Existence and uniqueness of optimal dynamic pricing and advertising controls without concavity

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Abstract: We consider a pricing and advertising dynamic-optimization problem where the goodwill dynamics evolve à la Nerlove-Arrow. The firm maximizes its profit over a finite-planning horizon corresponding to the product’s lifespan, and it turns out that the Hamiltonian is non-concave. We show the existence and uniqueness of an optimal solution under some mild conditions.

Keywords: Advertising, pricing, Goodwill Stock, Nerlove-Arrow

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1 Introduction

An investment in advertising has a long-lasting effect, that is, it not only boosts the firm’s demand and revenues today but also its future ones. The main reason for this is that advertising is a main input in building the brand’s reputation, which in turn, is a significant driver of current and future sales. This simple observation in itself explains the existence of an extensive literature on dynamic advertising models and decisions, which started more than five decades ago and is still ongoing.

In this paper, we consider a continuous-time dynamic-optimization model à la Nerlove-Arrow and determine optimal pricing and advertising decisions. The cornerstone piece in a Nerlove-Arrow (N-A) model is the goodwill stock, a variable that positively affects sales. The firm can raise its goodwill stock, which is also referred to in the literature and in practice as brand reputation or brand equity, by investing in advertising, while part of this goodwill is lost as a result of consumers forgetting of the advertisement messages. In the parlance of dynamic optimization, the goodwill is a state variable that summarizes, in a compact way, the firm’s current and past advertising outlays on its sales, which can also depend on other decision variables such as price and quality.

Our contribution to the Nerlove-Arrow class of advertising models is threefold. First, in any monopoly-profit-maximization model, one expects the price to be a decision variable, not a given parameter. Indeed, there is no valid conceptual reason to assume both a monopolistic environment and an exogenously given price during the whole planning horizon. By letting the price be a control variable, we add some realism to the stream of literature that only considered advertising. Second, as any product has a finite lifespan, we believe that the model must also have a finite terminal date. This reasoning especially holds when one assumes away any quality improvements over time for the product, which has been the norm rather than the exception in the dynamic advertising literature (see El Ouardighi and Pasin (2006) and Reddy et al. (2016) for a discussion). The advantage of an infinite planning horizon resides in the fact that the optimal (or equilibrium in a competitive model) solution is stationary, which is typically easier to compute than a time-varying solution. Here, we stick to a finite horizon, and by the same token, provide insights into the firm’s advertising and pricing trajectories in a more realistic context, and beyond the steady-state values that are often the focal point of the analysis in infinite-horizon models. The third contribution concerns the existence and uniqueness of an optimal solution. In our case, it turns out that the Hamiltonian function corresponding to the profit-maximization problem is not concave. Consequently, the usual sufficient conditions of optimality cannot be applied. We show, under some mild conditions, that the non-oscillating interior pricing and advertising solution is indeed optimal.

We shall refrain from extensively reviewing the literature and refer the reader to the comprehensive surveys in Feichtinger et al. (1994) and Huang et al. (2012). Instead, we focus on the literature that is directly relevant to our paper, namely, the contributions that used an N-A model, which is referred to as a capital stock advertising model in Feichtinger et al. (1994), and highlight our contribution.

Table 1 provides an updated list of the papers covered in the survey in Huang et al. (2012). Each paper is characterized in terms of four features: (i) the price being or not being a decision variable; (ii) the planning horizon (finite or infinite); (iii) the type of strategic interactions; and finally (iv) a brief description of the main topic (additionally to the determination of optimal (or equilibrium) advertising policies). A first observation based on Table 1 is that few papers have investigated pricing issues. Indeed, only 6 of the 36 listed papers included price as a decision variable, and in all these cases, the planning horizon was infinite. The conclusion here is that we do not know much about optimal pricing policies in the (probably more realistic) case of a finite terminal date. Note that the previous literature surveys in Sethi (1977), Feichtinger et al. (1994) and Erickson (1995) included 19 papers using N-A dynamics, and only two of them had price as a decision variable, and both retained an infinite planning horizon. As price is clearly profit-relevant, we do believe that including it as a decision variable in a finite horizon setting fills an important gap in the literature. A second observation is that the N-A framework has been used in a wide variety of topics, both in an oligopoly/monopoly settings and in supply chains, which signals the framework’s broad appeal.

