Les Cahiers du GERAD ISSN: 0711–2440

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G–2018–11

March 2018

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2018
– Bibliothèque et Archives Canada, 2018

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

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An interval observer for discrete-time SEIR epidemic models

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March 2018
Les Cahiers du GERAD
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Abstract: This paper focuses on designing a state estimator for a discrete-time SEIR epidemic model of an influenza-like illness. It is assumed that only sets of admissible values are known for the model's disturbances, uncertainties and parameters, except for the time-varying transmission rate from the "susceptible" to the "exposed" stage, whose bounding values are unavailable. An interval observer is designed to estimate the set of possible values of the state, and a sufficient condition guaranteeing the asymptotic stability of the proposed estimator is formulated in terms of a linear matrix inequality. The performance of the proposed approach is demonstrated by numerical simulations.

Keywords: Interval estimation, epidemic model, modelling dynamics, bounding methods, diseases

Acknowledgments: This work was supported by NSERC under Grant RGPIN 435905-13 and by the Fonds de recherche du Québec – Nature et technologies (FRQNT) international internships fellowship (Energy/Digital/Aerospace).

1 Introduction

Seasonal influenza epidemics usually cause three to five million cases of severe illness and result in about 250,000 to 500,000 deaths worldwide every year, according to the World Health Organization [32]. Infectious disease surveillance plays a major role in analyzing epidemics' causes, dynamics and spread. Public Health Services (PHS) rely on surveillance data collected by agencies such as the Centers for Disease Control and Prevention in the United States to estimate these infectious diseases' activity levels, prepare intervention strategies and design policy recommendations.

Mathematical modelling of epidemics has become an essential part in the sentinel role played by public health response planning and early outbreak detection systems [13, 6, 25, 16, 24]. Kermack and McKendrick proposed the first modern mathematical epidemiology models in [26]: a Susceptible-Infectious-Recovered (SIR) model was used to model the plague (London 1665-1666, Bombay 1906) and cholera (London 1865) epidemics. The basic SIR model assumes that a fixed population, at any time, can be divided into three compartments: susceptible people (those who are not infected but could become infected), infectious people (those who have the disease and are able to infect others), and recovered people (those who were infected by the disease and are now immune). It is assumed that the total number of people, N, is constant. Homogeneous mixing is also assumed by these models. That is, each individual is equally likely to come in contact with any other [6].

In the case of influenza, one needs to extend the standard SIR model and introduce a fourth compartment corresponding to the disease's latency period, when a person is infected but not yet able to infect others. This extension is called the Susceptible-Exposed-Infected-Recovered (SEIR) model [18]. Several estimators have been previously designed for SEIR models [23, 6, 1]. In the existing literature, strong assumptions on the disturbances or uncertain parameters in these models can enable the design of estimators converging to the true state values. However, the problem of observer design for SEIR models becomes very challenging when one has to take into account the presence of disturbances or uncertain parameters whose values are only known to belong to an interval or polytope. Such issues can be addressed by an interval estimation approach [11, 4, 5, 12, 8]. Using input-output measurements, an observer has to estimate the set of admissible values (interval) for the state at each instant of time [15]. A major advantage of interval estimation is that it allows many types of uncertainties to be taken into account in the system [9].

This paper presents an interval estimator for a discrete-time SEIR epidemic model of an Influenza-Like Illness (ILI). We are interested in estimating the four compartment states, to support the prediction of epidemic outbreaks. An interval observer design was proposed for the first time for an epidemic model in [3]. However, it applies to SIR rather than SEIR models, and so it does not consider the fourth compartment of the population that corresponds to the incubation stage for diseases such as influenza. Furthermore, it assumes continuous-time dynamics, whereas we focus on discrete-time epidemic models, which have gained substantial importance during the last decade [22, 29]. We also assume that the PHS have only access to noisy measurements, whereas [3] considered that perfect measurements were available. Importantly, [3] assumed that the time-varying transmission rate $\beta(t)$ from the "susceptible" to the "exposed" stage is bounded by two functions $\overline{\beta}(t)$ and $\underline{\beta}(t)$, available to the PHS in real-time. The time-varying observer proposed in [3] provided accurate results in simulated models, but the transmission rate $\beta(t)$ is a highly uncertain parameter that cannot be estimated by biological considerations [2, 21] and its bounds are generally unknown in epidemiology models [4]. Here, we assume that neither the value of $\beta(t)$ nor its bounding values are available, which makes the estimation problem more complicated. Finally, [3] assumed that the recovery rate γ_t is constant, whereas in our work it is time-varying and only its interval of admissible values is available. On the other hand, one drawback of our observer is that it is not perfectly causal, namely, it produces state estimates with a delay of two periods. The interval estimation approach described in this paper can be extended to higher/lower order discrete-time epidemic models, when the model has more/less than 4 compartments, such as SEIR and SIR models with several parallel infective stages [28].

