^*, \sigma, \varphi).$$

Plugging all these expressions into (44) and evaluating at $\varphi = 0$, when it holds true that $g^* = \tilde{g}^*$, $\Omega = \tilde{\Omega}$, $\bar{J} = \tilde{\bar{J}}$ and $\Delta = \tilde{\Delta}$, then

$$H'^{\frac{1}{\sigma}}\left\{\left[\frac{d\tilde{g}^*}{d\varphi} - \frac{dg^*}{d\varphi}\right]\frac{\bar{J}}{\sigma} - 1\right\},$$

where

$$\frac{d\tilde{g}^*}{d\varphi} - \frac{dg^*}{d\varphi} = \frac{\sigma}{\sigma\Omega + (1-\sigma)\bar{J}},$$

and, therefore,

$$H'^{\frac{1}{\sigma}}\frac{\sigma(\bar{J}-\Omega)}{\sigma\Omega + (1-\sigma)\bar{J}} = \Omega^{\frac{1}{\sigma}}\frac{-Cov(g^*, \sigma, 0)\sigma}{1-(1-\sigma)Cov(g^*, \sigma, 0)}.$$

Finally, given that $h(0) > 0$ and $h(A) < A$, then the uniqueness of the equilibrium guarantees that $h'(g^*) < 1$. From (45) this inequality is equivalent to $\sigma\Omega + (1-\sigma)\bar{J} > 0$ i.e. $1-(1-\sigma)Cov(g^*, \sigma, 0) > 0$. Therefore, $H'(0) > 0$, which concludes the proof.

Case $\sigma = 1$.

Following similar reasoning, the gap between the value functions under the market and the centralized solutions can be obtained from (17), (35) and (36):

$$\widetilde{W}^1(K_t, K_t) - W^1(K_t) = H^1(\varphi) = \varphi \bar{J}^1 + \frac{\ln(1-\varphi)}{\hat{\rho}}.$$

This function satisfies $H^1(0) = 0$ and

$$(H^1)'(0) = \bar{J}^1 - \frac{1}{\hat{\rho}} = \bar{J}^1 - \Omega(g^{*1}, 1, 0) = -\Omega(g^{*1}, 1, \varphi)Cov(g^{*1}, 1, 0) > 0,$$

which concludes the proof. □

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