1An early survey of the literature is Sethi (1977). See also the survey in Erickson (1995) and the books by Erickson (1991) and Jørgensen and Zaccour (2014).
Most papers on supply chains have dealt with coordination strategies with a clear interest in cooperative advertising. Finally, we note that 20 papers retained an infinite horizon, whereas 16 had a finite terminal date. In the later case, the focus has often been on the introduction of a new product or the management of advertising for a perishable (seasonal) product.

Our main results show that the optimal pricing policy follows the goodwill stock and is time-invariant, that is, its dependency with respect to time is only indirect through the value of the goodwill stock. Further, the advertising trajectory is convex and monotone. Depending on the parameter values, advertising expenditures and the goodwill stock can be increasing or decreasing over time.

The rest of the paper is organized as follows: Section 2 introduces the model. Section 3 presents the optimal non-oscillating interior solution, whose existence and uniqueness are shown in Section 4 under some mild conditions. Section 5 briefly concludes.

### 2 Model

We consider a planning horizon \([0, T]\), with time \(t\) running continuously. The initial date corresponds to the introduction of a new product by the firm, and \(T\) to the end of the selling season. After \(T\), the product loses its appeal because of, e.g., a change of season for fashion apparel, or the arrival of a new version for software. Denote by \(p(t)\) the product’s price at time \(t \in [0, T]\), and by \(G(t)\) the goodwill stock (brand reputation or brand equity). The demand function is given by

\[
q(t) = \alpha G(t) - \beta p(t),
\]

where \(\alpha\) and \(\beta\) are strictly positive parameters. Following a long tradition in economics and management science, our demand function is affine, however with the additional feature that the market potential at any \(t\) is proportional to the brand goodwill.

**Remark 1** Our demand function is microfounded. Indeed, let the utility function of the representative consumer be given by the following quadratic function:

\[
U(q, y) = \phi q - \frac{\kappa q^2}{2} + y,
\]

where \(q\) is the demand for the firm’s product, \(y\) is a composite good whose price is normalized to one, and \(\phi\) and \(\kappa\) are positive parameters. The budget constraint is given by \(pq + y = I\), where \(p\) is the price of the product and \(I\) the income. Maximizing \(U(q, y)\) subject to the budget constraint yields the demand

\[
q = \frac{\phi - p}{\kappa}.
\]

It suffices to set \(\alpha G = \frac{\phi}{\kappa}\) and \(\beta = \frac{1}{\kappa}\) to get (1).

The goodwill stock evolves à la Nerlove-Arrow (1962), that is,

\[
\frac{dG}{dt}(t) = \dot{G}(t) = ka(t) - \delta G(t), \quad G(0) = G_0 > 0,
\]

where \(a(t)\) is the advertising investment at time \(t\), \(k > 0\) is the marginal efficiency of advertising, and \(\delta\) is the decay rate. Following a broad literature on dynamic advertising models (see, e.g., the book by Jørgensen and Zaccour (2004) and the surveys by Feichtinger et al. (1994) and Huang et al. (2012)), we assume that the advertising cost is convex increasing and given by the quadratic function

\[
C(a) = \frac{\omega}{2}a^2(t),
\]

where \(\omega\) is a positive parameter. Without any loss of generality, we suppose that the marginal production cost is constant and set equal to zero. An implication of this assumption is that the price can also be interpreted as a profit margin.
Table 1: Articles using goodwill dynamics