In Section 2, we present the problem statement and some results from interval estimation theory. Section 3 describes the application of these results to design an interval observer for a discrete-time SEIR epidemic model. Input-to-state stability of the proposed interval observer is also proven in order to guarantee that it has bounded solutions for any bounded input. Finally, numerical simulations demonstrating the performance of the the observer are presented in Section 4.

Notation: The real numbers are denoted by $\mathbb{R}$, the integers by $\mathbb{Z}$, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. For a vector-valued signal $u : \mathbb{Z}_+ \to \mathbb{R}^n$, the L_∞ norm $||u||_{L_\infty}$ is defined as $||u||_{L_\infty} = \sup_{t\in[0,+\infty]} \|u_t\|_\infty$, where $\|u_t\|_\infty := \max_{i\in\{1,\dots,n\}} |u_{t,i}|$. We denote by $\mathcal{L}_\infty^n$ the set of such signals u with the property $||u||_{L_\infty} < \infty$. We denote the ℓ_p-norm of a vector $x \in \mathbb{R}^k$ by $|x|_p := (\sum_{i=1}^k |x_i|^p)^{1/p}$, for $p \in [1, \infty]$. The symbols I_n, $\Lambda_{n\times m}$ and Λ_p denote the $n \times n$ identity matrix and the matrices with all elements equal to 1 and dimensions $n \times m$ and $p \times 1$, respectively. For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood element-wise. The notation $P \prec 0$ ($P \succ 0$) means that the matrix $P \in \mathbb{R}^{n\times n}$ is symmetric and negative (positive) definite. A matrix $A \in \mathbb{R}^{n\times n}$ is called Schur stable if all its eigenvalues have absolute value less than one. It is called nonnegative if all its elements are nonnegative, i.e., if $A \geq 0$.

2 Problem statement and preliminary results

Figure 1 illustrates the discrete-time SEIR epidemic model, a discretization of the classic influenza continuous-time dynamics proposed in [18]

$$\begin{aligned}
S_{t+1} &= (1-\mu_t)S_t - \beta_t S_t I_t + \mu_t,\\
E_{t+1} &= (1-\alpha_t-\mu_t)E_t + \beta_t S_t I_t,\\
I_{t+1} &= (1-\gamma_t-\mu_t)I_t + \alpha_t E_t,\\
R_{t+1} &= (1-\mu_t)R_t + \gamma_t I_t,
\end{aligned} \tag{1}$$

where S_t, E_t, I_t, R_t are nonnegative state variables, $\alpha : \mathbb{Z}_+ \to [\underline{\alpha}, \overline{\alpha}]$ with $\underline{\alpha} > 0$, $\gamma : \mathbb{Z}_+ \to [\underline{\gamma}, \overline{\gamma}]$ and $\mu : \mathbb{Z}_+ \to [\underline{\mu}, \overline{\mu}]$ are unknown signals taking values in known intervals (i.e., the parameters $\underline{\alpha}, \overline{\alpha}, \underline{\gamma}, \overline{\gamma}, \underline{\mu}, \overline{\mu} \in \mathbb{R}_+$ are given). The parameter $\beta : \mathbb{Z}_+ \to \mathbb{R}$ is highly uncertain and time-varying. Note that we achieve non-dimensionalization by setting the population size $N = 1$ in the model (1). It is assumed that all the four compartments experience the same constant death rate, equal to the birth rate μ_t. Indeed,

$$\begin{aligned}
&S_{t+1} + E_{t+1} + I_{t+1} + R_{t+1} =\\
&(1-\mu_t)(S_t + E_t + I_t + R_t) + \mu_t = (1-\mu_t) + \mu_t = 1.
\end{aligned}$$

The parameters α_t, β_t and γ_t stand for the time-varying transition rates from one disease stage to the next, while μ_t represents the time-varying natural birth and death rate. The disease transmissions that arise from contacts between susceptible and infectious people is described by the first equation of (1). In the original continuous-time SEIR model [18], the pathogen is transmitted by each infectious individual to β individuals per unit time. However, a new disease case occurs only if the contact is made with a susceptible person, with probability S_t. Hence, at time t, people in the compartment S migrate to the "susceptible but not yet infectious" compartment E at the rate $\beta_t I_t$. People in compartment E move to the infectious compartment I at the rate α_t per unit time, while infectious individuals migrate to the recovered compartment R at the rate γ_t per unit time.

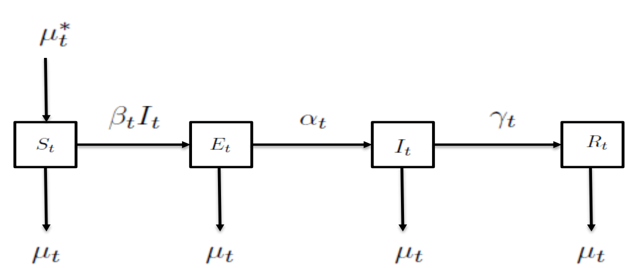

Figure 1: The non-dimensionalized ($N = 1$) classic time-varying Susceptible-Exposed-Infected-Recovered (SEIR) model.

