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Selection of a Markov perfect Nash equilibrium in a class of differential games

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Abstract: This note revisits the problem of how to select an equilibrium in a differential game in the case of multiplicity of Nash equilibria. Most of the previous applied dynamic games literature has considered preplay negotiations between players, implicitly or explicitly, with the aim of reaching an agreement on the selection of the pair of strategies. The main objective of this note is to analyze which would be the most likely equilibrium without preplay communications. We study the linear and nonlinear Markov perfect Nash equilibria for a class of well-known models in the literature if preplay communications are eliminated. We analyze both symmetric and nonsymmetric strategies. We show that the nonlinear strategies are not always the optimal strategies implemented when cheap talk is removed. We conclude that in the presence of multiple equilibria and without cheap talk the most likely equilibria are symmetric piecewise linear Markov perfect Nash equilibria at least for a range of initial values of the state variable.

Keywords: Multiple equilibria, differential games, Markovian strategies, nonlinear strategies, international pollution control

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1 Introduction

This note revisits the problem of how to select an equilibrium in a differential game when there is an infinite number of equilibria. Most of the previous applied dynamic games literature has considered preplay negotiations between players, implicitly or explicitly, with the aim of reaching an agreement on the selection of the pair of strategies. The main objective of this note is to analyze which would be the most likely equilibrium without preplay communications.

A considerably important issue in the area of dynamic games is the sustainability of cooperative strategies. In the dynamic games literature it is well-known that the Pareto optimum is difficult to achieve because cooperation can be frustrated by the prisoners' dilemma type of situation. If one player stands to receive a lower payoff in the coordinated solution than what he would get in a noncooperative solution, he will find optimal to deviate; he may have an incentive to cheat on the agreement, that is, to choose a different course of action than that prescribed by the agreement.[1]

Different mechanisms have been proposed in the literature to ensure that the cooperative outcome is sustained by a pair of equilibrium strategies (see, for example, Dockner et al. (2000), Ch. 6 for the use of trigger strategies and De Frutos and Martín-Herrán (2015) and the references therein for the use of incentive equilibrium strategies). One option often used in the applied dynamic games literature to implement cooperative solutions by means of noncooperative play is through nonlinear Markov-perfect strategies. The idea is to sustain over time the cooperative outcome or to approach the cooperative outcome as much as possible looking for nonlinear Markov-perfect strategies. These nonlinear strategies have received a great attention in the applied dynamic games literature after the publication of the seminal paper by Tsutsui and Mino (1990). Our paper tries to contribute to this literature.

Tsutsui and Mino (1990) provide a method (different from the guessing method) to construct stationary Markov feedback equilibria for differential games with one state variable and infinite time horizon. They discuss nonlinear strategies in a linear-quadratic game of duopolistic competition with sticky prices. One of the main purposes of the paper was to examine whether it was possible to construct a more efficient feedback equilibrium (more efficient with respect to the linear feedback equilibrium derived using the guessing method) (see footnote 2 of Tsutsui and Mino's paper). This same general purpose has been behind the use of nonlinear strategies in the analysis of a large number of economic and managerial problems. One important question related to the use of nonlinear strategies is that there are many equilibria, in fact an infinite number, depending on which pair of nonlinear strategies is chosen. The lack of a boundary condition in the problem implies that multiple feedback Nash equilibria exist. The multiplicity of equilibria requires some criterion for selecting equilibria. Equilibrium selection is the topic of this paper.

As far as we know, most of the literature on multiple equilibria and nonlinear strategies in linear-quadratic differential games has overtook the problem of the selection of equilibria assuming that the agents will be able to coordinate on the best feedback equilibrium. The choice of the most efficient strategy play is the result of preplay negotiations or communications, also called cheap talk. The selection of a specific equilibrium is determined in a preplay phase where agents have cheap talks and agree on a specific equilibrium.[2] The negotiations that lead to an implicit and partial cooperation are not included in the original model and as pointed out by Haurie et al. (2012) (page 274), presumably, if they were included, all strategies would be different that those obtained for the original criterion. Our main objective in this paper is to analyze which would be the most likely equilibrium if cheap talk is removed.

If there are not preplay communications the problem of equilibrium selection comes together the multiplicity of equilibria. Different criteria for selecting equilibria have been proposed in the literature of dynamic games, see, for example, Başar (1977), Driskil (1997), and Cartigny and Michel (2003). The criterion pro-

[1]The problem of the sustainability over time of the coordinated outcomes is already solved when the efficient solution is in itself an equilibrium, as in Chiarella et al. (1984), Rincón-Zapatero et al. (2000) and Martín-Herrán and Rincón-Zapatero (2005).

[2]One exception is Zagonari (1998) where no agreement on the selection of the pair of strategies is required because unilateral initiatives are considered. The paper analyzes a model where two group of countries differ in their preferences for consumption goods as well as in their attention to environmental issues. Another exception is Tasneem et al. (2017) where the empirical relevance of the nonlinear equilibria in a two-player common property resource game is examined. Their results show that nonlinear equilibria cannot be ruled out as irrelevant on behavioral grounds.

posed in these papers removes the multiplicity of equilibria through the definition of some desirable properties. Driskill (1997) avoids the problem of nonuniqueness by considering the equilibrium strategies that result from the game with finite horizon T and letting T tends to infinity. In the same line, for a one-dimensional linear-quadratic infinite horizon game Cartigny and Michel (2003) proves that the unique Nash equilibrium of each finite-horizon game converges to a unique Nash equilibrium of the given infinite-horizon game. Furthermore, they prove that the finite-horizon games admit equilibrium strategies that converge to a steady state for all fixed data and this steady state corresponds precisely to the linear strategies of one equilibrium in infinite horizon. Başar (1977) for linear-quadratic differential games proposes an optimal unique selection of an element of the Nash equilibrium set which exhibits a robust behavior by being insensitive to additive random perturbation in the state dynamics, i.e. the unique noncooperative solution of a particular stochastic differential game.