<table>
<thead>
<tr>
<th>Papers in chronological order</th>
<th>Price</th>
<th>Horizon</th>
<th>Interaction</th>
<th>Main topic</th>
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<tr>
<td>Jørgensen, Sigué and Zaccour (2000)</td>
<td>N</td>
<td>Inf</td>
<td>V</td>
<td>Cooperative advertising</td>
</tr>
<tr>
<td>Jørgensen, Sigué and Zaccour (2001)</td>
<td>Y</td>
<td>Inf</td>
<td>V</td>
<td>Cooperative advertising</td>
</tr>
<tr>
<td>Jørgensen, Taboubi and Zaccour (2001)</td>
<td>N</td>
<td>Inf</td>
<td>V</td>
<td>Non-price promotion</td>
</tr>
<tr>
<td>Cellini and Lambertini (2003)</td>
<td>N</td>
<td>Inf</td>
<td>H</td>
<td>Quantity competition</td>
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<tr>
<td>Jørgensen, Taboubi and Zaccour (2003)</td>
<td>N</td>
<td>Inf</td>
<td>V</td>
<td>Cooperative advertising</td>
</tr>
<tr>
<td>Jørgensen and Zaccour (2003)</td>
<td>N</td>
<td>F</td>
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<td>Non-price promotion</td>
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<td>Lambertini (2005)</td>
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<td>Location choice</td>
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<td>Martín-Herrán, Taboubi and Zaccour (2005)</td>
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<td>Viscolani and Zaccour (2009)</td>
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<td>Buratto, Grosset and Viscolani (2006)</td>
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<tr>
<td>Nair and Narasimhan (2006)</td>
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<tr>
<td>Bass, Bruce, Majumdar and Murthi (2007)</td>
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<td>Wearout effects of advertising themes</td>
</tr>
<tr>
<td>Buratto, Grosset and Viscolani (2007)</td>
<td>N</td>
<td>F</td>
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<td>Launching a new product</td>
</tr>
<tr>
<td>Amrouche, Martín-Herrán and Zaccour (2008)</td>
<td>Y</td>
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<td>Marinelli and Savin (2008)</td>
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<td>Zaccour (2008)</td>
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<td>Sigüé and Chintagunta (2009)</td>
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<td>Bertuzzi and Lambertini (2010)</td>
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<td>De Giovanni (2011)</td>
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<td>Quality vs advertising support</td>
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<tr>
<td>Martín-Herrán, Sigué and Zaccour (2011)</td>
<td>N</td>
<td>Inf</td>
<td>V+H</td>
<td>Franchising</td>
</tr>
<tr>
<td>Lambertini and Zaccour (2015)</td>
<td>Y</td>
<td>Inf</td>
<td>H</td>
<td>Market power and investment incentives</td>
</tr>
</tbody>
</table>

Legend: F (finite); Inf (infinite); V (vertical interaction, supply chain); H (horizontal interaction, oligopoly); M (monopoly, no strategic interaction).

Assuming profit-optimizing behavior, the firm maximizes its stream of profits over the planning horizon, that is,

\[
\max_{p(t), a(t)} \pi = \int_0^T \left( p(t) (\alpha G(t) - \beta p(t)) - \frac{\omega}{2} a^2(t) \right) dt + S(G(T)), \tag{3}
\]

subject to (2),

where \(S(G(T))\) is the salvage value of the brand at \(T\), which measures the future profits that the firm can obtain from marketing products under the same brand name. We suppose that \(S(G(T))\) can be approximated by a linear function, that is, \(S(G(T)) = sG(T)\), where \(s\) is a positive parameter.

To recapitulate, we have a finite-horizon optimal-control problem with two control variables (price and advertising) and one state variable (brand reputation).

3 Optimal solution

We will highlight below that, since the optimization problem at hand is non-concave and the control set is unbounded, the existence and uniqueness of an optimal interior solution are far from assured. We will proceed in two steps. First, we present the optimal pricing and advertising decisions assuming the existence of an interior solution. Second, we provide a set of sufficient conditions that guarantees the existence and uniqueness of an optimal interior solution.
3.1 An interior solution

We start by making the following assumption, which will imply that the solution is not oscillating.

