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a one-time entertainment event**

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Optimal pricing and advertising policies for a one-time entertainment event

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Abstract: We consider the problem of pricing and advertising a one-time entertainment event. We assume that the organizers want to sell all available tickets. Three pricing policies are characterized and contrasted, namely, dynamic price (DP), constant price (CP) and two-market price (TMP). In this last scenario, the selling season is composed of a regular price period and a last-minute price period, with the switching date between the two markets being determined endogenously.

We show that the price is monotonically increasing over time in the DP scenario and that the last-minute price is larger than the regular price in the TMP scenario. In all three cases, advertising is non-increasing over time, which is a feature often encountered in finite-horizon dynamic optimization advertising models. Finally, we compute the textitcost of simplification, which is the difference in profits under dynamic pricing and constant pricing. Among other results, we obtain that this loss is independent of the market size and increasing in the number of available tickets.

Keywords: Entertainment, advertising, pricing, aapacity planning, optimal control problems

Résumé: On considère le problème de détermination de politiques optimales de publicité et de prix pour un spectacle. On suppose que les organisateurs désirent que tous les billets soient vendus pour ne pas froisser les artistes. On caractérise et on compare trois politiques de prix, à savoir, un prix dynamique qui varie continuellement dans le temps, un prix constant et une tarification avec deux prix constants (prix régulier et prix de dernière minute), la date de changement est déterminée d'une manière endogène.

On montre que le prix dynamique est croissant dans le temps et que le prix de dernière minute est plus élevé que le prix régulier. Dans les trois scénarios, les dépenses publicitaires ne sont pas croissantes dans le temps, un résultat typique dans le cas d'un modèle dynamique à horizon fini. Enfin, on calcule le coût de simplification de la tarification, qui est obtenu par la différence entre les profits dans le cas de prix dynamique et celui de prix constant. Entre autres, on montre que ce coût est indépendant de la taille du marché et qu'il est croissant dans le nombre de billets disponibles.

Mots clés: Divertissement, publicité, tarification, planification de capacité, commande optimale

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1 Introduction

The current research deals with the design of optimal pricing and advertising decisions for an entertainment event, e.g., a classical, pop or rock concert, an opera or a ballet, or a theatre performance. Our primary aim is to provide recommendations to the organizers who must decide what will be their advertising and ticket pricing policies. Assuming that the organizers have decided the duration of the selling season, three pricing scenarios are investigated:

Dynamic Pricing (DP): The price can be varied continuously throughout the selling season;

Constant Pricing (CP): The price is constant throughout the selling season;

Two-Market Pricing (TMP): Tickets are sold in two, non-overlapping markets, namely, *regular market* and *last-minute market*. In each of the two markets, the price is constant, and the switching date is endogenously determined.

DP is a policy in which the price of a product/service can be changed continuously, taking into account past and current demand and supply information. In many industries, including the entertainment industry, short-term capacity is fixed and dynamic pricing then can be used to balance supply and demand. Dynamic pricing policies have, as far as we know, not been applied to the sales of tickets for entertainment events. CP and TMP policies are those most often used in practice, reflecting a certain “price stickiness” (Courty (2000)). To illustrate, organizers of performances that are sold-out for long periods do not increase prices to exploit the high demand. Tickets prices for performances that do not sell very many tickets are not lowered to stimulate demand. The TMP policy has a long history in markets for perishable (seasonal) products where firms very often discount their prices towards the end of the season. The rationale of a markdown obviously is to sell as much as possible of a stock that will have little or no value by the end of the season. This feature also applies to ticket sales in the entertainment industry.

The objectives of our research are to:

1. Characterize the optimal pricing and advertising strategies in each of the three scenarios.
2. Assess and compare the profits in the three pricing scenarios.
3. Assess and compare consumer’s surplus in the different pricing strategies.

We note from the outset that as the constant pricing and the two-market pricing are constrained instances of the dynamic pricing optimization problem, it is clear that the organizers will realize the highest profit with DP, followed by TMP and CP. Comparing the profits under DP and CP is of interest as it allows to compute what we will call the *cost of simplicity (CoS)*. A fully dynamic pricing strategy may not be easy to implement for a number of reasons, e.g., administrative costs and the risk of frustrating some consumers. Therefore, our comparison gives a measure of the *opportunity loss* by not following an optimal (sophisticated) pricing strategy.

To construct the model of ticket sales we need to specify the characteristics of the event. The type of event we have in mind is unique, i.e., it cannot (will not) be duplicated. Many events are duplicated but there are also one-time events, often featuring top performers (pop, rock, or opera stars). It is plausible to assume that such an event has no close substitutes and we shall assume that the organizers can act as a monopolist. Demand for tickets is supposed to be deterministic.¹ Finally, we assume that the organizers wish that the event eventually is sold out, for the sake of their profits and to avoid disappointment of the performers. In a deterministic setting, this can always be accomplished.

The paper proceeds as follows. Section 2 provides a review of the relevant literature. The review is brief, from the simple reason that the literature in this area is sparse. Section 3 presents our model of ticket sales. Sections 4, 5, and 6 deal with dynamic pricing (DP), constant pricing (CP) and two-market pricing (TMP). Section 7 compares the three types of pricing and Section 8 concludes.

¹This assumption is a simplification that the current research shares with many other works in the area.

2 Literature review

The current research is related to work in management and marketing science, operations research, and revenue management dealing with perishable assets and pricing. This literature has been reviewed in Weatherford and Bodily (1992), Elmaghraby et al. (2002), Elmaghraby and Keskinocak (2003), Talluri and Van Ryzin (2004), and Philips (2005). This section provides a brief overview of literature to which the current research is related. We are not aware of research dealing with the problem of designing optimal advertising and pricing policies for ticket sales in the entertainment industry, given that the event organizer has a choice among alternative pricing policies.

Courty (2000) studied the entertainment industry and described the ticket pricing practices that are encountered in the industry. The paper also discussed whether pricing theory is consistent with what can be observed in real ticket markets. Talluri and Van Ryzin (2004, pp. 567-574) provide an illustrative overview of various practices in real life ticket sales for theaters and sporting events.

Gallego and Van Ryzin (1994) studied the problem of designing a pricing policy for a perishable product under stochastic demand. It was shown that the value function (i.e., optimal expected revenue) is increasing and concave in the initial stock (capacity) and in the duration of the selling season. Thus, more stock and/or time will increase expected revenue.² Moreover, at any instant of time the optimal price would be smaller if the initial inventory were larger. For a fixed initial inventory, the optimal price rises if the duration of the selling season increases. These findings are intuitive.

The authors also studied a problem in which price cannot be varied continuously over time. They wished to develop heuristics that lead to “stable” policies being easy to apply. For this purpose, a model with deterministic demand was formulated and its solution was used to construct a simple fixed-price heuristic that, when the expected sales volume is large, is nearly optimal in many cases. When the firm has “many” items to sell it disregards the possibility of running out of stock and ends up with unsold units. If the firm has “few” items in stock, it raises the price to a level such that all items are sold.³ The case of “few” items and a zero stock at the horizon date is what the current research also will address.

Feng and Gallego (1995) studied the problem of selling a fixed stock over a finite horizon. Demand is stochastic and depends on price. The manager knows the expected revenue at certain price levels and her problem is to determine optimally (i) the time to change the price from one predetermined level to another, also predetermined, level and (ii) the direction of the change (markup or markdown). The authors showed that it is optimal to decrease (increase) the price from its initial level as soon as remaining time goes below (above) a threshold that depends on the current inventory. In our TMP problem we do not assume that prices are predetermined; They will be optimally determined.

Smith and Achabal (1998) considered a deterministic optimal control problem in continuous time. This paper seems to be the one which comes closest to the current research. Since the demand function has some similarity with the one we shall choose, it will be presented briefly. The demand rate is $x(t)$ where t is real time, price is $p(t)$, and the current inventory is $I(t)$. The authors employed the following multiplicatively separable demand specification which has convenient properties:

$$x(t) = \kappa(t)y(I(t))e^{-\gamma p(t)},$$

in which $\kappa(t)$ is a seasonal component. Function $y(I)$ is specified in such a way that demand is decreased by low inventory levels but is unaffected by high inventory levels. The reason for the decrease is incomplete assortments, reduced merchandise selection, and the stock being insufficient to make an attractive in-store display of the product. The authors showed that an optimal policy is to adjust price such that demand is proportional to the seasonal component $\kappa(t)$ at any instant of time. They also identified the pricing policy that will sell all of the initial stock. This is what the current research also will do.

Elmaghraby et al. (2002) were concerned with the design of an optimal markdown pricing mechanism. Thus the price decrease over time follows a schedule that is known to the consumers in the market. Pricing

²The model does not include any costs.

³In both cases the solution recommends a constant price throughout.

policies are periodic in the sense that prices are updated at fixed time intervals. Given that the firm's inventory is limited, buyers who wish to purchase at the low price may face the possibility that they cannot be served. The authors' objective is to investigate the structure of optimal markdown mechanisms in the presence of strategic buyers. For complete as well as incomplete information settings the firm's profits resulting from a policy of markdown prices and a single price are compared.

Jørgensen et al. (2009) divided the selling season into two sub-periods, the regular market and the last-minute market. Advertising is done in the regular market only and prices in the two markets are constant. There may be either a markup or a markdown at the start of the last-minute market. In the regular market the demand rate depends on advertising, price, and cumulative demand. The latter is included to model demand learning (a diffusion effect) and the hypothesis here is that as the number of sold tickets increase, and potential customers learn about this, demand increases. The authors found that advertising should be decreased over time. The switching time between the two sub-periods initially is given but, in a simplified version of the model, the switching time is optimally determined. The current research will use the assumption that demand depends on the cumulative number of tickets sold and will determine an optimal switching time (whenever it exists).

3 A model of ticket sales

The model is constructed to represent ticket sales dynamics for a unique event featuring top performers. Considerable excess demand is often observed for such events but, since the number of tickets is fixed, backlogging is impossible. Customers are served on a first-come-first-served basis. It is an explicit objective of the organizers that the event will be sold out. (Given the type of event we have in mind, this may happen very early). The organizers use advertising to (i) create awareness of the event and (ii) to provide an incentive to participate.⁴

Our model has the following elements:

1. The number of tickets offered for sale is denoted $q > 0$ (for mathematical convenience, a real number) and corresponds to the capacity (typically, the number of seats) of the location where the event takes place. The number q is fixed.
2. The decision to have the event has been made and is irreversible. The cost of providing capacity (e.g., rental cost of the location and salaries to performers) then is sunk and will be disregarded. It seems plausible to assume that the variable costs of having the event are negligible compared to the sunk costs. Variable costs will be disregarded.
3. Time t is continuous and the event takes place at time $t = T > 0$. The date T is fixed and the planning period of the organizers then is the time interval $[0, T]$. Ticket sales start at time $t = 0$ where the organizers advertise the date T to the public. For an event like the one we have in mind, tickets are expensive and we assume that any customer who bought a ticket will show up.
4. In practice, some seats often are more expensive than others, students and senior citizens may obtain a discount when they buy a ticket, and seats are priced differently on different days of the week. Taking features like these into consideration would complicate the model considerably and we shall assume that at any instant of time the same price applies to all seats and all buyers.
5. In the case of a TMP strategy we suppose that during the initial selling period, potential attendees do not know - or do not care - whether there will be a last-minute sale
6. Our demand function will feature a simple kind of strategic behavior, in the sense that any potential customer observes the current inventory of tickets and base her purchase decision on this information. The hypothesis that customers are able to figure out how many tickets that are currently in stock is not unrealistic. It will apply in situations where tickets are sold online and the organizers' website - where tickets are sold - offers a plan over the location, showing which seats are still available by time t . This is not uncommon in, e.g., opera houses and theaters.
7. Other profits, e.g., from sales of food, drinks, or merchandise, are disregarded.

