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Detecting periodicity from the trajectory of a random walk in random environment

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Abstract: For nearest neighbor univariate random walks in a periodic environment, where the probability of moving depends on a periodic function, we show how to estimate the period and the function. For random walks in non-periodic environments, we find that the asymptotic limit of the estimator is constant in the ballistic case, when the random walk is transient and the law of large numbers holds with a non zero limit. Numerical examples are given in the recurrent case, and the sub-ballistic case, where the random walk is transient but the law of large numbers yields a zero limit.

Keywords: Random walks, random environments, periodic functions, ergodic theory

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1 Introduction

Given no other information than the trajectory of a random walk in random environment (RWRE), one would like to estimate the law of the environment. While this is usually not feasible, there are simple instances where various modeling choices will generate the full distribution, at least asymptotically in some sense. Our purpose is solely to detect periodicity from observing a single trajectory of an RWRE.

All random variables are built on an ambient complete probability space $(E, \mathcal{E}, P)$. A random environment is a measurable two-sided sequence $\alpha(e) = \alpha(\cdot, e) \in (0,1)^{\mathbb{Z}}$ indexed by $e \in E$ and we write $\mu = P \circ \alpha^{-1}$ for its distribution on the Borel subsets of $[0,1]^{\mathbb{Z}}$. It will be convenient to write $\rho_i(e) = \frac{1-\alpha(i,e)}{\alpha(i,e)}$ and we will occasionally drop parameter e whenever there is no ambiguity in doing so. Let $X = \{X_t\}_{t\geq 0}$ be a (nearest neighbor) RWRE on $\mathbb{Z}$, to wit

$$\mathbb{P}^e\left(\Delta X_t = X_t - X_{t-1} = 1 |\, X_{t-1} = i\right) = \mathbb{P}\left(X_t = i+1 |\, X_{t-1} = i, \mathcal{E}\right)(e) = \alpha(i,e), \tag{1}$$

with $\mathbb{P}^e(\Delta X_t = -1 |\, X_{t-1} = i) = 1 - \alpha(i,e)$, for every choice of $i \in \mathbb{Z}$ and positive integer t. This simply means that given a realization e of the environment and a starting point X_0 for the walk, the successive locations $\{X_t : t \geq 1\}$ form a time homogeneous irreducible Markov chain on $\mathbb{Z}$ under $\mathbb{P}^e$ and the conditional law of the whole walk is a probability $\mathbb{P}^e$ on the power set over $\mathbb{Z}^{\mathbb{N}}$ known as the quenched law for the RWRE. The whole process is therefore encapsulated in the family of joint probability laws $\mathcal{P}_\mu$ defined by $\mathcal{P}_\mu(F \times A) = \int_A \mathbb{P}^e(F)\, \mu(de)$, and its first marginal on $\mathbb{Z}^{\mathbb{N}}$ is known as the annealed law for the RWRE. Note that X is not in general a Markov process under $\mathcal{P}_\mu$.

In order to use such models, it is necessary to be able to estimate the process $\mathcal{A} = \{\alpha(i)\}_{i\in\mathbb{Z}}$. Of course, this is not possible in general, but if there is some sort of parametric structure, estimation is possible. For instance Comets et al. (2014) study the asymptotic distributions for an M-estimator of their choice that is very close to the MLE for the solution to (1) above. The authors consider that $\{\alpha(i)\}_{i\in\mathbb{Z}}$ are independent and identically distributed (iid) with a parameterized marginal distribution, in the so-called ballistic right-transient case $\mathbb{E}(\log \rho_0) < 0$ and $\mathbb{E}(\rho_0) < 1$. The parameter of interest is estimated via a M-estimator of the sequence $X_0, \ldots, X_{\tau_n}$, where τ_n is the time of the first visit to n. The corresponding sub-ballistic case, i.e., $\mathbb{E}(\log \rho_0) < 0$ and $\mathbb{E}(\rho_0) \geq 1$, was studied in Falconnet et al. (2014), while the recurrent case $\mathbb{E}(\log \rho_0) = 0$ is examined in Comets et al. (2016), when the distribution of the iid sequence $\{\alpha(i)\}_{i\in\mathbb{Z}}$ has finite support. Further generalization was achieved in the ballistic case by Andreoletti et al. (2015) where a tractable class of hidden Markov models is assumed for $\{\alpha(i)\}_{i\in\mathbb{Z}}$. An alternative estimation procedure based on the moments of the random environment was proposed earlier on by Adelman and Enriquez (2004), which also applies in more general contexts than those considered here. Even in these examples, where in each case $\{\alpha(i)\}_{i\in\mathbb{Z}}$ itself is a very simple stochastic process, the resulting RWRE X in (1) is still not a Markov process.

In the present paper, we shall first analyze the MLE for the unobserved environment $\mathcal{A} = \{\alpha(i)\}_{i\in\mathbb{Z}}$ in (1) under the restricted context where this environment is known to be spatially periodic. We shall then investigate conditions on $\mathcal{A}$ under which the MLE obtained in the periodic case, allows for the detection of absence of periodicity.

2 Estimation of a periodic environment

Let $\mathcal{P}_d$ be the set of all periodic functions on $\mathbb{Z}$ with values in $(0,1)$ and period d. For a given $p \in \mathcal{P}_d$, set $E_p = \{e_j = p(\cdot + j); 1 \leq j \leq d\}$. The resulting environment $\{\alpha(\cdot, e_j) = p(\cdot + j); j \geq 0\}$ is a deterministic sequence (save for the possibly random starting point $e_0 \in E_p$) with a unique invariant probability, namely the uniform distribution over the finite set E_p, meaning that the probability of e_j is $1/d$, for every $j \in S_d = \{1, \ldots, d\}$. This particular case of a random environment leads to the definition of a random walk in a periodic environment (RWPE for short) introduced in Pyke (2003). We say that X is a RWPE if there exists $p \in \mathcal{P}_d$, for some $d \in \mathbb{N}$, so that $X = \{X_t\}_{t\geq 0}$ is the (nearest neighbor) random walk defined by

$$\mathbb{P}\left(\Delta X_t = 1 |\, X_{t-1} = i\right) = p(i),\ i \in \mathbb{Z},\ t \geq 1. \tag{2}$$

Of course, if $p \in \mathcal{P}_d$, then $p \in \mathcal{P}_{kd}$, for any $k \in \mathbb{N}$. Clearly X is an irreducible Markov chain. The main aim of this section is to estimate the associated periodic function p and the smallest d_0 so that $p \in \mathcal{P}_{d_0}$, using a single trajectory $X_0, \ldots, X_n$ of the RWPE X. The justification of using the class of RWPE instead of the more general RWRE models defined by (1) comes from a celebrated result due to Parthasarathy (1961) which says that the law μ of the stationary ergodic process $\mathcal{A} = \{\alpha(i)\}_{i\in\mathbb{Z}}$ is the limit in distribution of a sequence of processes $\{\alpha_n(i) = p_n(i+\cdot)\}_{i\in\mathbb{Z}}$, where $p_n \in \bigcup_{d\geq 1} \mathcal{P}_d$. Henceforth we use the notation $(x)_d = i$ to mean that $x = i \mod (d)$, the residual class modulo d. Note that the stochastic process $\{(X_t)_d\}_{t\geq 0}$ valued in S_d is a Markov chain only when d is a multiple of d_0. It is this fact which requires some extra care in the treatment of periodic environments.

