

CONICOPF: A tight-and-cheap conic relaxation with accuracy metrics for single-period and and multi-period ACOPF problems

C. Bingane, M.F. Anjos,
S. Le Digabel

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CONICOPF: A tight-and-cheap conic relaxation with accuracy metrics for single-period and multi-period ACOPF problems

Christian Bingane^{a,b}

Miguel F. Anjos^{a,b,c}

Sébastien Le Digabel^{a,b}

^a GERAD, HEC Montréal, Montréal (Québec), Canada, H3T 2A7

^b Department of Mathematics and Industrial Engineering, Polytechnique Montréal (Québec), Canada, H3C 3A7

^c School of Mathematics, University of Edinburgh, Edinburgh (Scotland), UK, EH9 3FD

christian.bingane@polymtl.ca

anjios@stanfordalumni.org

sebastien.le-digabel@polymtl.ca

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Abstract: Computational speed and global optimality are a key need for practical algorithms of the OPF problem. Recently, we proposed a tight-and-cheap conic relaxation for the ACOPF problem that offers a favourable trade-off between the standard second-order cone and the standard semidefinite relaxations for large-scale meshed networks in terms of optimality gap and computation time. In this paper, we show theoretically and numerically that this relaxation can be exact and can provide a global optimal solution for the ACOPF problem. Thereafter, we propose a multi-period tight-and-cheap relaxation for the multi-period ACOPF problem. Computational experiments using MATPOWER test cases with up to 500 buses show that this new relaxation is promising for real-life applications.

Keywords: Global optimization, multi-period optimal power flow, power systems, semidefinite programming.

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Nomenclature

Notation	
\mathbb{R}/\mathbb{C}	Set of real/complex numbers,
\mathbb{H}^n	Set of $n \times n$ Hermitian matrices,
j	Imaginary unit,
a/a	Real/complex number,
\mathbf{a}/\mathbf{a}	Real/complex vector,
A/A	Real/complex matrix.
Operators	
$\text{Re}(\cdot)/\text{Im}(\cdot)$	Real/imaginary part operator,
$(\cdot)^*$	Conjugate operator,
$ \cdot $	Magnitude or cardinality set operator,
$\angle(\cdot)$	Phase operator,
$(\cdot)^H$	Conjugate transpose operator,
$\text{rank}(\cdot)$	Rank operator.
Input data	
$\mathcal{P} = (\mathcal{N}, \mathcal{L})$	Power network,
\mathcal{N}	Set of buses,
$\mathcal{G} = \bigcup_{k \in \mathcal{N}} \mathcal{G}_k$	Set of generators,
\mathcal{G}_k	Set of generators connected to bus k ,
\mathcal{L}	Set of branches,
p_{Dk}^τ/q_{Dk}^τ	Active/reactive power demand at bus k at period τ ,
g_k^τ/b_k^τ	Conductance/susceptance of shunt element at bus k ,
$y_\ell^{-1} = r_\ell + jx_\ell$	Series impedance of branch ℓ ,
b_ℓ'	Total shunt susceptance of branch ℓ .
t_ℓ	Turns ratio of branch ℓ .
Variables	
p_{Gg}^τ/q_{Gg}^τ	Active/reactive power generation by generator g at period τ ,
v_k	Complex (phasor) voltage at bus k at period τ ,
$p_{f\ell}^\tau/q_{f\ell}^\tau$	Active/reactive power flow injected along branch ℓ by its <i>from</i> end at period τ ,
$p_{t\ell}^\tau/q_{t\ell}^\tau$	Active/reactive power flow injected along branch ℓ by its <i>to</i> end at period τ .

1 Introduction

The optimal power flow (OPF) problem, first formulated in [1], seeks to find a network operating point that optimizes an objective function subject to power flow equations and other operational constraints [2, 3, 4, 5]. The continuous classical version with AC power flow equations, which is nonconvex and NP-hard [6], is generally also called AC optimal power flow (ACOPF) problem.

In recent years, convex relaxations of the ACOPF problem, such as the second-order cone relaxation (SOCR) [7], the semidefinite relaxation (SDR) [8], the quadratic convex relaxation [9], and others [10, 11, 12, 13, 14], have attracted a significant interest for several reasons. First, they can lead to global optimality. Second, because they are relaxations, they provide a bound on the global optimal value of the ACOPF problem. Third, if one of them is infeasible, then the ACOPF problem is infeasible. We should note that, according to [5], convex relaxations of the OPF problem are aimed at complementing nonlinear (local) solvers with valuable information about the quality of the solution obtained, rather than at replacing them.

For general meshed networks, SDR is stronger than SOCR but requires heavier computation. Therefore, the chordal relaxation (CHR) was proposed in [15] in order to exploit the fact that power networks are not densely connected, thus reducing data storage and increasing computation speed. However, even CHR remains expensive to solve compared to SOCR for large-scale power systems. On the other hand, for radial networks, SOCR is tantamount to SDR. In this case, one would normally solve the first one rather than the second one due to the difference in computation time. A full literature review on these three relaxations can be found in [16, 17].

According to [3], high computational speed is a key need for practical OPF algorithms, especially in real-time applications and when dealing with large-scale power systems. In fact, in real-time applications, an OPF problem is run every few minutes to update device and resource settings in response to the constantly changing conditions of power systems [18]. This need motivated the choice of the tight-and-cheap conic relaxation (TCR), first proposed in [19], that offers a favourable trade-off between SOCR and SDR for large-scale meshed instances of ACOPF in terms of optimality gap and computation time. Indeed, TCR was proven to be stronger than SOCR and nearly as tight as SDR. Moreover, computational experiments on standard test cases with up to 6515 buses showed that solving TCR for large-scale instances is much less expensive than solving CHR.