The measured output consists of noisy counts of ILI visits at emergency departments

$$y_t = I_t + v_t, \tag{2}$$

where $v \in \mathcal{L}_\infty$ is the measurement noise, with $||v||_{L_\infty} \leq V$ for some known $V > 0$. The dynamics of system (1)–(2) can be rewritten as follows

$$\begin{aligned} x_{t+1} &= A_t x_t + F\zeta_t + H\mu_t, \\ y_t &= Cx_t + v_t, \end{aligned} \tag{3}$$

where $x_t = \begin{bmatrix} S_t & E_t & I_t & R_t \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^4_+$ is the state vector and $\zeta_t := \beta_t S_t I_t$ is an uncertain input. In contrast to [17] and [30], the matrix A_t is time-varying in this paper. Moreover, it should be pointed out that only unknown inputs that have no impact on the output are considered in [30]. Since the uncertain input of (3) can be rewritten as $\zeta_t = \beta_t S_t(y_t - v_t)$, we consider here unknown inputs that affect the output. The time-varying matrix A_t and constant matrices C, F and H are defined as follows

$$\begin{aligned} C &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \\ F &= \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \\ H &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \\ A_t &= \begin{bmatrix} 1-\mu_t & 0 & 0 & 0 \\ 0 & 1-\alpha_t-\mu_t & 0 & 0 \\ 0 & \alpha_t & 1-\mu_t-\gamma_t & 0 \\ 0 & 0 & \gamma_t & 1-\mu_t \end{bmatrix}. \end{aligned}$$

For different ILIs, the values of the parameters α_t, β_t and γ_t are different and vary with time for a patient. No given confidence interval is assumed for β_t. The instant value of A_t is also unavailable, but we have the the bounds

$$\underline{A} \leq A_t \leq \overline{A}, \quad \forall t \geq 0,$$

for

$$\begin{aligned} \underline{A} &= \begin{bmatrix} 1-\overline{\mu} & 0 & 0 & 0 \\ 0 & 1-\overline{\alpha}-\overline{\mu} & 0 & 0 \\ 0 & \underline{\alpha} & 1-\overline{\gamma}-\overline{\mu} & 0 \\ 0 & 0 & \underline{\gamma} & 1-\overline{\mu} \end{bmatrix}, \\ \overline{A} &= \begin{bmatrix} 1-\underline{\mu} & 0 & 0 & 0 \\ 0 & 1-\underline{\alpha}-\underline{\mu} & 0 & 0 \\ 0 & \overline{\alpha} & 1-\underline{\gamma}-\underline{\mu} & 0 \\ 0 & 0 & \overline{\gamma} & 1-\underline{\mu} \end{bmatrix}. \end{aligned}$$

The goal of this paper is to design an interval observer, i.e., state signal bounds $\underline{x}_t \leq x_t \leq \overline{x}_t$, which can contribute to design a decision rule for disease outbreak detection in an interval approach framework. Note that at the disease-free equilibrium (when $I_t = 0$), the system (1) is detectable, but not observable (we refer the reader to [31] for the definition of detectability of nonlinear systems). An advantage of interval observers is that they can be designed even if the system is only detectable. Next, we review some basic facts from the theory of interval estimation.

2.1 Interval relations

Given a matrix $A \in \mathbb{R}^{m\times n}$, let us define $A^+ = \max\{0, A\}$ applied elementwise, $A^- = A^+ - A$ (the same for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

Lemma 1 [7] *Let $x \in \mathbb{R}^n$ be a vector with $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$. If $A \in \mathbb{R}^{m\times n}$ is a matrix, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{4}$$

2.2 Nonnegative discrete-time linear systems

A system

$$x_{t+1} = Ax_t + B\omega_t,\ \omega : \mathbb{Z}_+ \to \mathbb{R}^m_+,\ t \in \mathbb{Z}_+,$$
$$y_t = Cx_t + D\omega_t,$$

with $x_t \in \mathbb{R}^n, y \in \mathbb{R}^p$ and nonnegative matrices $A \in \mathbb{R}^{n\times n}_+$ and $B \in \mathbb{R}^{n\times m}_+$ is called cooperative or nonnegative [19]. Its solution is elementwise nonnegative for all $t \in \mathbb{Z}_+$ provided that $x_0 \geq 0$ [20]. Also, the output solution y_t of such a system is nonnegative if $C \in \mathbb{R}^{p\times n}_+$ and $D \in \mathbb{R}^{p\times q}_+$.

Lemma 2 *[14] A matrix $A \in \mathbb{R}^{n\times n}_+$ is Schur stable if and only if there exists a diagonal matrix P with positive diagonal elements such that $A^TPA - P \prec 0$.*

3 Interval observer design

In this section we design an interval observer for the SEIR model (1)–(2). We assume here that neither β_t nor its bounding values are available, which makes the estimation problem more complicated than in [3] but more realistic [4, 2, 21]. First, we determine upper and lower bounds for the uncertain input ζ_t . Next we design an interval observer for the system (1)–(2). We prove the inclusion relation $0 \leq \underline{x_t} \leq x_t \leq \overline{x_t}$, $\forall t \geq 0$, and the asymptotic stability of the error bounds. Finally, the boundedness of the interval observer's solutions is proven.