2.1 Estimation of p with d known

If $p \in \mathcal{P}_d$ then, assuming $(X_t)_{t=0}^n$ satisfies (2), the maximum likelihood estimator (MLE) $p_n^{(d)}$ of p is given by

$$p_n^{(d)}(j) = A_{n,j}^{(d)} \Big/ \left(A_{n,j}^{(d)} + B_{n,j}^{(d)}\right), \quad j \in \{1, \ldots, d\}, \tag{3}$$

where $A_{n,j}^{(d)} = \sum_{t=1}^n \mathbb{I}\{(X_{t-1})_d = j, \Delta X_t = 1\}$, $B_{n,j}^{(d)} = \sum_{t=1}^n \mathbb{I}\{(X_{t-1})_d = j, \Delta X_t = -1\}$, and $A_{n,j}^{(d)} + B_{n,j}^{(d)} = \sum_{t=1}^n \mathbb{I}\{(X_{t-1})_d = j\}$. The associated log-likelihood is given by

$$L_{n,d} \;=\; \sum_{j=1}^d \left[A_{n,j}^{(d)} \log\{p_n^{(d)}(j)\} + B_{n,j}^{(d)} \log\{1 - p_n^{(d)}(j)\}\right] = -\sum_{j=1}^d \left(A_{n,j}^{(d)} + B_{n,j}^{(d)}\right) H\left\{p_n^{(d)}(j)\right\}, \tag{4}$$

where for any $x \in (0,1)$, $H(x) = -x\log x - (1-x)\log(1-x) \geq 0$ is the well-known Boltzmann entropy function for the Bernoulli distribution with parameter x. Note that for any multiple of d, one should get a consistent estimator as well, since $p \in \mathcal{P}_d$ entails that $p \in \mathcal{P}_{kd}$, for any $k \in \mathbb{N}$. The following result, proven in A.1, shows that the MLE is consistent.

Proposition 1 *Suppose $p \in \mathcal{P}_d$ and let π be the unique invariant probability measure of the irreducible Markov chain $(X_t)_d$ on S_d, associated with p. Then, as $n \to \infty$, for any $j \in S_d$, μ almost surely, $A_{n,j}^{(d)}\big/n \to \pi_j p(j)$, $B_{n,j}^{(d)}\big/n \to \pi_j\{1 - p(j)\}$, so $p_n^{(d)}(j) \to p(j)$, and $L_{n,d}\big/n \to \mathcal{L}_d = -\sum_{j=1}^d \pi_j H\{p(j)\}$.*

The problem is to find d. The following example illustrates what one can expect.

Example 1 *We generated a random walk of length $n = 10000$ satisfying (2), starting from $X_0 = 100$, with $p = (0.099, 0.749, 0.749)$. The results of the estimation for $d = 3$ are $p_n = (0.0954, 0.7593, 0.7430) \in \mathcal{P}_3$, with a log-likelihood of -4.6976×10^3. Figure 1 shows the behavior of the log-likelihood for periods $d \in \{1, \ldots, 10\}$. As expected, the first local maximum is reached at $d = 3$, with other local maxima at $6, 9$, which are multiples of $d = 3$.*

2.2 Estimation of the least period

First, assume that $p \in \mathcal{P}_{d_0}$. We need to know what happens to $p_n^{(d)}$ when $d \neq d_0$. The following result is proven in A.2.

Proposition 2 *Suppose $p \in \mathcal{P}_{d_0}$ with minimal d_0 and let π be the unique invariant probability measure of the Markov chain $(X_t)_{d_0}$ associated with p. If $m = (d, d_0)$ is the largest common divisor between d and d_0, then, as $n \to \infty$, for any $i \in S_d$, $A_{n,i}^{(d)}\big/n \stackrel{a.s.}{\to} \frac{m}{d} \sum_{j\in S_{d_0},\,(j)_m=(i)_m} \pi_j p(j)$, $B_{n,i}^{(d)}\big/n \stackrel{a.s.}{\to} \frac{m}{d} \sum_{j\in S_{d_0},\,(j)_m=(i)_m} \pi_j \{1 - p(j)\}$,*

$$p_n^{(d)}(i) \stackrel{a.s.}{\to} p^{(d)}(i) = \sum_{j\in S_{d_0},\,(j)_m=(i)_m} \pi_j p(j) \Big/ \sum_{j\in S_{d_0},\,(j)_m=(i)_m} \pi_j, \tag{5}$$

$$L_{n,d}\Big/n \xrightarrow{a.s.} \mathcal{L}_d = -\frac{m}{d}\sum_{i=1}^{d}\sum_{j\in S_{d_0},\,(j)_m=(i)_m}\pi_j H\left\{p^{(d)}(i)\right\}. \tag{6}$$

In particular, if $m=1$, *then* $A^{(d)}_{n,i}\Big/n$ *converges almost surely to* $\frac{1}{d}\sum_{j=1}^{d_0}\pi_j p(j)$ *and* $p^{(d)}_n(i)$ *converges almost surely to* $\sum_{j=1}^{d_0}\pi_j p(j)$, *which are both independent of* $i\in S_d$.

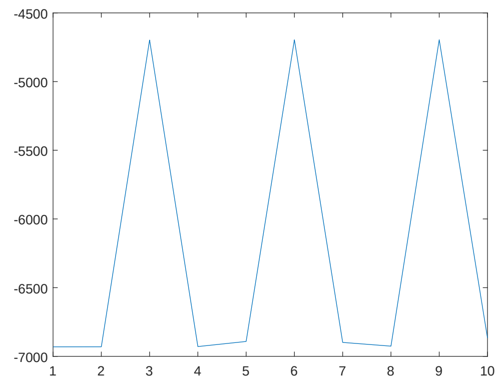

Figure 1: Log-likelihoods corresponding to estimated probabilities of period $d\in\{1,\ldots,10\}$**.**

For the estimation of d_0, we propose the following method: for $d=1,\ldots$, assume that the model defined by (2) holds with $d_0=d$ and estimate $p^{(d)}$ and $\mathcal{L}_d$ according to (3)–(4). $\hat{d}_0$ is then the first local maximum of $\mathcal{L}_d$. Then we set $p_n=\hat{p}_{\hat{d}_0}$. We prove in A.3 that this method works.

Theorem 1 *Suppose* $p\in\mathcal{P}_{d_0}$ *and* d_0 *is minimal. Then for any* $d\in\mathbb{N}$, $\mathcal{L}_d<\mathcal{L}_{d_0}$ *whenever* $(d,d_0)<d_0$. *If* $(d,d_0)=d_0$, *then* $\mathcal{L}_d=\mathcal{L}_{d_0}$.

3 Behavior under non-periodic environments

What if the model is not a RWPE? From now on $X=(X_t)_{t\ge 0}$ is a full fledged RWRE satisfying (1). We investigate next how our estimator behaves when the environment is no longer periodic. Henceforth the right shift operator on $(0,1)^{\mathbb{Z}}$ is denoted by T.