Convex relaxations can lead to global optimality of the original ACOPF problem when they are exact, i.e., the optimality gap is null. For instance, [20] provided numerical examples on several IEEE benchmarks systems where SDR is exact. On the other hand, when a convex relaxation is not exact, it only provides a lower bound on the objective value, and its solution is not even feasible for the original problem [21, 22]. In this case, different methods, discussed in [23], have been proposed in the literature to obtain a feasible solution of the ACOPF problem from an inexact convex relaxation.

SDR or CHR is exact when its optimal solution fulfills the rank-one condition. In this paper, we show that TCR is exact when its optimal solution also fulfills a similar condition. Unlike SOCR, an additional cycle condition is not necessary for meshed networks [17]. We should note that, for many test cases, convex relaxations of the ACOPF problem are inexact even though optimality gaps are close to zero [9, 19, 24]. The optimality gap is thus insufficient as metric of the exactness of a convex relaxation [23]. To assess the exactness of TCR, we consider two other metrics: the exactness error and the optimality distance. With these metrics, we show that TCR is exact and provides a global optimal solution to the ACOPF problem for some MATPOWER test cases.

Thereafter, we propose a multi-period TCR for the multi-period ACOPF problem. A multi-period OPF problem is a sequence of ordinary OPF problems strung together by dynamic costs and constraints [18]. We consider a 24-period ACOPF problem and computational experiments using MATPOWER test cases with up to 118 buses show that this new relaxation is promising for real-life applications.

The remainder of this paper is organized as follows. In Section 2, we recall TCR and we show that it can be exact. In Section 3, we formulate the mathematical model of the multi-period ACOPF problem (without loss of generality) and we derive the multi-period TCR. We report in Section 4 computational results on the exactness of TCR and the efficiency of the multi-period TCR. Section 5 concludes the paper.

2 ACOPF: Tight-and-cheap relaxation

Consider a typical power network $\mathcal{P} = (\mathcal{N}, \mathcal{L})$ where $\mathcal{N} = \{1, 2, \dots, n\}$ and $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$ denote respectively the set of buses and the set of branches (transmission lines, transformers and phase shifters). Each branch $\ell \in \mathcal{L}$ has a *from* end k (on the *tap side*) and a *to* end m as modeled in [25]. We note $\ell = (k, m)$. The ACOPF problem is given as:

$$\min \sum_{g \in \mathcal{G}} c_{g2} p_{Gg}^2 + c_{g1} p_{Gg} + c_{g0} \quad (1a)$$

over variables $\mathbf{p}_G, \mathbf{q}_G \in \mathbb{R}^{|\mathcal{G}|}$, $\mathbf{p}_f, \mathbf{q}_f, \mathbf{p}_t, \mathbf{q}_t \in \mathbb{R}^{|\mathcal{L}|}$, and $\mathbf{v} \in \mathbb{C}^{|\mathcal{N}|}$, subject to

- Power balance equations:

$$\sum_{g \in \mathcal{G}_k} p_{Gg} - p_{Dk} - g'_k |\mathbf{v}_k|^2 = \sum_{\ell=(k,m) \in \mathcal{L}} p_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} p_{t\ell} \quad \forall k \in \mathcal{N}, \quad (1b)$$

$$\sum_{g \in \mathcal{G}_k} q_{Gg} - q_{Dk} + b'_k |\mathbf{v}_k|^2 = \sum_{\ell=(k,m) \in \mathcal{L}} q_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} q_{t\ell} \quad \forall k \in \mathcal{N}, \quad (1c)$$

- Line flow equations:

$$p_{f\ell} + jq_{f\ell} = \frac{v_k}{t_\ell} \left[\left(j \frac{b'_\ell}{2} + y_\ell \right) \frac{v_k}{t_\ell} - y_\ell v_m \right]^* \quad \forall \ell = (k, m) \in \mathcal{L}, \quad (1d)$$

$$p_{t\ell} + jq_{t\ell} = v_m \left[-y_\ell \frac{v_k}{t_\ell} + \left(j \frac{b'_\ell}{2} + y_\ell \right) v_m \right]^* \quad \forall \ell = (k, m) \in \mathcal{L}, \quad (1e)$$

- Generator power capacities:

$$\underline{p}_{Gg} \leq p_{Gg} \leq \bar{p}_{Gg}, \quad \underline{q}_{Gg} \leq q_{Gg} \leq \bar{q}_{Gg} \quad \forall g \in \mathcal{G}, \quad (1f)$$

- Line thermal limits:

$$|p_{f\ell} + jq_{f\ell}| \leq \bar{s}_\ell, \quad |p_{t\ell} + jq_{t\ell}| \leq \bar{s}_\ell \quad \forall \ell \in \mathcal{L}, \quad (1g)$$

- Voltage magnitude limits:

$$\underline{v}_k \leq |v_k| \leq \bar{v}_k \quad \forall k \in \mathcal{N}, \quad (1h)$$

- Reference bus constraint:

$$\angle v_1 = 0. \quad (1i)$$

The objective function (1a) is the cost of conventional generation commonly used in the literature. Constraints (1b)–(1e) are derived from Kirchhoff's laws and represent power flows in the network. Constraint (1i) specifies bus $k = 1$ as the reference bus. We assume that $\underline{v}_k > 0$ for all $k \in \mathcal{N}$ in (1h), and that the generation cost $c_{g2}p_{Gg}^2 + c_{g1}p_{Gg} + c_{g0}$ is a convex function for all $g \in \mathcal{G}$.