Unlike the papers cited in the preceding paragraph in this paper we do not impose any additional property to the equilibrium in order to remove the multiplicity of equilibria. We study the linear and nonlinear Markov perfect Nash equilibria for a class of well-known models in the literature if preplay communications are eliminated. In Dockner and Long (1993), one of the first works to apply Tsutsui and Mino's method to compute the nonlinear Markov-perfect strategies of an international pollution control differential game, the authors already highlighted different situations (asymmetric players or incomplete information, for example) which would complicate the process of choosing among Nash equilibria.

The literature that focusses on the comparison of the performance of linear and nonlinear strategies with respect to the Pareto solution could be divided in two major streams.[3] The papers that establish that nonlinear solutions are Pareto-superior to the linear strategies belong to the first stream, while those papers showing the opposite result belong to the second stream. This technique has been applied to a variety of settings, including industrial organization, environmental economics and public economics. All these papers have considered preplay negotiations between the players, implicitly or explicitly, with the aim of reaching an agreement on the selection of the pair of strategies. The following is a no exhaustive list of works which can be classified in the first stream of the literature: Dockner and Long (1993), Feichtinger and Wirl (1993), Wirl (1996), Piga (2000), Ihori and Itaya (2001), Rubio and Casino (2002), Wirl and Feichtinger (2002), Benchekroun and Long (2008), Fujiwara (2008, 2009, 2010), Fujiwara and Matsueda (2009). A list of papers belonging to the second stream of the literature includes: Wirl (1994), Wirl and Dockner (1995), Shibata (2002), Rubio and Casino (2003), Wirl (2007).

In this paper we analyze a well-known linear-quadratic differential game for clarity and simplicity in the exposition, although our arguments extend to other more general models, even for games outside the class of linear-quadratic games. In the case of more general models Markov-perfect strategies would not be computed explicitly, but numerically. In this paper we study the differential game proposed in Dockner and Long (1993) to analyze a transboundary pollution problem in a symmetric two-player setting.

Following Rowat (2007) we focus on the case where the strategies are required to be defined over the whole state space[4] and there are an infinite number of equilibria. We make this assumption to avoid the discussion about whether our results depend on the local or global character of the strategies. We would like to stay out of the discussion global versus local strategies. Taking into account all the optimality conditions established in Rowat (2007) that must be imposed to the strategies in order to ensure Nash equilibria, including the smooth pasting condition, does not imply uniqueness for the game, but an infinite number of Nash equilibria exists. Therefore, our research question of how to choose an equilibrium is still relevant.

The answer to our main research question can be summarized as follows: The nonlinear strategies are not always the implemented optimal strategies when cheap talk is removed. This result is valid even in the case of identical symmetric players. We analyze both symmetric and nonsymmetric strategies. When symmetric strategies are considered, the results are obtained analytically, while in the case of nonsymmetric strategies the results are characterized numerically.

[3]Most of the works in this literature study different problems formulated as linear-quadratic differential games. However, there are some exceptions like Kossioris et al. (2008) that analyze the shallow lake pollution control differential game presenting a nonlinear dynamics.

[4]Dockner and Wagener (2014) consider strategies with a local support.

The note is organized as follows. In next section we briefly recall the model in Dockner and Long (1993) and present some preliminary results on the characterization of the symmetric and nonsymmetric nonlinear feedback strategies. Section 3 analyzes what we call the reduced game and collects our main results. Section 4 concludes the note.

2 The model and preliminary results

We consider an infinite time horizon noncooperative two-player differential game. Player i's objective is to maximize

$$W_i(u_1, u_2, x_0) := \int_0^\infty f_i(x, u_1, u_2) e^{-\rho t}\, dt, \quad i = 1, 2, \tag{1}$$

$$\text{s.t.: } \dot{x} = g(x, u_1, u_2), \quad x(0) = x_0. \tag{2}$$

For simplicity in the exposition and with the objective to compare our results with those obtained in previous literature, we focus on a particular linear-quadratic model that has been extensively studied in the environmental economics literature. The formulation is borrowed from Dockner and Long (1993). The control variables, denoted by u_i, $i = 1, 2$, are the emissions of the two players (countries). A natural assumption is that $u_i \geq 0$. The state variable, denoted by x, represents the stock of pollution, and is assumed to be positive. Its dynamics is defined by the linear ordinary differential equation:

$$\dot{x} = g(x, u_1, u_2) := u_1 + u_2 - \alpha x, \quad x(0) = x_0, \tag{3}$$

where parameter $\alpha > 0$ denotes the natural absorption rate. The objective of player i is defined by functional (1) with

$$f_i(x, u_i, u_j) := u_i\big(A - \frac{1}{2}u_i\big) - \frac{1}{2}\varphi x^2, \tag{4}$$

where A and φ are positive parameters. We are interested in Markovian strategies and, given that the problem is autonomous and the game is played over an infinite time horizon, we restrict ourselves to consider stationary (time-independent) strategies of the form $u_i = u_i(x)$.