**Assumption 1** The parameter values satisfy the following condition:

\[ \delta^2 - \frac{\alpha^2 k^2}{2\beta \omega} > 0. \]  

(4)

**Proposition 1** Under condition (4), assume that there exists an interior solution; it is then given by

\[ p(t) = \frac{\alpha G(t)}{2\beta}, \]  

(5)

\[ a(t) = \frac{k (2\beta s ((\delta + v)e^{v(T + t)} - (\delta - v)e^{v(T - t)}) + \alpha^2 G_0 (e^{v(2T - t)} - e^{-v}))}{2\beta \omega ((\delta + v)e^{2vT} - (\delta - v))}, \]  

(6)

and the brand goodwill by

\[ G(t) = \frac{2\beta s (\delta^2 - v^2) (e^{v(T + t)} - e^{v(T - t)}) - \alpha^2 G_0 ((\delta - v)e^{vt} - e^{v(2T - t)}(\delta + v))}{\alpha^2 ((\delta + v)e^{2vT} - (\delta - v))}, \]  

(7)

where

\[ v = \sqrt{\delta^2 - \frac{\alpha^2 k^2}{2\beta \omega}}. \]

**Proof.** To solve the firm’s optimal control problem, introduce the Hamiltonian

\[ H(p(t), a(t), G(t), \lambda(t)) = p(t) (\alpha G(t) - \beta p(t)) - \frac{\omega}{2} a^2(t) + \lambda(t) (ka(t) - \delta G(t)), \]  

(8)

where \( \lambda(t) \) is the adjoint variable appended to the state dynamics. Assuming an interior solution, the first-order optimality conditions are

\[ \frac{\partial H}{\partial p} = 0 \Leftrightarrow p(t) = \frac{\alpha G(t)}{2\beta}, \]  

(9)

\[ \frac{\partial H}{\partial a} = 0 \Leftrightarrow a(t) = \frac{\lambda(t) k}{\omega}, \]  

(10)

\[ \dot{\lambda}(t) = -\frac{\partial H}{\partial G} = -\alpha p(t) + \lambda(t) \delta, \quad \lambda(T) = s, \]  

(11)

\[ \dot{G}(t) = \frac{\partial H}{\partial \lambda} = ka(t) - \delta G(t), \quad G(0) = G_0. \]  

(12)

Substituting for the controls from (9)–(10) in the state and adjoint equations, we obtain the following linear-differential system:

\[ \dot{\lambda} = \delta \lambda - \frac{\alpha^2}{2\beta} G, \quad \lambda(T) = s, \]

\[ \dot{G} = \frac{k^2}{\omega} \lambda - \delta G, \quad G(0) = G_0, \]

which can be written \( \dot{Y} = AY \), where

\[ Y = \begin{pmatrix} \lambda \\ G \end{pmatrix}, \dot{Y} = \begin{pmatrix} \dot{\lambda} \\ \dot{G} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \delta & -\frac{\alpha^2}{2\beta} \\ \frac{k^2}{\omega} & -\delta \end{pmatrix}. \]

We have

\[ \det(A) = -\delta^2 + \frac{\alpha^2 k^2}{2\beta \omega} \quad \text{and} \quad \text{Trace}(A) = 0. \]
Let $v_1$ and $v_2$ be the two eigenvalues of $A$; then,

$$v_1v_2 = \det(A) = -\delta^2 + \frac{\alpha^2 k^2}{2\beta \omega} \quad \text{and} \quad v_1 + v_2 = \text{Trace}(A) = 0.$$ 

Therefore, $v_2 = -v_1$ and $-v_1^2 = -\delta^2 + \frac{\alpha^2 k^2}{2\beta \omega}$. Set $v_1 = v$.

Under (4) the two real eigenvalues are $v$ and $-v$, where $v = \sqrt{\delta^2 - \frac{\alpha^2 k^2}{2\beta \omega}}$. Solving $(A - vI)V = 0$ and $(A + vI)V = 0$ provides the eigenvectors

$$V_1 = \left( \frac{1}{2\beta (\delta - v)} \right), \quad V_2 = \left( \frac{1}{2\beta (\delta + v)} \right).$$

Therefore, the general solution of the linear-differential system is given by

$$Y(t) = r_1 e^{vt} V_1 + r_2 e^{-vt} V_2,$$

that is,

$$\lambda(t) = r_1 e^{vt} + r_2 e^{-vt},$$

$$G(t) = r_1 \frac{2\beta (\delta - v)}{\alpha^2} e^{vt} + r_2 \frac{2\beta (\delta + v)}{\alpha^2} e^{-vt}.$$