Problem (1) is highly nonconvex and NP-hard [6] due to the nonconvex constraints (1d)–(1e). Applying local methods to this problem provides no guarantee about the optimality of any solution found. Moreover, it is intractable to solve to global optimality for large-scale instances.

With $V := \mathbf{v}\mathbf{v}^H$, the ACOPF problem (1) can be reformulated as follows

minimize (1a)

subject to (1f), (1g), (1i),

$$\sum_{g \in \mathcal{G}_k} p_{Gg} - p_{Dk} - g'_k V_{kk} = \sum_{\ell=(k,m) \in \mathcal{L}} p_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} p_{t\ell} \quad \forall k \in \mathcal{N}, \quad (2a)$$

$$\sum_{g \in \mathcal{G}_k} q_{Gg} - q_{Dk} + b'_k V_{kk} = \sum_{\ell=(k,m) \in \mathcal{L}} q_{f\ell} + \sum_{\ell=(m,k) \in \mathcal{L}} q_{t\ell} \quad \forall k \in \mathcal{N}, \quad (2b)$$

$$p_{f\ell} + jq_{f\ell} = \frac{1}{|t_\ell|^2} \left(-j \frac{b'_\ell}{2} + y_\ell^* \right) V_{kk} - \frac{y_\ell^*}{t_\ell} V_{km} \quad \forall \ell = (k, m) \in \mathcal{L}, \quad (2c)$$

$$p_{t\ell} + jq_{t\ell} = -\frac{y_\ell^*}{t_\ell^*} V_{km} + \left(-j \frac{b'_\ell}{2} + y_\ell^* \right) V_{mm} \quad \forall \ell = (k, m) \in \mathcal{L}, \quad (2d)$$

$$\underline{v}_k^2 \leq V_{kk} \leq \bar{v}_k^2 \quad \forall k \in \mathcal{N}, \quad (2e)$$

$$V = \mathbf{v}\mathbf{v}^H. \quad (2f)$$

The nonconvexity of (2) is captured by the constraint (2f). We can show that $V = \mathbf{v}\mathbf{v}^H$ if and only if $V \succeq 0$ and $\text{rank}(V) = 1$. The standard semidefinite relaxation (SDR), first introduced in [8], is obtained by dropping the rank constraint. If we relax the constraint $V \succeq 0$ in SDR by $|\mathcal{L}|$ constraints of the form

$$V_{\{k,m\}} := \begin{bmatrix} V_{kk} & V_{km} \\ V_{km}^* & V_{mm} \end{bmatrix} \succeq 0 \quad \forall (k, m) \in \mathcal{L}, \quad (3)$$

we obtain the standard second-order cone relaxation (SOCR) [7], which is equivalent to SDR for radial networks.

SDR can be very expensive to solve for large-scale instances and SOCR remains weaker than SDR for meshed networks. We then consider a cheaper relaxation, called *tight-and-cheap relaxation* (TCR), given in Model 1 and obtained as follows. We replace (2f) by

$$\begin{bmatrix} 1 & \mathbf{v}_k^* & \mathbf{v}_m^* \\ \mathbf{v}_k & V_{kk} & V_{km} \\ \mathbf{v}_m & V_{km}^* & V_{mm} \end{bmatrix} \succeq 0 \quad \forall \ell = (k, m) \in \mathcal{L}, \quad (4a)$$

and we add the following constraints

$$\operatorname{Re}(\mathbf{v}_1) \geq \frac{V_{11} + \underline{v}_1 \bar{v}_1}{\underline{v}_1 + \bar{v}_1}, \quad (4b)$$

$$\operatorname{Im}(\mathbf{v}_1) = 0, \quad (4c)$$

corresponding to the reference bus $k = 1$.

Model 1 Tight-and-cheap relaxation (TCR)

Variables:

$$\begin{aligned} \mathbf{p}_G, \mathbf{q}_G &\in \mathbb{R}^{|\mathcal{G}|}, \\ \mathbf{p}_f, \mathbf{q}_f, \mathbf{p}_t, \mathbf{q}_t &\in \mathbb{R}^{|\mathcal{L}|}, \\ \mathbf{v} &\in \mathbb{C}^{|\mathcal{N}|}, \\ V &\in \mathbb{H}^{|\mathcal{N}|}. \end{aligned}$$

Minimize: (1a)

Subject to: (1f), (1g), (2a)–(2e), (4).

TCR was first proposed in [19] for the ACOPF problem. It was shown in [19] that TCR is stronger than SOCR and nearly as tight as SDR. Moreover, computational experiments on standard test cases with up to 6515 buses showed that solving TCR for large-scale instances is much less expensive than solving the chordal relaxation, a SDP relaxation technique that exploits the sparsity of power networks.

Lemma 1 Let $(\mathbf{x}, X) \in \mathbb{C}^n \times \mathbb{H}^n$ and let

$$Y = \begin{bmatrix} 1 & \mathbf{x}^H \\ \mathbf{x} & X \end{bmatrix} \in \mathbb{H}^{n+1}.$$

Then Y is a rank-one matrix if and only if $X = \mathbf{x}\mathbf{x}^H$.

