Les Cahiers du GERAD ISSN: 0711–2440

Exploitation of a productive asset in the presence of strategic behavior and pollution externalities

B. Vardar,
G. Zaccour

G–2018–43

June 2018

GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Exploitation of a productive asset in the presence of strategic behavior and pollution externalities

Baris Vardar [a]
Georges Zaccour [b]

[a] *GERAD, HEC Montréal*

[b] *Chair in Game Theory and Management, GERAD, HEC Montréal*

baris.vardar@gerad.ca
georges.zaccour@gerad.ca

June 2018
Les Cahiers du GERAD
G–2018–43

Abstract: This paper studies the strategic behavior of firms competing in the exploitation of a common-access productive asset, in the presence of pollution externalities. We consider a differential game with two state variables (asset stock and pollution stock), and, by using a piecewise-linear approximation of the nonlinear asset growth function, we provide a tractable characterization of the symmetric feedback-Nash equilibrium, which is globally asymptotically stable. The results show that the firm's strategy takes three forms depending on the pair of state variables, and that different options for the model parameters lead to contrasting outcomes in both the short- and long-run equilibria.

Keywords: Productive asset, oligopoly, pollution externalities, dynamic games

Résumé : Cet article étudie le comportement stratégique de firmes concurrentes dans l'exploitation d'un actif productif d'accès commun, en présence d'externalités de pollution. Nous considérons un jeu différentiel à deux variables d'état (stock de l'actif et stock de pollution). L'approximation linéaire par morceaux de la fonction de croissance non-linéaire de l'actif, nous permet une caractérisation tractable de l'équilibre en rétroaction symétrique de Nash, qui est globalement asymptotiquement stable. Les résultats montrent que la stratégie de chaque firme peut prendre trois formes dépendamment des variables d'état, et que différentes valeurs des paramètres du modèle mènent à des résultats contrastés dans les équilibres à court et à long terme.

Mots clés : Actif productif, oligopole, externalités de pollution, jeux dynamiques

1 Introduction

There is an extensive literature in economics, operations research, game theory and dynamic optimization dealing with the exploitation of renewable resources (e.g., a fishery or a forest). One main question shared by all parties, namely, firms, governments (regulators), and citizens, is how to exploit these resources in a sustainable way. In the vast dynamic games literature, where players (firms and regulators) interact strategically over time, the models have very often focused on the resource itself, without any other considerations. Typically these models start with a dynamic system describing the evolution of the stock, and next, they characterize the equilibrium strategies under different assumptions related to (i) the information structure (e.g., open-loop, feedback, or closed-loop with or without memory) and (ii) to the players' behavior (cooperative or noncooperative).

There is ample evidence showing that the evolution of renewable resources depends not only on natural variations and on human intervention (harvesting, deforestation, etc.), but also on the accumulated pollution. An illustrative example is the recent discovery—devastating for the biomass—of over 5 trillion pieces of plastic, weighing a total of 250,000 tons, afloat on the Pacific Ocean. More than half of this island of plastic is made of fishing gear, i.e., is the result of the resource exploitation (Eriksen et al. (2014)). Scientists from various disciplines have proposed models to analyze the influence of other (state) variables—e.g., pollution stock, marine environmental quality, habitat quality—on the resource, omitting, however, the strategic interactions between the various parties involved (see, e.g., Botsford et al. (1997) and Ryan et al. (2014)).

In this paper, we consider an oligopoly exploiting a renewable resource (a productive asset), and contribute to the literature by having a more realistic model where strategic behavior and pollution externalities are present. Indeed, strategic thinking has often been ignored in large-scale ecosystem models and in representative-agent frameworks, and pollution has been disregarded in games of renewable-resource exploitation.

To represent the habitat's limited carrying capacity, the rate of growth of the productive asset is typically modeled as a nonlinear, inverted U–shaped function of the asset stock (see e.g., Clark (1990)). Benchekroun (2003, 2008) approximated this nonlinear rate of growth by an inverted-V function, which allowed for a tractable characterization of the equilibrium strategies and payoffs. We adopt a differential game framework that extends the productive-asset model in Benchekroun (2008) by introducing a second state variable, namely, pollution stock, while retaining his approximation approach. The game is played by n identical firms competing à la Cournot over an infinite planning horizon. Each player aims at maximizing her stream of discounted profit, taking into account the market structure, the initial size of each stock, and their dynamics. The problem is technically an infinite horizon linear-quadratic n-player differential game with two state variables and one control variable for each player, which influences the two dynamic processes.

We characterize, fully analytically, the symmetric feedback-Nash equilibrium that is globally asymptotically stable. The firm's strategy is a piecewise-linear function in the two state variables, whose shape depends on the position of the game in the state space. We show that this space is divided into three regions, namely, scarcity, abundance, and no-exploitation. Three equilibrium cases are identified and are shown to depend on the relationship between the asset growth rate, the discount rate, and the pollution decay rate. The equilibrium converges to a stationary state with positive exploitation for any given pair of initial asset–pollution stocks. When there are at least two players, there may be either one or multiple long-run equilibria; however when the industry is monopolistic, the long-run equilibrium is unique.

Our work belongs quite naturally to the literature on the exploitation of productive assets and the management of resources under pollution control. Early contributions in this area include Benhabib and Radner (1992) and Dockner and Sorger (1996), both of which explored the set of equilibria in a dynamic game framework. The characterization of the equilibrium in Benchekroun (2003, 2008) showed that the firm's strategy and the value function take a piecewise-linear form, which depends on the asset stock level. There can be a unique or multiple long-run stationary equilibria, depending on the asset growth rate, and the decision of a single firm to unilaterally decrease its exploitation may result in a decrease in the asset stock. These contributions led to extensive research on various topics related to the strategic exploitation of common access resources that also took into account the nonlinearity of growth rule. The literature has examined issues such as optimal taxation (Kossioris et al. (2011)), losses from competition (Fujiwara (2011)), the role of property rights and convergence to the Cournot equilibrium (Colombo and Labrecciosa (2013a,b)). More recently, Lambertini and Mantovani (2014) showed results on pre-emption, voracity, and exhaustion. Lambertini and Mantovani (2016) show that nonlinear feedback strategies are unstable in a dynamic duopoly game with renewable resource exploitation. The effects of mergers are analyzed in Benchekroun and Gaudet (2015), the impact of social status concern in these industries are studied in Benchekroun and Long (2016), and Grilli and Bisceglia (2017) investigate the incentives in a duopoly by considering a finite planning horizon.

In parallel to this dynamic-games literature on resources, a significant literature has dealt with the strategic behavior of agents under pollution externalities (see the surveys in Jørgensen et al. (2010) and Long (2011)). The pioneering contributions are van der Ploeg and de Zeeuw (1992) and Dockner and Long (1993), where both noncooperative and cooperative solutions are characterized and contrasted. Many papers have ensued, focusing on issues such as taxation (e.g., Benchekroun and Long (1998), Rubio and Escriche (2001)); sustainability and uncertainty (e.g., Wirl (1994, 2008)); international environmental agreements (e.g., Germain et al. (2003), Petrosjan and Zaccour (2003)); and technical change and R&D (e.g., Xepapadeas (1995a)).

Some studies have integrated pollution accumulation in the exploitation of a renewable resource (see, e.g. Tahvonen (1991), Xepapadeas (1995b), Wirl (2004)). However, these contributions have not accounted for strategic behavior, or when they have, they omitted the feature that the resource growth rate is not linear. In a nutshell, our paper attempts to integrate within the same framework the exploitation of a productive asset, in the presence of pollution externalities and strategic behavior. We approximate the nonlinear growth function by a piecewise linear function, which allows us to have a tractable linear-quadratic dynamic game, whose equilibria can be characterized analytically.

The rest of the paper is organized as follows: Section 2 introduces the model, Section 3 characterizes the equilibrium and shows its properties, and Section 4 concludes.

2 The model

We consider an n-player infinite-horizon differential game, with the asset stock S and the pollution stock Z being the two state variables. The model extends the framework in Benchekroun (2008) by introducing pollution externalities in the firms' decision-making problem. At each date $t \in [0, +\infty)$, firms exploit the common-access asset in quantities $q_i(t)$, $i = 1, \ldots, n$, and compete à la Cournot. We consider the transformation rate of the asset to the final product to be one-to-one, and the unit cost of exploitation to be zero for simplicity. The price p is determined by the linear inverse-demand function given by $p(Q) = a - bQ$, where $Q = \sum_{i=1}^n q_i$ is the total quantity of supply, and where the parameters $a > 0$ and $b > 0$.

The growth rate of the productive asset (e.g., a fishery, a forest, etc.) is assumed to be nonlinear, in an inverted U shape in the asset stock. Following the literature, we adopt the following piecewise-linear approximation:

$$f(S) = \begin{cases} \delta S & \text{if } S \leq \frac{S_{max}}{2}, \\ \delta(S_{max} - S) & \text{if } S > \frac{S_{max}}{2}, \end{cases} \tag{1}$$

where $\delta > 0$ denotes the intrinsic growth rate of the asset, and $S_{max} > 0$ is the carrying capacity of the habitat. Note that $f(S) = 0$ if $S = \{0, S_{max}\}$, $f(S) > 0$ if $S \in (0, S_{max})$, and $f(S) < 0$ if $S > S_{max}$. The quantity $\frac{S_{max}}{2} = S_y$ is the so-called maximum sustainable yield.

Taking into account the firms' exploitation, the change in the asset stock at date t is governed by the following differential equation:

$$\frac{dS(t)}{dt} = \dot{S}(t) = f(S(t)) - \sum_{i=1}^{n} q_i(t). \tag{2}$$

The firms' activities generate emissions as a by-product, which add up to the pollution stock, which evolves over time as follows:

$$\frac{dZ(t)}{dt} = \alpha \sum_{i=1}^{n} q_i(t) - kZ(t), \tag{3}$$

where $\alpha > 0$ denotes the amount of emissions resulting from exploiting a unit of asset, and $k > 0$ is the pollution decay rate.

Denote by $d(Z)$ the (symmetric) damage cost of player i. We suppose that this environmental cost is convex increasing in the pollution stock Z and satisfies the property $d(0) = 0$. For tractability, we adopt the quadratic functional form $d(Z) = \frac{\phi}{2}Z^2$ where $\phi > 0$.

Assuming that each player maximizes her discounted stream of profit, the optimization problem of player i is then as follows:

$$\begin{aligned}&\max_{q_i(t)} \int_{t=0}^{+\infty} e^{-rt}\Big(p\left(\sum_{i=1}^{n} q_i(t)\right) q_i(t) - d(Z(t))\Big)dt, \\ &\text{subject to } (2),\ (3), \text{ and } q_i(t) \geq 0, \\ &\text{with } S(0) = S_0 > 0, Z(0) = Z_0 \geq 0 \text{ given.}\end{aligned} \tag{4}$$

where $r > 0$ denotes the common discount rate.

3 The equilibrium

We consider the equilibrium where firms use feedback information in their decision-making. This is a subgame-perfect equilibrium (also called a Markovian Perfect Nash Equilibrium (MPNE)), in which the firm's strategy is state dependent and strongly time consistent (Dockner et al. (2000), Haurie et al. (2012)). Denote by $V_i(S(t), Z(t))$ the value function of firm i, which is the discounted sum of profits the firm obtains in the game starting in state $(S(t), Z(t))$. Unless an ambiguity arises, we shall from now on omit the time argument. Introduce the Hamilton-Jacobi-Bellman (HJB) equation associated to firm i's maximization problem, that is,

$$rV_i(S, Z) = \max_{q_i} \left\{ p\left(\sum_{i=1}^{n} q_i\right) q_i - d(Z) + \frac{\partial V_i(S, Z)}{\partial S}\left(f(S) - \sum_{i=1}^{n} q_i\right) + \frac{\partial V_i(S, Z)}{\partial Z}\left(\alpha \sum_{i=1}^{n} q_i - kZ\right)\right\}, \tag{5}$$

for $i = \{1, .., n\}$, where the partial derivative $\partial V_i(S, Z)/\partial S$ represents the shadow price (or value) of the asset stock (also called scarcity rent), and $\partial V(S, Z)/\partial Z$ denotes the shadow value of the pollution stock. Taking into account the nonnegativity restriction on q_i in problem (4), maximizing the right-hand side of (5) yields the following condition:

$$q_i \geq 0; \quad a - b\sum_{i=1}^{n} q_i - bq_i \leq \frac{\partial V_i(S, Z)}{\partial S} - \alpha \frac{\partial V_i(S, Z)}{\partial Z}. \tag{6}$$