In order to center the discussion on the objective of this paper we focus on Markov Perfect Nash equilibria (MPNE) defined over the whole state space.[5] Let us denote by $\mathbb{R}_+$ the set of nonnegative real numbers. The set of admissible controls $\mathcal{U}_i$ is defined as the set of measurable functions $u_i = \phi_i(x)$ defined in $\mathbb{R}_+$ with values in $\mathbb{R}_+$, such that for all $\phi_i \in \mathcal{U}_i$, $i = 1, 2$, the differential equation (3) with $u_i = \phi_i(x)$, $i = 1, 2$, possesses a unique absolute continuous solution defined in $[0, +\infty)$ for all $x_0 \in \mathbb{R}_+$.

The value function V_i of player $i = 1, 2$, solves the following Hamilton-Jacobi-Bellman (HJB) system

$$\rho V_i(x) = \max_{u_i \in \mathbb{R}_+} \Big\{ u_i\big(A - \frac{1}{2}u_i\big) - \frac{\varphi}{2}x^2 + \frac{d}{dx}V_i(x)\big(u_i + \phi_j - \alpha x\big) \Big\}, \quad i \neq j. \tag{5}$$

For each solution (V_1, V_2) of (5), the first-order condition

$$A - u_i + \frac{d}{dx}V_i(x) = 0, \quad i = 1, 2, \tag{6}$$

jointly with the nonnegativity condition $u_i \geq 0$ defines a unique pair of strategies

$$\phi_i(x) = \max\big(0, A + \frac{d}{dx}V_i(x)\big), \quad x \in \mathbb{R}_+, \quad i = 1, 2,$$

that are candidates to constitute a MPNE. Furthermore, if the following transversality condition is satisfied

$$\limsup_{t \to \infty} e^{-\rho t} V(x(t)) \leq 0, \tag{7}$$

[5] We focus on globally defined strategies to avoid the discussion about whether our results depend or not on the local or global character of the strategies.

where $x(t)$ is the solution of (3) with $u_i = \phi_i(x)$, $i = 1, 2$, then the pair $(\phi_1(x), \phi_2(x))$ is a MPNE.

In Rincón-Zapatero et al. (1998) and Martín-Herrán and Rincón-Zapatero (2002), it has been proved that, as long as the strategies remain interior, that is $\phi_i > 0$, $i = 1, 2$, the strategies ϕ_i can be computed by means of the following system of ordinary differential equations:

$$M(h_1, h_2, x) \begin{bmatrix} h_1' \\ h_2' \end{bmatrix} = \begin{bmatrix} \rho + \alpha & 0 \\ 0 & \rho + \alpha \end{bmatrix} \begin{bmatrix} h_1 - A \\ h_2 - A \end{bmatrix} + \begin{bmatrix} \varphi x \\ \varphi x \end{bmatrix}, \tag{8}$$

where,

$$M(h_1, h_2, x) = \begin{bmatrix} h_1 + h_2 - \alpha x & h_1 - A \\ h_2 - A & h_1 + h_2 - \alpha x \end{bmatrix}. \tag{9}$$

Let us remark that given an initial state x_0, there exists a solution of (8) for each choice of initial values

$$[h_1(x_0), h_2(x_0)]^T = [\eta_1, \eta_2]^T \in \mathcal{H}(x_0),$$

where, for each $x_0 \geq 0$, $\mathcal{H}(x_0)$ is the set of regular points defined by

$$\mathcal{H}(x_0) = \left\{ [\eta_1, \eta_2]^T \in \mathbb{R}^2 | \det M(h_1, h_2, x_0) \neq 0 \right\}.$$

Each one of the solutions of (8) for a given initial state x_0 defines a MPNE if the transversality condition (7) is satisfied.

The symmetric case, $h_1(x) = h_2(x) = h(x)$, has been studied in Dockner and Long (1993), Rubio and Casino (2002) and Rowat (2007), where it is proved that, if condition

$$\alpha^2 + 3\alpha\rho + 2\rho^2 < \varphi \tag{10}$$

is satisfied, then an infinite number of symmetric globally defined MPNE exists. In this note we consider that the model parameters satisfy condition (10).

Imposing symmetry, $h_1(x) = h_2(x) = h(x)$, system (8) reduces to the single ordinary differential equation

$$(3h - A - \alpha x)h' = (\rho + \alpha)(h - A) + \varphi x. \tag{11}$$

Equation (11) possesses (Dockner and Long (1993)), two linear solutions defined by

$$h_a(x) = B_a + C_a x, \quad h_b(x) = B_b + C_b x,$$

where

$$B_j = \frac{A}{3} - F_j \frac{E}{D}, \quad C_j = F_j + \frac{\alpha}{3}, \quad j = a, b.$$

Coefficients F_j, D and E are given by

$$D = \frac{\rho\alpha + \alpha^2 + 3\varphi}{3}, \qquad E = \frac{2A(\rho + \alpha)}{3},$$

$$F_a = \frac{\rho + \sqrt{\rho^2 + 12D}}{6}, \qquad F_b = \frac{\rho - \sqrt{\rho^2 + 12D}}{6}.$$

Each one of the linear solutions defines two symmetric piecewise linear (affine) MPNE, $\phi_1 = \phi_2 = \phi_b$ and $\phi_1 = \phi_2 = \phi_{ab}$ with

$$\phi_b(x) = \max(h_b(x), 0), \tag{12}$$

$$\phi_{ab}(x) = \max(\min(h_a(x), h_b(x)), 0). \tag{13}$$

Note that functions h_a and h_b intersect at point $x^* = E/D$, and therefore, $\phi_{ab}(x) = \phi_b(x)$ for $x \geq x^*$. The piecewise MPNE ϕ_{ab} has been identified in Rowat (2007).