Proof. Consider the Schur complement $X - \mathbf{x}\mathbf{x}^H$ of 1 in Y . By the Guttman rank additivity formula, $\operatorname{rank}(Y) = 1 + \operatorname{rank}(X - \mathbf{x}\mathbf{x}^H)$. Then $\operatorname{rank}(Y) = 1$ iff $X = \mathbf{x}\mathbf{x}^H$. \square

Proposition 1 If the optimal solution (\mathbf{v}, V) of TCR in Model 1 is such that $V_{kk} = |\mathbf{v}_k|^2$ for all $k \in \mathcal{N}$, then TCR is exact. Moreover, the TCR solution $\mathbf{v} \in \mathbb{C}^n$ is a global optimal solution for the ACOPF problem (1).

Proof. Let (\mathbf{v}, V) be the optimal solution of TCR in Model 1. We show that if $V_{kk} = |\mathbf{v}_k|^2$ for all $k \in \mathcal{N}$, then $V_{km} = \mathbf{v}_k \mathbf{v}_m^*$ for all $(k, m) \in \mathcal{L}$.

For all $\ell = (k, m) \in \mathcal{L}$, the semidefinite constraint (4a) is equivalent to

$$\begin{bmatrix} V_{kk} - |\mathbf{v}_k|^2 & V_{km} - \mathbf{v}_k \mathbf{v}_m^* \\ V_{km}^* - \mathbf{v}_k^* \mathbf{v}_m & V_{mm} - |\mathbf{v}_m|^2 \end{bmatrix} \succeq 0.$$

If $V_{kk} = |\mathbf{v}_k|^2$ or $V_{mm} = |\mathbf{v}_m|^2$, then $V_{km} = \mathbf{v}_k \mathbf{v}_m^*$. Therefore, if $V_{kk} = |\mathbf{v}_k|^2$ for all $k \in \mathcal{N}$, then $V_{km} = \mathbf{v}_k \mathbf{v}_m^*$ for all $(k, m) \in \mathcal{L}$. It follows that the TCR solution $\mathbf{v} \in \mathbb{C}^n$ is a global optimal solution of the ACOPF problem (1) since it is feasible for (2), which is equivalent to (1). \square

Lemma 1 and Proposition 1 prove that the TCR optimal solution \mathbf{v} is a global optimal solution for the ACOPF problem (1) if the positive semidefinite matrix in (4a) is rank-one at optimality for all branches $\ell = (k, m) \in \mathcal{L}$. We note that, when TCR is exact, global optimal voltages of the ACOPF problem are directly given by the TCR optimal solution \mathbf{v} , unlike SDR or SOCR where we have to recover them from the optimal solution \mathbf{V} .

3 MP-ACOPF: Multi-period TCR

Many power system applications that require solving an OPF problem are multi-period because of the evolution of market prices, of the ramping limits of generation units and of the behavior of the demand [26]. An OPF problem is run to meet the requirements of a time horizon optimally in every period. Then, the multi-period model must be adjusted to ensure that decisions in one period are consistent with the next one [27].

Consider a set $\{1, 2, \dots, \bar{\tau}\}$ of time periods τ . All parameters in the ACOPF problem (1) remain the same for all periods except demand which varies in each period. Without loss of generality, the MP-ACOPF problem is given as:

$$\min \sum_{\tau=1}^{\bar{\tau}} \sum_{g \in \mathcal{G}} c_{g2} p_{Gg}^{\tau 2} + c_{g1} p_{Gg}^{\tau} + c_{g0} \quad (5a)$$

over variables $\mathbf{p}_G^{\tau}, \mathbf{q}_G^{\tau} \in \mathbb{R}^{|\mathcal{G}|}$, $\mathbf{p}_f^{\tau}, \mathbf{q}_f^{\tau}, \mathbf{p}_t^{\tau}, \mathbf{q}_t^{\tau} \in \mathbb{R}^{|\mathcal{L}|}$, and $\mathbf{v}^{\tau} \in \mathbb{C}^{|\mathcal{N}|}$ for all $\tau = 1, \dots, \bar{\tau}$, subject to

- Power balance equations:

$$\sum_{g \in \mathcal{G}_k} p_{Gg}^{\tau} - p_{Dk}^{\tau} - g'_k |v_k^{\tau}|^2 = \sum_{\ell=(k,m) \in \mathcal{L}} p_{f\ell}^{\tau} + \sum_{\ell=(m,k) \in \mathcal{L}} p_{t\ell}^{\tau} \quad \forall k \in \mathcal{N}, \forall \tau = 1, \dots, \bar{\tau}, \quad (5b)$$

$$\sum_{g \in \mathcal{G}_k} q_{Gg}^{\tau} - q_{Dk}^{\tau} + b'_k |v_k^{\tau}|^2 = \sum_{\ell=(k,m) \in \mathcal{L}} q_{f\ell}^{\tau} + \sum_{\ell=(m,k) \in \mathcal{L}} q_{t\ell}^{\tau} \quad \forall k \in \mathcal{N}, \forall \tau = 1, \dots, \bar{\tau}, \quad (5c)$$