The left-hand-side is the marginal revenue of firm i for given quantities of competitors. The right-hand side represents the marginal (opportunity) cost of a unit of exploitation, which comprises the shadow prices of the asset stock and pollution stock. Since we are considering costless exploitation, the opportunity cost consists only of the shadow prices of the state variables. We focus on a symmetric equilibrium in which all firms exploit the same quantities ($q_i = q$ for all $i = \{1, .., n\}$). Consequently, (6) becomes

$$q^*(S, Z) = \max\left\{0, \frac{1}{b(n+1)}\left(a - \frac{\partial V(S, Z)}{\partial S} + \alpha \frac{\partial V(S, Z)}{\partial Z}\right)\right\}, \tag{7}$$

for $i = \{1, .., n\}$. Condition (7) results in two possibilities for the equilibrium strategy, that is, $q^*(S, Z) > 0$ or $q^*(S, Z) = 0$. There may exist different cases where $q^*(S, Z) > 0$ depending on the signs of the partial derivatives of the function $V(S, Z)$. We consider the cases in which the asset has a scarcity rent ($\partial V(S, Z)/\partial S > 0$), or not ($\partial V(S, Z)/\partial S = 0$), and we consider all possible cases for the effect of pollution on the value of the firm, thus $sign(\partial V(S, Z)/\partial Z)$ is free. By using (7), we write these cases in the following definition:

Definition 1 *The three regions are as follows:*

Scarcity region (R_S): $q^*(S,Z) > 0$ *with* $\partial V(S,Z)/\partial S > 0$*:*

$$R_S = \left\{ (S,Z) \mid \left(a > \frac{\partial V(S,Z)}{\partial S} - \alpha \frac{\partial V(S,Z)}{\partial Z} \right) \text{ and } \left(\frac{\partial V(S,Z)}{\partial S} > 0 \right) \right\}.$$

Abundance region (R_A): $q^*(S,Z) > 0$ *with* $\partial V(S,Z)/\partial S = 0$*:*

$$R_A = \left\{ (S,Z) \mid \left(a > -\alpha \frac{\partial V(S,Z)}{\partial Z} \right) \text{ and } \left(\frac{\partial V(S,Z)}{\partial S} = 0 \right) \right\}.$$

No-exploitation region (R_0): $q^*(S,Z) = 0$*:*

$$R_0 = \left\{ (S,Z) \mid a \leq \frac{\partial V(S,Z)}{\partial S} - \alpha \frac{\partial V(S,Z)}{\partial Z} \right\}.$$

In region R_S, the pair of state variables is such that it is profitable to exploit the asset, and the asset has a scarcity rent. The firms view the level of the asset stock as scarce, and they consider their impact on the asset stock in their strategy. In region R_A, the asset stock is too high, so having an additional unit in the stock brings no value to firms. Players consider only the pollution externality as an intertemporal effect of exploitation, and the asset stock does not play a role in their decision. In region R_0, the marginal revenue of an initial asset supply (given by the price $P(0) = a$) is lower than its marginal cost, which depends on the shadow prices of the asset stock and pollution stock. Hence, exploitation is not dynamically profitable, and the equilibrium strategy is to wait for the asset to replenish and for pollution to decline.

A priori, since the function V is not known, it is not clear beforehand whether any of these three cases exist or not. Theorem 1 shows that the symmetric feedback equilibrium exists and is unique under certain conditions on the model parameters, which are incorporated into the following assumptions:

Assumption 1

$$(a)\ \frac{r\left(1+n^2\right)}{2} < \delta \leq r + k \text{ and } \phi < \phi_1, \quad (b)\ r + k < \delta < \frac{k\left(n^2+1\right)}{n^2-1} \text{ and } \phi < \phi_2,$$

where the terms ϕ_1 *and* ϕ_2 *are given in* (36) *and* (38)*, and* $\phi_2 < \phi_1$*.*

Assumption 2 $S_2 < S_y$ *where* S_2 *is given in* (31)*.*

Theorem 1 *Suppose that Assumptions 1 and 2 are satisfied. The strategy profile* $\{q_1(t),..,q_n(t)\} = \{q^*(S(t),Z(t)),..,q^*(S(t),Z(t))\}$ *for* $t \in [0,+\infty)$ *where*

$$q^*(S,Z) = \begin{cases} (a + c_0 + c_S S + c_Z Z)/b(n+1) & \text{if} \quad (S,Z) \in R_S, \\ (a + \bar{c}_0 + \bar{c}_Z Z)/b(n+1) & \text{if} \quad (S,Z) \in R_A, \\ 0 & \text{if} \quad (S,Z) \in R_0, \end{cases} \tag{8}$$

is the unique symmetric feedback-Nash equilibrium that is globally asymptotically stable. The terms $\{c_0, c_S, c_Z\}$ *and* $\{\bar{c}_0, \bar{c}_Z\}$*, which depend on the exogenous model parameters, are given in* (18) *to* (22)*, and the regions* $\{R_S, R_A, R_0\}$ *are defined in* (26)*,* (40)*, and* (64)*. The discounted sum of profits obtained by each firm is given by the following value function:*

$$V_i(S,Z) = \begin{cases} W(S,Z) & \text{if} \quad (S,Z) \in R_S, \\ \bar{V}(Z) & \text{if} \quad (S,Z) \in R_A, \\ V_0(S,Z) & \text{if} \quad (S,Z) \in R_0, \end{cases} \tag{9}$$

for $i = \{1,..,n\}$*, which is continuous and continuously differentiable* $\forall (S,Z) \in \mathbb{R}_+^2$*. The function* $W(S,Z)$ *is a polynomial of degree 2 in* S *and* Z *given in* (10)*, the function* $\bar{V}(Z)$ *is a polynomial of degree 2 in* Z *given in* (23)*, and the function* $V_0(S,Z)$ *is a piecewise nonlinear function given in* (76)*.*

Proof. The long proof is built throughout the paper and the details are provided in the Appendix, which has several subsections. The road map to complete the proof of the theorem is as follows:

1. In Subsection A.1, we state some preliminaries and introduce the methodological approach.
2. In Subsection A.2, we study the case $q^*(S,Z) > 0$. By guessing a polynomial form for the value function, and by applying the undetermined coefficient method, we obtain the functions $W(S,Z)$ and $\bar{V}(Z)$ and the solutions associated to their coefficients. Then, we analyze the boundary cases and their positions in (S,Z).
3. Lemma 1 in Subsection A.2.1 shows that under Assumptions 1 and 2, the function $W(S,Z)$ satisfies the HJB equation $\forall(S,Z) \in R_S$, and that strategy profile $q_i = q^*\ \forall i$ satisfies (7) with $q^*(S,Z) > 0$ and $\partial W(S,Z)/\partial S > 0$, which constitutes a symmetric feedback-Nash equilibrium.
4. Lemma 2 in Subsection A.2.2 shows that under Assumptions 1 and 2, the function $V(S,Z) = \{W(S,Z)$ if $(S,Z) \in R_S, \bar{V}(Z)$ if $(S,Z) \in R_A\}$ is continuous and continuously differentiable in S and Z and satisfies the HJB equation $\forall(S,Z) \in R_S \cup R_A$. The strategy profile $q_i = q^*$, $\forall i$ satisfies (7) with $q^*(S,Z) > 0$ and $\partial V(S,Z)/\partial S \geq 0$, which constitutes a symmetric feedback-Nash equilibrium that is asymptotically stable.
5. Subsection A.3 looks at the case $q^* = 0$. Lemma 3 obtains the function $V_0(S,Z)$, and shows that it is continuous and continuously differentiable in S and Z, $\forall(S,Z) \in R_0$, and on the boundary cases of R_0.
6. Combining these results, we conclude that the piecewise function $V(S,Z)$ given in (9) satisfies HJB equation (5), and the strategy profile $q_i = q^*$, $\forall i$ satisfies the condition in (7) $\forall(S,Z) \in \mathbb{R}^2_+$ and constitutes a feedback-Nash equilibrium that is globally asymptotically stable.

□

Theorem 1 characterizes the symmetric equilibrium strategy of firms for any given pair (S,Z). The strategy $q^*(S,Z)$ is a piecewise-linear function in S and Z with coefficients (c_0, c_S, c_Z) and $(\bar{c}_0, \bar{c}_Z)$ that correspond to the coefficients of the marginal cost function given in (6) (see (25) in Subsection A.2 in the Appendix).

3.1 The properties of the equilibrium

We briefly explain the methods used for obtaining the equilibrium strategies, and then investigate their properties. Using a linear-quadratic model with the piecewise-linear approximation of the asset growth function enables us to guess the form of the value function as a polynomial of degree 2 in S and Z within an interior solution. We obtain the six-dimensional equation system associated to the coefficients of $W(S,Z)$, then reduce it into a system of two equations in (c_S, c_Z) given in (16) and (17). This system yields four solutions: two include $\partial W(S,Z)/\partial S \neq 0$ and the other two include $\partial W(S,Z)/\partial S = 0$. Among all solutions, only one pair makes it possible to characterize a global and stable equilibrium. This solution is equivalent to the one given in Benchekroun (2008) in the limit case where the damage function parameter $\phi \to 0$ (see Remark 1 in Subsection A.2 in the Appendix).

We use the solutions for the strategies and the value function to derive the analytical formulation of the case R_S in (26). We then obtain the linear functions associated to its boundary cases where $q^*(S,Z) = 0$ and $\partial W(S,Z)/\partial S = 0$, and then study their positions in (S,Z) by analyzing the signs of (c_S, c_Z), which results in three cases that differ in the relationship among the dynamic model parameters, i.e. $sign(\delta - r - k)$. In all cases, $c_S > 0$ and $\bar{c}_Z < 0$, but the sign of c_Z differs, i.e., $sign(c_Z) = sign(\delta - r - k)$. As will be shown, this difference leads to contrasting results in the equilibrium behavior in R_S.

We analyze the properties of the equilibrium by using the diagram in Figure 1, which shows the shapes and positions of the cases given in Definition 1 in the (S,Z) plane.

(i) There are four regions. The value function takes a different form in each region.
(ii) $q^* > 0$ in $R_S \cup R_A$, and $q^* = 0$ in $R_0^W \cup R_0^{\bar{V}}$.
(iii) $\partial V/\partial S > 0$ in $R_S \cup R_0^W$, and $\partial V/\partial S = 0$ in $R_A \cup R_0^{\bar{V}}$.

(iv) The boundary case $Z = Z_0^S(S)$ given in (27) denotes the threshold where $q^* = 0$ and $\partial W/\partial S > 0$, beyond which the firms voluntarily cease exploitation, and wait for the asset to replenish and pollution to decline. The sign of its slope depends on $sign(r + k - \delta)$.

(v) The boundary case $Z = Z_S^A(S)$ given in (29) denotes the threshold where $q^* > 0$ and $\partial W/\partial S = 0$. It is decreasing in S in all cases.

(vi) The boundary $Z = Z_0^A$ given in (41), which is a positive constant, denotes the threshold level of pollution where $q^* = 0$ and $\partial V/\partial S = 0$. The firms refrain from any exploitation if the level of pollution is above this threshold.

(vii) The boundary cases of R_S intersect with the $Z = 0$ axis at points $(S_1, 0)$ and $(S_2, 0)$, and their intersection point is denoted by (S_3, Z_3), where the closed-form solutions of S_1, S_2, and (S_3, Z_3) are given in (30) to (33). When 1 and 2 are satisfied, the ordering of these points is given as follows: $0 < S_1 < S_2 < S_y$, and $S_3 > 0$, $Z_3 > 0$.

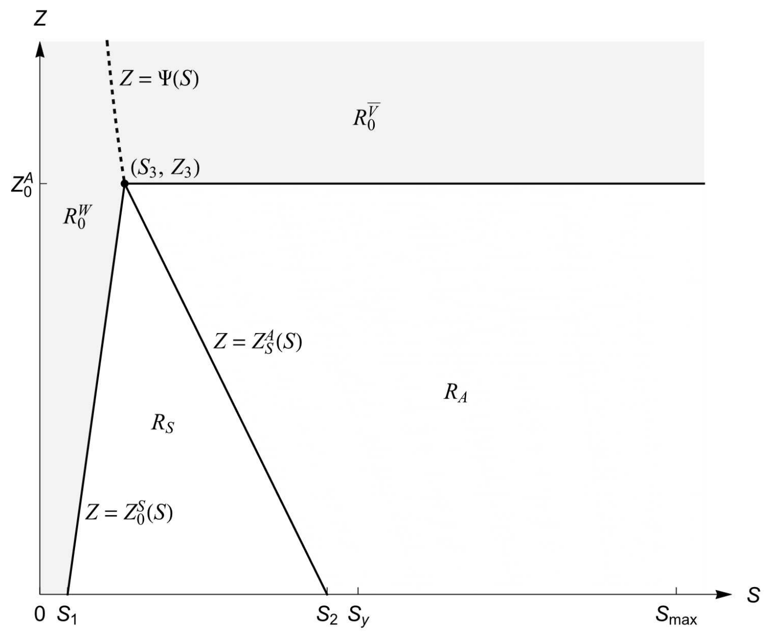

Figure 1: Illustration of the regions and their boundaries in (S, Z)

The equilibrium in R_A remains the same in all cases, as the firm's strategy and the value function do not depend on δ. Within that region, the equilibrium behavior is that of the dynamic oligopoly with pollution externalities. Since $dZ_S^A(S)/dS < 0$ in all cases, then, for a fixed point on this boundary, $\partial V/\partial S > 0$ for lower levels of pollution, and $\partial V/\partial S = 0$ if pollution is high.