Let us denote by $h_c(x)$ the solution of (11) satisfying $h_c(0) = 0$ and let define $\phi_c(x) = \max(h_c(x), 0)$. The strategies $\phi_1 = \phi_2 = \phi_c$ constitute a symmetric nonlinear MPNE.

For each $x_0 \in \mathbb{R}_+$ let us define the set

$$H(x_0) = \{\eta \in \mathbb{R}_+ \mid \phi_c(x_0) \leq \eta \leq \phi_{ab}(x_0)\}. \tag{14}$$

Given an initial state $x_0 \in \mathbb{R}_+$, for each choice of $\eta \in H(x_0)$ there exists a (globally defined) symmetric MPNE $\phi_1 = \phi_2 = \max(h(x), 0)$ satisfying the transversality condition (7) with h the solution of (11) with $h(x_0) = \eta$. The nonlinear strategies defined by condition $h(x_0) = \eta$ can be explicitly computed (see Dockner and Long (1993), Rubio and Casino (2002), Rowat (2007)). Outside of region $H(x_0)$ the symmetric nonlinear strategies are locally defined and/or singular. As previously noted, we rule out these strategies and focus exclusively on globally defined MPNE. The set $H(x_0)$ defined in (14) is represented in Figure 1 with continuous line in the shaded region. In this figure we also represent the piecewise linear MPNE ϕ_b and ϕ_{ab}, and the nonlinear MPNE ϕ_c.

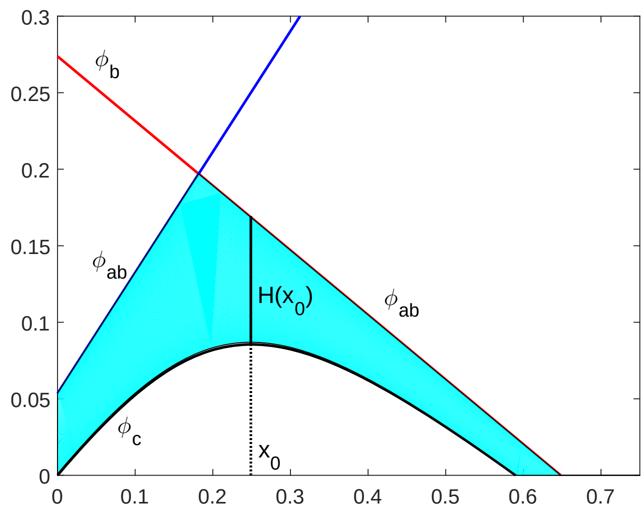

Figure 1: The set $H(x_0)$ defined in (14) (continuous line in the shaded region). The MPNE ϕ_b, ϕ_{ab} and ϕ_c for the values of the parameters $A = 0.5$, $\phi = 1$, $\rho = 0.1$ and $\alpha = 0.5$.

We remark that obviously, there exist nonsymmetric solutions of (8). For given data $\eta_1 \in H(x_0)$, $\eta_2 \in H(x_0)$ with $x_0 \in \mathbb{R}_+$, let $(h_1(x), h_2(x))$ be the solution of (8) with $h_1(x_0) = \eta_1$ and $h_2(x_0) = \eta_2$. We assume, without loss of generality, that $\eta_1 \leq \eta_2$. Denoting by $x_1^* > 0$ the point satisfying $h_1(x_1^*) = 0$, assumption $\eta_1 \leq \eta_2$ implies that $h_2(x_1^*) \geq 0$. Let $\tilde{h}_2(x)$ be the solution of the differential equation

$$(\tilde{h}_2 - \alpha x)\tilde{h}_2' = (\rho + \alpha)(\tilde{h}_2 - A) + \varphi x, \tag{15}$$

with $\tilde{h}_2(x_1^*) = h_2(x_1^*)$. Let denote by x_2^* $(x_2^* \geq x_1^*)$ the point with $\tilde{h}_2(x_2^*) = 0$. We define

$$(\phi_1(x), \phi_2(x)) = \begin{cases} (h_1(x), h_2(x)), & 0 \leq x \leq x_1^*, \\ (0, \tilde{h}_2(x)), & x_1^* \leq x \leq x_2^*, \\ (0, 0), & x \geq x_2^*. \end{cases} \tag{16}$$

It is possible to show numerically that if $\eta_i \in H(x_0)$, $i = 1, 2$, equation (3) with $u_i = \phi_i(x)$, defined by (16) has an asymptotically stable positive steady-state, so that the transversality condition (7) is satisfied and the pair $(\phi_1(x), \phi_2(x))$ constitutes a nonsymmetric MPNE defined for all $x \in \mathbb{R}_+$.