⁴Awareness and incentives to participate can also be created through word-of-mouth. Of increasing importance here is communication through social media.

Price and advertising efforts are the decision variables of the organizers. In the TMP problem, the organizers also decide the time at which they switch from one constant price to another.

To construct the model, let $p \geq 0$ denote the price of a ticket.⁵ In the DP model, price can have any value during the selling season and is required to be at least a piecewise continuous function of time, denoted $p(t)$. In the CP model we have

$$p = \bar{p} = \text{const.} \geq 0 \text{ for all } t,$$

and in the TMP model

$$p = \begin{cases} \bar{p}_1 = \text{const.} \geq 0, & \text{for } t \in [0, t_1), \\ \bar{p}_2 = \text{const.} \geq 0, & \text{for } t \in (t_1, T], \end{cases}$$

where \bar{p}_2 can be smaller, equal to, or larger than \bar{p}_1 .

Denote by $a(t) \geq 0$ the advertising rate (at least piecewise continuous). The cost of advertising, denoted $C(a)$, is a quadratic function such that $C(a) = ca^2/2$ where $c > 0$ is a constant. The use of quadratic advertising cost functions is quite common in the literature and reflects diminishing returns to scale.

To keep track of the utilization of capacity q , let the state variable $s(t)$ (a nonnegative real number) represent the inventory of tickets by time t . It holds that $s(0) = q$. It is an objective of the organizers that the constraint $s(T) = 0$ be satisfied. As already said, this may happen much earlier than time T . Most often, a unique event featuring top performers leads to a high demand for tickets such that tickets are sold out long before the date of the event. Indeed, some events are sold out in less than an hour after the start of tickets sales.⁶ Note that since $\dot{s}(t) \leq 0$ for all t , the path constraint $s(t) \geq 0$ is satisfied for all $t < T$.

Let $x(t)$ (a nonnegative real number) represent the demand rate. By definition, $x(t) = 0$ whenever $s(t) = 0$. The current inventory of tickets, $s(t)$, evolves according to the simple dynamics

$$\dot{s}(t) = -x(t), \quad s(0) = q, \tag{1}$$

and demand is affected by three factors: the price, the advertising rate, and the number of tickets still for sale ($s(t) > 0$). The evolution of demand is modelled as follows:

$$\text{DP:} \quad x(t) = \alpha - p(t) + ka(t) - \varphi s(t), \quad t \in [0, T],$$

$$\text{CP:} \quad x(t) = \alpha - p + ka(t) - \varphi s(t), \quad t \in [0, T],$$

$$\text{TMP:} \quad x(t) = \begin{cases} \alpha - p_1 + ka_1(t) - \varphi s(t), & t \in [0, t_1), \\ \alpha - p_2 + ka_2(t) - \varphi s(t), & t \in (t_1, T]. \end{cases}$$

where k, α and φ are time-invariant parameters. This assumption seems plausible in view of the duration of the period for which the organizers are planning. The parameter k measures the effectiveness of advertising in creating demand and φ reflects our assumption that potential attendees observe that available seats become scarcer which induces them to purchase now.⁷ Since $\varphi > 0$, demand is positively influenced as customers see the inventory of tickets decrease over time.

We shall assume

$$\alpha > \varphi q,$$

which means that demand is positive if the firm gives all the tickets away for free at the initial time and does not advertise the event. The assumption can be satisfied if α (the maximal demand for tickets) is sufficiently large compared to the number of tickets available for sale.

⁵If price p is zero, tickets are given away for free. This is seen, probably quite often, in reality.

⁶Many other types of events are not sold out, even if substantial discounts are offered at the end of the selling season. Courty (2000) noted that for a particular Broadway show, 12 out of 199 performances were sold out.

⁷An effect that works in the opposite direction is the ‘‘inventory-depletion’’ effect; see, for instance, Talluri and Van Ryzin (2004) in which demand decreases as the inventory level decreases. A similar effect has been noted in retail sales, typically supermarkets. The idea here is that a large displayed stock can somehow induce consumers to buy.

Discounting of future profits is omitted in view of the duration of the planning period. The objective functional, to be maximized by the organizers, is defined as follows:

$$\begin{aligned} \text{DP} : J(p, a) &= \int_0^T \left(p(t)x(t) - \frac{c}{2}a^2(t) \right) dt, \\ \text{CP} : J(p, a) &= \int_0^T \left(\bar{p}x(t) - \frac{c}{2}a^2(t) \right) dt, \\ \text{TMP} : J(p_1, p_2, a) &= \int_0^{t_1} p_1x(t)dt - \int_0^{t_1} \frac{c}{2}a_1^2(t) dt \\ &+ \int_{t_1}^T p_2x(t)dt - \int_{t_1}^T \frac{c}{2}a_2^2(t) dt. \end{aligned}$$

The optimization problem includes the following constraints: The inventory dynamics and its initial condition

$$\dot{s}(t) = -x(t) \text{ for } t \in [0, T]; \quad s(0) = q,$$

and nonnegativity of prices and advertising rates

$$\begin{aligned} p(t) \geq 0, \quad p \geq 0, \quad p_1 \geq 0, \quad p_2 \geq 0, \\ a(t) \geq 0, \quad a_1(t) \geq 0, \quad a_2(t) \geq 0. \end{aligned}$$

Since $\dot{s}(t)$ necessarily is nonpositive it follows that $x(t) \geq 0$ and $s(t) \geq 0$ for all t . Finally we have the terminal constraint $s(T) = 0$. Note that the constraint can be satisfied for some t smaller than T .

4 Dynamic pricing

In practice, prices of tickets for entertainment events most often do not change continuously over time. Therefore we shall see this scenario as a benchmark. If it were possible to adjust the price continuously, then a profit-maximizing firm would do so because this would increase its profits compared to any other solution in which constraints on price-changes are imposed. The following proposition characterizes the optimal solution. The optimal values are superscripted with D for 'dynamic' pricing.

Proposition 1 *In the dynamic pricing scenario, the optimal advertising and pricing policies are given by*

$$p^D(t) = \alpha + \frac{q(k^2 - 2c) + cq + cq\varphi(t - T)}{Tc}, \quad (2)$$

$$a^D(t) = \frac{kq}{Tc}. \quad (3)$$

The optimal demand rate and profit are

$$x^D(t) = \frac{q}{T}, \quad (4)$$

$$J^D = \frac{q(2q(k^2 - c) + Tc(2\alpha - q\varphi) - k^2q)}{2Tc}. \quad (5)$$

Proof. The Appendix contains the proofs of all propositions. □

The results in the proposition deserve some comments. The optimal price $p^D(t)$ increases over time, a policy that is often encountered in dynamic pricing and revenue management. The advertising rate is constant, i.e., the organizers should use the simple policy of even spending. Demand is constant over time which seems to be an effect of the demand function term φs that counterbalances the effect of the increasing price.

The stock $s(t)$ of remaining tickets decreases over time which has the implication that the shadow price of the stock $s(t)$ is non-decreasing. This is expected: as time passes, the value of getting an extra ticket to sell increases.

The proof of Proposition 1 assumes that an optimal solution is interior. This is in fact the case because it is easy to verify that the pairs $(a = 0, p > 0)$ and $(a = 0, p = 0)$ cannot be optimal. Hence it is suboptimal to refrain from advertising. What remains is the pair $(a > 0, p = 0)$, a policy in which the firm advertises and gives tickets away for free. Obviously, such a policy cannot be optimal because the organizers can do better by choosing $a = 0$.

5 Constant pricing

This section analyzes the situation in which the price of a ticket is constant. This is a non-discriminatory policy, in the sense that all buyers pay the same price no matter at what time they make their purchase. The policy is the one that traditionally has been used when selling tickets for entertainment events. The proposition below gives the optimal solution in which optimal values have superscript C for 'constant' pricing.

The profit functional is

$$J(p^C, a) = \int_0^T \left(p^C (\alpha - p + ka(t) - \varphi s(t)) - \frac{c}{2} a^2(t) \right) dt,$$

and the constraints are

$$\dot{s}(t) = -x(t), \quad s(0) = q, \quad s(T) = 0, \quad a(t) \geq 0, \quad p^C \geq 0.$$

Proposition 2 *Assuming an interior solution, the optimal constant price and the optimal advertising rate are given by*

$$p^C = \frac{2c(\alpha(1 - e^{-\varphi T}) - q\varphi) + q\varphi k^2(1 + e^{-\varphi T})}{2c(1 - e^{-\varphi T})}, \quad (6)$$

$$a^C(t) = \frac{k\varphi q e^{-\varphi t}}{c(1 - e^{-\varphi T})}. \quad (7)$$

The optimal demand rate and the profit are

$$x^C(t) = \frac{q\varphi(1 + e^{-\varphi T})(k^2 e^{-2\varphi t} + (2c - k^2)e^{-\varphi T})}{2c(e^{\varphi T} - e^{-\varphi T})e^{-\varphi(T+t)}}, \quad (8)$$

$$J^C = \frac{k^2 q^2 \varphi(1 + e^{-\varphi T}) + 4cq(\alpha(1 - e^{-\varphi T}) - q\varphi)}{4c(1 - e^{-\varphi T})}. \quad (9)$$

In contrast to the scenario where the price was time-variant and advertising effort constant over time, the price now is constant (by assumption) and the advertising rate $a^C(t)$ is strictly positive and decreases over time. Optimal advertising policies that recommend decreasing effort over time have been reported quite often in the literature, typically for the reason that the objective does not have a salvage value term at the horizon date.⁸ To have a positive price, parameters must satisfy

$$\text{Condition A : } \alpha > q\varphi \left(\frac{2c - k^2(1 + e^{-\varphi T})}{2c(1 - e^{-\varphi T})} \right), \quad (10)$$

that is, the market potential is "large" compared to the number of tickets available for sale. Recall that we earlier have assumed $\alpha > q\varphi$. Finally, the demand rate increases over time. To have positive demand for all t it suffices to verify that $x^C(0) > 0$ which is easily done.