3.1 Fixed environments

For a fixed environment e, the asymptotic behavior for X is well-known (Chung, 1960). For the sake of completeness, we formulate them in the following lemma. Recall that $\rho_k(e)=\frac{1-\alpha(k,e)}{\alpha(k,e)}$ and define $S(e)=1+\sum_{j=1}^{\infty}\prod_{k=1}^{j}\rho_k(e)$, $F(e)=1+\sum_{j=1}^{\infty}\prod_{k=1}^{j}\frac{1}{\rho_{-k}(e)}$, $C_0(e)\equiv 1$, and

$$C_j(e)=\begin{cases}\frac{\alpha(0,e)}{\alpha(j,e)}\prod_{k=1}^{j}\frac{1}{\rho_k(e)} & \text{for } j>0,\\[2ex] \frac{1-\alpha(0,e)}{1-\alpha(j,e)}\prod_{k=1}^{-j}\rho_{-k}(e) & \text{for } j<0.\end{cases} \tag{7}$$

Lemma 1 *Let* $X=(X_t)_{t\ge 0}$ *satisfy* (1) *for some fixed environment* $\alpha(\cdot,e)\in(0,1)^{\mathbb{Z}}$. *For any starting point* X_0, *one and only one of the following four statements holds. If* $S(e)<\infty$ *and* $F(e)<\infty$ *then* $\mathbb{P}^e\left(\liminf_{n\to\infty}|X_n|=+\infty\right)=1$. *If* $S(e)<\infty$ *and* $F(e)=\infty$ *then* $\mathbb{P}^e\left(\lim_{n\to\infty}X_n=+\infty\right)=1$. *If* $S(e)=\infty$ *and* $F(e)<\infty$ *then* $\mathbb{P}^e\left(\lim_{n\to\infty}X_n=-\infty\right)=1$. *If* $S(e)=\infty$ *and* $F(e)=\infty$ *then* $\mathbb{P}^e\left(\liminf_{n\to\infty}X_n=-\infty,\limsup_{n\to\infty}X_n=+\infty\right)=1$. *Further, if* $\sum_{j\in\mathbb{Z}}C_j(e)<\infty$, *then* $S(e)=\infty$, $F(e)=\infty$ *and* X *has a unique invariant measure* $\pi_j(e)=C_j(e)\pi_0(e)$, *with* $\pi_0(e)=1/\sum_{j\in\mathbb{Z}}C_j(e)$.

In the case where X is positive recurrent, prescribed by the finiteness of $\sum_{j\in\mathbb{Z}} C_j(e) < \infty$, the ergodic theorem for Markov chain $(X_t, X_{t+1})_{t\geq 0}$, which is also irreducible on $\{(j,k)\in\mathbb{Z}^2; k-j=\pm 1\}$, and positive recurrent as well, directly implies the following result, proven is given in A.4.

Proposition 3 *Let π be the unique invariant probability measure of the Markov chain X associated with an arbitrary but fixed environment α satisfying $\sum_{j\in\mathbb{Z}} C_j(e) < \infty$ in (7). As $n\to\infty$, there holds, for every integer pair $1\leq i\leq d<\infty$ and any starting point X_0,*

$$p_n^{(d)}(i,e) \overset{a.s.}{\to} \alpha^{(d)}(i,e) = \sum_{j\in\mathbb{Z},\ (j)_d=i} \pi_j(e)\alpha(j,e) \Big/ \sum_{j\in\mathbb{Z},\ (j)_d=i} \pi_j(e), \tag{8}$$

$$L_{n,d}(e)\Big/n \overset{a.s.}{\to} \mathcal{L}_d(e) = -\sum_{i=1}^{d} \sum_{j\in\mathbb{Z},\ (j)_d=i} \pi_j(e) H\left\{\alpha^{(d)}(i,e)\right\}. \tag{9}$$

Furthermore, for any $d\geq 1$ and $k>1$, we have

$$-\log 2 = \mathcal{L}_1(e) \leq \mathcal{L}_d(e) \leq \mathcal{L}_{kd}(e) \leq \mathcal{L}_\infty(e) = -\sum_{j\in\mathbb{Z}} \pi_j(e) H\left\{\alpha(j,e)\right\} \leq 0. \tag{10}$$

Remark 1 *Comparing these results with Proposition 2 and Theorem 1 shows that the sequence of log-likelihoods for large sample estimates $\{p_n^{(d)}; d\geq 1\}$, which exhibits a telltale seesaw motion under the hypothesis of periodicity, as shown in Figure 1, will instead display a tendency to increase in d from approximately $-\log 2 \sim -.693$ to an upper bound $\mathcal{L}_\infty$ when the environment in no longer periodic but such that the simple random walk X on the whole line is positive recurrent. The convexity of $-H$ yields the inequalities $L_{n,1}\leq L_{n,d}\leq L_{n,kd}$ for any $n\geq 1$, $d\geq 1$, $k>1$ and all environments, recurrent or not, hence any departure from periodicity in the environment is likely to be detected through this tendency to increase, simply by graphing the empirical log-likelihoods $(L_{n,kd})_{k\geq 1}$, for a fixed d.*

3.2 Random environments

From now on, assume that the sequence $\alpha(i,\cdot)$ is stationary and ergodic with respect to the measure μ and T is a μ measure preserving transformation. The asymptotic behavior of X is completely determined by the expectation of $\log\rho_0$, as proven in (Alili, 1999, Theorem 2.1).

Theorem 2 *Suppose that $u=\mathbb{E}(\log\rho_0)$ is well defined, with $u\in[-\infty,+\infty]$. If $u>0$, then μ a.s., $F(e)<\infty$, $S(e)=\infty$, and for any $i\in\mathbb{Z}$, $\mathbb{P}_i^e(\lim_{n\to\infty} X_n=-\infty)=1$. If $u<0$, then μ a.s., $S(e)<\infty$, $F(e)=\infty$, and for any $i\in\mathbb{Z}$, $\mathbb{P}_i^e(\lim_{n\to\infty} X_n=+\infty)=1$. Finally, if $u=0$, then μ a.s., $F(e)=\infty$, $S(e)=\infty$, and for any $i\in\mathbb{Z}$, $\mathbb{P}_i^e(\liminf_{n\to\infty} X_n=-\infty, \limsup_{n\to\infty} X_n=+\infty)=1$.*

Let us return to (1) and write $X_n=\sum_{t=1}^n\{\Delta X_t-2\alpha(X_{t-1})+1\}-n+2\sum_{t=1}^n \alpha(X_{t-1})$. Now, $\xi_t=\Delta X_t-2\alpha(X_{t-1})+1$ is a bounded martingale difference sequence, so $\frac{1}{n}\sum_{t=1}^n \xi_t$ converges to 0 almost surely, as $n\to\infty$. Hence the limiting behavior of $\bar{X}_n=X_n/n$ is the same as the limiting behavior of $-1+\dfrac{2}{n}\displaystyle\sum_{t=1}^n \alpha(X_t)$. It is shown in Alili (1999, Theorem 4.1) that $\bar{X}_n$ converges μ almost surely, an extension of the original iid case due to Solomon (1975).