- Line flow equations:

$$p_{f\ell}^{\tau} + j q_{f\ell}^{\tau} = \frac{v_k^{\tau}}{t_{\ell}} \left[\left(j \frac{b'_{\ell}}{2} + y_{\ell} \right) \frac{v_k^{\tau}}{t_{\ell}} - y_{\ell} v_m^{\tau} \right]^* \quad \forall \ell = (k, m) \in \mathcal{L}, \forall \tau = 1, \dots, \bar{\tau}, \quad (5d)$$

$$p_{t\ell}^{\tau} + j q_{t\ell}^{\tau} = v_m^{\tau} \left[-y_{\ell} \frac{v_k^{\tau}}{t_{\ell}} + \left(j \frac{b'_{\ell}}{2} + y_{\ell} \right) v_m^{\tau} \right]^* \quad \forall \ell = (k, m) \in \mathcal{L}, \forall \tau = 1, \dots, \bar{\tau}, \quad (5e)$$

- Generator power capacities:

$$\underline{p}_{Gg} \leq p_{Gg}^{\tau} \leq \bar{p}_{Gg}, \underline{q}_{Gg} \leq q_{Gg}^{\tau} \leq \bar{q}_{Gg} \quad \forall g \in \mathcal{G}, \forall \tau = 1, \dots, \bar{\tau}, \quad (5f)$$

- Line thermal limits:

$$|p_{f\ell}^{\tau} + j q_{f\ell}^{\tau}| \leq \bar{s}_{\ell}, |p_{t\ell}^{\tau} + j q_{t\ell}^{\tau}| \leq \bar{s}_{\ell} \quad \forall \ell \in \mathcal{L}, \forall \tau = 1, \dots, \bar{\tau}, \quad (5g)$$

- Voltage magnitude limits:

$$\underline{v}_k \leq |v_k^{\tau}| \leq \bar{v}_k \quad \forall k \in \mathcal{N}, \forall \tau = 1, \dots, \bar{\tau}, \quad (5h)$$

- Reference bus constraints:

$$\angle v_1^{\tau} = 0 \quad \forall \tau = 1, \dots, \bar{\tau}, \quad (5i)$$

- Ramp constraints:

$$\underline{\Delta}_g^{\tau} \leq p_{Gg}^{\tau+1} - p_{Gg}^{\tau} \leq \bar{\Delta}_g^{\tau} \quad \forall g \in \mathcal{G}, \forall \tau = 1, \dots, \bar{\tau} - 1. \quad (5j)$$

In (5), the ACOPF problem (1) was replicated in each period $\tau = 1, \dots, \bar{\tau}$, and coupled sequentially by the ramp constraints (5j). These constraints enforce the generation limits when the demand increases or falls sharply between two consecutive periods. Note that one can also consider other time-coupled constraints such as storage ones [18, 28]. Beyond the constraints that can be taken into account, the major concern of the MP-ACOPF problem (5) is the computational scalability because the number of variables and constraints in the ACOPF problem (1) has been multiplied by the number of periods [18].

We now propose a convex relaxation of the MP-ACOPF problem (5). For all $\tau = 1, \dots, \bar{\tau}$, let $V^\tau := \mathbf{v}^\tau \mathbf{v}^{\tau H}$. With the same reasoning as for the single-period ACOPF problem (1), we define in Model 2 a tight-and-cheap relaxation for the MP-ACOPF problem (5). We call this relaxation “*multi-period tight-and-cheap relaxation*” (MP-TCR). We should note that other convex relaxations of the MP-ACOPF problem were already considered in the literature, e.g., SDR in [28], SOCR in [29].

Model 2 Multi-period tight-and-cheap relaxation (MP-TCR)

Variables:

$$\begin{aligned} \mathbf{p}_G^\tau, \mathbf{q}_G^\tau &\in \mathbb{R}^{|\mathcal{G}|}, \\ \mathbf{p}_f^\tau, \mathbf{q}_f^\tau, \mathbf{p}_t^\tau, \mathbf{q}_t^\tau &\in \mathbb{R}^{|\mathcal{L}|}, \\ \mathbf{v}^\tau &\in \mathbb{C}^{|\mathcal{N}|}, \\ V^\tau &\in \mathbb{H}^{|\mathcal{N}|} \end{aligned}$$

for all $\tau = 1, \dots, \bar{\tau}$.

Minimize: (5a)

Subject to: (5f), (5g), (5j), and for all $\tau = 1, \dots, \bar{\tau}$,

$$\begin{aligned} \sum_{g \in \mathcal{G}_k} p_{Gg}^\tau - p_{Dk}^\tau - g'_k V_{kk}^\tau &= \sum_{\ell=(k,m) \in \mathcal{L}} p_{f\ell}^\tau + \sum_{\ell=(m,k) \in \mathcal{L}} p_{t\ell}^\tau & \forall k \in \mathcal{N}, \\ \sum_{g \in \mathcal{G}_k} q_{Gg}^\tau - q_{Dk}^\tau + b'_k V_{kk}^\tau &= \sum_{\ell=(k,m) \in \mathcal{L}} q_{f\ell}^\tau + \sum_{\ell=(m,k) \in \mathcal{L}} q_{t\ell}^\tau & \forall k \in \mathcal{N}, \\ p_{f\ell}^\tau + j q_{f\ell}^\tau &= \frac{1}{|t_\ell|^2} \left(-j \frac{b'_\ell}{2} + y_\ell^* \right) V_{kk}^\tau - \frac{y_\ell^*}{t_\ell} V_{km}^\tau & \forall \ell = (k, m) \in \mathcal{L}, \\ p_{t\ell}^\tau + j q_{t\ell}^\tau &= -\frac{y_\ell^*}{t_\ell^*} V_{mk}^\tau + \left(-j \frac{b'_\ell}{2} + y_\ell^* \right) V_{mm}^\tau & \forall \ell = (k, m) \in \mathcal{L}, \\ \underline{v}_k^2 &\leq V_{kk}^\tau \leq \bar{v}_k^2 & \forall k \in \mathcal{N}, \\ \begin{bmatrix} 1 & \mathbf{v}_k^{\tau*} & \mathbf{v}_m^{\tau*} \\ \mathbf{v}_k^\tau & V_{kk}^\tau & V_{km}^\tau \\ \mathbf{v}_m^\tau & V_{km}^{\tau*} & V_{mm}^\tau \end{bmatrix} &\succeq 0 & \forall \ell = (k, m) \in \mathcal{L}, \\ \operatorname{Re}(\mathbf{v}_1^\tau) &\geq \frac{V_{11}^\tau + \underline{v}_1 \bar{v}_1}{\underline{v}_1 + \bar{v}_1}, \\ \operatorname{Im}(\mathbf{v}_1^\tau) &= 0. \end{aligned}$$