Using Equation (3), one can write

$$\begin{aligned} y_{t+2} = &\ Cx_{t+2} + v_{t+2}, \\ = &\ CA_{t+1}A_tx_t + CA_{t+1}F\zeta_t + CF\zeta_{t+1} \\ &+ CA_{t+1}H\mu_t + CH\mu_{t+1} + v_{t+2}, \end{aligned}$$

so that we have

$$\begin{aligned} CA_{t+1}F\zeta_t = &\ -CA_{t+1}A_tx_t - CF\zeta_{t+1} - CH\mu_{t+1} \\ &- CA_{t+1}H\mu_t - v_{t+2} + y_{t+2}. \end{aligned}$$

The tructure of the model (1)–(2) implies that $CF = 0$, $CA_{t+1}H = 0$ and $CA_{t+1}F = \alpha_{t+1}$. Hence, we have

$$\alpha_{t+1}\zeta_t = y_{t+2} - CA_{t+1}A_tx_t - v_{t+2}.$$

Suppose $0 \leq \underline{x_t} \leq x_t \leq \overline{x_t}$, $\forall t \geq 0$, for some $\underline{x_t}, \overline{x_t} \in \mathbb{R}^4$. Notice that $A_{t+1}A_t - \underline{A_{t+1}A_t} \geq 0$, $\overline{A_{t+1}A_t} - A_{t+1}A_t \geq 0$ with $\underline{A_{t+1}A_t} = \underline{A}^2$ and $\overline{A_{t+1}A_t} = \overline{A}^2$. Hence using the inequality (4) of Lemma 1, we get the following relations for all $t \geq 0$

$$(\underline{A}^2)^+\underline{x_t} - (\underline{A}^2)^-\overline{x_t} \leq \underline{A}^2x_t \leq A_{t+1}A_tx_t$$

and

$$A_{t+1}A_tx_t \leq \overline{A}^2x_t \leq (\overline{A}^2)^+\overline{x_t} - (\overline{A}^2)^-\underline{x_t}.$$

We then obtain the following relations

$$\underline{\zeta_t} \leq \zeta_t \leq \overline{\zeta_t},$$

for all $t \geq 0$, where

$$\underline{\zeta_t} = \overline{\alpha}^{-1}(y_{t+2} - V - (C\overline{A}^2)^+\overline{x_t} + (C\overline{A}^2)^-\underline{x_t}),$$
$$\overline{\zeta_t} = \underline{\alpha}^{-1}(y_{t+2} + V - (C\underline{A}^2)^+\underline{x_t} + (C\underline{A}^2)^-\overline{x_t}).$$

An interval estimator's equations for (3) takes the form

$$\begin{array}{rcl} \underline{\chi_{t+1}} & = & \underline{A}\underline{\chi_t} + F^+\underline{\zeta_t} - F^-\overline{\zeta_t} + H\underline{\mu_t} \\ & & +\underline{L}(y_t - C\underline{\chi_t}) - \underline{L}^\star V, \\ \overline{\chi_{t+1}} & = & \overline{A}\overline{\chi_t} + F^+\overline{\zeta_t} - F^-\underline{\zeta_t} + H\overline{\mu_t} \\ & & +\overline{L}(y_t - C\overline{\chi_t}) + \overline{L}^\star V, \\ \underline{x_t} & = & \max\{0, \underline{\chi_t}\}, \\ \overline{x_t} & = & \max\{0, \overline{\chi_t}\}, \end{array} \tag{5}$$

where $\underline{x}_t \in \mathbb{R}^4$ and $\overline{x}_t \in \mathbb{R}^4$ are respectively the lower and the upper interval estimates for the state x_t, $\underline{\chi}_t, \overline{\chi}_t \in \mathbb{R}^4$ is the state of (5), $\underline{L}$ and $\bar{L}$ are some matrices, $\underline{L}^\star = |\underline{L}|\Lambda_{p\times 1}$ and $\overline{L}^\star = |\underline{L}|\Lambda_{p\times 1}$.

Assumption 1 *There exist matrices $\overline{L} \in \mathbb{R}^{4\times 1}$, $\underline{L} \in \mathbb{R}^{4\times 1}$ such that the matrices $(\overline{A} - \overline{L}C)$ and $(\underline{A} - \underline{L}C)$ are nonnegative.*

Assumption 2 *The state $x(t)$ is bounded (*i.e., $x \in \mathcal{L}^4_\infty$*) for $x(0) \in [\underline{x}(0), \overline{x}(0)]$, and $\underline{x}(0), \overline{x}(0) \in \mathbb{R}^4$ are given constants.*

We take matrices $\underline{L}$ and $\bar{L}$ satisfying Assumption 1 in order to enforce the positivity of the interval observer's error dynamics. Assumption 2 is common in the interval observer design literature and is satisfied here since the state components belong to $[0, 1]$.