6 Two-market pricing

This section analyzes the problem where the organizers have the option to create two markets. The price in each market is constant. We denote the switching time between the regular market and the last-minute

⁸See Jørgensen and Zaccour (2004).

market by $t_1 \in [0, T]$. If $t_1 = 0$ there is no regular market; if $t_1 = T$ there is no last-minute market. Notice a new feature of the organizers' optimization problem, the determination of an optimal switching position $(s(t_1), t_1) \triangleq (s_1, t_1)$.

It may be optimal to have one market only, that is, having the regular market for all $t \in [0, T]$ or starting the last-minute market at $t = 0$. Although these two situations formally are the same as the one in the constant-price scenario, there is an important difference. In the previous section, the organizers have decided in advance that one price only will be charged. If we get a one-price solution in the two-market problem, that is, $t_1 = 0$ or $t_1 = T$, it is because such a solution is optimal.

Remark 1 *The solution technique that we shall use to solve the two-stage optimal control problem was developed in Tomiyama (1985) and Amit (1986). Boucekkine et al. (2004, 2011) and Saglam (2011) applied it to problems of the adoption of new technology and pollution abatement. The technique relies on a dynamic programming argument and solves the problem backwards in time.*

The problem still has state $s(t)$ and control $a_i(t)$, $i = 1, 2$ but now there are three control parameters (real numbers) p_1, p_2 , and t_1 . These quantities must be determined so as to maximize the objective functional

$$\begin{aligned} J(a, p_1, p_2, t_1) &= J_1(a_1, p_1, t_1) + J_2(a_2, p_2, t_1), \\ &= \int_0^{t_1} \left(p_1 x(t) - \frac{c}{2} a_1^2(t) \right) dt + \int_{t_1}^T \left(p_2 x(t) - \frac{c}{2} a_2^2(t) \right) dt, \end{aligned}$$

subject to the state equations

$$\begin{aligned} \dot{s}(t) &= -x(t) = -\alpha + p_1 - ka_1(t) + \varphi s(t), \quad t \in [0, t_1]; \quad s(0) = q, \\ \dot{s}(t) &= -x(t) = -\alpha + p_2 - ka_2(t) + \varphi s(t), \quad t \in (t_1, T]; \quad s(T) = 0, \end{aligned}$$

and $p_i \geq 0, a_i(t) \geq 0, i \in \{1, 2\}$, for $t \in [0, T]$. The switching time t_1 must satisfy $0 \leq t_1 \leq T$. If it happens that $s_1 = q$ (no tickets were sold in the regular market), all tickets need to be sold in the last-minute market. If $s_1 = 0$, all tickets were sold in the regular market and there is nothing to sell in a last-minute market.

We briefly describe the procedure for finding an optimal switching position (s_1, t_1) . Let $\eta_1(t)$ and $\eta_2(t)$ be costate variables associated with the state in the regular and last minute markets, respectively. Let H_1^* and H_2^* denote the maximized Hamiltonians and note that the integrands of J_1 and J_2 as well as the right-hand sides of the dynamics are C^2 functions. A necessary condition for an interior optimal switching time $t_1^* \in (0, T)$ to exist is that the costate and the maximized Hamiltonian are continuous at $t = t_1^*$, that is,

$$\eta_1(t_1^*) = \eta_2(t_1^*), \quad (11)$$

$$H_1^*(s_1^*, t_1^*) = H_2^*(s_1^*, t_1^*). \quad (12)$$

If there is no interior solution, two corner solutions are candidates for optimality: No regular market ($t_1^* = 0$) if $H_1^*(q, 0) \leq H_2^*(q, 0)$, no last-minute market ($t_1^* = T$) if $H_1^*(0, T) \geq H_2^*(0, T)$. We shall see that the optimal payoff $J_2^*(s_1^*, t_1^*)$ is a C^2 function and hence it holds that

$$\frac{\partial J_2^*}{\partial s_1} = \eta_2(t_1^*); \quad \frac{\partial J_2^*}{\partial t_1} = H_2^*(s_1^*, t_1^*) \quad (13)$$

and the conditions in (11) and (12) are satisfied. The first equation in (13) states, as is well known, that the partial derivative of the value function with respect to state equals the costate. The second equation is the Hamilton-Jacobi-Bellman equation.

6.1 Last-minute market

Supposing that there was a switch from regular to last-minute market at time t_1 we have $t \in [t_1, T]$. This switching time is considered as fixed and then the state $s(t_1) = s_1$ also is fixed.⁹ Recall that $s(T) = 0$ is

⁹We assume $s_1 > 0$ since otherwise there is no problem to solve.

required. The optimization problem is

$$\max_{a_2 \geq 0, p_2 \geq 0} \left\{ J_2(a_2, p_2) = \int_{t_1}^T \left(p_2 x(t) - \frac{c}{2} a_2^2(t) \right) dt \right\}.$$

Proposition 3 *Assuming interior policies, the optimal price, advertising rate and profit in the last-minute market are given by*

$$p_2^* = \alpha - \varphi s_1 \left(\frac{1}{1 - e^{\varphi(t_1 - T)}} - \frac{k^2}{2c} \frac{1 + e^{-\varphi(T - t_1)}}{1 - e^{-\varphi(T - t_1)}} \right), \quad (14)$$

$$a_2^*(t) = \frac{k\varphi s_1}{c(e^{-\varphi t_1} - e^{-\varphi T})} e^{-\varphi t}, \quad (15)$$

$$J_2^*(s_1, t_1) = p_2^* s_1 + \frac{k^2 \varphi s_1^2 (e^{-2\varphi T} - e^{-2\varphi t_1})}{4c(e^{-\varphi t_1} - e^{-\varphi T})^2}. \quad (16)$$

A few comments on the results of Proposition 3 are in order. First, the optimal advertising rate is positive and decreases over time, which, as alluded to before, is a common feature in finite-horizon dynamic optimization problems with no salvage values. Second, for price to be positive, the following condition must be satisfied

$$\text{Condition B : } \alpha > \varphi s_1 \left(\frac{1}{(1 - e^{\varphi(t_1 - T)})} - \frac{k^2}{2c} \frac{1 + e^{\varphi(T - t_1)}}{1 - e^{\varphi(T - t_1)}} \right), \quad (17)$$

which, as in the constant-price scenario, requires that the market potential be sufficiently large. Moreover, the lower the number of remaining tickets to be sold in the last-minute market, the easier it is to satisfy the inequality. The time instant t_1 is a decision variable and Condition B can only be checked after having found t_1^* .

Finally, the term $p_2^* s_1$ in the optimal profit is the revenue gained by selling s_1 tickets. The term $(k^2 \varphi s_1^2 (e^{-2\varphi T} - e^{-2\varphi t_1})) / (4c(e^{-\varphi t_1} - e^{-\varphi T})^2)$ is negative and is the total advertising cost incurred in the last-minute market.

6.2 Regular market

Suppose that there is a regular market, starting at time zero. The optimization problem is

$$\max_{a_1, p_1, t_1} \left\{ \int_0^{t_1} \left[p_1 x(t) - \frac{c}{2} a_1^2(t) \right] dt + J_2^*(s_1, t_1) \right\},$$

subject to the usual constraints. The term $J_2^*(s_1, t_1)$, which plays the role of a salvage value, is given by (41). The initial inventory is given, $s(0) = q$, while the terminal inventory $s(t_1) = s_1$ is free.¹⁰

Proposition 4 *Assuming an interior solution, the optimal price, advertising rate and profit in the regular market are given by*

$$p_1^* = \alpha - \varphi \left(\frac{q - s_1}{1 - e^{\varphi t_1}} + s_1 \frac{4c - k^2(1 + e^{\varphi(t_1 - T)})}{2c(1 - e^{\varphi(t_1 - T)})} \right), \quad (18)$$

$$a_1^*(t) = \frac{k\varphi(q - s_1)}{c(e^{\varphi t_1} - 1)} e^{\varphi(t_1 - t)}, \quad (19)$$

$$J_1^*(s_1, t_1) = p_1^*(q - s_1) + \frac{k^2 \varphi (q - s_1)^2 (1 + e^{\varphi t_1})}{4c(1 - e^{\varphi t_1})}. \quad (20)$$

¹⁰We now have a problem with free-end point s_1 and free-terminal time t_1 .

As before, we obtain that the advertising rate is strictly positive and decreasing over time. Also, the larger the number of tickets that the organizers wish to sell in the regular market (i.e., $q - s_1$), the higher the advertising rate. We note that at time t_1 optimal advertising rates in the two markets are given by $a_1^*(t_1) = k(p_1 - \eta_1(t_1))/c$ and $a_2^*(t_1) = k(p_2 - \eta_2(t_1))/c$, respectively. Hence the optimal advertising trajectory is discontinuous at t_1 (unless prices are equal which is a hairline case).¹¹ The market with the highest price will have the highest advertising rate at t_1 .

Further, to have a positive price, and hence an interior solution, the following condition on the parameter values must be satisfied:

$$\text{Condition C : } \alpha > \varphi \left(\frac{q - s_1}{1 - e^{\varphi t_1}} + s_1 \frac{4c - k^2 (1 + e^{\varphi(t_1-T)})}{2c (1 - e^{\varphi(t_1-T)})} \right). \quad (21)$$

The maximized Hamiltonian can be derived from the transversality condition $H_1^*(s_1, t_1) + \partial J_2^*(s_1, t_1)/\partial t_1 = 0$ which becomes

$$(p_1^* - \eta_1) x^* - \frac{c}{2} (a^*)^2 + \frac{\partial}{\partial t_1} \left(p_2^* s_1 + \frac{k^2 \varphi s_1^2 (e^{-2\varphi T} - e^{-2\varphi t_1})}{4c (e^{-\varphi T} - e^{-\varphi t_1})^2} \right) = 0,$$

in which η_1, x^* , and a^* are evaluated at (s_1, t_1) . The derivative in the above equation is

$$\frac{\partial J_2^*(s_1, t_1)}{\partial t_1} = - \frac{k^2 \varphi^2 s_1^2 e^{-\varphi T} e^{-\varphi t_1}}{2c (e^{-\varphi T} - e^{-\varphi t_1})^2},$$

which shows, as expected, that optimal profits in the last-minute market decrease, the later the organizers switch to this market. Summarizing, we have

$$H_1^*(s_1, t_1) = - \frac{\partial J_2^*(s_1, t_1)}{\partial t_1} = \frac{k^2 \varphi^2 s_1^2 e^{-\varphi T} e^{-\varphi t_1}}{2c (e^{-\varphi T} - e^{-\varphi t_1})^2} > 0.$$

6.3 Optimal switching time

The maximized Hamiltonians are given by

$$H_2^*(s_1, t_1) = \alpha + \frac{2\varphi s_1}{e^{\varphi(t_1-T)} - 1} + \frac{k^2 \varphi s_1}{2c} \frac{e^{\varphi T} + e^{\varphi t_1}}{e^{\varphi T} - e^{\varphi t_1}}, \quad (22)$$

$$H_1^*(s_1, t_1) = \frac{k^2 \varphi^2 s_1^2 e^{-\varphi T} e^{-\varphi t_1}}{2c (e^{-\varphi T} - e^{-\varphi t_1})^2}. \quad (23)$$

To determine the optimal switching time, one considers the following possibilities:

$$\begin{aligned} \text{Regular market only} & : t_1^* = T \\ \text{Last-minute market only} & : t_1^* = 0 \\ \text{Both markets exist} & : t_1^* \in (0, T). \end{aligned}$$

The following proposition shows that both one-market solutions are suboptimal.