3.2.1 Invariant measure for $T^{X_t}e$

In order to find the limit of $\sum_{t=1}^n \alpha(X_t)\Big/n$, consider the Markov chain $T^{X_t}e$ on E. Its Markov operator $\mathcal{T}$ is given by

$$\mathcal{T}h(e) = \alpha(0,e)h(Te) + \{1-\alpha(0,e)\}h(T^{-1}e), \tag{11}$$

for any bounded measurable function h. Setting $X_t^x(e)$ for the chain starting at x from environment e, then $X_t^x(e)=x+X_t^0(T^xe)$, so $\alpha(X_t^x,e)=\alpha(X_t^0(T^xe),T^xe)$. The associated Markov operator, denoted $\mathcal{T}_x$, satisfies $\mathcal{T}_xh(e)=\mathbb{E}\left\{h(T^{X_1^x}e)\right\}=\mathcal{T}h(T^xe)$. If λ is an invariant measure for $\mathcal{T}$, the invariant measure λ_x for $\mathcal{T}_x$ is $\lambda_x(A)=\lambda(T^xA)$. Hence $\frac{1}{n}\sum_{t=1}^n \alpha(X_{t-1}^x,e)$ should converge to $\mathbb{E}_{\lambda_x}\{\alpha(x)\}=$

$\mathbb{E}_\lambda\{\alpha(0)\}$. Hence the limit is independent of x even if $\alpha(x)$ is not necessarily stationary with respect to λ. If λ is a stationary distribution for this Markov chain that is absolutely continuous with respect to μ, its density ϕ satisfies $\phi = \mathcal{T}^\star\phi$, where $\mathcal{T}^\star$ is the adjoint operator of $\mathcal{T}$ on $L^1(\mu)$ given by

$$\mathcal{T}^\star\phi(e) = \alpha(0, T^{-1}e)\phi(T^{-1}e) + \{1 - \alpha(0, Te)\}\phi(Te). \tag{12}$$

Suppose that $\mathbb{E}(S) < \infty$. Then $d\lambda = v\{1 + \rho_0(e)\}S(e)d\mu$ is an invariant ergodic measure for $T^{X_t}e$, so in this case $\bar{X}_n$ converges almost surely to $-1 + 2\mathbb{E}_\lambda(\alpha_0) = -1 + 2v\mathbb{E}(S) = v$, where $\alpha_0(e) = \alpha(0, e)$. Next, if $\mathbb{E}(F) < \infty$, $d\tilde{\lambda} = \tilde{v}\{1 + \rho_0^{-1}(e)\}F(e)d\mu$ is an invariant ergodic measure for $T^{X_t}e$, so in this case $\bar{X}_n$ converges almost surely to $-1 + 2\mathbb{E}_{\tilde{\lambda}}(\alpha_0) = 1 - 2\tilde{v}\mathbb{E}(F) = -\tilde{v}$. If $\mathbb{E}(S) = \mathbb{E}(F) = +\infty$, then $\bar{X}_n$ converges almost surely to 0, which is equivalent to $\displaystyle\frac{1}{n}\sum_{t=1}^n \alpha(X_t)$ converging almost surely to 1/2. That $\mathcal{T}$ is an ergodic operator goes back to Kozlov (1985), where a proof is sketched. We prove it in A.5.

The following section will be used for computing some important limiting values such as $\mathbb{E}_\lambda(\alpha_0)$.

3.2.2 Distribution of α_0 under the invariant measure λ in the ballistic case

Suppose we are in the right ballistic case, i.e., $\mathbb{E}(S) < \infty$. Then, the (unique) invariant measure λ has density vS/α_0 with respect to μ, and $v = \frac{1}{2\mathbb{E}(S)-1}$. It follows that $\mathbb{E}_\lambda(\alpha_0) = v\mathbb{E}(S) = \frac{1}{1+\rho}$. What is the distribution of α_0 under λ? In the iid case, S is independent of α_0 and $\mathbb{E}(S) = \frac{1}{1-\rho}$, so $v = \frac{1-\rho}{1+\rho}$, and $\mathbb{E}(1 + \rho_0) = \mathbb{E}(1/\alpha_0) = 1 + \rho$. Note also that $\mathbb{E}(\log\rho_0) < 0$ and $\mathbb{E}(\rho_0) = \rho < 1$. For any bounded measurable function H on $(0, 1)$,

$$\mathbb{E}_\lambda\{H(\alpha_0)\} \quad = \quad v\mathbb{E}\left\{H(\alpha_0)(1 + \rho_0)S\right\} = v\mathbb{E}(S)\mathbb{E}\left\{\frac{H(\alpha_0)}{\alpha_0}\right\} = \frac{1}{1+\rho}\mathbb{E}\left\{\frac{H(\alpha_0)}{\alpha_0}\right\}.$$

Hence the density of the distribution of α_0 under λ, with respect to the distribution of α_0 under μ is $\frac{1}{(1+\rho)x}$. In particular, if α_0 has density $g(x)$ under μ, then it has density $\frac{g(x)}{(1+\rho)x}$ under λ. Suppose now that we are in the left ballistic case, i.e., $\mathbb{E}(F) < \infty$. Then, the (unique) invariant measure $\tilde{\lambda}$ has density $vF/(1 - \alpha_0)$ with respect to μ, and $\tilde{v} = \frac{1}{2\mathbb{E}(F)-1}$. In the iid case, i.e., $\mathbb{E}(\log\rho_0) > 0$ and $\mathbb{E}(1/\rho_0) = \tilde{\rho} < 1$. Then under the (unique) invariant measure $\tilde{\lambda}$, $\mathbb{E}_{\tilde{\lambda}}(\alpha_0) = \frac{\tilde{\rho}}{1+\tilde{\rho}}$. Also, F is independent of α_0 and $\mathbb{E}(F) = \frac{1}{1-\tilde{\rho}}$, so $\tilde{v} = \frac{1-\tilde{\rho}}{1+\tilde{\rho}}$, and $\mathbb{E}(1+1/\rho_0) = \mathbb{E}\left(\frac{1}{1-\alpha_0}\right) = 1+\tilde{\rho}$. For any bounded measurable function H on $(0, 1)$, $\mathbb{E}_\lambda\{H(\alpha_0)\} = \tilde{v}\mathbb{E}\left\{H(\alpha_0)(1 + 1/\rho_0)F\right\} = \tilde{v}\mathbb{E}(F)\mathbb{E}\left\{\frac{H(\alpha_0)}{1-\alpha_0}\right\} = \frac{1}{1+\tilde{\rho}}\mathbb{E}\left\{\frac{H(\alpha_0)}{1-\alpha_0}\right\}$. Hence the density of the distribution of α_0 under $\tilde{\lambda}$, with respect to the distribution of α_0 under μ is $\frac{1}{(1+\tilde{\rho})(1-x)}$. In particular, if α_0 has density $g(x)$ under μ, then it has density $\frac{g(x)}{(1+\tilde{\rho})(1-x)}$ under $\tilde{\lambda}$. In this case, $\bar{X}_n$ converges also surely to $2\mathbb{E}_{\tilde{\lambda}}(\alpha_0) - 1 = \frac{\tilde{\rho}-1}{\tilde{\rho}+1} = -\tilde{v}$.

One can ask now if there is an invariant measure when $\mathbb{E}(S) = \mathbb{E}(F) = +\infty$ which is not treated by Alili (1999). We can give a complete answer to this question in the periodic case.

3.3 Invariant measure in the periodic case

Suppose that $E_p = \{T^k p; k \in S_d\}$, for some $p \in \mathcal{P}_d$. We noted earlier that $(X_t)_d$ forms an irreducible Markov chain with a unique ergodic distribution $\pi = \pi(p, X_0)$, one for each fixed $e \in E_p$. The unique solution of (12) is $\phi(T^k e) = d\pi_k(e)$, where $\pi(e)$ is the unique stationary distribution of $(X_t)_d$. As a result, $\mathbb{E}_\lambda(\alpha_0) = \sum_{k=1}^d \pi_k(p)\alpha(k, p)$. In fact the density of λ with respect to μ should be $c\phi(e)$, where $\phi(e) = \{1+\rho_0(e)\}\{1 + \rho_1(e) + \cdots + \rho_1(e)\cdots\rho_{d-1}(e)\}$, and one gets that $c = d/\left\{\sum_{k=1}^d \phi(T^k e)\right\}$. This means that we have a closed-form expression for the invariant measure π, namely, for any $k \in \{1, \ldots, d\}$, $\pi_k(e) = \phi(T^k e)/\left\{\sum_{j=1}^d \phi(T^j e)\right\}$. Note that the invariant measure exists even in the recurrent case, i.e., if $\mathbb{E}(S) = \mathbb{E}(F) = +\infty$. In this case, $\mathbb{E}(\log\rho_0) = 0$ means that $\frac{1}{d}\sum_{k=1}^d \log(\rho_k) = 0$, so $\rho_1 \cdots \rho_d = 1$.