Theorem 1 *Let Assumptions 1 and 2 be satisfied. Then the estimates $\underline{x}_t$ and $\overline{x}_t$ given by (5) yield the relations*

$$0 \le \underline{x}_t \le x_t \le \overline{x}_t \quad \forall t \ge 0, \tag{6}$$

provided that $0 \le \underline{x}_0 \le x_0 \le \overline{x}_0$.

Proof. Notice that $x_t \ge 0$ for all $t \ge 0$ and x_t is also bounded since it represents population proportions. We can rewrite the Equation (3) as follows

$$x_{t+1} = (A' - LC)x_t + (A_t - A')x_t + F\zeta_t + H\mu_t + Ly_t - Lv_t$$

for A' equal to $\underline{A}$ or $\overline{A}$ and L' equal to $\underline{L}$ or $\overline{L}$. Hence, the interval observer's errors $\underline{e_t} = x_t - \underline{\chi_t}$, $\overline{e_t} = \overline{\chi_t} - x_t$ satisfy the equations

$$\begin{array}{rcl} \underline{e_{t+1}} & = & (\underline{A} - \underline{L}C)\underline{e_t} + \underline{\phi_t}, \\ \overline{e_{t+1}} & = & (\overline{A} - \overline{L}C)\overline{e_t} + \overline{\phi_t}, \end{array} \tag{7}$$

where

$$\begin{array}{rcl} \underline{\phi_t} & = & (A_t - \underline{A})x_t + F\zeta_t - F^+\underline{\zeta_t} + F^-\overline{\zeta_t} \\ & & -\underline{L}v_t + \underline{L}^\star V, \\ \overline{\phi_t} & = & (\overline{A} - A_t)x_t + F^+\overline{\zeta_t} - F^-\underline{\zeta_t} - F\zeta_t \\ & & +\overline{L}v_t + \overline{L}^\star V. \end{array}$$

Under the introduced conditions, it can be inferred from Lemma 1 that $\underline{\phi}_t \ge 0$ and $\overline{\phi}_t \ge 0$, $\forall t \ge 0$. Therefore one deduces from Assumption 1 that $\underline{e_t} \ge 0$ and $\overline{e_t} \ge 0$ since $\underline{e_0} \ge 0$ and $\overline{e_0} \ge 0$ (the system (7) is cooperative). This implies that the order relation $\underline{\chi}_t \le x_t \le \overline{\chi}_t$ is satisfied for all $t \ge 0$. Hence the inequality (6) is true by construction of $\underline{x}, \overline{x}$ and due to the non-negativity of x. □

To state the next theorem, we introduce the notation

$$
\begin{aligned}
J &= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \text{ where} \\
J_{11} &= \overline{\alpha}^{-1}F^{+}(C\overline{A}^{2})^{-} + \underline{\alpha}^{-1}F^{-}(C\underline{A}^{2})^{+}, \\
J_{12} &= -\overline{\alpha}^{-1}F^{+}(C\overline{A}^{2})^{+} - \underline{\alpha}^{-1}F^{-}(C\underline{A}^{2})^{-}, \\
J_{21} &= -\underline{\alpha}^{-1}F^{+}(C\underline{A}^{2})^{+} - \overline{\alpha}^{-1}F^{-}(C\overline{A}^{2})^{-}, \\
J_{22} &= \underline{\alpha}^{-1}F^{+}(C\underline{A}^{2})^{-} + F^{-}\overline{\alpha}^{-1}(C\overline{A}^{2})^{+}.
\end{aligned}
$$

and let $\eta = \|J\|_2$, the induced 2-norm of J.

Theorem 2 *Let Assumptions 1 and 2 be satisfied. Suppose there exists a diagonal matrix $P \in \mathbb{R}^{8\times 8}$ with positive diagonal entries, two symmetric positive definite matrices $Q, K \in \mathbb{R}^{8\times 8}$, and a constant $\gamma > 0$ such that*

$$
\Xi = \begin{bmatrix} \mathcal{A}^{T}P\mathcal{A} - P + Q + \gamma\eta^{2}I_{8} & \mathcal{A}^{T}P & \mathcal{A}^{T}P \\ P\mathcal{A} & P - \gamma I_{8} & P \\ P\mathcal{A} & P & P - K \end{bmatrix} \preceq 0, \tag{8}
$$

for

$$
\mathcal{A} = \left[\begin{array}{cc} (\underline{A} - \underline{L}C) & 0 \\ 0 & (\overline{A} - \overline{L}C) \end{array} \right].
$$

Then $\underline{\chi}, \overline{\chi} \in \mathcal{L}_{\infty}^{4}$ and so $\underline{x}, \overline{x} \in \mathcal{L}_{\infty}^{4}$.