Proposition 5 *Choosing $t_1^* = T$ or $t_1^* = 0$ is not optimal.*

Hence, if there exists an optimal switching time, it must be interior switching time. Such a solution exists if the equation $H_1^*(s_1, t_1) = H_2^*(s_1, t_1)$ has a unique solution $t_1^* \in (0, T)$. It is easy to check that $H_1^*(s_1, t_1) - H_2^*(s_1, t_1)$ is a polynomial of degree 2 in t_1 . Solving $H_1^*(s_1, t_1) - H_2^*(s_1, t_1) = 0$ with respect to t_1 provides two candidates for an optimal interior switching time:

¹¹The discontinuity is not an issue as the advertising rate only needs to be piecewise continuous.

$$t_{1(1)} = T + \frac{1}{\varphi} \ln \frac{k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) + \varphi s_1 \Gamma}{2(2c\alpha - k^2 \varphi s_1)} \triangleq T + \frac{1}{\varphi} \ln \Omega,$$

$$t_{1(2)} = T + \frac{1}{\varphi} \ln \frac{k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) - \varphi s_1 \Gamma}{2(2c\alpha - k^2 \varphi s_1)} \triangleq T + \frac{1}{\varphi} \ln \Lambda,$$

where

$$\Gamma \triangleq \sqrt{4(2c - k^2)^2 + k^4 \varphi^2 s_1^2 + 8ck^2(\alpha - \varphi s_1)} > 0,$$

$$\Omega = \frac{k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) + \varphi s_1 \Gamma}{2(2c\alpha - k^2 \varphi s_1)},$$

$$\Lambda = \frac{k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) - \varphi s_1 \Gamma}{2(2c\alpha - k^2 \varphi s_1)}.$$

Under our assumption $\alpha > \varphi s(0) = \varphi q$ the numerator of Ω is positive. Therefore, the denominator must also be positive for $\ln \Omega$ to exist. A sufficient condition for the denominator to be positive is

$$\text{Condition D : } 2c - k^2 > 0, \quad (24)$$

which we shall assume is satisfied. The condition, which can be rewritten as

$$\frac{\partial^2 C(a)}{\partial a^2} \frac{\partial x}{\partial p} > \left(\frac{\partial x}{\partial a} \right)^2,$$

is rather mild. To start with, it is economically intuitive that $\left(\frac{\partial x}{\partial p} = 1 \right) > \left(\frac{\partial x}{\partial a} = k \right)$, i.e., the marginal impact of price on demand is larger than the marginal impact of advertising on demand. Therefore, Condition D “only” requires that $k = \frac{\partial x}{\partial a}$ to be less than $2c = \frac{\partial^2 C(a)}{\partial a^2}$, which is the rate of increase of the marginal cost of advertising. Put differently, Condition D is satisfied if advertising is ‘expensive’, price has a ‘significant’ impact on demand while advertising has a ‘less significant’ impact on demand.

Consider the root $t_{1(1)}$. This candidate can be excluded because $t_{1(1)} > T$ or, equivalently, $\Omega > 1$. Indeed,

$$\begin{aligned} \Omega &= \frac{k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) + \varphi s_1 \sqrt{4(2c - k^2)^2 + k^4 \varphi^2 s_1^2 + 8ck^2(\alpha - \varphi s_1)}}{2(2c\alpha - k^2 \varphi s_1)}, \\ &> \frac{k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) + \varphi s_1 \sqrt{4(2c - k^2)^2}}{2(2c\alpha - k^2 \varphi s_1)}, \\ &= \frac{k^2 \varphi^2 s_1^2 + 2(2c\alpha - k^2 \varphi s_1)}{2(2c\alpha - k^2 \varphi s_1)} > 1. \end{aligned}$$

For $t_{1(2)}$ to be interior, that is, $t_{1(2)} \in (0, T)$, we must have $0 < \Lambda < 1$ and $T + \frac{1}{\varphi} \ln \Lambda > 0$.

Lemma 1 *Under the assumption $\alpha > \varphi q$ and Condition D we have $0 < \Lambda < 1$.*

For $t_{1(2)}$ to be interior we still need to have $T + \frac{1}{\varphi} \ln \Lambda > 0$. Since Λ is independent of T , and in order to have a two-market problem, we assume that parameter values satisfy

$$\text{Condition E : } t_1^* = T + \frac{1}{\varphi} \ln \Lambda > 0.$$

The final expression of the optimal switching time then is

$$t_1^* = T + \frac{1}{\varphi} \ln \left(\frac{k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) - \varphi s_1 \sqrt{4(2c - k^2)^2 + k^4 \varphi^2 s_1^2 + 8ck^2(\alpha - \varphi s_1)}}{2(2c\alpha - k^2 \varphi s_1)} \right).$$

Not surprisingly, the above expression is too complicated to be amenable to a qualitative analysis. Still, two observations can be made. First, the longer the planning horizon, the later the date at which the last-minute market starts. Second, the switching date is independent of the initial number of tickets, q , while - as one would expect - it depends on the number of remaining tickets s_1 .

The only remaining point is the determination of the sales at the switching point, that is, $s(t_1^*) = s_1$. To determine s_1 , we solve for the dynamics

$$\dot{s}(t) = -\alpha + p_1 - ka_1(t) + \varphi s(t), \quad t \in [0, t_1]; \quad s(0) = q,$$

to get

$$s(t) = qe^{\varphi t} + \frac{(1 - e^{\varphi t})}{\varphi} (\alpha - p_1) + \frac{k^2 q (e^{-\varphi t} - e^{\varphi t})}{2c(1 - e^{-\varphi T})}.$$

In particular

$$s(t_1^*) = s_1 = qe^{\varphi t_1^*} + \frac{(1 - e^{\varphi t_1^*})}{\varphi} (\alpha - p_1^*) + \frac{k^2 q (e^{-\varphi t_1^*} - e^{\varphi t_1^*})}{2c(1 - e^{-\varphi T})}.$$

As t_1^* and p_1^* depend on s_1 , we obtain t_1^* as an implicit function which however does not seem to have an analytical solution. Therefore, we can determine s_1 only numerically.

Substituting for t_1^* in (14)–(16), and in (18)–(20), we obtain optimal price, advertising and profit in the last-minute market and regular market, respectively.

7 Evaluation of policies

After having characterized the optimal solutions in the three pricing strategies, which was our first objective, now we turn to our second and third objectives, namely, comparing dynamic pricing (DP), constant pricing (CP), and two-market pricing (TMP) scenarios, from the firm’s and consumer’s perspectives. Table 1 summarizes the results.

Table 1: Optimal advertising, prices and profits

	Advertising	price	Total Profit
DP	$\frac{kq}{Tc}$	$\alpha + \frac{q(k^2 - 2c) + cq + c\varphi(t - T)}{Tc}$	$\frac{q(2q(k^2 - c) + Tc(2\alpha - q\varphi) - k^2 q)}{2Tc}$
CP	$\frac{k\varphi q e^{-\varphi t}}{c(1 - e^{-T\varphi})}$	$\frac{2c(\alpha(1 - e^{-\varphi T}) - q\varphi) + q\varphi k^2(1 + e^{-\varphi T})}{2c(1 - e^{-\varphi T})}$	$\frac{k^2 q^2 \varphi(1 + e^{-\varphi T}) + 4cq(\alpha(1 - e^{-\varphi T}) - q\varphi)}{4c(1 - e^{-\varphi T})}$
TMP			
$0 \leq t \leq t_1^*$	$\frac{k\varphi(q - s_1)e^{\varphi(t_1^* - t)}}{c(e^{\varphi t_1^*} - 1)}$	$\alpha - \varphi \left(\frac{q - s_1}{1 - e^{\varphi t_1^*}} + s_1 \frac{4c - k^2(1 + e^{\varphi(t_1^* - T)})}{2c(1 - e^{\varphi(t_1^* - T)})} \right)$	$p_1^*(q - s_1) + \frac{k^2 \varphi (q - s_1)^2 (1 + e^{\varphi t_1^*})}{4c(1 - e^{\varphi t_1^*})}$
$t_1^* < t \leq T$	$\frac{k\varphi s_1 e^{-\varphi t}}{c(e^{-\varphi t_1^*} - e^{-\varphi T})}$	$\alpha - \varphi s_1 \left(\frac{1}{1 - e^{\varphi(t_1^* - T)}} - \frac{k^2}{2c} \frac{1 + e^{-\varphi(T - t_1^*)}}{1 - e^{-\varphi(T - t_1^*)}} \right)$	$p_2^* s_1 + \frac{k^2 \varphi s_1^2 (e^{-2\varphi T} - e^{-2\varphi t_1^*})}{4c(e^{-\varphi t_1^*} - e^{-\varphi T})^2}$

Propositions 6–10 characterize the advertising and pricing policies in the three scenarios.

Proposition 6 *Advertising is (i) constant in the dynamic-pricing scenario; (ii) monotonically decreasing over time in the constant-pricing; and (iii) monotonically decreasing during each time interval in the two-market pricing scenario.*

As already noted, advertising being non-increasing over time is a feature frequently encountered in finite-horizon optimization problems, typically in the absence of a salvage value which is the case in our problem. Note in the two-market pricing scenario that the decrease in advertising effort occurs in each market, but does not specify the shape of the overall advertising path. The next proposition completes the picture in this scenario.

Proposition 7 *At the switching date t_1^* in the TMP scenario we have*

$$a_2^*(t_1^*) - a_1^*(t_1^*) \text{ is } \begin{cases} < 0, & \text{for } t_1^* < \hat{t}, \\ = 0, & \text{for } t_1^* = \hat{t}, \\ > 0, & \text{for } t_1^* > \hat{t}, \end{cases}$$

where

$$\hat{t} = -\frac{1}{\varphi} \ln \frac{s_1(1 - e^{-T\varphi}) + qe^{-T\varphi}}{q} > 0.$$

Proposition 7 shows that there is a jump in advertising at t_1^* , unless $t_1^* = \hat{t}$ (which is a hairline case). The sign of the jump depends on the parameter values.

Proposition 8 *There exists an instant of time $\bar{t} \in (0, T)$ such that*

$$a^D(t) - a^C(t) \text{ is } \begin{cases} < 0, & \text{for } t < \bar{t}, \\ = 0, & \text{for } t = \bar{t}, \\ > 0, & \text{for } t > \bar{t}, \end{cases}$$

where

$$\bar{t} = -\frac{1}{\varphi} \ln \frac{1 - e^{-\varphi T}}{\varphi T}.$$

At any instant of time, the advertising rate is higher with DP than in CP.

Recalling that in the DP scenario the advertising policy is constant and is monotonically decreasing in the CP scenario, Proposition 8 shows that the DP advertising is “averaging” the advertising path with CP.