4 Convergence of the estimator in the ballistic case

Next, consider the Markov chain $\mathfrak{X}_t = ((X_t)_d, T^{X_t}e)$ on $S_d \times E$, and suppose that there are m closed classes $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$. Its Markov operator $\check{\mathcal{T}}$ is given, for any bounded measurable h on $S_d \times \mathcal{E}$, by

$$\check{\mathcal{T}}h(j,e) = \alpha(0,e)h(j+1,Te) + \{1-\alpha(0,e)\}h(j-1,T^{-1}e). \tag{13}$$

Let u_d be the uniform distribution on $\{1,\ldots,d\}$. It is then easy to check that if $\check{\lambda}^{(k)}$ is a stationary distribution for this Markov chain on $\mathfrak{C}_k$ which is absolutely continuous with respect to $u_d \otimes \mu$, then its density φ_k satisfies, for any $(j,e) \in \mathfrak{C}_k$, $\varphi_k = \check{T}^\star \varphi_k$, where $\check{T}^\star$ is the adjoint operator of $\check{T}$ defined by

$$\check{T}^\star\varphi_k(j,e) = \alpha(0,T^{-1}e)\varphi_k(j-1,T^{-1}e) + \{1-\alpha(0,Te)\}\varphi_k(j+1,Te), \quad \mu \text{ a.s.}, \tag{14}$$

and $\displaystyle\frac{1}{d}\sum_{j=1}^{d}\int \varphi_k(j,e)\mu(de) = 1$. Here it is assumed that $\varphi_k(j,e) = 0$ whenever $(j,e) \notin \mathfrak{C}_k$. If ϕ solves (12), then $\varphi_k(j,e) = \phi(e)\mathbb{I}_{\mathfrak{C}_k}(j,e)$ satisfies (14). In particular, if $m=1$, i.e., the Markov chain $\mathfrak{X}$ is irreducible, then $\varphi(j,e) = \phi(e)$ solves (13) if ϕ solves (12). We can also proves uniqueness in this case.

Theorem 3 *Assume $\mathfrak{X}$ is irreducible and that all powers of T are ergodic. Then $\varphi(j,e) = \phi(e)$ solves (13) if ϕ solves (12) and it is the unique invariant ergodic measure for the Markov chain $\mathfrak{X}$.*

Remark 2 *A sufficient condition for all powers of T to be ergodic is that T is weak mixing (Brown, 1976, p. 16), i.e., for any measurable sets A, B, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1}\left|\mu\left(A \cap T^{-k}B\right) - \mu(A)\mu(B)\right| = 0$. If T is weak mixing then T^k is also weak mixing for any k, yielding that T^k is ergodic.*

Using the ergodic theorem, we have the following interesting result.

Corollary 1 *Under the conditions of Theorem 3, for any (j,e), $A^{(d)}_{n,j}\big/n \stackrel{a.s.}{\to} \frac{1}{d}\mathbb{E}\{\varphi(j,\cdot)\alpha_0\} = \frac{1}{d}\mathbb{E}_\lambda(\alpha_0)$, $B^{(d)}_{n,j}\big/n \stackrel{a.s.}{\to} \frac{1}{d}\mathbb{E}\left\{\varphi(j,\cdot)(1-\alpha_0)\right\} = \frac{1}{d}\mathbb{E}_\lambda(1-\alpha_0)$, so $p^{(d)}_n(j) \stackrel{a.s.}{\to} \mathbb{E}_\lambda(\alpha_0)$. Also, $L_{n,d}/n \stackrel{a.s.}{\to} -H\left\{\mathbb{E}_\lambda(\alpha_0)\right\}$.*

Remark 3 *Suppose that d_0 is the least k so that $T^k = I$. Then we are in the periodic case, and it follows from the proof of the theorem that any harmonic function is constant on the closed classes*

$$\begin{aligned}\mathfrak{C}_i &= \left\{(j, T^{i+j-1+\ell d}p); j \in S_d, \ell = 0,\ldots,(d_0/m-1)\right\}\\ &= \left\{(j, T^{i+j-1+\ell m}p); j \in S_d, \ell = 0,\ldots,(d_0/m-1)\right\},\end{aligned}$$

$i \in \{1,\ldots,m\}$, with $m = (d,d_0)$. In this case the Markov chain is ergodic on each class $\mathfrak{C}_i$, with invariant density proportional to ϕ. Let ϕ be defined as in Section 3.3. Set $\tilde{d} = d_0 \times d/m$. Then $p \in \mathcal{P}_{\tilde{d}}$. As a result, since μ is the uniform measure on $E_p = \{Tp,\ldots,T^{d_0}p\}$, one gets for $(j,e) = \left(j, T^{i+j-1+\ell m}p\right) \in \mathfrak{C}_i$, $\varphi_i(j,e) = md_0\frac{\phi(e)}{\sum_{r=1}^{d_0}\phi(T^r p)} = md_0\pi_{i+j-1+\ell m}$, where $\pi_k = \pi_k(p)$ is the unique invariant measure of the Markov chain $(X_t)_{d_0}$ defined by (2). One then recovers the results of Proposition 2 using the ergodic theorem for the Markov chain $\mathfrak{X}$ on $\mathfrak{C}_i$, $i \in S_d$.

4.1 Numerical experiments

We provide three illustrations when the $\alpha(i)$ are iid with uniform distribution on $[a,b]$. First, we consider a right-ballistic case, where the assumptions of Theorem 3 are met. In the other two cases, we consider a sub-ballistic case, and a recurrent case. In the latter, the assumptions of Theorem 3 are not met.

4.1.1 Right-ballistic iid case

Suppose the $\alpha(i)$ are iid, with a uniform distribution over the interval (a,b). Then $\mathbb{E}(\rho_k) = \rho = -1 + \frac{\log(b/a)}{b-a} < 1$ is required for the right-ballistic case. Using the results of Section 3.2.2, the density of α_0 under λ is then $\frac{1}{x\log(b/a)}\mathbb{I}(a < x < b)$. The density of the invariant measure λ for T is $\phi(e) = \left(\frac{1-\rho}{1+\rho}\right)\frac{S(e)}{\alpha(0,e)}$, where $S(e) = 1 + \sum_{n=1}^{\infty}\prod_{k=1}^{n}\rho_k(e)$. In particular $\mathbb{E}_\lambda[\alpha(0)] = \frac{1}{1+\rho}$. As an example, take $a = .55$, $b = .65$. Hence, $\rho = -1 + 10\log(65/55) \approx .6705$, so $\frac{1}{1+\rho} = \frac{.1}{\log(13/11)} \approx 0.5986085$. Choosing estimator $p_n^{(d)}$ with $d \in \{1,\ldots,5\}$, and using a simulated trajectory of length 10^7, one obtains that $p_n^{(d)}(i) = .5954 \pm .0005$ for all $i \in \{1,\ldots,d\}$. These results are coherent with Corollary 1. The estimation of the distribution of α_0 and the graph of the log-likelihoods are displayed in Figure 2. The graphs of the likelihoods seems to vary but if we look closely at the scale, there is no significant variability.