Proof. Let us define

$$
\begin{aligned}
\chi_t =& [\underline{\chi_t}^{\mathrm{T}}\ \overline{\chi_t}^{\mathrm{T}}]^{\mathrm{T}},\ \epsilon_t = [\underline{\epsilon_t}^{\mathrm{T}}\ \overline{\epsilon_t}^{\mathrm{T}}]^{\mathrm{T}}, \\
\underline{\epsilon_t} =& \overline{\alpha}^{-1}F^{+}(y_{t+2} - V) - \underline{\alpha}^{-1}F^{-}(y_{t+2} + V) + H\underline{\mu} \\
&+ \underline{L}y_t - \underline{L}^{\star}V, \\
\overline{\epsilon_t} =& \underline{\alpha}^{-1}F^{+}(y_{t+2} + V) - \overline{\alpha}^{-1}F^{-}(y_{t+2} - V) + H\overline{\mu} \\
&+ \overline{L}y_t + \overline{L}^{\star}V.
\end{aligned}
$$

The dynamics of the interval observer can be rewritten as

$$
\chi_{t+1} = \mathcal{A}\chi_t + J\max\{0, \chi_t\} + \epsilon_t, \tag{9}
$$

where the matrix $\mathcal{A}$ is defined in the theorem. Consider a Lyapunov function $\mathcal{V}(\chi_t) = \chi_t^{\mathrm{T}}P\chi_t$ (By using Lemma 2, the matrix P can be chosen diagonal since the matrix $\mathcal{A}$ is non-negative), and let $\tau = \mathcal{V}(\chi_{t+1}) - \mathcal{V}(\chi_t)$. We have

$$
\begin{aligned}
\tau =& \chi_t^{\mathrm{T}}(\mathcal{A}^{\mathrm{T}}P\mathcal{A} - P)\chi_t + \chi_t^{\mathrm{T}}\mathcal{A}^{\mathrm{T}}PJ\max\{0, \chi_t\} \\
&+ \max\{0, \chi_t\}^{\mathrm{T}}J^{\mathrm{T}}PA\chi_t + \max\{0, \chi_t\}^{\mathrm{T}}J^{\mathrm{T}}PJ\max\{0, \chi_t\} \\
&+ 2\chi_t^{\mathrm{T}}\mathcal{A}^{\mathrm{T}}P\epsilon_t + 2\epsilon_t^{\mathrm{T}}PJ\max\{0, \chi_t\} + \epsilon_t^{\mathrm{T}}P\epsilon_t.
\end{aligned}
$$

Taking into account that $|J\max\{0, \chi_t\}|_2 \leq \eta|\chi_t|_2$, we get

$$
\begin{aligned}
\tau \leq& \chi_t^{\mathrm{T}}(\mathcal{A}^{\mathrm{T}}P\mathcal{A} - P)\chi_t + \chi_t^{\mathrm{T}}\mathcal{A}^{\mathrm{T}}PJ\max\{0, \chi_t\} \\
&+ \max\{0, \chi_t\}^{\mathrm{T}}J^{\mathrm{T}}PA\chi_t + \max\{0, \chi_t\}^{\mathrm{T}}J^{\mathrm{T}}PJ\max\{0, \chi_t\} \\
&+ 2\chi_t^{\mathrm{T}}\mathcal{A}^{\mathrm{T}}P\epsilon_t + 2\epsilon_t^{\mathrm{T}}PJ\max\{0, \chi_t\} + \epsilon_t^{\mathrm{T}}(P - K)\epsilon_t \\
&+ \epsilon_t^{\mathrm{T}}K\epsilon_t + \gamma\eta^{2}\chi_t^{\mathrm{T}}\chi_t - \gamma\max\{0, \chi_t\}^{\mathrm{T}}J^{\mathrm{T}}J\max\{0, \chi_t\} \\
&+ \chi_t^{\mathrm{T}}Q\chi_t - \chi_t^{\mathrm{T}}Q\chi_t,
\end{aligned}
$$

$$\leq \begin{bmatrix} \chi_t \\ J\max\{0,\chi_t\} \\ \epsilon_t \end{bmatrix}^{\mathrm{T}} \Xi \begin{bmatrix} \chi_t \\ J\max\{0,\chi_t\} \\ \epsilon_t \end{bmatrix} - \chi_t^{\mathrm{T}} Q \chi_t + \epsilon_t^{\mathrm{T}} K \epsilon_t,$$
$$\leq -\chi_t^{\mathrm{T}} Q \chi_t + \epsilon_t^{\mathrm{T}} K \epsilon_t,$$

because of (8). This inequality shows that the system (9) is input-to-state stable from ϵ to χ [27, 10], hence from y to χ. Since $\epsilon \in \mathcal{L}_\infty^8$ by construction, $\underline{\chi}$ and $\overline{\chi}$ stay bounded for all $t \geq 0$. □

In this paper, we select the observer gains $\overline{L}$, $\underline{L}$ manually. However, because of the diagonal structure of P, one can in fact also optimize the observer gains $\overline{L}$, $\underline{L}$ using semidefinite-programming to improve the observer's accuracy, following the approach of [10] to reformulate the matrix inequality (8).