By contrast, the behavior in R_S differs depending on $sign(\delta - r - k)$. In the following, we show the properties of the equilibrium strategies in each case.

Case 1. $\delta < r+k$, $c_S > 0$ and $c_Z < 0$ (Figure 1 and 2(a)): The equilibrium level of exploitation is faster if the asset stock is larger ($\partial q^*/\partial S > 0$), and it is slower if the level of pollution is higher ($\partial q^*/\partial Z < 0$). Since $dZ_0^S(S)/dS > 0$, for a fixed level of asset stock, firms will exploit the resource if pollution is sufficiently low, but will not exploit if pollution is too high.

Case 2. $\delta = r+k$, $c_S > 0$ and $c_Z = 0$: The firm's strategy includes only the asset stock and is independent of the level of pollution. The boundary case where $q^* = 0$ and $\partial W/\partial S > 0$ becomes a constant $S = S_1$.

Case 3. $\delta > r + k$, $c_S > 0$ and $c_Z > 0$ (Figure 2(b)): In this case, the slope of the threshold $Z = Z_0^S(S)$ becomes negative. Firms exploit the asset faster if the level of pollution is higher ($\partial q^*/\partial Z > 0$). Since $dZ_0^S(S)/dS < 0$, for a fixed a point on this threshold, firms exploit the asset if the level of pollution is high, and do not exploit it if pollution is low. The opportunity cost of exploitation given in (6) is decreasing in pollution.

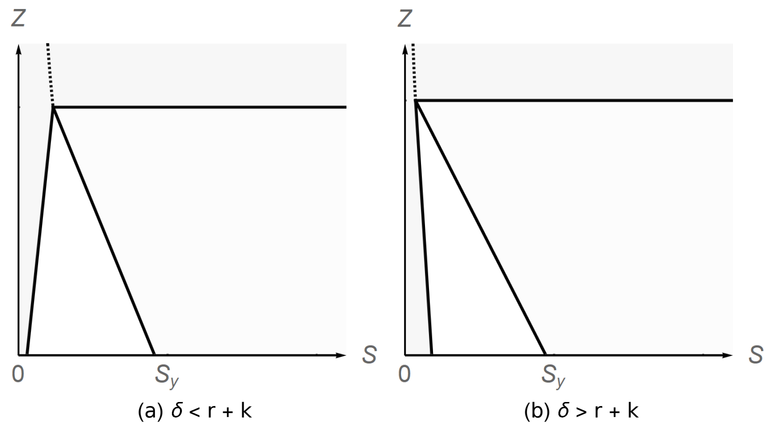

Figure 2: Two cases of region positioning under Assumptions 1 and 2

To characterize the value function in R_0 given in (64), we use the fact that in this region, $q_i = 0$ for all $i \in \{1, .., n\}$, thus the asset stock grows at rate δ and the pollution stock declines at rate k without an intervention. At a certain date $\hat{t}$, depending on the initial state $(S(0), Z(0))$, the pair $(S(t), Z(t))$ reaches either one of the boundary cases where firms begin exploitation. The value of the game starting at a point in R_0 depends on this boundary point associated to itself, which can be computed. The following steps are taken in Appendix A.3:

(i) We define an implicit function denoted by $(\hat{S}(S, Z), \hat{Z}(S, Z))$ in (66–67), which yields the point (and the date) at which the firms launch their exploitation for $(S, Z) \in R_0$.

(ii) Using this function, we obtain the curve denoted by $Z = \Psi(S)$ given in (73) associated to the intersection point of the three regions (S_3, Z_3) where $W(S_3, Z_3) = \bar{V}(Z_3)$.

(iii) This curve enables us to partition region R_0 into two parts, denoted by R_0^W and $R_0^{\bar{V}}$ given in (74–75), such that the boundary for launching exploitation is known for a given $(S, Z) \in R_0$.

Having obtained the boundary point associated to all points in $(S, Z) \in R_0$, in Lemma 3 we find $V_0(S, Z)$ that satisfies the HJB equation with $q_i^* = 0, \; \forall i$ and show its continuity.

The function $(\hat{S}(.), \hat{Z}(.))$ does not have an analytical solution for $(S, Z) \in R_0^W$, except in special cases $(\delta = k)$ and $(\delta = r + k)$ (see Remark 2 in Subsection A.3 in the Appendix), nevertheless the characterization of $V_0(S, Z)$ remains tractable. For any other parameter setting, this function has to be computed numerically in order to obtain the value of a point in R_0^W. In the other partition, $R_0^{\bar{V}}$, the value function has an analytical form.

3.2 The equilibrium dynamics, stationary points, and stability

In order to analyze the stability of the equilibrium, in Lemma 2 (see subsection A.2.2 in the Appendix) we derive the set of points such that $\dot{S}(t) = 0$ and $\dot{Z}(t) = 0$, respectively. Then, we obtain the stationary points and study their stability by using the methods provided in Takayama (1993). The diagram in Figure 3 illustrates the results.

(i) Under Assumptions 1 and 2, for $n \geq 2$, the stationary point ξ_S given in (47–48) always exists. The condition for stability is $\delta > r(1 + n)/2$, which is weaker than that of Assumption 1; thus, ξ_S is asymptotically stable under Assumption 1.

(ii) The existence of ξ_{A1} and ξ_{A2} given in (56-58) depends on condition (55).

- If (55) is true, then the point ξ_{A1} is unstable, whereas ξ_{A2} is asymptotically stable. In that case, the long-run stationary equilibria are multiple (ξ_S, ξ_{A2}), and the equilibrium to which a game converges depends on its initial state $(S(0), Z(0))$.
- If (55) is not true, then ξ_S is the unique stationary point.

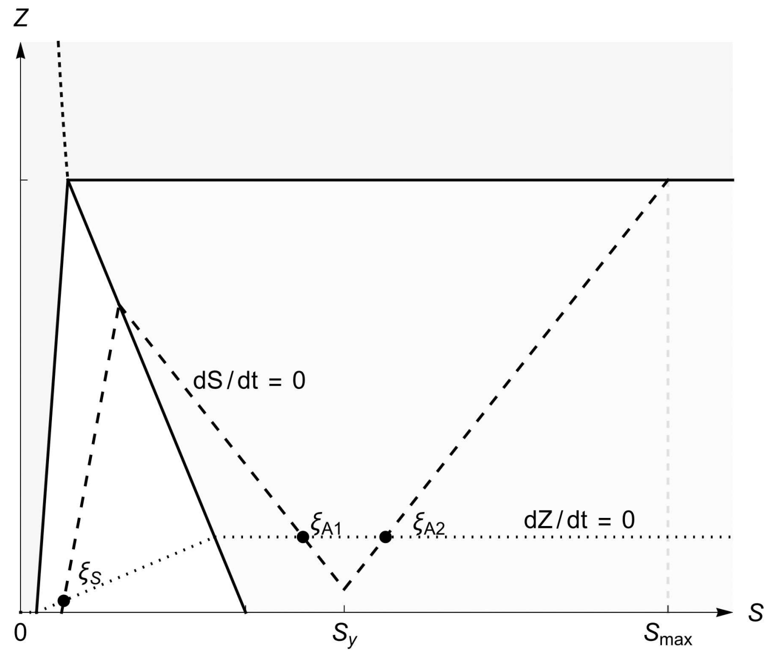

Figure 3: Sample stationary points in the case of multiple long-run equilibria

The cases of unique and multiple equilibria are illustrated in Figure 4, which shows the equilibrium trajectories obtained by using the differential equation system resulting from replacing $q^*(S, Z)$ in (8) into (2) and (3). For a given initial state, the equilibrium strategy may shift from one to another as the values of the state variables cross the thresholds between the regions.

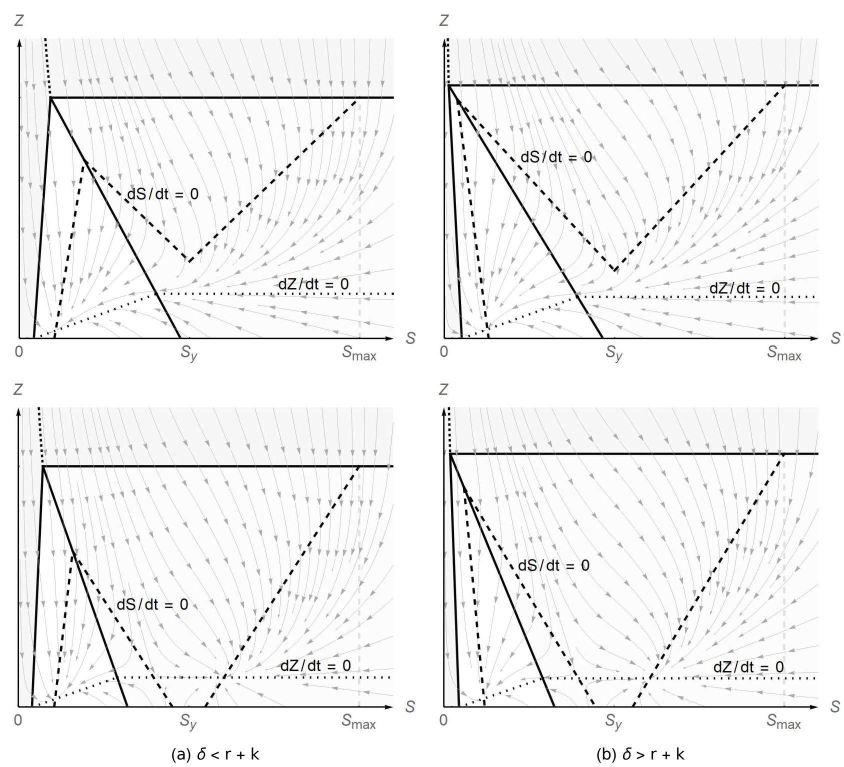

Figure 4: Sample equilibria with unique and multiple stationary states

The continuity of $V(S,Z)$ on the boundary cases ensures a smooth transition between the regions with continuous $q^*(S,Z)$; consequently, the strategies $q_i(t) = q^*(S(t),Z(t)),\ \forall i$ converge to a stationary state with $\lim\limits_{t\to+\infty} q^*(t) > 0$ for all initial states, and constitute a symmetric feedback-Nash equilibrium that is globally asymptotically stable.

In order to keep the focus of the work on the general properties of the equilibrium, we refrain from studying the effects of variations in the parameter values, and leave such an analysis for future research. A case worth mentioning is the monopoly case ($n = 1$), in which the stationary point ξ_S lies on the boundary $Z = Z_S^A(S)$, and coincides with the unstable point in R_A, i.e., $\xi_S = \xi_{A1}$. The analysis regarding the stability of ξ_{A2} remains valid. Therefore, for a monopoly, ξ_{A2} is the unique stationary point that is asymptotically stable.

4 Concluding remarks

We characterized the symmetric feedback-Nash equilibrium and showed its existence and uniqueness within a certain range of model parameters given in Assumptions 1 and 2. The equilibrium path always reaches a stationary state that is sustainable. For a set of parameters outside this range, there may still exist local equilibria for some levels of asset-pollution stock pairs.

The framework we present includes various simplifications and abstractions, which made the characterization of the equilibrium more conveniently tractable. We introduced the pollution externalities in a simple way in order to guarantee that variations in the equilibrium results between the outcomes with and without pollution externalities could be studied through a single exogenous parameter.

The methodology used to characterize the equilibrium can be applied in problems involving similar features by considering different objective functions and state dynamics. Some examples are the issues relating to the open-access fisheries shared by multiple countries, analysis of cooperation and stability of coalitions, welfare analysis, spillover effects, and the interactions between other possible state variables and the asset stock.

Appendix: Proof of Theorem 1

A.1 Some preliminaries

A symmetric equilibrium exists if there exists a function $V_i(S,Z) = V(S,Z)\ \forall i$ that is continuous and continuously differentiable in S and Z, which satisfies the HJB equation in (5) and the first-order condition in (7) (see Dockner et al. (2000), Haurie et al. (2012)). Due to the nonnegativity restriction on q_i in problem (4), we consider the two following cases:

$$q^*(S,Z) > 0 \text{ and } q^*(S,Z) = 0.$$

In order to characterize a global equilibrium strategy $q^*(S,Z)$ that is defined for the whole state space $(S,Z) \in \mathbb{R}^2_+$ that is stable (i.e., converges to a stationary state with $\lim\limits_{t\to+\infty}(q^*(S(t),Z(t)) > 0)$, we look for the parameter constellations under which the following two conditions are satisfied:

(1) $q^*(0,Z) = 0,\ \forall Z \geq 0$, which is to ensure global stability, and

(2) $\partial V(S,Z)/\partial S = 0,\ \forall S \geq S_y$, which is to ensure the continuity of $V(S,Z)$ on $S = S_y$ where the asset growth function $f(S_y)$ is not continuously differentiable.