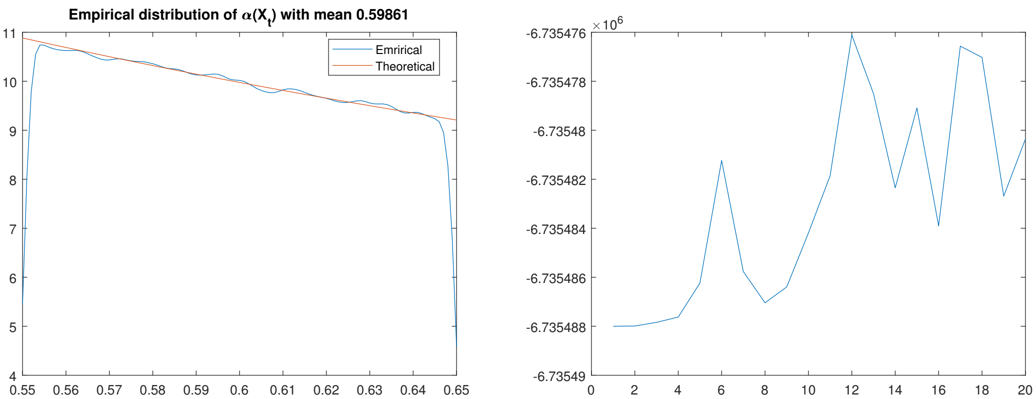

Figure 2: Estimation of the distribution of α_0 (left panel) and graph of the log-likelihoods for period $d \in \{1,\ldots,20\}$ (right panel).

4.1.2 Sub-ballistic iid case

Suppose again that the $\alpha(i)$ are iid, with a uniform distribution over the interval (a,b). It then follows that $\mathbb{E}(\log\rho_0) = \frac{1}{b-a}\{a\log a + (1-a)\log(1-a) - b\log b - (1-b)\log(1-b)\}$. Also, $\mathbb{E}(S) < \infty$ iff $\rho = \mathbb{E}(\rho_0) = -1 + \frac{\log(b/a)}{b-a} < 1$. In the latter case, $\mathbb{E}_\lambda(\alpha_0) = \frac{1}{1+\rho}$. By taking $a = .4$, $b = .61$, we are in the (right) sub-ballistic case. Indeed, $\rho = 1.0095$, so $\mathbb{E}(S) = +\infty$ but $S < \infty$ a.s. since $\mathbb{E}(\log\rho_0) = -0.0203$. We simulated a trajectory of 10^7 points. The trajectory and the graph of the log-likelihoods are displayed in Figure 3. According to Alili's result, the RWRE is transient with $X_n \to +\infty$, but $\bar{X}_n \to 0$ as $n \to \infty$, explaining the slow convergence of X_n. Here we see that the log-likelihoods behave erratically, showing that we are not in the periodic case, nor the ballistic case.

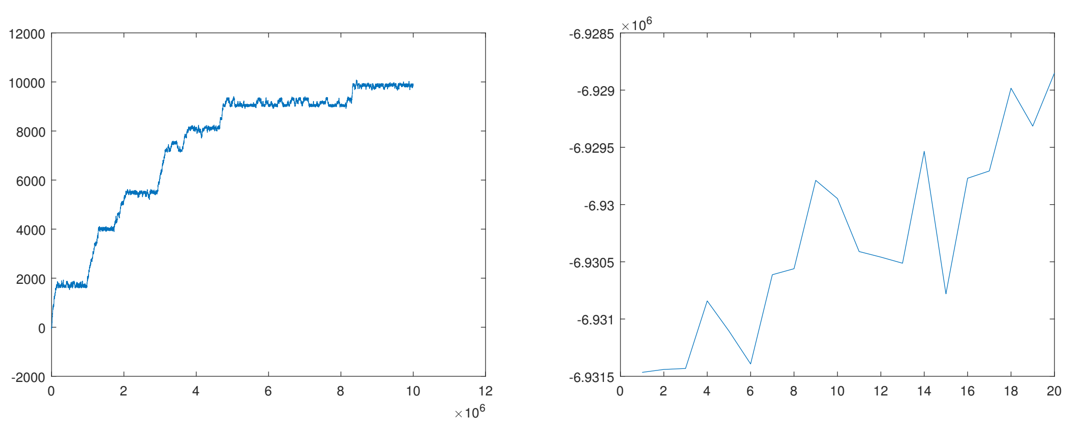

Figure 3: Trajectory of the RWRE (left panel) and graph of the log-likelihoods for period $d \in \{1,\ldots,20\}$ (right panel).

4.1.3 Recurrent iid case

Here we generated the so-called Sinai's random walk of length 10^7 satisfying (1), starting from $X_0 = 0$, with iid uniform probabilities over $(0, 1)$. We are the recurrent case. The trajectory and the graph of the log-likelihoods are displayed in Figure 4. Clearly the RWRE X in (1) is not a stationary process in this case. For more on those recurrent cases affording tractability, see Andreoletti (2011) and the references therein. Here we see that the log-likelihoods behave erratically, showing that we are not in the periodic case, nor the ballistic case.

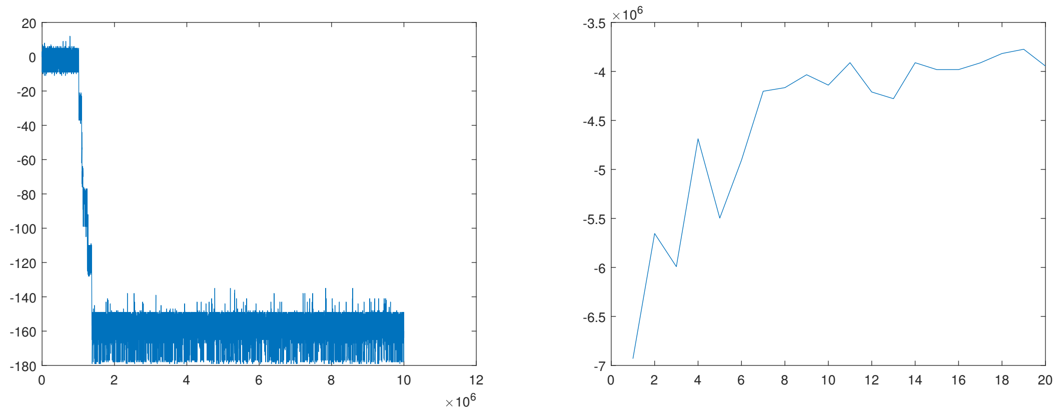

Figure 4: Trajectory of the RWRE (left panel) and graph of the log-likelihoods for period $d \in \{1, \ldots, 20\}$ (right panel).

5 Conclusion

For a RWPE, we can detect the period and estimate the associated probabilities, while in the ballistic non-periodic case, the limiting distribution of the probability estimators is constant, as well as the log-likelihoods, providing a way to detect that we are indeed in the ballistic non-periodic case. Finally, in the sub-ballistic and recurrent cases, the graphs of the log-likelihoods behave almost randomly.