4 Simulations

In this section, we illustrate the performance of the proposed interval observer. Consider a scenario where

$$\mu_t = \mu_0 + 0.05\sin(\frac{\pi}{4}t), \alpha_t = 0.2 + 0.1\sin^2(\frac{\pi}{4}t),$$
$$\gamma_t = \gamma_0 + 0.1\sin(\frac{\pi}{4}t), \beta_t = \beta_0(1+\kappa\cos(\frac{\pi}{4}t)),$$

with $\mu_0 = 0.4$/year, $\gamma_0 = 0.4$/year, $\beta_0 = 0.5$/year and the degree of seasonality $\kappa = 0.4$. The output measurements y_t are corrupted by noise such that $v_t = V\cos(\frac{\pi}{4}t)$ with $V = \frac{0.1}{1000}$. The state's initial conditions are $S_0 = \frac{852}{1000}$, $E_0 = \frac{120}{1000}$, $I_0 = \frac{5}{1000}$ and $R_0 = \frac{23}{1000}$. We select

$$\underline{L} = (1-\varrho)\,[0\;0\;l\;0\;0]^{\mathrm{T}},$$
$$\overline{L} = (1+\varrho)\,[0\;0\;l\;0\;0]^{\mathrm{T}},$$

with $l = \frac{1}{20}$. Assumption 1 holds for these choices of $\overline{L}$ and $\underline{L}$ and all conditions of Theorem 1 are satisfied. Figure 2 shows the results of the interval estimation for $\varrho = \frac{1}{10}$, where the solid lines represent the states x_k, $k = 1, 2, 3, 4$ and the dash lines are used for the interval estimates $\underline{x_k}$ and $\overline{x_k}$. Notice that the dynamics of S_t and its estimates can be deduced from Figure 2 by using the relation $S_t = 1 - E_t - I_t - R_t$.

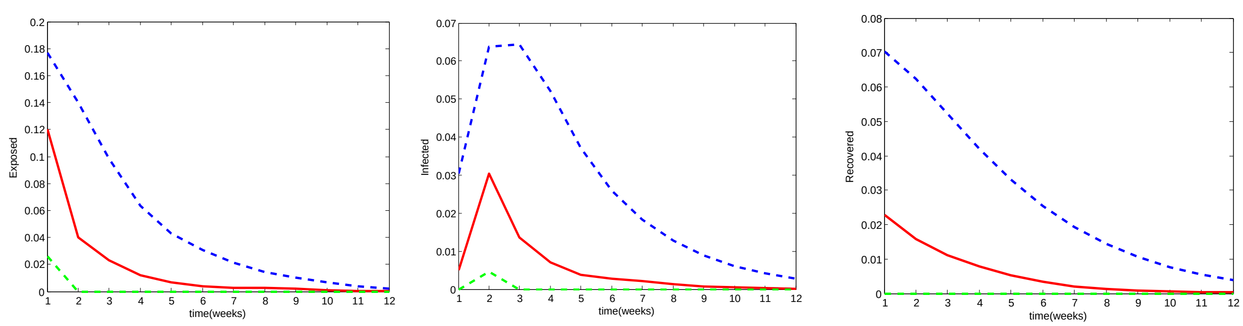

Figure 2: Evolution of the actual state and the observed bounds.

5 Conclusion

The problem of state-observer design for a discrete-time SEIR epidemic model of ILI has been considered in this paper. The proposed approach only requires sets of admissible values for the model's disturbances or uncertainties and parameters, and no information about the bounding values of the time-varying transmission rate from the "susceptible" to the "infected" stage. A new approach for the estimation of the the four compartments' state is proposed, where an interval observer is used instead of a point-wise one. Its performance is illustrated in simulation. Future work can focus on performance evaluation for real data collected by the Public health services.

References

[1] S. Alonso-Quesada, M. De la Sen, RP Agarwal, and A. Ibeas. An observer-based vaccination control law for an SEIR epidemic model based on feedback linearization techniques for nonlinear systems. Advances in Difference Equations, 2012(1):161, September 2012.

[2] D. Bichara, N. Cozic, and A. Iggidr. On the estimation of sequestered infected erythrocytes in plasmodium falciparum malaria patients. Mathematical Biosciences and Engineering, 11(4):741–759, August 2014.

[3] P.-A. Bliman and B. D'Avila Barros. Interval observer for SIR epidemic model subject to uncertain seasonality. In Proc. of the 5th International Symposium on Positive Systems Theory and Applications (POSTA 2016), Roma, Italy, September 2016.

[4] K. H. Degue, D. Efimov, and A. Iggidr. Interval estimation of sequestered infected erythrocytes in malaria patients. In Proc. 15th European Control Conference (ECC), Aalborg, Denmark, June 2016.

[5] K. H. Degue, D. Efimov, and J.-P. Richard. Stabilization of linear impulsive systems under dwell-time constraints: Interval observer-based framework. European Journal of Control, January 2018.

[6] V. Dukic, H. F. Lopes, and N. G. Polson. Tracking epidemics with google flu trends data and a state-space SEIR model. Journal of the American Statistical Association, 107(500):1410–1426, 2012.