More specifically,

- if $q^*(0,Z) = 0,\ \forall Z \geq 0$ is not satisfied, then $\exists Z \geq 0 \mid q^*(0,Z) > 0$ and then there may exist an initial state $(S(0),Z(0))$ such that $\lim\limits_{t\to+\infty}(S(t),Z(t)) = (0,0)$ and $\lim\limits_{t\to+\infty}(q^*(S(t),Z(t)) = 0$, then $q^*(S,Z)$ may not be a stable global equilibrium,
- if $(\partial V(S,Z)/\partial S = 0\ \forall S \geq S_y)$ is not satisfied, then the discontinuous point $f(S_y)$ given in (2) is included in the case $q^*(S,Z) > 0$ and $\partial V(S,Z)/\partial S > 0$. For this reason, we cannot find a function that is continuously differentiable in S, and $q^*(S,Z)$ does not satisfy the HJB equation for $S \geq S_y$; thus, in that case, $q^*(S,Z)$ would not be a global equilibrium.

In the following subsections, we use the methods developed in the literature to study the cases given in Definition 1. We focus on linear strategies, obtain the function $V(S,Z)$, and analyze a number of closed-form formulas to identify the restrictions on the model parameters under which the conditions discussed above are satisfied, which allows us to characterize the equilibrium.

A.2 Case with positive exploitation ($q^*(S,Z) > 0$)

Since the model is linear-quadratic, we make the informed guess that within an interior solution ($q^*(S,Z) > 0$), the value function is a polynomial of degree 2 in S and Z. We consider the function $W(S,Z)$ given by

$$W(S,Z) = A + \frac{B}{2}S^2 + CS + \frac{D}{2}Z^2 + EZ + FSZ. \tag{10}$$

The maximized HJB equation is obtained by replacing $q^*(S,Z)$ in (7) into (5). Using $W(S,Z)$ results in an equation that is a polynomial of degree 2 in S and Z. We then apply the method of undetermined coefficients (see Haurie et al. (2012)), by identification, and after simplifications, the system of equations in (A,B,C,D,E,F) is written as follows:

$$A = \frac{(a+\alpha E - C)\left(a + n^2(\alpha E - C)\right)}{b(n+1)^2 r}, \qquad B = \frac{2n^2(\alpha F - B)^2}{b(n+1)^2(r-2\delta)}, \tag{11}$$

$$C = \frac{(\alpha F - B)\left(a(n^2+1) + 2n^2(\alpha E - C)\right)}{b(n+1)^2(r-\delta)}, \qquad D = \frac{2n^2(\alpha D - F)^2 - b(n+1)^2\phi}{b(n+1)^2(2k+r)}, \tag{12}$$

$$E = \frac{(\alpha D - F)\left(a(n^2+1) + 2n^2(\alpha E - C)\right)}{b(n+1)^2(k+r)}, \qquad F = \frac{2n^2(\alpha D - F)(\alpha F - B)}{b(n+1)^2(k+r-\delta)}. \tag{13}$$

Introduce the following changes of variables:

$$c_0 = \alpha E - C, \qquad c_S = \alpha F - B, \qquad c_Z = \alpha D - F. \tag{14}$$

Replacing (14) into the equations for C and E, the term c_0 is written as a function of c_S and c_Z, and is given by

$$c_0(c_S, c_Z) = \frac{a\left(n^2+1\right)\left(c_S(k+r) + \alpha c_Z(\delta - r)\right)}{b(n+1)^2(k+r)(\delta-r) - 2n^2\left(c_S(k+r) + \alpha c_Z(\delta - r)\right)}. \tag{15}$$

Using the equations for B, D, and F in (11–13) with (14) enables us to write the system of equations for c_S and c_Z as follows:

$$c_S = \alpha \frac{2n^2 c_Z c_S}{b(n+1)^2(k+r-\delta)} - \frac{2n^2 c_S^2}{b(n+1)^2(r-2\delta)}, \tag{16}$$

$$c_Z = \alpha \frac{2n^2 c_Z^2 - b(n+1)^2\phi}{b(n+1)^2(2k+r)} - \frac{2n^2 c_Z c_S}{b(n+1)^2(k+r-\delta)}. \tag{17}$$

Hence, we reduced the six-dimensional equation system in (11-13) into a system of two equations in c_S and c_Z, which contain polynomials of degree 2.

Equation (16) has two solutions for c_S, i.e., $\left\{c_S = \frac{1}{2}(2\delta - r)\left(\frac{b(n+1)^2}{n^2} + \frac{2\alpha c_Z}{\delta - k - r}\right), c_S = 0\right\}$. Inserting these values into (17) yields four solutions to the system in (16–17):

- Solution A is written as follows:

$$c_S = \frac{(n+1)}{4n^2}\frac{(2\delta - r)}{(\delta + k)}\left(\lambda + b(n+1)(2\delta - r)\right), \tag{18}$$

$$c_Z = \frac{(n+1)}{4n^2\alpha}\frac{(\delta - r - k)}{(\delta + k)}\left(\lambda - b(n+1)(2k + r)\right), \tag{19}$$

  and $c_0 = c_0(c_S, c_Z)$, where the term λ is given by

$$\lambda = \sqrt{\left(b(n+1)(2k+r)\right)^2 + 8n^2\alpha^2 b\phi}, \tag{20}$$

  and $\lambda > b(n+1)(2k+r)$, which will be used in the sign analysis of the closed-form expressions.

- Solution B is written as follows:

$$\bar{c}_S = 0, \tag{21}$$

$$\bar{c}_Z = -\frac{(n+1)}{4n^2\alpha}\left(\lambda - b(n+1)(2k+r)\right) < 0, \tag{22}$$

  and $\bar{c}_0 = c_0(\bar{c}_S, \bar{c}_Z)$.

- Solution A' and Solution B', which are denoted by (c'_S, c'_Z) and $(\bar{c}'_S, \bar{c}'_Z)$, are written exactly as in (18–19) and (21–22) by inverting the sign of λ, i.e.,

$$\lambda_{A'} = \lambda_{B'} = -\sqrt{\left(b(n+1)(2k+r)\right)^2 + 8n^2\alpha^2 b\phi}.$$

In Solutions A and B, by studying the signs of the terms in (18–22), we obtain the following results: $sign(c_S) = sign(2\delta - r)$, $sign(c_Z) = sign(\delta - r - k)$, and $\bar{c}_Z < 0$. Using these results, for $\delta > r/2$, we write the following cases:

$$\begin{aligned}
&\text{Case 1. } \delta < r+k: \quad c_S > 0, \quad c_Z < 0, \quad \bar{c}_Z < 0,\\
&\text{Case 2. } \delta = r+k: \quad c_S > 0, \quad c_Z = 0, \quad \bar{c}_Z < 0,\\
&\text{Case 3. } \delta > r+k: \quad c_S > 0, \quad c_Z > 0, \quad \bar{c}_Z < 0.
\end{aligned}$$

The solutions given above consist only of the exogenously given model parameters. Therefore, all solutions for $W(S, Z)$ can be obtained by replacing (14) in (11–13) and inserting the solutions given in (18–22).

In Solutions A and A', $\partial W(S,Z)/\partial S \neq 0 \ \forall \delta \neq r/2$, thus they are candidates for the case $q^* > 0$ and $\partial V(S,Z)/\partial S > 0$. In Solutions B and B' we have $\partial W(S,Z)/\partial S = 0$, and they are candidates for the case $q^* > 0$ and $\partial V(S,Z)/\partial S = 0$. Therefore, we consider Solutions A and A' for the function $W(S,Z)$ given in (10); and for Solutions B and B', where $\bar{c}_S = \bar{c}'_S = 0$, we define the following function denoted by $\bar{V}(Z)$, which is a polynomial of degree 2 in Z, written as

$$\bar{V}(Z) = \bar{A} + \frac{\bar{D}}{2}Z^2 + \bar{E}Z, \tag{23}$$

where its coefficients of $\bar{V}(Z)$ are written as follows:

$$\bar{A} = \frac{(a+\bar{c}_0)\left(a+n^2\bar{c}_0\right)}{b(n+1)^2 r}, \quad \bar{D} = \frac{\bar{c}_Z}{\alpha}, \quad \bar{E} = \frac{\bar{c}_0}{\alpha}. \tag{24}$$

It can be verified that both functions $W(S,Z)$ and $\bar{V}(Z)$ satisfy the HJB equation in (5) for all choices of solutions.

Remark 1 *In the limit case where $\phi \to 0$, the problem reduces to a game with one state variable (S). Solution A reduces to $c_Z = 0$, $c_S = (2\delta - r)\frac{b(1+n)^2}{2n^2}$, and $c_0 = \frac{a\left(1+n^2\right)}{2n^2\delta}(r-2\delta)$, which results in $D = E = F = 0$, and $B = -c_S = (r-2\delta)\frac{b(1+n)^2}{2n^2}$, $C = -c_0 = \frac{a\left(1+n^2\right)}{2n^2\delta}(2\delta - r)$, $A = \frac{a^2\left(r\left(1+n^2\right)-2\delta\right)\left(r\left(1+n^2\right)-2n^2\delta\right)}{4brn^2(1+n)^2\delta^2}$. Solution B vanishes with $\bar{c}_S = \bar{c}_Z = \bar{c}_0 = 0$, which leads to the equilibrium outcome of the static Cournot oligopoly, i.e., $\bar{V}(Z) = \frac{a^2}{b(1+n)^2 r}$ and $q^* = \frac{a}{b(1+n)}$. These outcomes are identical to their corresponding terms in the solution provided in Benchekroun (2008).*

The equilibrium strategy is written by using (7) and (14) with $V(S,Z) = W(S,Z)$:

$$q^*(S,Z) = (a + c_0 + c_S S + c_Z Z)/b(n+1), \tag{25}$$

which is linear in S and Z. Note that by using (14), RHS(6) can be written as $MC(S,Z) = -(c_0 + c_S S + c_Z Z)$; hence, these terms correspond to the coefficients of the marginal cost function.

In the following sections, we use the functions $W(S,Z)$ and $\bar{V}(Z)$ to study the two cases where $\partial V/\partial S > 0$ and $\partial V/\partial S \geq 0$.

A.2.1 Case with $q^*(S,Z) > 0$ and $\partial V(S,Z)/\partial S > 0$

We first obtain the analytical formulation of R_S. For $q^*(S,Z) > 0$, from (25) we obtain $a > -c_0 - c_S S - c_Z Z$, and for $\partial W(S,Z)/\partial S > 0$ we use (11) to (14). Since $c_S > 0$ in all cases for $\delta > r/2$, writing both inequalities in S enables us to obtain the following region and its boundary cases:

(i) $q^*(S,Z) > 0$ and $\partial V(S,Z)/\partial S > 0$ for all $(S,Z) \in R_S$

where

$$R_S = \left\{(S,Z) \mid \left(S > -\frac{a+c_0}{c_S} - \frac{c_Z}{c_S}Z\right) \text{ and } \left(S < \frac{(2\delta - r)}{(k+r-\delta)}\frac{c_Z}{c_S}Z - \frac{(2\delta - r)}{(\delta - r)}\frac{\left(a\left(1+n^2\right)+2n^2 c_0\right)}{2n^2 c_S}\right)\right\}, \tag{26}$$

and the boundary line associated to the first inequality, which is denoted by $Z = Z_0^S(S)$, is written as follows:

$$Z = Z_0^S(S) = -\frac{c_S}{c_Z}S - \frac{a+c_0}{c_Z}, \tag{27}$$

where the sign of its slope is given by $sign(\frac{dZ_0^S(S)}{dS}) = sign(-\frac{c_S}{c_Z})$.

- if $r/2 < \delta < r+k$ then $\frac{dZ_0^S(S)}{dS} > 0$;
- if $\delta = r+k$, equation (27) reduces to $S = -\frac{a+c_0}{c_S}$;
- if $\delta > r+k$ then $\frac{dZ_0^S(S)}{dS} < 0$.

(ii) The set of points such that $q^*(S,Z)=0$ is defined as follows:

$$q^*(S,Z)=0 \text{ if } \left\{(S,Z) \mid S \leq -\frac{a+c_0}{c_S}-\frac{c_Z}{c_S}Z\right\}, \tag{28}$$

which is obtained by using $a \leq \frac{\partial W(S,Z)}{\partial S}-\alpha\frac{\partial W(S,Z)}{\partial Z}$.