Further, $r_1$ and $r_2$ are uniquely determined (using Cramer’s rule, for example) so that the solution satisfies the boundary conditions $\lambda(T) = s$ and $G(0) = G_0$. We obtain

$$r_1 = \frac{2\beta (\delta + v) se^{vT} - \alpha^2 G_0}{2\beta ((\delta + v) e^{2vT} - (\delta - v))},$$

$$r_2 = \frac{e^{2vT} \alpha^2 G_0 - 2\beta (\delta - v) se^{vT}}{2\beta ((\delta + v) e^{2vT} - (\delta - v))}.$$  \hspace{1cm} (13)  

Consequently, we get

$$G(t) = r_1 \frac{2\beta (\delta - v)}{\alpha^2} e^{vt} + r_2 \frac{2\beta (\delta + v)}{\alpha^2} e^{-vt},$$

$$p(t) = \frac{\alpha G(t)}{2\beta} = r_1 \frac{(\delta - v)}{\alpha} e^{vt} + r_2 \frac{(\delta + v)}{\alpha} e^{-vt},$$

$$a(t) = \frac{(r_1 e^{vt} + r_2 e^{-vt}) k}{\omega}.$$  \hspace{1cm} (15)  

Substituting for $r_1$ and $r_2$ into $G(t)$ and $a(t)$, we obtain the advertising and goodwill trajectories in the statement of the proposition.

The results in this proposition call for the following comments:

1. The pricing policy follows the goodwill: the higher is the goodwill, the higher is the price. This result is empirically observable where well-known brands do indeed command higher prices, and it can be explained by the fact that the market potential is increasing in the brand’s reputation. Note that although we have a finite-horizon model, the price does not depend explicitly on time, which is likely a by-product of not having the price in the state dynamics.

2. From (10), we note that the advertising investment is dictated by the familiar rule equating marginal cost (given by $wa$) to marginal revenue, which is measured by $k\lambda$, that is, the marginal efficiency of advertising in raising the goodwill times its shadow price. Observe that the advertising investment is time-varying and independent of the current value of the brand’s reputation. Recall that advertising only indirectly affects the demand, that is, through the brand’s reputation.
3. Specializing (7) for \( t = T \), we obtain the following terminal value for the brand’s reputation:

\[
G(T) = \frac{sk^2 (e^{2vT} - 1) + 2ve^{vT}\omega G_0}{\omega (\delta e^{2vT} - 1) + v(e^{2vT} + 1))},
\]

which is, as expected, increasing in the marginal valuation \( s \) of the goodwill stock at \( T \), and in the initial goodwill \( G_0 \).

Before showing the existence and uniqueness of the optimal solution under some mild conditions, we characterize the evolution over time of advertising, pricing and brand reputation.

3.2 Monotonicity of the optimal solution

**Proposition 2** If and only if

\[
s \leq \hat{s} = \frac{\alpha^2 G_0}{\beta (\delta + v)e^{vT} + (\delta - v)e^{-vT}},
\]

then advertising is monotonically decreasing over time for all \( t \in [0, T] \).

**Proof.** Recall that advertising is given by (see (6) and (15))

\[
a(t) = \frac{k(2\beta se^{vT}((\delta + v)e^{vT} - e^{-vt}(\delta - v)) + \alpha^2 G_0(e^{v(2T-t)} - e^{vt}))}{2\omega \beta ((\delta + v)e^{2vT} - (\delta - v))}
= \frac{(r_1 e^{vt} + r_2 e^{-vt})k}{\omega},
\]

which is strictly positive for all \( t \in [0, T] \).

Differentiating twice \( a(t) \) with respect to time, we get

\[
\dot{a}(t) = \frac{vk}{\omega} (r_1 e^{vt} - r_2 e^{-vt}),
\]
\[
\ddot{a}(t) = \frac{\omega^2 k}{\omega} (r_1 e^{vt} + r_2 e^{-vt}).
\]

Clearly, \( \ddot{a}(t) = v^2a(t) > 0 \) all \( t \in [0, T] \). Therefore, \( a(t) \) is convex in \( t \) and \( \dot{a}(t) \) is increasing on \([0, T]\). Substituting for \( r_1 \) and \( r_2 \) in \( \dot{a}(t) \) we obtain

\[
\ddot{a}(t) = \frac{vk(2\beta se^{vT}((\delta + v)e^{vT} + e^{-vt}(\delta - v)) - \alpha^2 G_0(e^{vt} + e^{v(2T-t)}))}{2\omega \beta ((\delta + v)e^{2vT} - (\delta - v))}.
\]