Proposition 9 *In the TMP scenario, the price in the last-minute market is higher than in the regular market.*

This result is a ‘simpler version’ of the result that was obtained in the DP scenario. The price jumps from a lower to a higher level, in contrast to the DP case where price increases continuously over time.

Proposition 10 *There exists an instant of time $\check{t} \in (0, T)$ such that*

$$p^D(t) - p^C \begin{cases} < 0, & \text{for } t < \check{t}, \\ = 0, & \text{for } t = \check{t}, \\ > 0, & \text{for } t > \check{t}, \end{cases}$$

where

$$\check{t} = \frac{2c(1 - e^{-T\varphi} - T\varphi e^{-T\varphi}) - 2k^2(1 - e^{-T\varphi}) + Tk^2\varphi(1 + e^{-T\varphi})}{2c\varphi(1 - e^{-T\varphi})}.$$

As for advertising, the above proposition shows that a constant price is ‘averaging’ the dynamic price $p^D(t)$ which starts out at a lower level than p^C and overtakes p^C at time \check{t} .

Now we turn to comparing profits and recall that the ordering of profits is as follows:

$$J^D > J^* > J^C.$$

It holds that $J^* > J^C$ because otherwise a constant price would have been the optimal solution of the TMP problem. (Recall that we proved that the boundary solutions for t_1 were suboptimal.) Similarly, $J^D > J^*$ because otherwise two constant prices would have been the optimal solution of the dynamic pricing problem.

The only issues that are pending issues are the determination of the differences between profits as well as an assessment of the impact that key parameters have on the differences in profits. As already said, the difference $J^D - J^C$ can be thought of as the cost of simplicity (*CoS*). Similarly, the difference $J^* - J^C$ can be seen as the benefit of having a last-minute market (*BLMM*).

Straightforward computations give

$$CoS = J^D - J^C = \frac{q^2 (2c - k^2)}{4Tc(1 - e^{-T\varphi})} (e^{-T\varphi} (2 + T\varphi) - (2 - T\varphi)),$$

and it is easy to confirm that CoS is positive. The fraction is positive due to Condition D. To show that the second term is also positive, define the function

$$g(\varphi T) = e^{-\varphi T} (2 + \varphi T) - (2 - \varphi T),$$

for which it holds that

$$\begin{aligned} g(0) &= 0, \\ g'(\varphi T) &= 1 - e^{-\varphi T} - \varphi T e^{-\varphi T} > 0. \end{aligned}$$

We know from the proof of Proposition 8 that $g'(T\varphi)$ is positive for all $T\varphi > 0$. Hence, the result.

The following proposition provides a sensitivity analysis of CoS with respect to key model parameters.

Proposition 11 *CoS is independent of α , increasing in q, c and φ , and decreasing in k .*

The market potential α factors out in CoS and is irrelevant for the evaluation of profit differences. The two advertising parameters play an opposite role: the cost parameter (c) has a positive impact on CoS while the impact of advertising on demand (k) has a negative effect. Further, the larger the initial number of tickets for sale (q), and the marginal impact on inventory on demand (φ), the larger the loss when using constant pricing.

Comparing the constant and dynamic pricing results to TMP is not analytically feasible and we shall proceed numerically.¹² Consider the following parameter values:

$$T = 12, q \in \{270, 230, 150\}, \alpha = 200, \varphi = 0.3, k = 1, c = 1, \quad (25)$$

that is, we have fixed all parameter values but q . For this parameter we used the three values stated above. This calibration satisfies Conditions A-E. Table 2 gives the optimal profits in the three scenarios for the three values of q .

Table 2: Profits in the three pricing scenarios

q	270	230	150
J^D	40,027.5	35,860.8	25,687.5
J^C	37,290.3	33,874.6	24,842.7
J^*	38,306.4	35,295.5	25,188.1
J^C/J^D (%)	93.1	94.4	96.7
J^*/J^D (%)	95.6	98.4	98.1

All three profits are increasing in q which is expected. The loss in profit due to the use of a constant price, instead of the dynamic price, is less than 10%. This loss diminishes if the organizers choose to have a two markets. In fact, for low values of q , the TMP profit is very close to the DP profit. These comments do not justify the use of non-optimal pricing policy, but are intended to give an indication of the magnitude of the loss. As alluded to above, there might be other reasons making the organizers to select CP or TMP, e.g., ease of implementation and tradition. Such factors are not accounted for in our model.

Figure 1 shows optimal advertising, price, and demand trajectories in the three pricing scenarios for the values of q listed in Table 2. The figure shows the following:

(a) Generally speaking, we observe that DP advertising, price, and sales trajectories 'average' the corresponding CP trajectories. Similarly, the TMP trajectories 'average' the DP trajectories; Here the approximation is done in two pieces.

¹²We ran many numerical experiments. In all cases we got the same qualitative results that are presented below. The Mathematica program used for the numerical analysis is available from the authors upon request.

(b) The larger the value of q , the earlier the switch to the last-minute market in the TMP scenario. This result may be counterintuitive in view of the fact that we require all tickets to be sold and $p_2^* > p_1^*$, that is, price is marked-up in the last-minute market. One explanation of this result is that in TNP the organizers increases advertising efforts in the last-minute market. This allows for charging a higher price.

(c) The advertising trajectory in CP scenario is between the two branches of advertising in TMP. When q is higher (lower), $a^C(t)$ is closer to $a_2^*(t)$ ($a_1^*(t)$). One explanation is that a higher q requires more advertising, which is the case with $a_2^*(t)$.

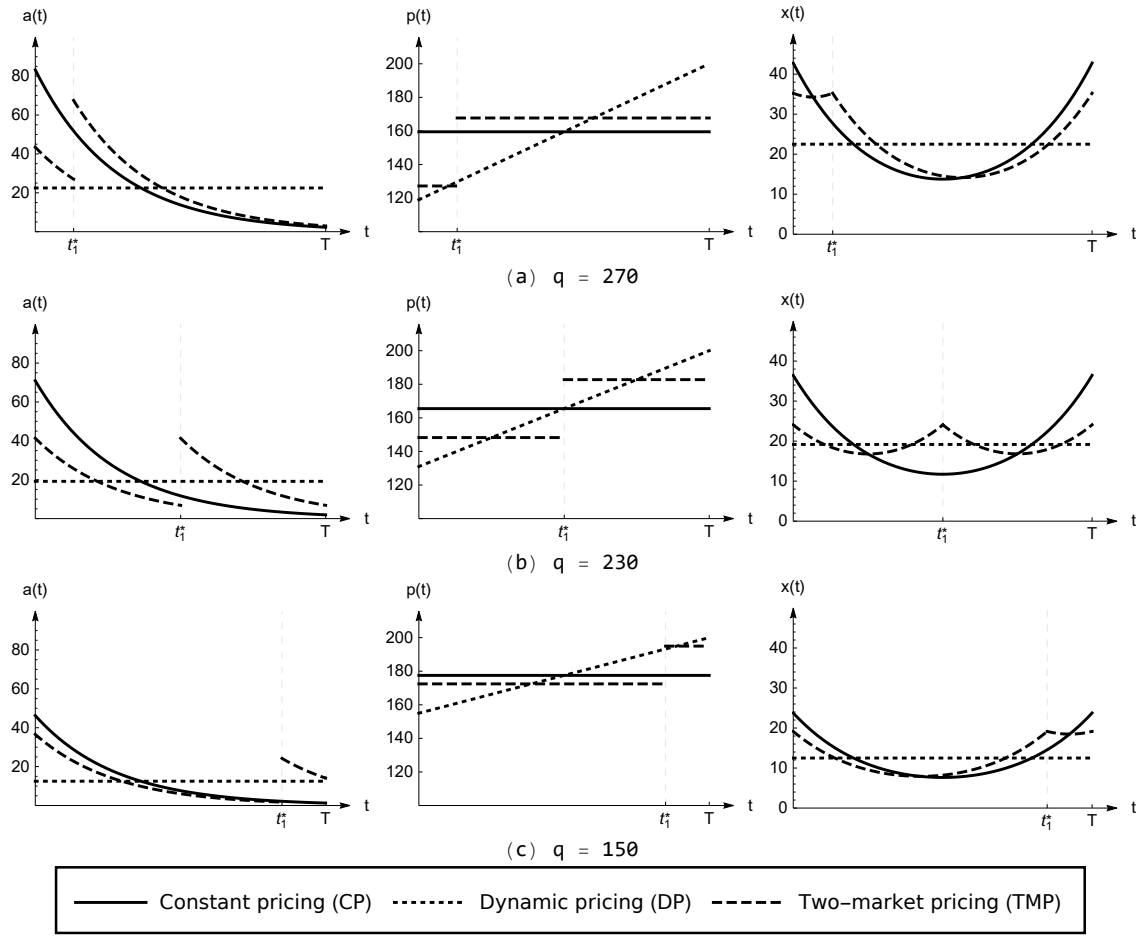


Figure 1: Trajectories of advertising, price and demand in the three scenarios

8 Conclusions

The current research has investigated a real-world problem of how to sell tickets to one-time entertainment events. In a simplified setup we considered the organizers of an event, acting so as to maximize profits, who must choose optimal pricing and advertising policies. Clearly, our setup is simple taking into consideration that organizers of an event must make many other decisions.

The objective of the research is to identify how three ticket pricing policies, all well-known from practice, would work in our setup. We analyzed the following policies: A single constant price (i.e., the traditional policy), a pricing policy such that price is constant within each of two periods, called the regular and the last-minute market (i.e., a two-level markup or markdown policy, known from the sales of seasonal or perishable items), and a dynamic policy where price changes continuously (known from revenue management, e.g., the sales of air-tickets).

The organizers' decision problem has been simplified in order to derive results by analytic methods. In some cases, however, this did not work and we had to resort to numerical simulations.

It is obvious that our setup can be extended in many different directions. Here we mention just a few.

- A crucial assumption of our model is that demand is deterministic. The implication is that the organizers can ensure that the event eventually is sold out. If one would take uncertainty into account the following is one option (among several). In optimal control applications in marketing a common practice is to replace deterministic demand dynamics by stochastic differential equations and maximize the expected value of the objective functional. The effect of introducing stochastic demand in the current research would surely be to complicate computations and make it harder or even impossible to provide meaningful interpretations and intuitions.
- Another extension of the setup would be to take into account that the organizers must choose the starting day for ticket sales and advertising as well as the date of the event, still having to decide pricing and advertising policies.
- The demand function could be modified to take into account consumer reactions to the policies employed by the organizers. For example, one could assume that customers form expectations of future prices and let these expectations influence their purchase behavior. Consumers acting in this way are known as strategic customers (e.g., Chatterjee (2009)).