In Figure 2 we represent two different nonsymmetric MPNE, (ϕ_1^1, ϕ_2^1) and (ϕ_1^2, ϕ_2^2), corresponding to the initial values $(\phi_1^1(0.1), \phi_2^1(0.1)) = (0.06, 0.1)$ and $(\phi_1^2(0.5), \phi_2^2(0.5)) = (0.04, 0.06)$, respectively. For reference

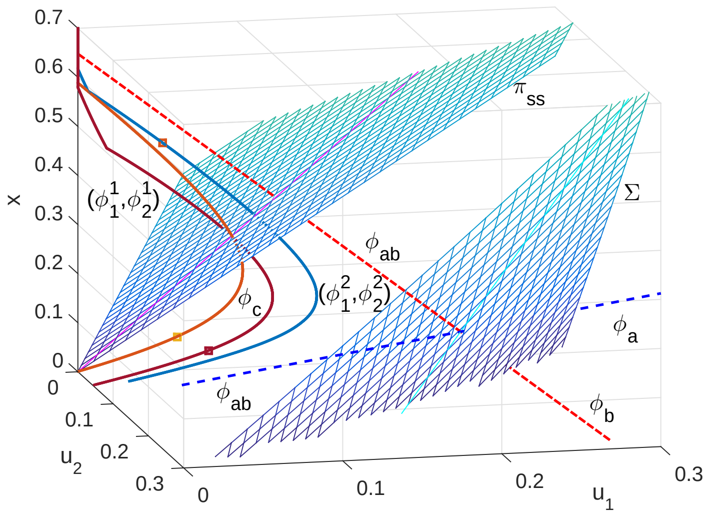

Figure 2: Nonsymmetric MPNE labelled (ϕ_1^r, ϕ_2^r), $r = 1, 2$. The plane of possible steady-states labelled Π_{ss} and part of the locus of singular points labelled Σ.

we also represent the symmetric MPNE ϕ_b, ϕ_{ab} and ϕ_c. This figure includes the plane Π_{ss} of possible stationary steady-states for the dynamics (3) defined by the equation $u_1 + u_2 - \alpha x = 0$, as well as part of the cone of singular points (labelled Σ) defined by $\det M(u_1, u_2, x) = 0$ where M is the matrix defined in (9). The possible (symmetric or nonsymmetric) MPNE remain in the interior of Σ.

For illustration purposes the left-hand chart in Figure 3 shows the optimal time trajectories of the controls and the right-hand chart the optimal time trajectory of the state when players use the strategies $u_i = \phi_i^1(x)$, $i = 1, 2$, where (ϕ_1^1, ϕ_2^1) is the MPNE represented in Figure 2 and $x_0 = 0.1$. We recall that (ϕ_1^1, ϕ_2^1) is univocally determined by conditions $(\phi_1^1(0.1), \phi_2^1(0.1)) = (0.06, 0.1)$. More precisely, the right-hand chart in Figure 3 represents $x(t)$, $t \geq 0$, the solution of (3) with $u_i = \phi_i^1(x)$, $i = 1, 2$ and $x(0) = 0.1$. The left-hand chart in Figure 3 presents $u_1(t) = \phi_1^1(x(t))$ the lower (continuous line) curve and $u_2(t) = \phi_2^1(x(t))$ the upper (dashed line) curve.

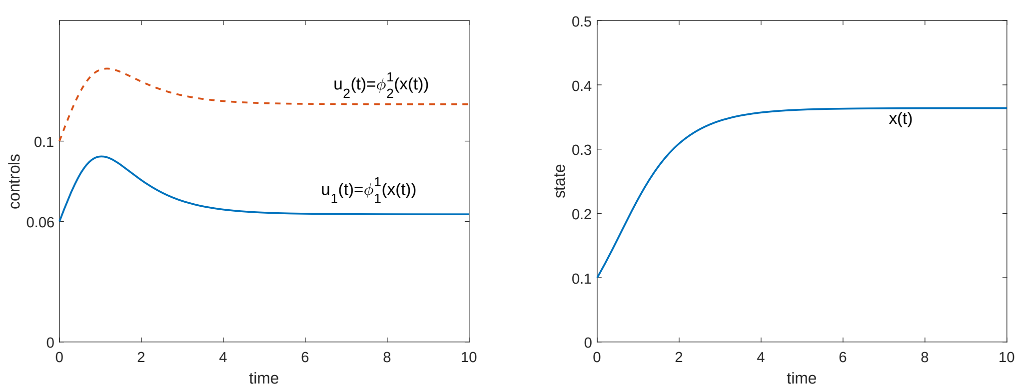

Figure 3: Optimal time rajectories of the controls (left) and of the state (right) corresponding to the MPNE (ϕ_1^1, ϕ_1^2) in Figure 2. On the left figure, with continuous line $u_1(t) = \phi_1^1(x(t))$ and with dashed line $u_2(t) = \phi_2^1(x(t))$.

The differential equation (15) can be obtained taking into account that if $\phi_1(x) = 0$, then the HJB equation (5) for player 2 particularizes as

$$\rho V_2(x) = \max_{u_2 \in \mathbb{R}_+} \Big\{ u_2\big(A - \frac{1}{2}u_2\big) - \frac{\varphi}{2}x^2 + \frac{d}{dx}V_2(x)\big(u_2 - \alpha x\big) \Big\}.$$

Using now the first-order condition (6) and differentiating the previous equation we get (15) (see Rowat, 2007).