(iiii) The set of points such that $\partial W(S,Z)/\partial S=0$ is given by the following linear function in S:

$$Z=Z_S^A(S)=\frac{(k+r-\delta)}{(2\delta-r)}\frac{c_S}{c_Z}S+\frac{(k+r-\delta)\left(a\left(1+n^2\right)+2n^2c_0\right)}{2n^2(\delta-r)c_Z}, \tag{29}$$

and the sign of its slope is given by $sign(\frac{dZ_S^A(S)}{dS})=sign(-\frac{(\delta-r-k)}{(2\delta-r)}\frac{c_S}{c_Z})$. Since $sign(c_Z)=sign(\delta-r-k)$ and $c_S>0$, we have $\frac{dZ_S^A(S)}{dS}<0$ in all cases where $\delta>r/2$.

We now calculate the points at which the linear functions obtained for the boundary cases intersect with the $Z=0$ axis and each other, i.e., $Z_0^S(S)=0$, $Z_S^A(S)=0$, and $Z=Z_0^S(S)=Z_S^A(S)$. These formulas are first written in terms of (c_0,c_S,c_Z), and then, after inserting the solutions in (18–22), they are written in closed form, using λ in (20) to study their signs.

(i) $(S_1,0)$: $Z_0^S(S_1)=0$ and $q^*(S,0)=0$ $\forall S\leq S_1$ where

$$\begin{aligned}S_1=&-\frac{a+c_0}{c_S},\\ =&\frac{2a(k+\delta)\left(2br\delta+\left(1+n^2\right)b\left(\delta(2k+r)-2r(k+r)\right)-(n-1)\delta\lambda\right)}{\delta(2\delta-r)(\lambda+b(1+n)(2\delta-r))(\lambda+b(1+n)r)}.\end{aligned} \tag{30}$$

(ii) $(S_2,0)$: $Z_S^A(S_2)=0$ where

$$\begin{aligned}S_2=&-\frac{(2\delta-r)\left(a\left(1+n^2\right)+2n^2c_0\right)}{2n^2(\delta-r)c_S},\\ =&\frac{4ab\left(1+n^2\right)(k+r)(k+\delta)}{\delta(\lambda+b(1+n)r)(\lambda+b(1+n)(2\delta-r))}>0 \text{ if } \delta>r/2.\end{aligned} \tag{31}$$

(iii) (S_3,Z_3): $Z_0^S(S_3)=Z_S^A(S_3)=Z_3$ is written as follows:

$$S_3=\frac{2a\left(2bk\left(n^2+1\right)(k+r)-b\delta\left(n^2-1\right)r-\delta\lambda(n-1)\right)}{\delta(b(n+1)r+\lambda)(\lambda+b(n+1)(2\delta-r))}, \tag{32}$$

$$Z_3=\frac{2a\alpha\left(b\left(2k\left(n^2+1\right)+r\left(3n^2+1\right)\right)+\lambda(n-1)\right)}{(\lambda+b(n+1)r)(\lambda-b(n+1)(2k+r))}>0. \tag{33}$$

By using (30) and (31), the difference S_2-S_1 is given by

$$S_2-S_1=\frac{2a\left(b\left(2k\left(1+n^2\right)+r\left(1+3n^2\right)\right)+(n-1)\lambda\right)(k+\delta)}{(2\delta-r)(\lambda+b(1+n)r)(\lambda+b(1+n)(2\delta-r))}>0 \text{ if } \delta>r/2, \tag{34}$$

thus, for $\delta>r/2$, $Z=Z_S^A(S)$ intersects with the $Z=0$ axis at $S_2>0$ and $S_2>S_1$. By using (30), we obtain the conditions under which $S_1>0$:

$$\delta>\frac{r\left(1+n^2\right)}{2}, \tag{35}$$

$$\phi<\frac{b(k+r)\left(2\delta-\left(n^2+1\right)r\right)\left(\delta\left(2kn^2+\left(n^2+1\right)r\right)-\left(n^2+1\right)r(k+r)\right)}{2(\delta\alpha n(n-1))^2}=\phi_1. \tag{36}$$

For $\delta\leq r+k$, $S_3>0$ if (35) and (36) are satisfied, and for $\delta>r+k$, $S_3>0$ if

$$\delta<\frac{k\left(n^2+1\right)}{n^2-1}, \tag{37}$$

$$\phi<\frac{bk(k+r)\left(k\left(n^2+1\right)-\delta\left(n^2-1\right)\right)\left(\left(n^2+1\right)(k+r)+\delta\left(n^2-1\right)\right)}{2(\delta\alpha n(n-1))^2}=\phi_2, \tag{38}$$

where condition (38) is stricter than the one in (36) ($\phi_2<\phi_1$).

Lemma 1 *Suppose that Assumptions 1 and 2 are satisfied. For all* $(S, Z) \in R_S$*, the function* $V(S, Z) = W(S, Z)$ *satisfies the HJB equation (5), and the strategies* $q_i = q^*(S, Z)$*,* $\forall i$ *given in (8) satisfy (7) with* $q^*(S, Z) > 0$ *and* $\partial V(S, Z)/\partial S > 0$*, which constitutes a symmetric feedback-Nash equilibrium.*

Proof. The sign analysis conducted in (30) to (38) shows that, if the conditions given in Assumptions 1 and 2 are satisfied, then $0 < S_1 < S_2 < S_y$ and $S_3 > 0,\ Z_3 > 0$. We study the slopes of the two boundary lines of R_S for each case. The slope of $Z = Z_0^S(S)$ depends on $sign(-\frac{c_S}{c_Z})$, and the slope $dZ_S^A(S)/dS < 0$ if $\delta > r/2$, then the three cases are written as follows:

Case 1: $\delta < r + k$: In this case $dZ_0^S(S)/dS > 0$ and $dZ_S^A(S)/dS < 0$. If $S_1 > 0$ ((35) and (36)) then $q^*(S, Z) = 0\ \forall(S, Z) \mid S \le S_1$. In addition, if $S_2 < S_y$ then $(S, Z) \notin R_S\ \forall(S, Z) \mid S \ge S_2$ (see Figures 1 and 2(a)). The function $W(S, Z)$ satisfies the HJB equation (5) and condition (7) with $q^*(S, Z) > 0\ \forall(S, Z) \in R_S$, and the strategies $q_i = q^*\ \forall i$ constitute a symmetric equilibrium.

Case 2: $\delta = r + k$: In this case $c_Z = 0$ and the boundary case $q^*(S, Z) = 0$ reduces to $(S_1, Z)\ \forall Z \le Z_3$. We have the same result as the previous case. If $S_1 > 0$ and $S_2 < S_y$, then the function $W(S, Z)$ satisfies the HJB equation (5) and condition (7) with $q^*(S, Z) > 0\ \forall(S, Z) \in R_S$, and the strategies $q_i = q^*\ \forall i$ constitute a symmetric equilibrium.

Case 3: $\delta > r + k$: In this case $dZ_0^S(S)/dS < 0$ and $dZ_S^A(S)/dS < 0$, and hence both boundary lines are decreasing in S. The difference in their slopes is given by $dZ_0^S(S)/dS - dZ_S^A(S)/dS = -\frac{c_S}{c_Z}\frac{(\delta+k)}{(2\delta-r)}$. Since for $\delta > r + k$ we have $c_S > 0$ and $c_Z > 0$, the slopes compare as follows:

$$\frac{dZ_0^S(S)}{dS} < \frac{dZ_S^A(S)}{dS} < 0 \text{ if } \delta > r/2, \tag{39}$$

thus, $Z_0^S(S)$ is steeper than $Z_S^A(S)$, and $Z_3 > 0$, which can also be seen in (33). If $S_3 > 0$ and $S_2 < S_y$ are satisfied, then $q^*(S, Z) = 0\ \forall S \le S_3$ and $\forall Z \ge 0$ (see Figure 2(b)), and the function $W(S, Z)$ satisfies the HJB equation (5) and condition (7) with $q^*(S, Z) > 0\ \forall(S, Z) \in R_S$, and the strategies $q_i = q^*,\ \forall i$ constitute a symmetric equilibrium.

□

Note that when using (26), we eliminate the possibility of choosing Solution A for $\delta < r/2$ (where $c_S < 0$), and Solution A' for $r/2 < \delta < r + k$ (where $c'_S < 0$ and $c'_Z > 0$). In these cases, the inequalities in (26) and (28) change their directions. And in step (39), we eliminate the possibility of choosing Solution A' for $\delta < r/2$ (where $c'_S > 0$ and $c'_Z > 0$), as it leads to $\frac{dZ_S^A(S)}{dS} < \frac{dZ_0^S(S)}{dS} < 0$, which does not allow to characterize a global equilibrium.

A.2.2 Case with $q^*(S, Z) > 0$ and $\partial V(S, Z)/\partial S \ge 0$

We now consider the function $V(S, Z) = \bar{V}(Z)$ given in (23), and by using $a > -\alpha\frac{\partial \bar{V}(Z)}{\partial Z}$, which is given by $a > -\alpha(\bar{D}Z + \bar{E})$ with (24), (26), and (29),

(i) $q^*(S, Z) > 0$ and $\partial V(S, Z)/\partial S = 0$ for all $(S, Z) \in R_A$ where

$$R_A = \left\{(S, Z) \mid \left(Z < Z_0^A\right) \text{ and } \left(S \ge \frac{(2\delta - r)}{(k + r - \delta)}\frac{c_Z}{c_S}Z - \frac{(2\delta - r)}{c_S}\frac{\left(a\left(1 + n^2\right) + 2n^2 c_0\right)}{2n^2(\delta - r)}\right)\right\}. \tag{40}$$

(ii) The set of points such that $q^*(S, Z) = 0$ and $\partial V(S, Z)/\partial S = 0$ is written as follows:

$$Z \ge Z_0^A = -\frac{a + \bar{c}_0}{\bar{c}_Z} = \frac{2a\alpha\left(b\left(2k\left(n^2 + 1\right) + r\left(3n^2 + 1\right)\right) + \lambda(n - 1)\right)}{(\lambda + b(n + 1)r)(\lambda - b(n + 1)(2k + r))} = Z_3, \tag{41}$$

which is obtained by using $a = -\alpha\frac{\partial \bar{V}(Z)}{\partial Z}$ and (24).

Lemma 2 *Suppose that Assumptions 1 and 2 are satisfied. For* $(S,Z) \in R_S \cup R_A$ *where* $V(S,Z) = \{W(S,Z)$ *if* $(S,Z) \in R_S, \bar{V}(Z)$ *if* $(S,Z) \in R_A\big\}$, *the strategies* $q_i = q^*$ $\forall i$ *given in (8) satisfy the HJB equation (5) and condition (7) with* $q^*(S,Z) > 0$, *the function* $V(S,Z)$ *is continuous and continuously differentiable* $\forall(S,Z) \in R_S \cup R_A$, *and* $q_i = q^*$ $\forall i$ *constitutes a symmetric feedback-Nash equilibrium that is asymptotically stable.*

Proof. We first show the continuity of $V(S,Z)$, and then study the stationary points and their stability. The function $W(S,Z)$ is a polynomial of degree 2, which is continuous and continuously differentiable in S and Z $\forall(S,Z) \in R_S$. The function $\bar{V}(Z)$ is a polynomial of degree 2, which is continuous and continuously differentiable in Z and $\frac{\partial \bar{V}(Z)}{\partial S} = 0$ $\forall(S,Z) \in R_A$. For continuity of $V(S,Z)$ on the boundary, as $(S,Z) \to (\hat{s},\hat{z}) \mid \hat{z} = Z_S^A(\hat{s})$:

$$\lim_{(S,Z)\to(\hat{s},\hat{z})} W(S,Z) = \left(A - \frac{C^2}{2B}\right) + \left(\frac{D}{2} - \frac{F^2}{2B}\right)\hat{z}^2 + \left(E - \frac{CF}{B}\right)\hat{z}, \tag{42}$$

$$\lim_{(S,Z)\to(\hat{s},\hat{z})} \bar{V}(Z) = \bar{A} + \frac{\bar{D}}{2}\hat{z}^2 + \bar{E}\hat{z}. \tag{43}$$

where (42) is obtained by using $S = -C/B - ZF/B$ (where $\partial W(S,Z)/\partial S = 0$). By using (11) to (24), it can be verified that

$$\bar{A} = A - \frac{C^2}{2B}; \quad \frac{\bar{D}}{2} = \left(\frac{D}{2} - \frac{F^2}{2B}\right); \quad \bar{E} = \left(E - \frac{CF}{B}\right), \tag{44}$$

therefore, $\lim_{(S,Z)\to(\hat{s},\hat{z})} W(S,Z) = \lim_{(S,Z)\to(\hat{s},\hat{z})} \bar{V}(Z)$ for $\hat{z} = Z_S^A(\hat{s})$, thus $V(S,Z)$ is continuous on $\big\{(S,Z) \mid Z = Z_S^A(S)\big\}$. The four solutions to the system in (16–17) are coupled such that (44) is true for Solutions (A and B) and (A' and B') given in (18) to (22), while (44) is not true for (A' and B) and (A and B'); therefore, in order to have a function that is continuous and continuously differentiable on the boundary where $\lim_{(S,Z)\to(\hat{s},\hat{z})} \partial W(S,Z)/\partial S = 0$, either one of the pair of solutions must be selected.