Proposition 2 For all $\tau = 1, \dots, \bar{\tau}$, if the optimal solution $(\mathbf{v}^\tau, V^\tau)$ of MP-TCR in Model 2 is such that $V_{kk}^\tau = |\mathbf{v}_k^\tau|^2$ for all $k \in \mathcal{N}$, then MP-TCR is exact. Moreover, for each time period $\tau = 1, \dots, \bar{\tau}$, the MP-TCR solution $\mathbf{v}^\tau \in \mathbb{C}^n$ is a global optimal solution for the MP-ACOPF problem (5).

Proof. The result follows from Proposition 1. □

4 Computational results

4.1 Exactness of TCR

In this section, we evaluate the exactness of TCR. We tested Model 1 on standard test cases available from MATPOWER [25, 30, 31].

We solved TCR in MATLAB using CVX 2.1 [32] with the solver MOSEK 8.0.0.60 and `default precision` (tolerance $\epsilon = 1.49 \times 10^{-8}$). All the computations were carried out on an Intel Core i7-6700 CPU @ 3.40 GHz computing platform. TCR and others relaxations defined in [19] were implemented as a MATLAB package, which is available on GitHub [33]. It requires that MATPOWER and CVX be installed and the input test case be in MATPOWER format.

We considered two different objective functions: the generation cost [\$/h] (1a) and the active loss [MW] where $c_{g2} = 0$, $c_{g1} = 1$ and $c_{g0} = 0$ for all $g \in \mathcal{G}$ in (1a). Both objective functions of test cases from [31] are the same.

We assess the exactness of TCR using three metrics:

1. the *exactness error* (which derives from Proposition 1) measured as

$$\varepsilon := \max_{k \in \mathcal{N}} \left(1 - \frac{|v_k^{\text{TCR}}|}{\sqrt{V_{kk}^{\text{TCR}}}} \right) \times 100\%,$$

where $(\mathbf{v}^{\text{TCR}}, \mathbf{V}^{\text{TCR}})$ is the optimal solution of TCR;

2. the *optimality gap*

$$\gamma := \left(1 - \frac{v^{\text{TCR}}}{v^{\text{MAT}}} \right) \times 100\%,$$

where v^{MAT} is the upper bound provided by the MATPOWER-solver (MIPS) and v^{TCR} is the TCR optimal value; and

3. the *optimality distance* defined as

$$\rho := \frac{\|\mathbf{v}^{\text{MAT}} - \mathbf{v}^{\text{TCR}}\|}{\|\mathbf{v}^{\text{MAT}}\|} \times 100\%,$$

where \mathbf{v}^{MAT} and \mathbf{v}^{TCR} represent the optimal bus voltages provided by MIPS and TCR, respectively.

Table 1 and Table 2 summarize these three metrics for cost minimization and loss minimization respectively. The results highlight the following key points:

1. TCR is exact for the `case6ww` instance in both cost and loss minimizations, and for the `case14` instance in cost minimization. TCR optimal voltages correspond exactly to optimal voltages provided by MIPS.
2. TCR is exact for the `LMBD3_60` instance in loss minimization, even though TCR optimal voltages do not match to optimal voltages provided by MIPS.
3. TCR is near-exact (i.e., $\varepsilon < 0.1\%$ and $\rho < 0.1\%$) for the `case24_ieee_rts` and `case_illinois200` instances in both cost and loss minimizations, for the `case_ieee30` instance in cost minimization and for the `case5` and `case_ACTIV_SG_500` instances in loss minimization.
4. TCR is on average more accurate in loss minimization than in cost minimization.

4.2 MP-TCR

Next we assess the accuracy and the computational efficiency of MP-TCR. We considered a 24-period ACOPF problem, i.e., $\bar{\tau} = 24$ in the MP-ACOPF problem (5). Experimental settings are the same