We also remark that, as long as $\phi_i > 0$, $i = 1, 2$, any MPNE should be of the form $\phi_i = h_i(x)$, $i = 1, 2$, with $(h_1(x), h_2(x))$ a solution of (8). Although there is no explicit solution of system (8) available, it can be numerically solved without difficulty.

3 The reduced game

In view of the analysis of the preceding section it is apparent that, for a given initial state $x_0 > 0$, the players should choose initial conditions for their control variables $\eta_i \in H(x_0)$, $i = 1, 2$ in order to determine which MPNE (ϕ_1, ϕ_2) defined by (16) will be played. We restrict our attention to the region defined by $\eta_i \in H(x_0)$, $x_0 > 0$, because is in this region where we can guarantee the existence of globally defined (not necessarily symmetric) MPNE with $\phi_i(x_0) = \eta_i$.

We observe that once a choice of initial data $\eta_i \in H(x_0)$, $i = 1, 2$, is made there is a unique MPNE (ϕ_1, ϕ_2) such that $\phi_i(x_0) = \eta_i$, $i = 1, 2$. Being a MPNE, the players have no incentive to deviate unilaterally from the feedback strategy $u_i = \phi_i(x)$. Therefore, the equilibrium $(\phi_1(x), \phi_2(x))$ actually played in the differential game is completely determined by the choice of η_i, $i = 1, 2$.

This fact raises the main research question of this paper, that is, which η_i will player i possibly choose in absence of preplay communication?

In order to be more precise, in the sequel the MPNE given by (16) when the initial choice of controls is $\phi_i(x_0) = \eta_i \in H(x_0)$, $i = 1, 2$ is denoted by $(\phi_1(x, \eta_1, \eta_2), \phi_2(x, \eta_1, \eta_2))$.

We can reformulate the problem of choosing (η_1, η_2), given $x_0 > 0$, as a symmetric game in normal form (which we call the reduced game). The set of strategies of the two players is $H(x_0)$, and the payoff functions are defined as $V_i(x_0, \cdot, \cdot) : H(x_0) \times H(x_0) \to \mathbb{R}$ with $V_i(x_0, \eta_1, \eta_2)$ the value function of player i, $i = 1, 2$ when the players follow the strategy $u_i = \phi_i(x, \eta_1, \eta_2)$, $i = 1, 2$, and the initial state is $x(0) = x_0$. The reduced game is played in pure strategies.

We have the following results

Proposition 1 *Let $x_0 > 0$. If $\eta_1 \le \eta_2$, $\eta_i \in H(x_0)$, $i = 1, 2$, then*

$$V_i(x_0, \eta_1, \eta_1) \ge V_i(x_0, \eta_2, \eta_2), \quad i = 1, 2. \tag{17}$$

Proof. The symmetry of the game implies $V_1(x_0, \eta, \eta) = V_2(x_0, \eta, \eta)$ for all $\eta \in H(x_0)$. Therefore, it is enough to prove the proposition for player 1. Let $(\phi(x, \eta, \eta), \phi(x, \eta, \eta))$ be the (symmetric) solution of (8) with $\phi(x_0, \eta, \eta) = \eta$.

For $\eta \in H(x_0)$, the strategies $u_1 = u_2 = \phi(x, \eta, \eta)$ are positive along the solution of (3) with $u_1 = u_2 = \phi(x, \eta, \eta)$. Then, using the first-order condition (6) in the HJB equations (5) one has

$$V_i(x_0, \eta_i, \eta_i) = \frac{1}{\rho}\Big(\eta_i\big(A - \frac{\eta_i}{2}\big) - \frac{\varphi}{2}x_0^2 + (\eta_i - A)(2\eta_i - \alpha x_0)\Big), \quad i = 1, 2.$$

Then,

$$V_1(x_0, \eta_1, \eta_1) - V_1(x_0, \eta_2, \eta_2) = (\eta_1 - \eta_2)\left(\frac{3}{2}(\eta_1 + \eta_2) - (A + \alpha x_0)\right).$$

Taking into account that for $\eta_1, \eta_2 \in H(x_0)$ the inequality

$$\frac{3}{2}(\eta_1 + \eta_2) - (A + \alpha x_0) \le 0$$

is satisfied, we conclude that $V_1(x_0, \eta_1, \eta_1) - V_1(x_0, \eta_2, \eta_2) \ge 0$. □

Remark 1 *Proposition 1 shows that if the players restrict themselves to play symmetric strategies, then the MPNE leading to the greatest payoff requires choosing the lowest initial value of the control variable, that is, strategy ϕ_c. Note that this requires previous communication (cheap talk) between the two players in order to agree on the initial data to be chosen. This is the most common point of view in the literature when cheap talk is used to implement a MPNE as close as possible to the cooperative solution. However, as next propositions show if cheap talk is not allowed and depending on the initial value of the state, the outcome could be different.*