By the definition of $Z = Z_S^A(S)$ given in (29), $\lim_{(S,Z)\to(\hat{s},\hat{z})} \frac{\partial W(S,Z)}{\partial S} = 0$, and $\frac{\partial \bar{V}(Z)}{\partial S} = 0$; hence $\partial V(S,Z)/\partial S$ is continuous on $\big\{(S,Z) \mid Z = Z_S^A(S)\big\}$. Further, $\lim_{(S,Z)\to(\hat{s},\hat{z})} \frac{\partial W(S,Z)}{\partial Z} = \lim_{(S,Z)\to(\hat{s},\hat{z})} \frac{\partial \bar{V}(Z)}{\partial Z}$ since $(D - F^2/B) = \bar{D}$ by (44), and thus $\partial V(S,Z)/\partial Z$ is continuous on $Z = Z_S^A(S)$, and $q^*(S,Z)$ is continuous on $Z = Z_S^A(S)$, which can also be shown by using the solutions given in (18) to (22).

By using Lemma 1, $dZ_S^A(S)/dS < 0$ in all cases with $S_2 > 0$. If $S_2 < S_y$, then $(S_y, Z) \in R_A$ $\forall Z \in \big[0, Z_0^A\big)$. For $(S,Z) \in R_A$ we have $V(S,Z) = \bar{V}(Z)$, which does not depend on S; hence the point $S = S_y$ where $f'(S)$ is not continuous does not affect the continuity of $V(S,Z)$. Therefore, the function $V(S,Z)$ is continuous and continuously differentiable in S and Z, satisfies the HJB equation (5) and condition (7) with $q^*(S,Z) > 0$ $\forall(S,Z) \in R_S \cup R_A$, and the strategies $q_i = q^*$ $\forall i$ constitute a symmetric feedback-Nash equilibrium.

- *Stationary points and stability:*

In the following, we derive the set of points such that $\dot{S}(t) = 0$ and $\dot{Z}(t) = 0$, respectively, and then obtain the stationary points and study their stability.

We begin with $(S,Z) \in R_S$. The locus $\dot{S}(t) = 0$ is given by $\delta S = n(a + c_0 + c_S S + c_Z Z)/b(n+1)$. By solving for Z, we obtain the following linear function in S:

$$Z = \overset{0}{S}_S(S) = S\left(\frac{b\delta(n+1)}{nc_Z} - \frac{c_S}{c_Z}\right) - \frac{a + c_0}{c_Z}. \tag{45}$$

The locus $\dot{Z}(t) = 0$ is given by $kZ = \alpha n(a + c_0 + c_S S + c_Z Z)/b(n+1)$ and defined by the following linear function in S:

$$Z = \overset{0}{Z}_S(S) = \frac{\alpha n}{bk(1+n) - \alpha n c_Z}(a + c_0 + c_S S). \tag{46}$$

The point at which the two loci intersect, i.e., $(S,Z) \in R_S$ such that $\dot{S}(t) = 0$ and $\dot{Z}(t) = 0$, is denoted by $\xi_S = \{\ (s_S^\infty, z_S^\infty) \mid z_S^\infty = \overset{0}{S}_S(s_S^\infty) = \overset{0}{Z}_S(s_S^\infty)\ \}$ and written as follows:

$$\begin{aligned} s_S^\infty &= \frac{kn\,(a+c_0)}{b\delta k(n+1)-n\,(\alpha\delta c_Z+kc_S)}, \\ &= \frac{2ak\left(-b\delta\left(2k\left(n^2+1\right)+\left(n^2+3\right)r\right)+2b\left(n^2+1\right)r(k+r)+\delta\lambda(n-1)\right)}{\delta(b(n+1)r+\lambda)\left(b\delta(2k(n-1)+r(n+1))-b(n+1)r^2-\lambda(\delta-r)\right)}, \end{aligned} \tag{47}$$

$$\begin{aligned} z_S^\infty &= -\frac{\alpha\delta n\,(a+c_0)}{n\,(\alpha\delta c_Z+kc_S)-b\delta k(n+1)}, \\ &= \frac{2a\alpha\left(b\delta\left(2k\left(n^2+1\right)+\left(n^2+3\right)r\right)-2b\left(n^2+1\right)r(k+r)-\delta\lambda(n-1)\right)}{(b(n+1)r+\lambda)\left(-b\delta(2k(n-1)+r(n+1))+b(n+1)r^2+\lambda(\delta-r)\right)}. \end{aligned} \tag{48}$$

We analyze the stability of the equilibrium strategies in (8) by studying the signs of the determinants and traces of the Jacobian matrix associated to the differential equation system in the neighborhood of the stationary points (Takayama (1993)); see Jun and Vives (2004) for a similar analysis. The Jacobian matrix associated to the point ξ_S is given by

$$J_S = \begin{pmatrix} \delta - \frac{nc_S}{b(n+1)} & -\frac{nc_Z}{b(n+1)} \\ \frac{n\alpha c_S}{b(n+1)} & \frac{n\alpha c_Z}{b(n+1)} - k \end{pmatrix}. \tag{49}$$

Its determinant and trace are written as follows:

$$\begin{aligned} det(J_S) &= \frac{n(kc_S+\delta\alpha c_Z)}{b(1+n)} - k\delta, \\ &= \frac{(\delta-r)(\lambda-b(n+1)r)-2b\delta k(n-1)}{4bn}, \end{aligned} \tag{50}$$

$$\begin{aligned} tr(J_S) &= \delta - k - \frac{n\,(c_S-\alpha c_Z)}{b(1+n)}, \\ &= -\frac{\lambda+b(4\delta+2k(n-1)-3(1+n)r)}{4bn}. \end{aligned} \tag{51}$$

It can be verified that $det(J_s) > 0$ and $tr(J_S) < 0$ if $\delta > r(1+n)/2$. This is a weaker condition than the first part of Assumption 1; thus, ξ_S is a stable stationary point if Assumption 1 is satisfied.

In the step above, we eliminate the possibility of using Solution A' for $\delta > r$, since replacing λ with $-\lambda$ in (50) leads to $det(J_S) < 0$ where the point ξ_S is not stable. Thus, combining with the previous results in (26), (39), and (44), the only choices that can make it possible to characterize a stable equilibrium are Solutions A and B.

We now turn to $(S,Z) \in R_A$. The locus $\dot{S}(t) = 0$ is given by the following piecewise-linear function in S:

$$Z = \overset{0}{S}_{A1}(S) = S\frac{b\delta(n+1)}{n\bar{c}_Z} - \frac{a+\bar{c}_0}{\bar{c}_Z}, \quad S \le S_y, \tag{52}$$

$$Z = \overset{0}{S}_{A2}(S) = (S_{max}-S)\frac{b\delta(n+1)}{n\bar{c}_Z} - \frac{a+\bar{c}_0}{\bar{c}_Z}, \quad S \ge S_y. \tag{53}$$

Since $\bar{c}_Z < 0$, we have $d\overset{0}{S}_{A1}(S)/dS < 0$, and $d\overset{0}{S}_{A2}(S)/dS > 0$. Thus, for $(S,Z) \in R_A$, the minimum level of Z such that $\dot{S}(t) = 0$ occurs at $Z = \overset{0}{S}_{A1}(S_y) = \overset{0}{S}_{A2}(S_y)$. The locus $\dot{Z}(t) = 0$ is given by $kZ = \alpha n(a+\bar{c}_0+\bar{c}_Z Z)/b(n+1)$, which results in the following constant:

$$\begin{aligned} Z = \overset{0}{Z}_A &= \frac{\alpha n\,(a+\bar{c}_0)}{bk(n+1)-\alpha n\bar{c}_Z}, \\ &= \frac{2a\alpha\left(b\left(2k\left(n^2+1\right)+\left(3n^2+1\right)r\right)+\lambda(n-1)\right)}{(b(n+1)r+\lambda)(b(2k(n-1)-(n+1)r)+\lambda)} > 0. \end{aligned} \tag{54}$$

Then, in order to have $\dot{S}(t) = 0$ and $\dot{Z}(t) = 0$, $\overset{0}{S}_{A1}(S_y) = \overset{0}{S}_{A2}(S_y) < \overset{0}{Z}_A$ must be satisfied:

$$\frac{S_{max}}{2}\frac{b\delta(n+1)}{n\bar{c}_Z} - \frac{a+\bar{c}_0}{\bar{c}_Z} < \overset{0}{Z}_A. \tag{55}$$

When (55) is true, the intersection points denoted by $\xi_{A1} = \{ (s_{A1}^{\infty}, z_{A1}^{\infty}) \mid z_{A1}^{\infty} = \overset{0}{S}_{A1}(s_{A1}^{\infty}) = \overset{0}{Z}_A \}$ and $\xi_{A2} = \{ (s_{A2}^{\infty}, z_{A2}^{\infty}) \mid z_{A2}^{\infty} = \overset{0}{S}_{A2}(s_{A2}^{\infty}) = \overset{0}{Z}_A \}$ are given by

$$\begin{aligned} s_{A1}^{\infty} &= \frac{kn(a+\bar{c}_0)}{\delta(bk(n+1) - n\alpha\bar{c}_Z)}, \\ &= \frac{2ak\left(b\left(2k\left(n^2+1\right)+3n^2r+r\right)+\lambda(n-1)\right)}{\delta(b(n+1)r+\lambda)(2bk(n-1)-b(n+1)r+\lambda)}, \end{aligned} \tag{56}$$

$$\begin{aligned} s_{A2}^{\infty} &= S_{max} - \frac{kn(a+\bar{c}_0)}{\delta(bk(n+1) - n\alpha\bar{c}_Z)}, \\ &= S_{max} - \frac{2ak\left(b\left(2k\left(n^2+1\right)+3n^2r+r\right)+\lambda(n-1)\right)}{\delta(b(n+1)r+\lambda)(2bk(n-1)-b(n+1)r+\lambda)}, \end{aligned} \tag{57}$$

$$z_{A1}^{\infty} = z_{A2}^{\infty} = \overset{0}{Z}_A, \tag{58}$$

where $\overset{0}{Z}_A$ is given in (54). To check stability, we obtain the Jacobian matrices, written as follows, that are associated to these stationary points:

$$J_{A1} = \begin{pmatrix} \delta & -\frac{n\bar{c}_Z}{b(n+1)} \\ 0 & \frac{n\alpha\bar{c}_Z}{b(n+1)} - k \end{pmatrix}; \quad J_{A2} = \begin{pmatrix} -\delta & -\frac{n\bar{c}_Z}{b(n+1)} \\ 0 & \frac{n\alpha\bar{c}_Z}{b(n+1)} - k \end{pmatrix}. \tag{59}$$

For the first matrix we have

$$\begin{aligned} det(J_{A1}) &= \frac{\alpha\delta n\bar{c}_Z}{b(n+1)} - \delta k, \\ &= -\frac{\delta(2bk(n-1)+\lambda-b(n+1)r)}{4bn} < 0, \end{aligned} \tag{60}$$

$$\begin{aligned} tr(J_{A1}) &= \frac{\alpha n\bar{c}_Z}{b(n+1)} + \delta - k, \\ &= \delta - \frac{k(n-1)}{2n} - \frac{\lambda - b(n+1)r}{4bn}. \end{aligned} \tag{61}$$

Since $det(J_{A1}) < 0$, ξ_{A1} is not a stable stationary point. For the second one,

$$\begin{aligned} det(J_{A2}) &= \delta k - \frac{n\alpha\delta\bar{c}_Z}{b(n+1)}, \\ &= \frac{\delta(2bk(n-1)+\lambda-b(n+1)r)}{4bn} > 0, \end{aligned} \tag{62}$$

$$\begin{aligned} tr(J_{A2}) &= \frac{n\alpha\bar{c}_Z}{b(n+1)} - \delta - k, \\ &= -\left(\delta + \frac{k(n-1)}{2n} + \frac{\lambda - b(n+1)r}{4bn}\right) < 0, \end{aligned} \tag{63}$$

therefore, ξ_{A2} is a stable stationary point.

We can conclude that, under Assumptions 1 and 2, the long-run stationary states are either unique $\{\xi_S\}$ or multiple $\{\xi_S, \xi_{A2}\}$ depending on condition (55). In both cases, $q^*(t)$ converges to a stationary state with $q^*(S(t), Z(t)) > 0$, $\forall (S(0), Z(0)) \in R_S \cup R_A$, thus the equilibrium is asymptotically stable. □

Hence, we characterized the equilibrium for the case where $q^*(S, Z) > 0$, and showed that it is asymptotically stable. For global stability, the strategies are required to be defined for the whole state space; hence we study the case $q^*(S, Z) = 0$ in the next section.