9 Appendix

Proof of Proposition 1 To solve the dynamic pricing problem, introduce the Hamiltonian

$$H(s, p, a, \lambda) = (p - \lambda)(\alpha - p + ka - \varphi s) - \frac{c}{2}a^2,$$

where $\lambda = \lambda(t)$ is the costate associated with state $s(t)$. Necessary optimality conditions are as follows. Suppose that the triple $(a^D(t), p^D(t), s^D(t))$ solves the optimal control problem. Then, there exists a piecewise continuously differentiable function $\lambda(t)$ such that for all $t \in [0, T]$, except at points where $a^D(t)$ and/or $p^D(t)$ are discontinuous, it holds that

$$H(s^D, p^D, a^D, \lambda) = \max_{a \geq 0, p \geq 0} H(s^D, p, a, \lambda),$$

$$\dot{\lambda}(t) = -\frac{\partial H^D}{\partial s} = \varphi(p^D(t) - \lambda(t)). \quad (26)$$

The first-order-optimality conditions for advertising rate and price are

$$\frac{\partial H}{\partial a} = k(p - \lambda) - ca^D \begin{cases} \leq 0 & \text{if } a^D = 0, \\ = 0 & \text{if } a^D > 0, \end{cases} \quad (27)$$

$$\frac{\partial H}{\partial p} = x - p^D + \lambda \begin{cases} \leq 0 & \text{if } p^D = 0, \\ = 0 & \text{if } p^D > 0. \end{cases}$$

It is readily shown that H is strictly concave in a and p . Then, $a^D(t)$ and $p^D(t)$ are uniquely determined and continuous.

Assuming an interior solution one obtains from (27)

$$p^D(t) = \frac{\alpha + \lambda(t) + ka^D(t) - \varphi s(t)}{2}; \quad a^D(t) = \frac{k(p^D(t) - \lambda(t))}{c},$$

and solving these equations for p^D and a^D yields the optimal controls as functions of state and costate:

$$p^D(s, \lambda) = \frac{c(\varphi s - \alpha - \lambda) + k^2\lambda}{k^2 - 2c}, \quad (28)$$

$$a^D(s, \lambda) = \frac{k(\varphi s - \alpha + \lambda)}{k^2 - 2c}. \quad (29)$$

Substituting $p^D(s, \lambda)$ and $a^D(s, \lambda)$ into the demand function yields

$$x^D = \frac{c(\varphi s - \alpha + \lambda)}{k^2 - 2c}. \quad (30)$$

Inserting $p^D(s, \lambda)$ and $a^D(s, \lambda)$ from (28), (29) into the state and costate equations yields

$$\dot{s}(t) = \frac{c(\alpha - \varphi s(t) - \lambda(t))}{k^2 - 2c}, \quad (31)$$

$$\dot{\lambda}(t) = -\frac{\varphi c(\alpha - \varphi s(t) - \lambda(t))}{k^2 - 2c}, \quad (32)$$

from which it follows that

$$\dot{\lambda}(t) = -\varphi \dot{s}^D(t). \quad (33)$$

The state and costate equations are solved using the boundary conditions $s(0) = q, s^D(T) = 0$. The unique solution is

$$s^D(t) = \frac{q(T-t)}{T}; \quad \lambda(t) = \frac{k^2 q - 2cq - Tcq\varphi + cqt\varphi + Tc\alpha}{Tc}, \quad (34)$$

With these results, optimal price, advertising, and demand time-paths can be found:

$$\begin{aligned} p^D(t) &= \alpha + \frac{q(k^2 - 2c) + cq + cqt\varphi(t-T)}{Tc}, \\ a^D(t) &= \frac{kq}{cT}, \quad x^*(t) = \frac{q}{T}. \end{aligned} \quad (35)$$

Finally, the optimal profit is

$$J^D = \frac{q(2q(c + k^2 - 2c) + Tc(2\alpha - q\varphi) - k^2 q)}{2Tc}.$$

Proof of Proposition 2 The Hamiltonian is

$$H(s, p, a, \mu) = (p - \lambda)(\alpha - p + ka - \varphi s) - \frac{c}{2}a^2,$$

where $\mu = \mu(t)$ is the costate associated with state $s(t)$. With the exception of the determination of the price, the proof follows the same steps as in the proof in the dynamic pricing scenario and we shall skip some details.

Assuming an interior solution, necessary optimality conditions are¹³

$$\begin{aligned} \frac{\partial H}{\partial a} &= k(p - \mu) - ca = 0, \\ \int_0^T \frac{\partial H}{\partial p} dt &= \int_0^T (\alpha - 2p + ka(t) - \varphi s(t) + \mu(t)) dt = 0. \end{aligned}$$

The above conditions are equivalent to

$$a(t) = \frac{k}{c}(p - \mu(t)), \quad q = \int_0^T (p - \mu(t)) dt.$$

The costate equation $\dot{\mu}(t) = \varphi(p - \mu(t))$ has the solution

$$\lambda(t) = p + C_1 e^{-\varphi t}; \quad C_1 = \text{constant}. \quad (36)$$

Substituting $a(t)$ into the state equation, and using (36), provides

$$\dot{s}(t) = -x(t) = -\alpha + p + \frac{k^2}{c}C_1 e^{-\varphi t} + \varphi s(t),$$

¹³Note that the optimality condition for the price is not the usual one, i.e., $\frac{\partial H}{\partial p} = 0$; see, for example Léonard and Long (1992).

which has the solution

$$s(t) = C_2 e^{\varphi t} - \frac{1}{2c\varphi} (C_1 k^2 e^{-\varphi t} + 2c(p - \alpha)); \quad C_2 = \text{constant.}$$

The constants C_1 and C_2 can be found by using the boundary conditions $s(0) = q, s(T) = 0$ and we get

$$\begin{aligned} s(t) &= \frac{q\varphi (e^{\varphi(T-t)} - e^{-\varphi(T-t)})}{\varphi (e^{\varphi T} - e^{-\varphi T})} \\ &\quad + \frac{(\alpha - p) (e^{\varphi T} (1 - e^{-\varphi t}) - e^{-\varphi T} (1 - e^{\varphi t}) + e^{-\varphi t} - e^{\varphi t})}{\varphi (e^{\varphi T} - e^{-\varphi T})}, \\ \mu(t) &= p - \frac{2c((\alpha - p)(1 - e^{\varphi T}) + q\varphi e^{\varphi T})}{k^2 (e^{\varphi T} - e^{-\varphi T})} e^{-\varphi t}. \end{aligned}$$

From the optimality condition $q = \int_0^T (p - \mu(t)) dt$ we obtain the optimal price

$$p^C = \frac{2c(\alpha(1 - e^{-\varphi T}) - q\varphi) + q\varphi k^2(1 + e^{-\varphi T})}{2c(1 - e^{-\varphi T})},$$

and substituting p^C into the expressions for $\mu(t), s(t)$ and $a(t)$ yields

$$\begin{aligned} \mu^C(t) &= \frac{2c\alpha(1 - e^{-\varphi T}) - 2c\varphi q(1 + e^{-\varphi t}) + q\varphi k^2(1 + e^{-\varphi T})}{2c(1 - e^{-\varphi T})}, \\ s^C(t) &= \frac{qe^{-\varphi t}(e^{\varphi t} - e^{\varphi T})(k^2(e^{\varphi t} - 1) - 2ce^{\varphi t})}{2c(e^{\varphi T} - 1)}, \\ a^C(t) &= \frac{k\varphi q e^{-\varphi t}}{c(1 - e^{-\varphi T})}. \end{aligned}$$

Finally, substituting these optimal paths into the demand and profit functions provides

$$\begin{aligned} x^C(t) &= \frac{q\varphi(1 + e^{-\varphi T})(k^2 e^{-2\varphi t} - (k^2 - 2c)e^{-\varphi T})}{2c(e^{\varphi T} - e^{-\varphi T})e^{-\varphi(T+t)}}, \\ J^C &= \frac{k^2 q^2 \varphi(1 + e^{-\varphi T}) + 4c q(\alpha(1 - e^{-\varphi T}) - q\varphi)}{4c(1 - e^{-\varphi T})}. \end{aligned}$$

Proof of Proposition 3 The Hamiltonian is

$$H_2(s, a_2, p_2, \eta_2) = (p_2 - \eta_2)x - \frac{c}{2}a_2^2,$$

where $\eta_2 = \eta_2(t)$ is the costate variable associated with $s(t)$. Suppose that the optimal price and advertising rate are positive for all t . Then $a_2^*(t) = k(p_2 - \eta_2(t))/c$ and the optimal price p_2^* can be found from

$$\int_{t_1}^T \frac{\partial H_2}{\partial p_2} dt = 0 \Leftrightarrow s_1 = \int_{t_1}^T (p_2 - \eta_2(t)) dt. \quad (37)$$

State and costate equations are

$$\begin{aligned} \dot{s}(t) &= -\alpha + p_2 - \frac{k^2(p_2 - \eta_2(t))}{c} + \varphi s(t); \quad s(t_1) = s_1, s(T) = 0 \\ \dot{\eta}_2(t) &= \varphi(p_2 - \eta_2(t)) \end{aligned}$$

in which s_1 is fixed. The unique solution of these equations is

$$\begin{aligned}\eta_2(t) &= p_2 - \frac{2c}{k^2} \frac{(\alpha - p_2)(e^{\varphi t_1} - e^{\varphi T}) + \varphi s_1 e^{\varphi T}}{e^{\varphi T} e^{-\varphi t_1} - e^{-\varphi T} e^{\varphi t_1}} e^{-\varphi t}, \\ s^*(t) &= \frac{\alpha - p_2}{\varphi} - \frac{(\alpha - p_2)(e^{-\varphi t_1} - e^{-\varphi T}) + \varphi s_1 e^{-\varphi T}}{\varphi(e^{\varphi(T-t_1)} - e^{-\varphi(T-t_1)})} e^{\varphi t} \\ &\quad + \frac{(\alpha - p_2)(e^{\varphi t_1} - e^{\varphi T}) + \varphi s_1 e^{\varphi T}}{\varphi(e^{\varphi(T-t_1)} - e^{-\varphi(T-t_1)})} e^{-\varphi t}.\end{aligned}\tag{38}$$

To determine the optimal price p_2 , substitute $\eta_2(t)$ into (37):

$$\begin{aligned}s_1 + \int_{t_1}^T (\eta_2(t) - p_2) dt &= 0 \Leftrightarrow \\ s_1 + \frac{2c}{\varphi k^2} \frac{(\alpha - p_2) e^{\varphi t_1} + (\varphi s_1 - \alpha + p_2) e^{\varphi T}}{e^{\varphi(T-t_1)} - e^{-\varphi(T-t_1)}} (e^{-\varphi T} - e^{-\varphi t_1}) &= 0,\end{aligned}$$

the solution of which is

$$p_2^* = \alpha - \varphi s_1 \left(\frac{1}{1 - e^{\varphi(t_1-T)}} - \frac{k^2}{2c} \frac{1 + e^{\varphi(T-t_1)}}{1 - e^{\varphi(T-t_1)}} \right).\tag{39}$$

Using (39) and (38) yields the optimal advertising rate

$$a_2^*(t) = \frac{k\varphi s_1}{c(e^{-\varphi t_1} - e^{-\varphi T})} e^{-\varphi t}.\tag{40}$$

Substituting for p_2^* and $a_2^*(t)$ in the profit function, we obtain

$$J_2^*(s_1, t_1) = p_2^* s_1 + \frac{k^2 \varphi s_1^2 (e^{-2\varphi T} - e^{-2\varphi t_1})}{4c(e^{-\varphi t_1} - e^{-\varphi T})^2}.\tag{41}$$

Finally, inserting p_2^* into $\eta_2(t_1)$ (given by (38)) yields

$$\eta_2(t_1) = \alpha + \frac{\varphi s_1 (k^2 (1 + e^{\varphi(t_1-T)}) - 4c)}{2c(1 - e^{\varphi(t_1-T)})}\tag{42}$$

which verifies the equality $\partial J_2^*(s_1, t_1)/\partial s_1 = \eta_2(t_1)$ in (13).