[7] D. Efimov, L.M. Fridman, T. Raïssi, A. Zolghadri, and R. Seydou. Interval estimation for LPV systems applying high order sliding mode techniques. Automatica, 48:2365–2371, 2012.

[8] D. Efimov, W. Perruquetti, T. Raïssi, and A. Zolghadri. On interval observer design for time-invariant discrete-time systems. In Proc. European Control Conference (ECC) 2013, Zurich, 2013.

[9] D. Efimov and T. Raïssi. Design of interval observers for uncertain dynamical systems. Automation and Remote Control, 76:1–29, 2015.

[10] D. Efimov, T. Raïssi, W. Perruquetti, and A. Zolghadri. Design of interval observers for estimation and stabilization of discrete-time LPV systems. IMA Journal of Mathematical Control and Information, 33(4):1051–1066, 2016.

[11] D. Efimov, T. Raïssi, and A. Zolghadri. Control of nonlinear and LPV systems: interval observer-based framework. IEEE Trans. Automatic Control, 58(3):773–782, 2013.

[12] Denis Efimov, Andrey Polyakov, E. M. Fridman, Wilfrid Perruquetti, and Jean-Pierre Richard. Delay-dependent positivity: Application to interval observers. In Proc. ECC 2015, Linz, 2015.

[13] J. J. Fallas-Monge, J. Chavarría-Molina, and G. Alpízar-Brenes. Combinatorial metaheuristics applied to infectious disease models. In Proc. of the 2016 IEEE 36th Central American and Panama Convention (CONCAPAN XXXVI), November 2016.

[14] L. Farina and S. Rinaldi. Positive Linear Systems: Theory and Applications. Wiley, New York, 2000.

[15] J.L. Gouzé, A. Rapaport, and M.Z. Hadj-Sadok. Interval observers for uncertain biological systems. Ecological Modelling, 133:46–56, 2000.

[16] N. C. Grassly and C. Fraser. Mathematical models of infectious disease transmission. Nature Reviews Microbiology, 6:477–487, June 2008.

[17] D. Gucik-Derigny, T. Raïssi, and A. Zolghadri. A note on interval observer design for unknown input estimation. International Journal of Control, 89(1):25–37, June 2016.

[18] H. W. Hethcote. The mathematics of infectious diseases. SIAM Review, 42(4):599–653, 2000.

[19] M. W. Hirsch. Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere. SIAM Journal on Mathematical Analysis, 16(3):423–439, 1985.

[20] M. W. Hirsch and H. L. Smith. Monotone maps: a review. J. Difference Equ. Appl., 11(4–5):379–398, 2005.

[21] G. Hooker, S. P. Ellner, L. D. V. Roditi, and D. J. D. Earn. Parameterizing state-space models for infectious disease dynamics by generalized profiling: measles in Ontario. Journal of the Royal Society Interface, 8(60):961–974, July 2011.

[22] Z. Hu, Z. Teng, and H. Jiang. Stability analysis in a class of discrete sirs epidemic models. Nonlinear Analysis: Real World Applications, 13(5):2017–2033, 2012.

[23] A. Ibeas, M. de la Sen, S. Alonso-Quesada, I. Zamani, and M. Shafiee. Observer design for seir discrete-time epidemic models. In Proc. of the 2014 13th International Conference on Control Automation Robotics Vision (ICARCV), pages 1321–1326, December 2014.

[24] E. H. Kaplan, D. L. Craft, and L. M. Wein. Emergency response to a smallpox attack: The case for mass vaccination. Proceedings of the National Academy of Sciences of the United States of America, 99(16):10935–10940, August 2002.

[25] M. J. Keeling and Rohani P. Modeling infectious diseases in humans and animals. Princeton and Oxford: Princeton University Press, 2008.

[26] W. O. Kermack and A. G. McKendrick. A contribution to the mathematical theory of epidemics. Proc. of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 115(772):700–721, 1927.

[27] H. K. Khalil. Nonlinear Systems. Prentice Hall PTR, 3rd edition, 2002.

[28] A. Korobeinikov. Global properties of SIR and SEIR epidemic models with multiple parallel infectious stages. Bulletin of Mathematical Biology, 71(1):75–83, January 2009.

[29] R. E. Mickens. Numerical integration of population models satisfying conservation laws: Nsfd methods. Journal of Biological Dynamics, 1(4):427–436, 2007.

[30] E. I. Robinson, J. Marzat, and T. Raïssi. Interval observer design for unknown input estimation of linear time-invariant discrete-time systems. IFAC-PapersOnLine, 50(1):4021 – 4026, 2017. 20th IFAC World Congress.

[31] E. D. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. Systems & Control Letters, 29(5):279–290, 1997.

[32] L. Vaillant, G. La Ruche, A. Tarantola, and P. Barboza. Epidemiology of fatal cases associated with pandemic H1N1 influenza 2009. Eurosurveillance, 14(33):1–6, August 2009.