A Proofs of the main results

A.1 Proof of Proposition 1

Suppose $j \in S_{d_0}$ is given. Then

$$\begin{aligned}\frac{A_{n,j}^{(d_0)}}{n} &= \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}\{(X_{t-1})_{d_0} = j, \Delta X_t = 1\} \\ &= \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}\{(X_{t-1})_{d_0} = j\}\{\mathbb{I}(\Delta X_t = 1) - \alpha(X_{t-1})\} + \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}\{(X_{t-1})_{d_0} = j\}\alpha(X_{t-1}).\end{aligned}$$

Now, $\xi_t = \mathbb{I}\{(X_{t-1})_{d_0} = j\}\{\mathbb{I}(\Delta X_t = 1) - \alpha(X_{t-1})\}$ is a bounded martingale difference sequence, so $\frac{1}{n}\sum_{t=1}^{n} \xi_t$ converges to 0 almost surely, as $n \to \infty$. Hence the limiting behavior of $A_{n,j}^{(d_0)}/n$ is the same as the limiting behavior of $\frac{\tilde{A}_{n,j}^{(d_0)}}{n} = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}\{(X_{t-1})_{d_0} = j\}\alpha(X_{t-1})$. Similarly, the limiting behavior of $B_{n,j}^{(d_0)}/n$ is the same as the limiting behavior of $\frac{\tilde{B}_{n,j}^{(d_0)}}{n} = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}\{(X_{t-1})_{d_0} = j\}\{1-\alpha(X_{t-1})\}$. Now, by hypothesis, $\alpha(X_{t-1}) = p(j)$ whenever $(X_{t-1})_{d_0} = j$. As a result, $\frac{\tilde{A}_{n,j}^{(d_0)}}{n} = p(j)\frac{1}{n}\sum_{t=1}^{n} \mathbb{I}\{(X_{t-1})_{d_0} = j\} \stackrel{a.s.}{\to} p(j)\pi(j)$. Similarly, $\frac{\tilde{B}_{n,j}^{(d_0)}}{n} = \{1-p(j)\}\frac{1}{n}\sum_{t=1}^{n} \mathbb{I}\{(X_{t-1})_{d_0} = j\} \stackrel{a.s.}{\to} \{1-p(j)\}\pi(j)$. Thus $p_n^{(d_0)}(j)$ converges almost surely to $p(j)$. The almost sure convergence of $L_{n,d_0}/n$ follows.

A.2 Proof of Proposition 2

Recall that $m = (d, d_0)$ and set $\tilde{d} = d_0 \times d/m$. If $m = 1$, for any $i \in \{1, \dots, d\}$ and any $j \in \{1, \dots, d_0\}$, there is a unique $T(i,j) \in \{1, \dots, \tilde{d}\}$ so that $x = T(i,j) \mod (\tilde{d})$. If $m > 1$, for any $i \in \{1, \dots, d\}$ and any $j \in \{1, \dots, d_0\}$ such that $i = j \mod (m)$, then there is a unique $T(i,j) \in \{1, \dots, \tilde{d}\}$ so that $x = T(i,j) \mod (\tilde{d})$. If $(j-i)_m \neq 0$, then there is no solution $T(i,j)$. Set $a = d/m$ and $b = d_0/m$. Then $(a,b) = 1$. For a given $i \in \{1, \dots, d\}$ such that $(i)_m = \beta \in \{1, \dots, m\}$, then for any $j = (l-1)m + \beta$, $l \in \{1, \dots, b\}$, there is a solution $T(i,j)$. As a result, for any $i \in \{1, \dots, d\}$,

$$A_{n,i}^{(d)} = \sum_{t=1}^{n} \sum_{j \in S_{d_0},\, (j)_m = (i)_m} \mathbb{I}\{(X_{t-1})_d = i, (X_{t-1})_{d_0} = j, \Delta X_t = 1\} = \sum_{j \in S_{d_0},\, (j)_m = (i)_m} A_{n,T(i,j)}^{(\tilde{d})}.$$

Now, as $n \to \infty$, it follows from Proposition 1 that $A_{n,T(i,j)}^{(\tilde{d})}/n \overset{a.s.}{\to} \pi_j p(j)$, since the invariant measure for the Markov chain $\{(X_t)_{\tilde{d}} : t \geq 0\}$ satisfies $\tilde{\pi}_l = \tilde{\pi}_j = \pi_j/a$ whenever $l = j \mod (d_0)$. As result, $A_{n,i}^{(d)} \overset{a.s.}{\to} \frac{1}{a} \sum_{j \in S_{d_0},\, (j)_m = (i)_m} \pi_j p(j)$. Similarly, $B_{n,i}^{(d)} \overset{a.s.}{\to} \frac{1}{a} \sum_{j \in S_{d_0},\, (j)_m = (i)_m} \pi_j \{1 - p(j)\}$. The almost sure convergence of $p_n^{(d)}(i)$ and $L_{n,d}/n$ then follows. Finally, to complete the proof, note that for any i, $\{j \in S_{d_0},\, (j)_1 = (i)_1\} = S_{d_0}$.

A.3 Proof of Theorem 1

From Proposition 1, $L_{n,d_0}/n \overset{a.s.}{\to} \mathcal{L}_{d_0} = -\sum_{j=1}^{d_0} \pi_j H\{p(j)\}$. Introduce the following notation for the relative entropy. For any $x, y \in (0,1)$, set $h(x|y) = x \log(x/y) + (1-x) \log\{(1-x)/(1-y)\}$. Then $h(x|y) \geq 0$ with equality iff $x = y$. We will show that $\mathcal{L}_{d_0} > \mathcal{L}_d$ if $m = (d, d_0) < d_0$. Set $a = d/m$ and $b = d_0/m$. It follows from (5) that for any $i = \beta + lm$, with $\beta \in \{1, \dots, m\}$ and $l \in \{0, \dots, a-1\}$, $p^{(d)}(\beta + lm) = p^{(d)}(\beta) = \sum_{k=0}^{b-1} \pi_{km+\beta}\, p(km+\beta) \Big/ \sum_{k=0}^{b-1} \pi_{km+\beta}$. It then follows from the previous equation and (6) that

$$\begin{aligned}
\mathcal{L}_d &= \frac{1}{a} \sum_{\beta=1}^{m} \sum_{l=0}^{a-1} \sum_{k=0}^{b-1} \pi_{km+\beta} \left[p^{(d)}(\beta) \log\left\{p^{(d)}(\beta)\right\} + \left\{1 - p^{(d)}(\beta)\right\} \log\left\{1 - p^{(d)}(\beta)\right\} \right] \\
&= \sum_{\beta=1}^{m} \sum_{k=0}^{b-1} \pi_{km+\beta} \left[p^{(d)}(\beta) \log\left\{p^{(d)}(\beta)\right\} + \left\{1 - p^{(d)}(\beta)\right\} \log\left\{1 - p^{(d)}(\beta)\right\} \right].
\end{aligned}$$

Using the previous computations, one gets
$\mathcal{L}_{d_0} = \sum_{\beta=1}^{m} \sum_{k=0}^{b-1} \pi_{km+\beta} \left[p(km+\beta) \log\{p(km+\beta)\} + \{1 - p(km+\beta)\} \log\{1 - p(km+\beta)\} \right]$. Hence,

$$\begin{aligned}
\mathcal{L}_{d_0} - \mathcal{L}_d &= \sum_{\beta=1}^{m} \sum_{l=0}^{b-1} \pi_{lm+\beta}\, p(lm+\beta) \left[\log\{p(lm+\beta)\} - \log\left\{p^{(d)}(\beta)\right\} \right] \\
&\quad + \sum_{\beta=1}^{m} \sum_{l=0}^{b-1} \pi_{lm+\beta} \{1 - p(lm+\beta)\} \times \left[\log\{1 - p(lm+\beta)\} - \log\left\{1 - p^{(d)}(\beta)\right\} \right] \\
&= \sum_{\beta=1}^{m} \sum_{l=0}^{b-1} \pi_{lm+\beta}\, h\left\{p(lm+\beta) | p^{(d)}(\beta)\right\} \geq 0,
\end{aligned}$$

with equality iff $m = d_0$. This shows that $\mathcal{L}_d < \mathcal{L}_{d_0}$ for all $d \in \{1, \dots, 2d_0 - 1\} \setminus \{d_0\}$. Hence, if n is large enough, it follows from Proposition 2 that $L_{n,d} < L_{n,d_0}$ for all $d \in \{1, \dots, 2d_0 - 1\} \setminus \{d_0\}$. This also explains the local maxima pictured in Figure 1. Note that $d_{0,n} \to d_0$ implies $d_{0,n} = d_0$ when n is large enough so $p_n^{(d_{0,n})}(j) = p_n^{(d_0)}(j)$. Hence the "randomness" of $d_{0,n}$ is not important in the limit.