Proposition 2 *Let $x_0 > 0$. If $\eta_1 \leq \eta_2$, $\eta_i \in H(x_0)$ and $(\eta_1 + \eta_2)/2 \geq -A + \alpha x_0$, then*

$$V_1(x_0, \eta_1, \eta_2) \leq V_2(x_0, \eta_1, \eta_2). \tag{18}$$

Proof. For $\eta_i \in H(x_0)$, $i = 1, 2$, the strategies $u_i = \phi_i(x, \eta_1, \eta_2)$ are positive along the solution of (3) with $u_i = \phi_i(x, \eta_1, \eta_2)$. Then, using the first-order condition (6) in the HJB equations (5) one has

$$V_i(x_0, \eta_1, \eta_2) = \frac{1}{\rho}\Big(\eta_i\big(A - \frac{\eta_i}{2}\big) - \frac{\varphi}{2}x_0^2 + (\eta_i - A)(\eta_1 + \eta_2 - \alpha x_0)\Big), \quad i = 1, 2.$$

The expressions for the value function can be easily manipulated to arrive to

$$V_2(x_0, \eta_1, \eta_2) - V_1(x_0, \eta_1, \eta_2) = \frac{1}{\rho}\Big(\frac{\eta_1 + \eta_2}{2} + A - \alpha x_0\Big)(\eta_2 - \eta_1).$$

Therefore, if $(\eta_1 + \eta_2)/2 \geq -A + \alpha x_0$ and $\eta_1 \leq \eta_2$, inequality (18) is proved. □

Remark 2 *Condition $(\eta_1 + \eta_2)/2 \geq -A + \alpha x_0$ is satisfied for all $\eta_1, \eta_2 \in H(x_0)$ if the model parameters satisfy the following two conditions*

$$-3 + \alpha^2 + \alpha\rho < 0, \quad -3 + \alpha^4 + (6 - \rho^2)\alpha^2 + 4\alpha\rho < 0.$$

The first condition is satisfied if $\alpha < 1$ and $\rho < 1$. The second condition is satisfied if α is moderately small (more precisely if $\alpha < 0.47$). These two conditions guarantee that the point x_b where $\phi_b(x_b) = \phi_{ab}(x_b) = 0$ satisfies $x_b = -B_b/C_b \leq A/\alpha$. Therefore, if $x_0 \leq x_b$, then $\eta \geq -A + \alpha x_0$ for all $\eta \in H(x_0)$, and as a consequence the condition on the mean of η_1 and η_2 in the statement of the proposition is fulfilled. If $x_0 > x_b$ the only choice leading to a MPNE is $\eta_1 = \eta_2 = 0$.

Remark 3 *Proposition 2 shows that the player who gets the highest payoff is the player who initially chooses the highest control variable (η). From Proposition 2 we infer that, in absence of preplay communication, player i, $i = 1, 2$ has an incentive to choose the MPNE defined with η_i as high as possible, regardless of the choice of his/her opponent. Note that by the symmetry of the game, if $\eta_1 \leq \eta_2$, then $V_1(x_0, \eta_1, \eta_2) \leq V_2(x_0, \eta_1, \eta_2) = V_1(x_0, \eta_2, \eta_1)$ and similarly for player 2. That is, each player is better off if he or she chooses an initial $\eta \in H(x_0)$ bigger than the choice of his/her opponent.*

The following proposition compares the payoffs for nonsymmetric MPNE in more detail.

Proposition 3 *Let $x_0 > 0$ and $\eta_1 \in H(x_0)$. If $\eta_{2,1} \leq \eta_{2,2}$ with $\eta_{2,1}, \eta_{2,2} \in H(x_0)$, then*

$$V_2(x_0, \eta_1, \eta_{2,1}) \leq V_2(x_0, \eta_1, \eta_{2,2}) \tag{19}$$

if and only if

$$(\eta_{2,1} + \eta_{2,2})/2 + \eta_1 - \alpha x_0 \geq 0. \tag{20}$$

Proof. Reasoning as in Proposition 2 we arrive to

$$V_2(x_0, \eta_1, \eta_{2,1}) - V_2(x_0, \eta_1, \eta_{2,2}) = \frac{1}{\rho}(\eta_{2,1} - \eta_{2,2})\left(\frac{1}{2}(\eta_{2,1} + \eta_{2,2}) + \eta_1 - \alpha x_0\right).$$

Therefore, if $\eta_{2,1} \leq \eta_{2,2}$, then $V_2(x_0, \eta_1, \eta_{2,1}) - V_2(x_0, \eta_1, \eta_{2,2}) \leq 0$ if and only if $(\eta_{2,1} + \eta_{2,2})/2 + \eta_1 - \alpha x_0 \geq 0$. □

Remark 4 *Condition (20) is satisfied if both players choose initial data* $\eta_1, \eta_{2,1}, \eta_{2,2}$ *above the plane* Π_{ss} *of possible steady states for the dynamics (3). The plane* Π_{ss} *is defined by the equation* $u_1 + u_2 - \alpha x = 0$*. If* $(\phi_1(x), \phi_2(x))$ *is the MPNE played, then the steady state* x_{ss} *is characterized by* $\phi_1(x_{ss}) + \phi_2(x_{ss}) - \alpha x_{ss} = 0$.