A.3 Case with no exploitation ($q^*(S,Z) = 0$)

By combining the results in (28) and (41), we obtain the set of points such that $q^*(S,Z) = 0$ (R_0) given by

$$R_0 = \left\{(S,Z) \mid \left(S \leq -\frac{a+c_0}{c_S} - \frac{c_Z}{c_S}Z\right) \text{ and } \left(Z \geq Z_0^A\right)\right\}. \tag{64}$$

The value function for this region, denoted by $V_0(S,Z)$, must satisfy the HJB equation in (5) with $q_i = 0$ for $i \in \{1,..,n\}$, i.e.,

$$rV_0(S,Z) = -\frac{\phi}{2}Z^2 + \delta S\frac{\partial V_0(S,Z)}{\partial S} - kZ\frac{\partial V_0(S,Z)}{\partial Z}. \tag{65}$$

The above equation is a first-order linear partial differential equation (PDE), and $V_0(S,Z)$ must be continuous and continuously differentiable in R_0 and on its boundary cases. In order to obtain this function, we begin by using the model's deterministic feature to derive the boundary case associated to a given point in R_0.

Under 1 and 2, for $(S(0), Z(0)) \in R_0$, $\exists \hat{t} \geq 0$ such that $Z(\hat{t}) = Z_0(S(\hat{t}))$. This point is denoted by $(\hat{s}, \hat{z}) = (S(\hat{t}), Z(\hat{t}))$ and found by solving the following system of equations:

$$\hat{z} = Z(\hat{t}) = Ze^{-k\hat{t}},\ \hat{s} = S(\hat{t}) = Se^{\delta\hat{t}}, \tag{66}$$
$$\text{such that } Z(\hat{t}) = Z_0(S(\hat{t})) \text{ and } \hat{t} \geq 0, \tag{67}$$

where (66) is found by using $q_i = 0\ \forall i$, which implies $\dot{S}(t) = \delta S(t)$ and $\dot{Z}(t) = -kZ(t)$. By using (64), there are three cases in which equation (67) holds true:

(i) $\hat{z} = Z_0^S(\hat{s})$, then the point $(\hat{s}, \hat{z})$ is given by the following system of equations:

$$\hat{S}(S,Z) = S\left(\frac{Z}{\hat{Z}(S,Z)}\right)^{\frac{\delta}{k}}, \quad \hat{Z}(S,Z) = -\frac{c_S}{c_Z}\hat{S}(S,Z) - \frac{a+c_0}{c_Z}. \tag{68}$$

(ii) $\hat{z} = Z_0^A$ with $S \geq S_3$, the point $(\hat{s}, \hat{z})$ is given by

$$\hat{S}(S,Z) = S\left(\frac{Z}{Z_0^A}\right)^{\frac{\delta}{k}} \geq S_3, \quad \hat{Z}(S,Z) = Z_0^A. \tag{69}$$

(iii) $Z(0) = 0$, then the point $(\hat{s}, \hat{z})$ is given by

$$\hat{S}(S,0) = -\frac{a+c_0}{c_S}, \quad \hat{Z}(S,0) = 0 \text{ for } S \leq -\frac{a+c_0}{c_S}. \tag{70}$$

The system in (68) does not have an analytical solution, and the pair of equations are written as implicit functions denoted by $\hat{S}(S,Z)$ and $\hat{Z}(S,Z)$. For a fixed point on the boundary $z_0 = Z_0^S(s_0)$, $\lim_{(S,Z)\to(s_0,z_0)}(\hat{S}(S,Z), \hat{Z}(S,Z)) = (s_0, z_0)$, which will be used later. The functions $\hat{S}(S,Z)$ and $\hat{Z}(S,Z)$ have to be computed numerically except some special cases given below:

Remark 2 *There are special cases where (68) has an analytical solution:*

$$\textit{if } \delta = k \textit{ then } \begin{cases} \hat{S}(S,Z) = \left(\sqrt{(a+c_0)^2 - 4SZc_Zc_S} - (a+c_0)\right)/(2c_S), \\ \hat{Z}(S,Z) = -\left(\sqrt{(a+c_0)^2 - 4SZc_Zc_S} + (a+c_0)\right)/(2c_Z), \end{cases} \tag{71}$$

$$\textit{if } \delta = r+k \textit{ then } \begin{cases} \hat{S}(S,Z) = -\frac{a+c_0}{c_S}, \\ \hat{Z}(S,Z) = Z\left(-\frac{Sc_S}{a+c_0}\right)^{k/\delta}, \end{cases} \tag{72}$$

Since $Z(t)$ is decreasing (by (66)), for $(S(0), Z(0)) \in R_0$ where $Z(0) < Z_3$, $\exists \hat{t} \geq 0$ such that $Z(\hat{t}) = \hat{Z}(S(0), Z(0)) = Z_0^S(\hat{S}(S(0), Z(0)))$. For $Z(0) \geq Z_3$, depending on the position of $(S(0), Z(0))$, $Z(\hat{t})$ may lie on $Z = Z_0^S(S)$ or $Z = Z_0^A$. In order to precisely determine the boundary associated to every point in

R_0, we first obtain the curve associated to the intersection of the boundary cases of R_S. The set of points $(S(0), Z(0)) \in R_0$ such that $\exists \hat{t} \geq 0$ where $(S(\hat{t}), Z(\hat{t})) = (S_3, Z_3)$ is given by the following curve:

$$Z = \Psi(S) = Z_3 \left(\frac{S_3}{S}\right)^{\frac{k}{\delta}}, \; S \leq S_3, \tag{73}$$

which is found by solving (66) with $(\hat{s}, \hat{z}) = (S_3, Z_3)$, and $d\Psi(S)/dS = -\frac{k}{\delta}\frac{S_3 Z_3}{S^2}\left(\frac{S_3}{S}\right)^{\frac{k}{\delta}-1} < 0$.

By using (73), we define the following regions in (S, Z):

$$R_0^W = \{(S, Z) \in R_0 \mid Z < Z_3 \;\text{ or }\; Z_3 \leq Z < \Psi(S)\}, \tag{74}$$

$$R_0^{\bar{V}} = \{(S, Z) \in R_0 \mid Z \geq Z_3 \;\text{ and }\; Z \geq \Psi(S)\}. \tag{75}$$

Consider a point $(S(0), Z(0)) = (s_0, z_0)$ such that $z_0 = \Psi(s_0)$, where $s_0 < S_3$ and $z_0 > Z_3$. It satisfies $(\hat{S}(s_0, z_0), \hat{Z}(s_0, z_0)) = (S_3, Z_3)$. Denote by $t = t_0$ such that $(S(t_0), Z(t_0)) = (S_3, Z_3)$. By using (66), we obtain $t_0 = \frac{1}{k}\log\left(\frac{z_0}{Z_3}\right) = \frac{1}{\delta}\log\left(\frac{S_3}{s_0}\right) > 0$. Then:

- for $(S(0), Z(0)) = (s', z_0) \in R_0^W$ where $s' < s_0$, we have $(S(t_0), Z(t_0)) = (S(t_0), Z_3)$ with $S(t_0) < S_3$; hence $(S(t_0), Z_3) \in R_0$, $(S(t_0), Z_3) \notin R_S$, and $(S(t_0), Z_3) \notin R_A$. Then $\hat{z} = Z_0^S(\hat{s})$, the point $(\hat{s}, \hat{z})$ is given by (68).
- for $(S(0), Z(0)) = (s'', z_0) \in R_0^{\bar{V}}$ where $s'' > s_0$, we have $(S(t_0), Z(t_0)) = (S(t_0), Z_3)$ with $S(t_0) > S_3$; hence the point $(\hat{s}, \hat{z})$ is given by (69).

To summarize, if $(S, Z) \in R_0^W$ then $\hat{Z}(S, Z) = Z_0^S(\hat{S}(S, Z))$, and if $(S, Z) \in R_0^{\bar{V}}$ then $\hat{Z}(S, Z) = Z_0^A$.

Since we obtained the boundary case associated to every point in R_0, by applying the method of characteristics (see Melikyan (2012)), we obtain the solution to (65) given in the following lemma:

Lemma 3 *The value function in R_0 is written as follows:*

$$V_0(S, Z) = \left(\frac{Z}{\hat{Z}(S, Z)}\right)^{-\frac{r}{k}} \Theta\left(S\left(\frac{Z}{\hat{Z}(S, Z)}\right)^{\frac{\delta}{k}}, \hat{Z}(S, Z)\right) - \frac{\phi Z^2}{2(2k+r)}\left(1 - \left(\frac{Z}{\hat{Z}(S, Z)}\right)^{-\frac{2k+r}{k}}\right), \tag{76}$$

where

$$\Theta(S, Z) = \begin{cases} W(S, Z) & \text{if} \quad (S, Z) \in R_0^W, \\ \bar{V}(Z) & \text{if} \quad (S, Z) \in R_0^{\bar{V}}. \end{cases} \tag{77}$$

The function $V_0(S, Z)$ is continuous and continuously differentiable in S and Z $\forall (S, Z) \in R_0$ and on the boundary cases of R_0.

Proof. We first consider equation (65) such that $V(S, \hat{z}) = \Theta(S, \hat{z})$ with a constant $\hat{z}$, where $\Theta(S, Z)$ is an analytical function of S and Z, which is a Cauchy problem. Suppose that a function u, which is parametrized by τ, is a solution to equation (65) such that $u(\tau) = u(S(\tau), Z(\tau)) = V(S(\tau), Z(\tau))$. Introduce the system of ordinary differential equations (ODEs), which are called the characteristic system for the PDE in (65):

$$\frac{du}{d\tau} = ru + \frac{\phi}{2}Z^2; \quad \frac{dS}{d\tau} = \delta S; \quad \frac{dZ}{d\tau} = -kZ \text{ such that,} \tag{78}$$

$$u(0) = \Theta(\hat{s}, \hat{z}); \quad S(0) = \hat{s}; \quad Z(0) = \hat{z}. \tag{79}$$

Solving the last two equations in (78) with their initial conditions results in

$$S(\tau) = \hat{s}e^{\delta\tau}, \quad Z(\tau) = \hat{z}e^{-k\tau}. \tag{80}$$

By solving for e^τ and $\hat{s}$, we obtain $\tau(S, Z)$ and $\hat{S}(S, Z)$ that satisfy (80) :

$$e^\tau = \left(\frac{Z(\tau)}{\hat{z}}\right)^{-\frac{1}{k}}; \quad \hat{s} = S(\tau)e^{-\delta\tau} = S(\tau)\left(\frac{Z(\tau)}{\hat{z}}\right)^{\frac{\delta}{k}}. \tag{81}$$

We insert $Z(\tau)$ given in (80) into the first ODE in (78) and obtain the following equation:

$$\frac{du}{d\tau} = ru + \frac{\phi}{2}\hat{z}^2 e^{-2k\tau}. \tag{82}$$

Solving (82) with its initial condition in (79) yields

$$u(\tau) = e^{r\tau}\Theta(\hat{s},\hat{z}) + e^{r\tau}\frac{\phi\hat{z}^2}{2(2k+r)}\left(1 - e^{-(2k+r)\tau}\right). \tag{83}$$

Replacing (81) into (83) yields

$$u(S(\tau),Z(\tau)) = \left(\frac{Z(\tau)}{\hat{z}}\right)^{-\frac{r}{k}}\Theta\left(S(\tau)\left(\frac{Z(\tau)}{\hat{z}}\right)^{\frac{\delta}{k}},\hat{z}\right) + \frac{\phi Z(\tau)^2}{2(2k+r)}\left(\left(\frac{Z(\tau)}{\hat{z}}\right)^{-\frac{2k+r}{k}} - 1\right); \tag{84}$$

hence we obtained the solution to (65), which is verified for any analytical function $\Theta(S,Z)$. Further, in order to satisfy the boundary conditions in (66)–(67) $\forall(S,Z)\in R_0$, we replace $\hat{z} = \hat{Z}(S,Z)$ and obtain (76). We then write (65) by using (76), which simplifies to the following equation:

$$\left(\frac{2}{k\hat{Z}(.)}\left(\frac{Z}{\hat{Z}(.)}\right)^{-\frac{r}{k}}\right)\left(kZ\frac{\partial\hat{Z}(.)}{\partial Z} - \delta S\frac{\partial\hat{Z}(.)}{\partial S}\right)$$
$$\left(r\Theta(\hat{S}(.),\hat{Z}(.)) - \left(\delta\hat{S}(.)\frac{\partial\Theta(\hat{S}(.),\hat{Z}(.))}{\partial S} - k\hat{Z}(.)\frac{\partial\Theta(\hat{S}(.),\hat{Z}(.))}{\partial Z} - \frac{\phi}{2}\hat{Z}(.)^2\right)\right) = 0. \tag{85}$$

By the analysis in (74) and (75), for $(S,Z)\in R_0^W$ where $\Theta(S,Z) = W(S,Z)$, we have $\hat{Z}(S,Z) = Z_0^S(\hat{S}(S,Z))$. For $\hat{z} = Z_0^S(\hat{s})$, $W(\hat{s},\hat{z})$ satisfies the HJB equation (5) with $q^*(\hat{s},\hat{z}) = 0$, i.e.,

$$rW(\hat{s},\hat{z}) = \delta\hat{s}\frac{\partial W(\hat{s},\hat{z})}{\partial S} - k\hat{z}\frac{\partial W(\hat{s},\hat{z})}{\partial Z} - \frac{\phi}{2}\hat{z}^2, \tag{86}$$

and for $(S,Z)\in R_0^{\bar{V}}$ where $\Theta(S,Z) = \bar{V}(Z)$, we have $\hat{Z}(S,Z) = Z_0^A$. When $\hat{z} = Z_0^A$, $\bar{V}(\hat{z})$ satisfies (5) with $q^*(\hat{s},\hat{z}) = 0$,

$$r\bar{V}(\hat{z}) = -k\hat{z}\frac{\partial\bar{V}(\hat{z})}{\partial Z} - \frac{\phi}{2}\hat{z}^2. \tag{87}$$

For both cases, the third term in (85) is zero and the equation holds true; hence the function $V_0(S,Z)$ given in (76) satisfies the HJB equation in (5) with $q_i^* = 0$ $\forall i$ and $\forall(S,Z)\in R_0$.