Using the other equality in (13), i.e., $\partial J_2^*(s_1, t_1)/\partial t_1 = H_2^*(s_1, t_1)$, yields the maximized Hamiltonian for the last minute market:

$$H_2^*(s_1, t_1) = \alpha + \frac{2\varphi s_1}{e^{\varphi(t_1-T)} - 1} + \frac{k^2 \varphi s_1}{2c} \frac{e^{\varphi T} + e^{\varphi t_1}}{e^{\varphi T} - e^{\varphi t_1}}.$$

Proof of Proposition 4 The Hamiltonian is

$$H_1(s, a_1, \bar{p}_1, \lambda_1) = (p_1 - \eta_1) x - ca_1^2/2,$$

where $\eta_1 = \eta_1(t)$ is the costate variable. Suppose that price and advertising rate are positive. The optimal advertising rate is given by $a_1^*(t) = k(p_1 - \eta_1(t))/c$ and the optimal price p_1^* is the solution of the equation

$$\int_0^{t_1} \frac{\partial H_1}{\partial p_1} dt = 0 \Leftrightarrow q - s_1 = \int_0^{t_1} (p_1 - \eta_1(t)) dt,\tag{43}$$

in which $q - s_1$ is the number of tickets sold in the regular market.

The costate equation $\dot{\eta}_1(t) = \varphi(p_1 - \eta_1(t))$ has the solution $\eta_1(t) = p_1 + C_1 e^{-\varphi t}$ and we determine C_1 such that the costate matching (or continuity) condition $\eta_1(t_1) = \eta_2(t_1)$ in (11) is satisfied. The right-hand side of this equation is

$$\eta_2(t_1) = \alpha + \varphi s_1 \frac{k^2 (1 + e^{\varphi(t_1-T)}) - 4c}{2c(1 - e^{\varphi(t_1-T)})},$$

and the costate $\eta_1(t)$ then is

$$\eta_1(t) = p_1 - \left(p_1 - \alpha + \varphi s_1 \frac{k^2 (e^{-\varphi(T-t_1)} + 1) - 4c}{2c (e^{-\varphi(T-t_1)} - 1)} \right) e^{\varphi(t_1-t)}. \quad (44)$$

Using this result, the optimal price can be found from (43):

$$p_1^* = \frac{\varphi(q - s_1)}{e^{\varphi t_1} - 1} - \varphi s_1 \frac{k^2 (e^{\varphi(t_1-T)} + 1) - 4c}{2c (e^{\varphi(t_1-T)} - 1)} + \alpha, \quad (45)$$

and, using (44) and (45), the optimal advertising rate is

$$a_1^*(t) = \frac{k\varphi(q - s_1)}{c(e^{\varphi t_1} - 1)} e^{\varphi(t_1-t)}. \quad (46)$$

To determine the optimal state trajectory one inserts p_1^* and $a^*(t)$ from (45) and (46) into the state equation:

$$\begin{aligned} \dot{s}(t) &= -\alpha + p_1^* - ka^*(t) + \varphi s(t), \\ &= \frac{\varphi(q - s_1)}{e^{\varphi t_1} - 1} - \varphi s_1 \frac{k^2 (e^{\varphi(t_1-T)} + 1) - 4c}{2c (e^{\varphi(t_1-T)} - 1)} \\ &\quad - \frac{k^2 \varphi (q - s_1)}{c (e^{\varphi t_1} - 1)} e^{\varphi(t_1-t)} + \varphi s(t). \end{aligned}$$

Solving this equation with initial condition $s(0) = q$ yields

$$\begin{aligned} s^*(t) &= \frac{q - s_1}{e^{\varphi t_1} - 1} \left(e^{\varphi t} - 1 + k^2 \frac{e^{\varphi t_1} (e^{-\varphi t} - e^{\varphi t})}{2c} \right) \\ &\quad + \frac{s_1 (e^{\varphi t} - 1)}{e^{\varphi(t_1-T)} - 1} \left(2 - \frac{k^2 (e^{\varphi(t_1-T)} + 1)}{2c} \right) + qe^{\varphi t}. \end{aligned} \quad (47)$$

Now we can determine the optimal state at which the switch between markets occur (still supposing that a switch occurs). Using (47) provides the optimal terminal state in the regular market

$$\begin{aligned} s^*(t_1) &= \frac{q - s_1}{e^{\varphi t_1} - 1} \left(e^{\varphi t_1} - 1 + k^2 \frac{e^{\varphi t_1} (e^{-\varphi t_1} - e^{\varphi t_1})}{2c} \right) \\ &\quad + \frac{s_1 (e^{\varphi t_1} - 1)}{e^{\varphi(t_1-T)} - 1} \left(2 - \frac{k^2 (e^{\varphi(t_1-T)} + 1)}{2c} \right) + qe^{\varphi t_1}, \end{aligned}$$

and from (??) we have $s^*(t_1) = s_1$ as the (arbitrary) initial state in the last minute market. Since the state variable must be continuous for all t , we equate these two values of s and solve for s_1 :

$$s_1^* = q \frac{e^{\varphi(t_1-T)} - 1}{2(e^{-\varphi T} - 1)} (e^{-\varphi t_1} + 1). \quad (48)$$

Finally, optimal profits in the regular market are

$$J_1^*(s_1, t_1) = \bar{p}_1^* (q - s_1) + \frac{k^2 \varphi (q - s_1)^2 (e^{\varphi t_1} + 1)}{4c (1 - e^{\varphi t_1})}. \quad (49)$$

Proof of Proposition 5 For $t_1^* = T$ we have $s_1 = 0$. For $t_1^* = T$ to be optimal, it is necessary that $H_1^*(0, T) \geq H_2^*(0, T)$. Setting $s_1 = 0$ and $t_1 = T$ in (22) and 23), we obtain $H_2^*(0, T) = 0$ and $H_1^*(0, T) = \alpha$. Therefore, $H_1^*(0, T) \geq H_2^*(0, T)$ cannot be true and we conclude that $t_1 = T$ is suboptimal. For $t_1^* = 0$ we have $s_1 = q$ and it is necessary that $H_1^*(q, 0) \leq H_2^*(q, 0)$. To see if this inequality is satisfied we compute

$$\begin{aligned} H_1^*(q, 0) - H_2^*(q, 0) &= \frac{k^2 \varphi^2 q^2 e^{-\varphi T}}{2c(e^{-\varphi T} - 1)^2} - \left(\alpha + \frac{2\varphi q}{e^{-\varphi T} - 1} + \frac{k^2 \varphi q e^{\varphi T} + 1}{2c e^{\varphi T} - 1} \right) \\ &= \frac{1}{2c(e^{-\varphi T} - 1)^2 (e^{\varphi T} - 1)} (k^2 \varphi^2 q^2 e^{-\varphi T} (e^{\varphi T} - 1) - \\ & (e^{-\varphi T} - 1) ((2\alpha c (e^{-\varphi T} - 1) + 4\varphi q c) (e^{\varphi T} - 1) + k^2 \varphi q (e^{\varphi T} + 1) (e^{-\varphi T} - 1))) \end{aligned}$$

and using the assumption $\alpha > \varphi q$ yields

$$\begin{aligned} H_1^*(q, 0) - H_2^*(q, 0) &= \frac{1}{2c(e^{-\varphi T} - 1)^2 (e^{\varphi T} - 1)} (k^2 \varphi^2 q^2 e^{-\varphi T} (e^{\varphi T} - 1) - \\ & (e^{-\varphi T} - 1) (2\alpha c (e^{-\varphi T} - 1) (e^{\varphi T} - 1) + 4\varphi q c (e^{\varphi T} - 1) + k^2 \varphi q (e^{\varphi T} + 1) (e^{-\varphi T} - 1))), \\ & \geq \frac{1}{2c(e^{-\varphi T} - 1)^2 (e^{\varphi T} - 1)} (k^2 \varphi^2 q^2 e^{-\varphi T} (e^{\varphi T} - 1) - \\ & - (e^{-\varphi T} - 1) (2\varphi q c (e^{-\varphi T} - 1) (e^{\varphi T} - 1) + 4\varphi q c (e^{\varphi T} - 1) + k^2 \varphi q (e^{\varphi T} + 1) (e^{-\varphi T} - 1))), \\ & = \frac{1}{2c(e^{-\varphi T} - 1)^2 (e^{\varphi T} - 1)} (k^2 \varphi^2 q^2 e^{-\varphi T} (e^{\varphi T} - 1) - \\ & (e^{-\varphi T} - 1) (2\varphi q c (e^{\varphi T} - 1) (1 + e^{-\varphi T}) + k^2 \varphi q (e^{\varphi T} + 1) (e^{-\varphi T} - 1))) \geq 0. \end{aligned}$$

Therefore, $H_1^*(q, 0) \leq H_2^*(q, 0)$ cannot be true and we conclude that $t_1 = 0$ cannot be optimal.

Proof of Lemma 1 First we compute

$$\begin{aligned} \Lambda - 1 &= \frac{1}{2(2c\alpha - k^2 \varphi s_1)} (k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) - 2(2c\alpha - k^2 \varphi s_1) \\ & - \varphi s_1 \sqrt{4(2c - k^2)^2 + k^4 \varphi^2 s_1^2 + 8ck^2(\alpha - \varphi s_1)}) \\ &= \frac{\varphi s_1 \left(k^2 \varphi s_1 - 2(2c - k^2) - \sqrt{4(2c - k^2)^2 + k^4 \varphi^2 s_1^2 + 8ck^2(\alpha - \varphi s_1)} \right)}{2(2c\alpha - k^2 \varphi s_1)} \\ &< \frac{\varphi s_1}{2(2c\alpha - k^2 \varphi s_1)} \left(k^2 \varphi s_1 - 2(2c - k^2) - \sqrt{k^4 \varphi^2 s_1^2} \right) \\ &= -\frac{\varphi s_1 (2c - k^2)}{2(2c\alpha - k^2 \varphi s_1)} < 0. \end{aligned}$$

Now, $\Lambda > 0$ is equivalent to

$$\begin{aligned} \Lambda > 0 &\Leftrightarrow k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) > \varphi s_1 \sqrt{4(2c - k^2)^2 + k^4 \varphi^2 s_1^2 + 8ck^2(\alpha - \varphi s_1)} \\ &\Leftrightarrow (k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1))^2 > \varphi^2 s_1^2 (4(2c - k^2)^2 + k^4 \varphi^2 s_1^2 + 8ck^2(\alpha - \varphi s_1)) \\ &\Leftrightarrow 4c^2 \alpha^2 - 8c^2 \alpha \varphi s_1 + k^2 \varphi^2 s_1^2 (4c - k^2) > 0. \end{aligned}$$

Define a function g by

$$g(\alpha) \triangleq 4c^2 \alpha^2 - 8c^2 \alpha \varphi s_1 + k^2 \varphi^2 s_1^2 (4c - k^2).$$

It is easy to verify that the roots of this polynomial of degree two are positive and given by $\alpha_1 = \frac{\varphi s_1 (4c - k^2)}{2c}$ and $\alpha_2 = \frac{\varphi s_1 k^2}{2c}$. Therefore, $g(\alpha)$ is positive for $\alpha < \alpha_1$ and for $\alpha > \alpha_2$. Using the assumptions $\alpha > \varphi q$ and $2c > k^2$ shows that $\alpha > \alpha_2$ and hence $g(\alpha)$ is positive for all admissible values of α . Consequently, $\Lambda > 0$.