Table 1: Exactness of TCR: Cost minimization

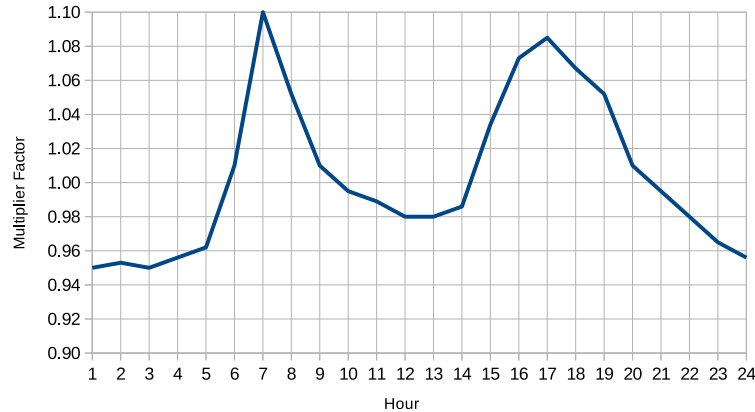
Test case	v^{MAT} [\$/h]	v^{TCR} [\$/h]	ε [%]	γ [%]	ρ [%]
<i>Small-scale instances</i>					
LMBD3_50	5 812.64	5 769.87	3.52	0.74	6.05
LMBD3_60	5 707.11	5 707.01	0.26	0.00	0.16
case5	17 551.89	15 313.38	1.04	12.75	6.55
case6ww	3 143.97	3 143.97	0.00	0.00	0.00
case9	5 296.69	5 296.69	0.82	0.00	0.00
case14	8 081.53	8 081.52	0.00	0.00	0.00
case24_ieee_rts	63 352.21	63 352.15	0.07	0.00	0.07
case30	576.89	576.50	0.65	0.07	0.65
case_ieee30	8 906.15	8 906.02	0.01	0.00	0.01
case39	41 864.18	41 861.91	1.83	0.01	0.32
case57	41 737.79	41 735.28	0.21	0.01	0.23
case89pegase	5 819.81	5 817.66	0.50	0.04	1.47
Average			0.58	0.23	0.74
<i>Medium-scale instances</i>					
case118	129 660.70	129 618.42	2.28	0.03	0.81
case_ACTIV_SG_200	27 557.57	27 557.33	0.16	0.00	0.17
case_illinois200	36 748.39	36 747.94	0.01	0.00	0.02
case300	719 725.11	719 547.51	6.46	0.02	3.16
case_ACTIV_SG_500	72 578.30	69 391.48	11.81	4.39	7.80
Average			6.18	1.67	3.78

Table 2: Exactness of TCR: Loss minimization

Test case	v^{MAT} [MW]	v^{TCR} [MW]	ε [%]	γ [%]	ρ [%]
<i>Small-scale instances</i>					
LMBD3_50	317.38	317.38	0.06	0.00	0.51
LMBD3_60	316.75	316.75	0.00	0.00	0.09
case5	1 001.06	1 001.06	0.01	0.00	0.01
case6ww	216.84	216.84	0.00	0.00	0.00
case9	317.32	317.32	0.77	0.00	0.00
case14	259.55	259.55	0.17	0.00	0.29
case24_ieee_rts	2 875.75	2 875.74	0.05	0.00	0.05
case30	191.09	191.07	0.49	0.01	0.21
case_ieee30	284.77	284.77	0.13	0.00	0.29
case39	6 284.15	6 283.90	1.57	0.00	0.50
case57	1 262.10	1 262.07	0.20	0.00	0.15
case89pegase	5 819.81	5 817.66	0.50	0.04	1.47
Average			0.48	0.01	0.58
<i>Medium-scale instances</i>					
case118	4 251.23	4 250.99	0.98	0.01	0.47
case_ACTIV_SG_200	1 483.92	1 483.91	0.19	0.00	0.19
case_illinois200	2 246.49	2 246.43	0.03	0.00	0.07
case300	23 737.72	23 735.69	9.64	0.01	2.47
case_ACTIV_SG_500	7 817.46	7 817.31	0.04	0.00	0.06
Average			2.33	0.00	0.67

as in the previous section. We tested Model 2 on MATPOWER instances. All instances' parameters remain the same for all periods except demand which varies in each period.

For each node $k \in \mathcal{N}$, demand $p_{Dk} + jq_{Dk}$ for an ACOPF problem was multiplied by a factor given in Figure 1, to obtain demand $p_{Dk}^\tau + jq_{Dk}^\tau$ in each period $\tau = 1, \dots, \bar{\tau}$ for an MP-ACOPF problem. The time varying multiplier factor was estimated based on a real power demand curve for a typical winter day provided by Hydro-Quebec [27].

**Figure 1: Time varying demand profile**

For all $\tau = 1, \dots, \bar{\tau} - 1$ and for all $g \in \mathcal{G}$, bounds $\bar{\Delta}_g^\tau$ and $\underline{\Delta}_g^\tau$ on ramp constraints (5j) were estimated as $\Delta^\tau \pm 0.1|\Delta^\tau|$, where

$$\Delta^\tau = \frac{1}{|\mathcal{G}|} \sum_{k \in \mathcal{N}} p_{Dk}^{\tau+1} - p_{Dk}^\tau.$$

Table 3 and Table 4 summarize the optimal values $v^{\text{MP-TCR}}$ of MP-TCR for cost minimization and loss minimization, respectively. MP-TCR was infeasible for following instances: LMBD3_50, LMBD3_60, case6ww, case24_ieee_rts, case89pegase, and all medium-scale instances except case118. The infeasibility of MP-TCR for these instances is explained by the fact that the demand varies at

Table 3: MP-TCR: Cost minimization

Test case	$v^{\text{MP-TCR}}$ [\$/h]	Exactness error [%]			Time [s]
		Min	Avg	Max	
<i>Small-scale instances</i>					
case5	391 803.23	0.56	0.77	1.03	0.26
case9	128 004.73	0.64	0.83	0.97	0.27
case14	194 952.01	0.00	0.00	0.00	0.43
case30	14 747.65	3.71	5.51	6.52	2.23
case_ieee30	*214 819.16	0.03	0.04	0.05	1.34
case39	1 039 085.79	4.93	5.35	5.82	2.36
case57	*1 007 169.75	0.20	0.24	0.25	3.94
Average		1.76	2.17	2.46	2.36
<i>Medium-scale instance</i>					
case118	*3 128 441.17	2.27	2.32	2.35	7.56