A.4 Proof of Proposition 3

Irreducible Markov chain $X = (X_t)_{t \geq 0}$ satisfies (1) for some arbitrary deterministic environment $\alpha(\cdot, e) \in (0,1)^{\mathbb{Z}}$, periodic or not. By definition any invariant positive measure π will satisfy the usual balance equation, which is written, for all $j \in \mathbb{Z}$, as $\pi_{j+1} = \pi_{j+1}\alpha(j+1,e) + \pi_j \alpha(j,e)$, from which there ensues $\sum_{j\in\mathbb{Z}} \pi_j \alpha(j,e) = \frac{1}{2}\sum_{j\in\mathbb{Z}} \pi_j$ as soon as either sum converges. This also implies both $\mathcal{L}_1 = -\log 2$ and the validity of (7) since $\displaystyle\sum_{j\in\mathbb{Z}} C_j(e) < \infty$ holds. The convergence of $\displaystyle\sum_{j\in\mathbb{Z}} C_j(e) < \infty$ ensures that the Markov chain $(X_t, X_{t+1})_{t\geq 0}$ on $\{(j,k) \in \mathbb{Z}^2; k-j = \pm 1\}$ is also irreducible and positive recurrent with unique invariant probability measure $\Pi = (\Pi_{j,k})$ given by

$$\Pi_{j,k} = \begin{cases} \pi_j \alpha(j,e) = \pi_{j+1}\{1-\alpha(j+1,e)\} & \text{for } k = j+1, \\ \\ \pi_j\{1-\alpha(j,e)\} = \pi_{j-1}\alpha(j-1,e) & \text{for } k = j-1. \end{cases} \tag{15}$$

The ergodic theorem for irreducible Markov chain $(X_t, X_{t+1})_{t\geq 0}$ immediately yields the asymptotic behavior provided in (8) for the sequence of statistics $p_n^{(d)}$ and in (9) for the sequence of likelihoods $L_{n,d}$ defined by (4). The convexity of $-H$ successively yields the inequalities $\mathcal{L}_d \leq \mathcal{L}_{kd}$ and $\mathcal{L}_1 \leq \mathcal{L}_d$, for all $d \geq 1$ and $k > 1$, through an application of Jensen's inequality:

$$-H\left\{\alpha^{(d)}(i,e)\right\} \leq -\sum_{\ell=0}^{k-1} H\left\{\alpha^{(kd)}(i+\ell d, e)\right\} \frac{\sum_{j\in\mathbb{Z},\ (j)_{kd}=(i+\ell d)_{kd}} \pi_j}{\sum_{j\in\mathbb{Z},\ (j)_d=(i)_d} \pi_j},$$

which implies

$$\mathcal{L}_d \leq -\sum_{i=1}^{d}\sum_{\ell=0}^{k-1} \sum_{j\in\mathbb{Z},\ (j)_{kd}=(i+\ell d)_{kd}} \pi_j H\left\{\alpha^{(kd)}(i+\ell d, e)\right\} = \mathcal{L}_{kd}.$$

The convergence of $\sum_{j\in\mathbb{Z}} C_j < \infty$ implies $\lim_{d\to\infty} \mathcal{L}_d = \mathcal{L}_\infty$, completing the proof.

A.5 Proof of the ergodicity of $\mathcal{T}$

Let h be a bounded harmonic function for $\mathcal{T}$ defined by (11). Since $\mathcal{T}h = h$ and $\mathcal{T}^\star\phi = \phi$, one gets

$$\begin{aligned}
\int \phi(e)\alpha(0,e)\left\{h(Te)-h(e)\right\}^2\mu(de) + \int \phi(e)\{1-\alpha(0,e)\}\left\{h(T^{-1}e)-h(e)\right\}^2\mu(de) \\
= \int \phi(e)\alpha(0,e)\left\{h^2(Te) - 2h(e)h(Te) + h^2(e)\right\}\mu(de) \\
+ \int \phi(e)\{1-\alpha(0,e)\}\left\{h^2(T^{-1}e) - 2h(e)h(T^{-1}e) + h^2(e)\right\}\mu(de) \\
= \int \phi(e)\left[\alpha(0,e)h^2(Te) + \{1-\alpha(0,e)\}h^2(T^{-1}e)\right]\mu(de) \\
-2\int \phi(e)h(e)\left[\alpha(0,e)h(Te) + \{1-\alpha(0,e)\}h(T^{-1}e)\right]\mu(de) + \int \phi(e)h^2(e)\mu(de) \\
= \int \phi(e)\mathcal{T}h^2(e)\mu(de) - 2\int \phi(e)h(e)\mathcal{T}h(e)\mu(de) + \int \phi(e)h^2(e)\mu(de) \\
= \int \mathcal{T}^\star\phi(e)h^2(e)\mu(de) - \int \phi(e)h^2(e)\mu(de) \\
= \int \phi(e)h^2(e)\mu(de) - \int \phi(e)h^2(e)\mu(de) = 0.
\end{aligned} \tag{16}$$

Hence $h(Te) = h(e)$ μ a.-s. so h is constant by the ergodic property of T. This is a necessary and sufficient condition for a Markov chain to be ergodic (Brown, 1976, p. 14).

A.6 Proof of Theorem 3

Suppose h is a bounded harmonic function for $\check{\mathcal{T}}$. Then we have the analog of (16).

$$\sum_{j=1}^{d} \int \phi(e)\alpha(0,e)\left\{h(j+1,Te)-h(j,e)\right\}^2 \mu(de) \\ + \int \phi(e)\{1-\alpha(0,e)\}\left\{h(j-1,T^{-1}e)-h(e)\right\}^2 \mu(de) = 0. \quad (17)$$

As a result, $h(j+1,Te) = h(j,e)$ μ-a.-s. Moreover, $\sum_{j=1}^{d} h(j,e)$ is a bounded harmonic function for $\mathcal{T}$ so it is constant. Next, set $z(e) = (h(1,e),\ldots,h(d,e))^\top$. Then $Az(Te) = z(e)$, where A is a permutation matrix on $\{1,\ldots,d\}$, with $A_{ij} = 1$ iff $j = i+1 \mod d$, while $A^{-1}_{ij} = 1$ iff $j = i-1 \mod d$, $i,j \in \{1,\ldots,d\}$. In particular, $A^d = I$. It then follows that $z(T^d e) = z(e)$. Since T^d is ergodic, it follows that z is constant so h is constant as well. It then follows from Brown (1976, p.14) that the Markov chain $\mathfrak{X}$ is ergodic with unique invariant measure having density ϕ with respect to $m_d \times \mu$.

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