Proof of Proposition 6 It suffices to compute the time derivatives to obtain

$$\begin{aligned}\dot{a}^D(t) &= 0, \\ \dot{a}^C(t) &= -\frac{k\varphi^2 q e^{-\varphi t}}{c(1 - e^{-\varphi T})} < 0, \text{ for } t \in [0, T] \\ \dot{a}_1^*(t) &= -\frac{k\varphi^2 (q - s_1) e^{\varphi(t_1^* - t)}}{c(e^{\varphi t_1} - 1)} < 0, \text{ for } 0 \leq t < t_1 \\ \dot{a}_2^*(t) &= -\frac{k\varphi^2 s_1 e^{-\varphi t}}{c(e^{-\varphi t_1^*} - e^{-\varphi T})} < 0, \text{ for } t_1 < t \leq T.\end{aligned}$$

Proof of Proposition 7 The difference between the advertising rates for the regular and last-minute market, evaluated at the switching instant t_1^* , is given by

$$a_2^*(t_1^*) - a_1^*(t_1^*) = \frac{k\varphi e^{-\varphi t_1^*} (s_1 - s_1 e^{-\varphi T} - q e^{-\varphi t_1^*} + q e^{-\varphi T})}{c(1 - e^{-\varphi t_1^*}) (e^{-\varphi t_1^*} - e^{-\varphi T})}.$$

Define a function $f(z)$ by

$$f(z) = s_1 - s_1 e^{-\varphi T} - q e^{-\varphi z} + q e^{-\varphi T}.$$

We have

$$\begin{aligned}f'(z) &= \varphi q e^{-\varphi z} > 0, \quad f''(z) = -\varphi^2 q e^{-\varphi z} < 0 \\ f(z) &= 0 \Leftrightarrow \hat{z} = -\frac{1}{\varphi} \ln \frac{s_1(1 - e^{-\varphi T}) + q e^{-\varphi T}}{q} \\ f(0) &= (s_1 - q)(1 - e^{-\varphi T}) < 0, \quad f(T) = s_1(1 - e^{-\varphi T}) > 0.\end{aligned}$$

Function $f(z)$ is convex and increasing. It takes negative values on $\left[0, -\frac{1}{\varphi} \ln \frac{s_1(1 - e^{-\varphi T}) + q e^{-\varphi T}}{q}\right)$ and positive values for $\left(-\frac{1}{\varphi} \ln \frac{s_1(1 - e^{-\varphi T}) + q e^{-\varphi T}}{q}, T\right]$. Consequently,

$$a_2^*(t_1^*) - a_1^*(t_1^*) \begin{cases} < 0 & \text{for } t_1^* < \hat{x} \\ = 0 & \text{for } t_1^* = \hat{x} \\ > 0 & \text{for } t_1^* > \hat{x} \end{cases}$$

and

$$\begin{aligned}t_1^* &= T + \frac{1}{\varphi} \ln \left(\frac{k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) - \varphi s_1 \sqrt{4(2c - k^2)^2 + k^4 \varphi^2 s_1^2 + 8ck^2(\alpha - \varphi s_1)}}{2(2c\alpha - k^2 \varphi s_1)} \right) \\ &\times \frac{s_1(1 - e^{-\varphi T}) + q e^{-\varphi T}}{q} \\ &\times \frac{k^2 \varphi^2 s_1^2 + 4c(\alpha - \varphi s_1) - \varphi s_1 \sqrt{4(2c - k^2)^2 + k^4 \varphi^2 s_1^2 + 8ck^2(\alpha - \varphi s_1)}}{2(2c\alpha - k^2 \varphi s_1)}.\end{aligned}$$

Proof of Proposition 8 It holds that

$$a^D(t) - a^C(t) = \frac{kq}{cT(1 - e^{-\varphi T})} (1 - e^{-\varphi T} - \varphi T e^{-\varphi t}).$$

Defining function $g(t) = 1 - e^{-\varphi T} - \varphi T e^{-\varphi t}$ it is easy to verify that for

$$t = \bar{t} = -\frac{1}{\varphi} \ln \frac{1 - e^{-\varphi T}}{\varphi T},$$

we have $g(\bar{t})$. Noting that $g'(t)$ is positive completes the proof.

Proof of Proposition 9 Compute the difference between prices to obtain

$$\begin{aligned} p_2^* - p_1^* &= \alpha - \varphi s_1 \left(\frac{1}{1 - e^{\varphi(t_1 - T)}} - \frac{k^2}{2c} \frac{1 + e^{-\varphi(T - t_1)}}{1 - e^{-\varphi(T - t_1)}} \right) \\ &\quad - \left(\alpha - \varphi \left(\frac{q - s_1}{1 - e^{\varphi t_1}} + s_1 \frac{4c - k^2 (1 + e^{\varphi(t_1 - T)})}{2c (1 - e^{\varphi(t_1 - T)})} \right) \right) \\ &= \frac{-\varphi e^{\varphi t_1}}{(1 - e^{-\varphi(T - t_1)}) (e^{\varphi t_1} - 1)} (q (e^{-\varphi t_1} - e^{-\varphi T}) + s_1 (e^{-\varphi T} - 1)). \end{aligned}$$

We have $q > s_1$. Consequently, to show that $p_2^* - p_1^* > 0$ it suffices to show that $(e^{-\varphi t_1} - e^{-\varphi T}) > (e^{-\varphi T} - 1)$. This is true. Indeed

$$\begin{aligned} (e^{-\varphi t_1} - e^{-\varphi T}) - (e^{-\varphi T} - 1) &= e^{-\varphi t_1} - 2e^{-\varphi T} + 1 > \\ e^{-\varphi T} - 2e^{-\varphi T} + 1 &= -e^{-\varphi T} + 1 > 0. \end{aligned}$$

Proof of Proposition 10 Compute the difference

$$p^D(t) - p^C = -\frac{q((2k^2 - 2c(\varphi T + 1) + 2\varphi ct + \varphi T k^2) e^{-\varphi T} + 2c - 2k^2 - 2\varphi ct + \varphi T k^2)}{2Tc(1 - e^{-\varphi T})}$$

and observe that

$$\frac{d}{dt}(p^D - p^C) = \frac{q\varphi}{T} > 0.$$

Suppose there exists an instant of time, say \check{t} , such that $p^D(\check{t}) - p^C = 0$. Then

$$p^D(\check{t}) - p^C = 0 \Leftrightarrow \check{t} = \frac{2c(1 - e^{-\varphi T} - \varphi T e^{-\varphi T}) - 2k^2 + 2k^2 e^{-\varphi T} + \varphi T k^2 + \varphi T k^2 e^{-\varphi T}}{2c\varphi(1 - e^{-\varphi T})}.$$

We wish to show that $\check{t} \in (0, T)$. First we shall show that $\check{t} > T$. Compute

$$\begin{aligned} \check{t} - T &= \frac{2c(1 - e^{-\varphi T} - T\varphi e^{-\varphi T}) - 2k^2 + 2k^2 e^{-\varphi T} + \varphi T k^2 + \varphi T k^2 e^{-\varphi T}}{2c\varphi(1 - e^{-\varphi T})} - T \\ &= \frac{1}{2c\varphi(1 - e^{-\varphi T})} ((2c - k^2)(1 - e^{-\varphi T} - \varphi T) - k^2(1 - e^{-\varphi T} - \varphi T e^{-\varphi T})). \end{aligned}$$

In the proof of Proposition 8 we showed that $1 - e^{-\varphi T} - \varphi T e^{-\varphi T} > 0$. Define function $f(\varphi T) = 1 - e^{-\varphi T} - \varphi T$. Clearly,

$$f(0) = 0, \quad f'(\varphi T) = e^{-\varphi T} - 1 < 0$$

which implies $f(\varphi T) \leq 0$ for all $\varphi T \geq 0$. Combining this result with the assumption $2c - k^2 > 0$ shows that $\check{t} < T$.

Next we show that $\check{t} > 0$. We have

$$\check{t} = \frac{2c(1 - e^{-\varphi T} - \varphi T e^{-\varphi T}) - k^2(2 - 2e^{-\varphi T} - \varphi T - \varphi T e^{-\varphi T})}{2c\varphi(1 - e^{-\varphi T})}$$

and have to show that

$$f(\varphi T) = 2 - 2e^{-\varphi T} - \varphi T - \varphi T e^{-\varphi T},$$

is negative for all $\varphi T > 0$. Indeed, this is true because

$$f(0) = 0, \quad f'(\varphi T) = e^{-\varphi T} - 1 + \varphi T e^{-\varphi T} < 0.$$

Proof of Proposition 11 The derivatives of CoS with respect to the parameters are given by

$$\begin{aligned}\frac{\partial CoS}{\partial q} &= \frac{q(2c - k^2)}{2Tc(1 - e^{-\varphi T})} (e^{-\varphi T}(2 + \varphi T) - (2 - \varphi T)) > 0, \\ \frac{\partial CoS}{\partial c} &= \frac{q^2 k^2}{4Tc^2(1 - e^{-\varphi T})} (e^{-\varphi T}(2 + \varphi T) - (2 - \varphi T)) > 0, \\ \frac{\partial CoS}{\partial k} &= -\frac{q^2 k}{2Tc(1 - e^{-\varphi T})} (e^{-\varphi T}(2 + \varphi T) - (2 - \varphi T)) < 0, \\ \frac{\partial CoS}{\partial \varphi} &= \frac{q^2(2c - k^2)}{4c(1 - e^{-\varphi T})^2} (1 - e^{-2\varphi T} - 2\varphi T e^{-\varphi T}) > 0.\end{aligned}$$

To establish the sign of the first three derivatives, it suffices to consider the function

$$g(\varphi T) = e^{-\varphi T}(2 + \varphi T) - (2 - \varphi T),$$

and note that $g(0) = 0$ and $g'(T\varphi) > 0$. Similarly for the last derivative.

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