Table 4: MP-TCR: Loss minimization

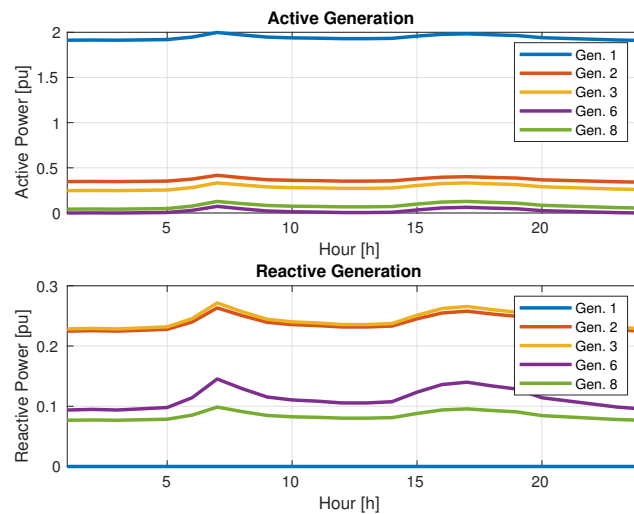
Test case	$v^{\text{MP-TCR}}$ [MW]	Exactness error [%]			Time [s]
		Min	Avg	Max	
<i>Small-scale instances</i>					
case5	24 118.98	0.02	0.02	0.02	0.15
case9	7 644.56	0.59	0.75	0.84	0.26
case14	6 252.62	0.16	0.17	0.18	0.40
case30	4 771.54	3.02	5.02	6.03	2.04
case_ieee30	6 860.80	0.13	0.14	0.15	0.84
case39	151 478.20	1.94	3.10	5.14	1.82
case57	30 431.56	0.18	0.20	0.30	1.93
Average		1.02	1.61	2.24	1.50
<i>Medium-scale instance</i>					
case118	102 418.57	0.98	0.99	1.05	5.42

each period (hour) while the generation bounds remain unchanged for all periods. For some instances marked with “*” in Table 3, MOSEK ended its computation with message `Mosek error: MSK_RES_TRM_STALL()`.

For each feasible instance, and for each period $\tau = 1, \dots, \bar{\tau}$, we calculated the exactness error ε^τ , and we report the minimum, the average, and the maximum values in Table 3 and Table 4. The results support the following key points:

1. MP-TCR is exact for the `case14` instance in cost minimization. Optimal active and reactive outputs of each generator and optimal voltage magnitudes of each bus are given in Figure 2 and Figure 3, respectively.
2. MP-TCR is near-exact for the `case5` instance in loss minimization.
3. MP-TCR is on average more accurate in loss minimization than in cost minimization.

We solved MP-TCR without any decomposition and computation times reported by MOSEK are shown in Table 3 and Table 4. We note that, for a 24-period problem, MP-TCR is computationally cheap and may be promising for large-scale instances with a decomposition technique.

**Figure 2: Cost minimization of case14 instance: Optimal active and reactive outputs of each generator**

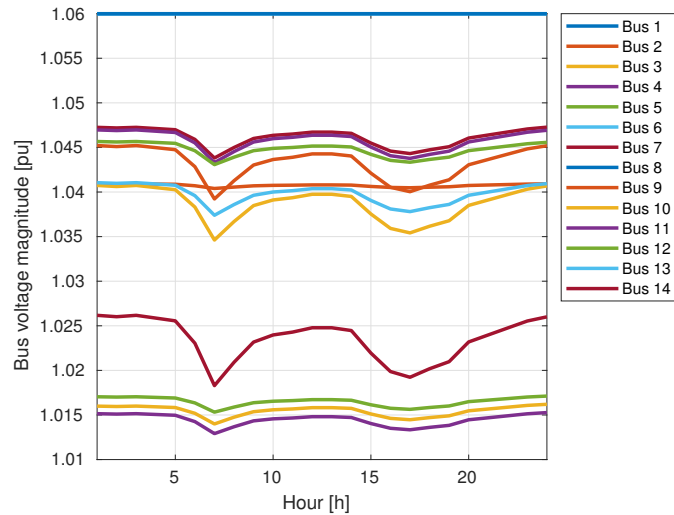


Figure 3: Cost minimization of case14 instance: Voltage magnitudes of each bus

5 Conclusion

In this paper, we showed theoretically that the tight-and-cheap conic relaxation (TCR) of the ACOPF problem can be exact. To assess the exactness of TCR, we used three metrics: the exactness error, the optimality gap and the optimality distance. In a new result, TCR provided a global optimal solution for 2 test cases: `case6ww` for both cost and loss minimization, `case14` for cost minimization and `LMBD3.60` for loss minimization. Experiments on other MATPOWER instances show that TCR optimal solutions are near-global optimal.

Thereafter, we defined the multi-period TCR (MP-TCR) for the multi-period ACOPF (MP-ACOPF) problem. MP-TCR was also shown to be exact for the `case14` instance in cost minimization. Experiments on MATPOWER instances with up to 118 buses show that, without any sophisticated decomposition algorithm, MP-TCR is computationally cheap and is promising for large-scale power systems in real-life applications.

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