For the partial derivatives of $V_0(S,Z)$ with respect to S and Z, by collecting the terms multiplied with $\partial\hat{Z}(S,Z)/\partial S$ and $\partial\hat{Z}(S,Z)/\partial Z$, using (86) and (87) allows us to simplify to the following:

$$\frac{\partial V_0(S,Z)}{\partial S} = \left(\frac{Z}{\hat{Z}(.)}\right)^{\frac{\delta-r}{k}}\frac{\partial\Theta(\hat{S}(.),\hat{Z}(.))}{\partial S}, \tag{88}$$

$$\frac{\partial V_0(S,Z)}{\partial Z} = \left(\frac{Z}{\hat{Z}(.)}\right)^{-\frac{k+r}{k}}\frac{\partial\Theta(\hat{S}(.),\hat{Z}(.))}{\partial Z} - \frac{\phi Z}{2k+r}\left(1 - \left(\frac{Z}{\hat{Z}(.)}\right)^{-\frac{2k+r}{k}}\right). \tag{89}$$

The function $V_0(S,Z)$ is continuous and continuously differentiable in S and Z $\forall(S,Z)\in R_0^W$, $\forall(S,Z)\in R_0^{\bar{V}}$, and $\forall(S,Z) \mid Z = \Psi(S)$ since on this curve $\hat{Z}(S,Z) = Z_3$, and $W(S_3,Z_3) = \bar{V}(Z_3)$.

- *Continuity of $V(S,Z)$ on the boundary cases of R_0:*

There are two cases: (a) $(S,Z)=(S,Z_0^S(S))$, (b) $(S,Z)=(S,Z_0^A)$ with $S \geq S_3$.

(a) $(S,Z)=(S,Z_0^S(S))$
On this boundary case, $\hat{z}=Z_0^S(\hat{s})$, and we have $\lim_{(S,Z)\to(\hat{s},\hat{z})}(\hat{S}(\hat{s},\hat{z}),\hat{Z}(\hat{s},\hat{z}))=(\hat{s},\hat{z})$,

$$\lim_{(S,Z)\to(\hat{s},\hat{z})} V_0(S,Z)=W(\hat{s},\hat{z}), \tag{90}$$

and thus $V(S,Z)$ is continuous on $Z=Z_0^S(S)$. For continuity of its partial derivatives, by using (88) and (89), we obtain

$$\lim_{(S,Z)\to(\hat{s},\hat{z})}\frac{\partial V_0(S,Z)}{\partial S}=\frac{\partial W(\hat{s},\hat{z})}{\partial S}, \quad \lim_{(S,Z)\to(\hat{s},\hat{z})}\frac{\partial V_0(S,Z)}{\partial Z}=\frac{\partial W(\hat{s},\hat{z})}{\partial Z}; \tag{91}$$

hence, $V(S,Z)$ is continuous and continuously differentiable in S and Z on $Z=Z_0^S(S)$.

(b) $(S,Z)\in(S,Z_0^A)$ with $S\geq S_3$:
On this boundary case we have $\lim_{(S,Z)\to(\hat{s},\hat{z})}(\hat{S}(\hat{s},\hat{z}),\hat{Z}(\hat{s},\hat{z}))=(\hat{s},Z_0^A)$ with $\hat{s}\geq S_3$. For continuity, we have

$$\lim_{(S,Z)\to(\hat{s},\hat{z})} V_0(S,Z)=\bar{V}(\hat{z}), \tag{92}$$

and thus $V(S,Z)$ is continuous on $Z=Z_0^A$. For continuity of its partial derivatives, by using (89) we have,

$$\lim_{(S,Z)\to(\hat{s},\hat{z})}\frac{\partial V_0(S,Z)}{\partial S}=\frac{\partial \bar{V}(\hat{z})}{\partial S}=0, \quad \lim_{(S,Z)\to(\hat{s},\hat{z})}\frac{\partial V_0(S,Z)}{\partial Z}=\frac{\partial \bar{V}(\hat{z})}{\partial Z}, \tag{93}$$

and thus, $V(S,Z)$ is continuous and continuously differentiable on Z_0^A with $S\geq S_3$.
Therefore, the function $V_0(S,Z)$ is continuous and continuously differentiable in S and Z in R_0 and both its boundary cases.

□

Consequently, by using Lemmas 1–3, under Assumptions 1 and 2, the function $V(S,Z)$ given in (9) is continuous and continuously differentiable in S and Z and satisfies the HJB equation in (5) $\forall (S,Z)\in\mathbb{R}_+^2$. The strategy profile $q_i(t)=q^*(S(t),Z(t))$ $\forall i$ given in (8) satisfies condition (7) for all $t\in[0,+\infty)$ and converges to a stationary state with $q^*(S(t),Z(t))>0$. Therefore, $q_i=q^*$ $\forall i$ is a symmetric feedback-Nash equilibrium that is globally asymptotically stable, which proves the statement of Theorem 1.

References

[1] Benchekroun, H. (2003). Unilateral production restrictions in a dynamic duopoly. Journal of Economic Theory, 111(2), 214–239.

[2] Benchekroun, H. (2008). Comparative dynamics in a productive asset oligopoly. Journal of Economic Theory, 138(1), 237–261.

[3] Benchekroun, H., and Gaudet, G. (2015). On the effects of mergers on equilibrium outcomes in a common property renewable asset oligopoly. Journal of Economic Dynamics and Control, 52, 209–223.

[4] Benchekroun, H., and Long, V.N. (1998). Efficiency inducing taxation for polluting oligopolists. Journal of Public Economics, 70(2), 325–342.

[5] Benchekroun, H., and Long, V.N. (2016). Status concern and the exploitation of common pool renewable resources. Ecological Economics, 125, 70–82.

[6] Benhabib, J., and Radner, R. (1992). The joint exploitation of a productive asset: a game-theoretic approach. Economic Theory, 2(2), 155–190.

[7] Botsford, L. W., Castilla, J. C., and Peterson, C. H. (1997). The management of fisheries and marine ecosystems. Science, 277(5325), 509–515.

[8] Clark, C. W. (1990). Mathematical Bioeconomics: The Optimal Management of Renewable Resources, Wiley, New York.

[9] Colombo, L., and Labrecciosa, P. (2013a). On the convergence to the Cournot equilibrium in a productive asset oligopoly. Journal of Mathematical Economics, 49(6), 441–445.

[10] Colombo, L., and Labrecciosa, P. (2013b). Oligopoly exploitation of a private property productive asset. Journal of Economic Dynamics and Control, 37(4), 838–853.

[11] Dockner, E. J., and Long, V. N. (1993). International pollution control: cooperative versus noncooperative strategies. Journal of Environmental Economics and Management, 25(1), 13–29.

[12] Dockner, E. J., Jørgensen, S., Long, V. N., and Sorger, G. (2000). Differential games in economics and management science. Cambridge University Press.

[13] Dockner, E. J., and Sorger, G. (1996). Existence and properties of equilibria for a dynamic game on productive assets. Journal of Economic Theory, 71(1), 209–227.

[14] Eriksen, M., Lebreton, L. C., Carson, H. S., Thiel, M., Moore, C. J., Borerro, J. C., ... and Reisser, J. (2014). Plastic pollution in the world's oceans: more than 5 trillion plastic pieces weighing over 250,000 tons afloat at sea. PloS one, 9(12).

[15] Fujiwara, K. (2008). Duopoly can be more anti-competitive than monopoly. Economics Letters, 101(3), 217–219.

[16] Fujiwara, K. (2011). Losses from competition in a dynamic game model of a renewable resource oligopoly. Resource and Energy Economics, 33(1), 1–11.

[17] Germain, M., Toint, P., Tulkens, H., and de Zeeuw, A. (2003). Transfers to sustain dynamic core-theoretic cooperation in international stock pollutant control. Journal of Economic Dynamics and Control, 28(1), 79–99.

[18] Grilli, L., and Bisceglia, M. (2017). A Duopoly with Common Renewable Resource and Incentives. International Game Theory Review, 19(04).

[19] Haurie, A., Krawczyk, J. B., and Zaccour, G. (2012). Games and dynamic games. World Scientific Books.

[20] Jørgensen, S., Martín-Herrán, G., and Zaccour, G. (2010). Dynamic games in the economics and management of pollution. Environmental Modeling and Assessment, 15(6), 433–467.

[21] Jun, B., and Vives, X. (2004). Strategic incentives in dynamic duopoly. Journal of Economic Theory, 116(2), 249–281.

[22] Kossioris, G., Plexousakis, M., Xepapadeas, A., and de Zeeuw, A. (2011). On the optimal taxation of common-pool resources. Journal of Economic Dynamics and Control, 35(11), 1868–1879.

[23] Lambertini, L., and Mantovani, A. (2014). Feedback equilibria in a dynamic renewable resource oligopoly: pre-emption, voracity and exhaustion. Journal of Economic Dynamics and Control, 47, 115–122.

[24] Lambertini, L., and Mantovani, A. (2016). On the (in) stability of nonlinear feedback solutions in a dynamic duopoly with renewable resource exploitation. Economics Letters, 143, 9–12.

[25] Long, V. N. (2011). Dynamic games in the economics of natural resources: a survey. Dynamic Games and Applications, 1(1), 115–148.

[26] Melikyan, A. (2012). Generalized characteristics of first order PDEs: applications in optimal control and differential games. Springer Science and Business Media.

[27] Petrosjan, L., and Zaccour, G. (2003). Time-consistent Shapley value allocation of pollution cost reduction. Journal of Economic Dynamics and Control, 27(3), 381–398.

[28] Rubio, S. J., and Escriche, L. (2001). Strategic pigouvian taxation, stock externalities and polluting non-renewable resources. Journal of Public Economics, 79(2), 297–313.

[29] Ryan, R. W., Holland, D. S., and Herrera, G. E. (2014). Ecosystem externalities in fisheries. Marine Resource Economics, 29(1), 39–53.

[30] Tahvonen, O. (1991). On the dynamics of renewable resource harvesting and pollution control. Environmental and Resource Economics, 1(1), 97–117.

[31] Takayama, A. (1993). Analytical methods in economics. University of Michigan Press.

[32] Van der Ploeg, F., and de Zeeuw, A. J. (1992). International aspects of pollution control. Environmental and Resource Economics, 2(2), 117–139.

[33] Wirl, F. (1994). Pigouvian taxation of energy for flow and stock externalities and strategic, noncompetitive energy pricing. Journal of Environmental Economics and Management, 26(1), 1–18.

[34] Wirl, F. (2004). Sustainable growth, renewable resources and pollution: thresholds and cycles. Journal of Economic Dynamics and Control, 28(6), 1149–1157.

[35] Wirl, F. (2008). Tragedy of the Commons in a Stochastic Game of a Stock Externality. Journal of Public Economic Theory, 10(1), 99–124.

[36] Xepapadeas, A. (1995a). Induced technical change and international agreements under greenhouse warming. Resource and Energy Economics, 17(1), 1–23.

[37] Xepapadeas, A. (1995b). Managing the international commons: Resource use and pollution control. Environmental and Resource Economics, 5(4), 375–391.