Remark 5 *If condition (20) is satisfied, Proposition 3 establishes that regardless of the choice of the opponent, each player gets a greater payoff, the greater his/her choice of the initial control variable. Then, it is clear that the more likely outcome of the game is both players choosing* $\eta_1 = \eta_2 = \phi_{ab}(x_0)$*. That is, the more likely equilibrium played in the original differential game is the symmetric piecewise MPNE* ϕ_{ab}*. In fact if condition (20) is satisfied, then* $\eta_1 = \eta_2 = \phi_{ab}(x_0)$ *is the Nash equilibrium of the reduced game as next proposition shows.*

Proposition 4 *Let* $\tilde{x}_0$ *such that* $\phi_{ab}(\tilde{x}_0) = (\alpha/2)\tilde{x}_0$ *and* $x^* = E/D$ *the point at which fucntions* h_a *and* h_b *intersect. Then,*

1. *If* $x_0 \leq \tilde{x}_0$*, then* $\eta_1 = \eta_2 = \phi_{ab}(x_0)$ *is the only Nash equilibrium of the reduced game.*
2. *If* $x_0 > x^*$*, then* $\eta_1 = \eta_2 = \phi_c(x_0)$ *is the only Nash equilibrium of the reduced game.*

Proof. If $x_0 \leq \tilde{x}_0$, then condition (20) is satisfied for all $\eta_1, \eta_{2,1}, \eta_{2,2} \in H(x_0)$. Proposition 3 can be rephrased by saying that $\eta_{2,2}$ $(\eta_{2,2} > \eta_{2,1})$ is a dominant strategy for player 2 regardless of the strategy η_1 chosen by player 1. Using the symmetry of the game, if $\eta_{1,1}, \eta_{1,2}, \eta_2 \in H(x_0)$ with $\eta_{1,1} \leq \eta_{1,2}$, then

$$V_1(\eta_{1,1}, \eta_2) \leq V_1(\eta_{1,2}, \eta_2), \quad \forall \eta_2 \in H(x_0).$$

Therefore, $\eta_{1,2}$ $(\eta_{1,2} > \eta_{1,1})$ is also a dominant strategy for player 1 for all possible choices η_2 of player 2. Given that $\phi_{ab}(x_0) = \max H(x_0)$, $\eta_1 = \eta_2 = \phi_{ab}(x_0)$ constitutes the only Nash equilibrium of the reduced game.

If $x_0 > x^*$, then $(\eta_{2,1} + \eta_{2,2})/2 + \eta_1 - \alpha x_0 \leq 0$ and consequently from Proposition 3, $V_2(x_0, \eta_1, \eta_{2,1}) - V_2(x_0, \eta_1, \eta_{2,2}) \geq 0$. Now the dominant strategy for player 2 is $\eta_{2,1}$ $(\eta_{2,1} < \eta_{2,2})$ regardless of the choice η_1 of player 1. The conclusion is reached using the same reasoning as before. □

Remark 6 *The consequence of Proposition 3 is striking. Even if there is preplay communication and an agreement to implement* $\eta_1 = \eta_2 = \phi_c(x_0)$ *has been reached (which gives the highest payoff for both players with symmetric MPNE), if* $x_0 \leq \tilde{x}_0$ *the players have an incentive to deviate from the initial agreement in a sort of prisoner's dilemma. Both players have an incentive to choose a higher initial value of the control variable, and in consequence, the plausible implemented equilibrium is the symmetric MPNE* $(\phi_{ab}(x), \phi_{ab}(x))$ *(which gives the lowest payoff for both players with symmetric MPNE).*

If the objective of preplay communications is to implement the MPNE which gives an asymptotic steady-state as close as possible to the cooperative steady-state, from Proposition 3 we infer that the objective can only be reached for $x_0 \geq x^*$*. If this is the case, one can remove preplay communications and still have the desired outcome. However, if* $x_0 \leq \tilde{x}_0$*, the objective is unattainable.*

Remark 7 *It is worth noting that interestingly enough in all cases the players implement a symmetric MPNE without imposing artificially symmetric play.*

4 Concluding remarks

Multiple equilibria in dynamic games are a fundamental feature of infinite horizon models. This note analyzes the problem of how to select an equilibrium in a differential game when there is an infinite number of equilibria. Authors have suggested various selection criteria for choosing one among the many equilibria. In the applied dynamic games literature the problem of the selection of equilibria has been overtook by considering preplay negotiations or communications between the players, also called cheap talk. The selection of a specific equilibrium is determined in a preplay phase where agents have cheap talks and agree on the selection of the pair of strategies. The main objective of this note was to analyze which would be the most likely equilibrium without preplay communications in a well-known class of differential games.

In particular, we analyzed the linear-quadratic differential game proposed by Dockner and Long (1993) to study a transboundary pollution problem. Existing studies have showed that there exists a continuous of nonlinear symmetric equilibria. In this note we show that when symmetry is not imposed there is also a continuous of nonsymmetric nonlinear equilibria. Furthermore, if the initial state is small enough the nonlinear strategies are not likely to be used as optimal strategies regardless of the existence of preplay communications. We concluded that in the presence of multiple equilibria and without cheap talk the most likely equilibria are symmetric piecewise linear Markov perfect Nash equilibria. For large values of the initial state the most likely strategy is the symmetric nonlinear MPNE that gives the closest asymptotic steady-state to the cooperative steady-state.

It is worth mentioning that the techniques and results presented here are applicable not only to other linear-quadratic differential games, but to more general differential games.

Finally, we want to emphasize two points of our analysis. The first point concerns the possibility for players to play asymmetric strategies. The conclusion would be different if players were restricted to using symmetric strategies. The second point refers to the fact that although symmetric play has not been imposed, the most likely equilibria are symmetric.

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