The value of \( \dot{a} \) at \( T \) is given by

\[
\dot{a}(T) = \frac{vk(2\beta se^{vT}((\delta + v)e^{vT} + e^{-vT}(\delta - v)) - 2\alpha^2 G_0 e^{vT})}{2\omega \beta ((\delta + v)e^{2vT} - (\delta - v))}.
\]

Straightforward computations lead to

\[
\dot{a}(T) \leq 0 \Leftrightarrow s \leq \frac{\alpha^2 G_0}{\beta ((\delta + v)e^{vT} + (\delta - v)e^{-vT})}.
\]

Since \( \dot{a}(t) \) is increasing on \([0, T]\), \( \dot{a}(t) \leq \dot{a}(T) \), \( \forall t \in [0, T] \).

Hence, if, and only if,

\[
s \leq \frac{\alpha^2 G_0}{\beta ((\delta + v)e^{vT} + (\delta - v)e^{-vT})},
\]

then \( \dot{a}(t) \leq 0 \) on \([0, T]\), and the stated result follows. \( \square \)
The above result is intuitive. Indeed, if $s$ is sufficiently low, then the firm should start by advertising at a relatively high level and decrease it over time. The earlier the advertising investment is made, the longer the period during which the firm enjoys a high goodwill. Two particular cases are worth mentioning. First, if the brand is new on the market, i.e., $G_0 = 0$, then from (17) we have

$$\dot{a}(t) = \frac{vk (2\beta se^{vT} ((\delta + v)e^{vt} + e^{-vt} (\delta - v)))}{2\omega \beta ((\delta + v)e^{vT} - (\delta - v))},$$

which is clearly positive, and therefore, advertising is increasing over time for any positive value of $s$.

**Proposition 3** If and only if

$$s \leq \hat{s} = \frac{2\omega R_0 \delta}{k^2 (e^{vT} + e^{-vT})},$$

then the price and goodwill are monotonically decreasing over time for all $t \in [0, T]$.

**Proof.** For the price to be interior, we must have $G(t) > 0$ for $t \in [0, T]$. Further,

$$\dot{p}(t) = \frac{\alpha \dot{G}(t)}{2\beta}.$$ 

Differentiating twice $G(t)$ with respect to time gives

$$\ddot{G}(t) = \frac{2\beta v}{\alpha^2} \left( r_1 (\delta - v)e^{vt} - r_2 (\delta + v)e^{-vt} \right),$$

$$\ddot{G}(t) = \frac{2\beta v^2}{\alpha^2} \left( r_1 (\delta - v)e^{vt} + r_2 (\delta + v)e^{-vt} \right).$$

Clearly, $\ddot{G}(t) = v^2 G(t)$ and hence $\dot{G}(t)$ is positive for $t \in [0, T]$; therefore, $G(t)$ is a convex function and $\dot{G}(t)$ is an increasing function. Compute $\dot{G}(T)$ to get

$$\dot{G}(T) = \left[ \frac{2v \beta}{\alpha^2} \frac{e^{vT}}{2\beta ((\delta + v)e^{vT} - (\delta - v))} \right] \times$$

$$\left( 2 (\delta - v) \beta (\delta + v) se^{vT} + 2 (\delta - v) \beta (\delta + v) se^{-vT} - \alpha^2 G_0 (\delta - v) - \alpha^2 G_0 (\delta + v) \right).$$

The term in square brackets is positive. Therefore, we have

$$\dot{G}(T) \leq 0 \Leftrightarrow 2 (\delta - v) \beta (\delta + v) se^{vT} + 2 (\delta - v) \beta (\delta + v) se^{-vT} - 2\alpha^2 G_0 \delta \leq 0$$

$$
\Leftrightarrow s \leq \frac{\alpha^2 G_0 \delta}{\beta (\delta + v) (\delta - v) (e^{vT} + e^{-vT})} = \frac{\alpha^2 G_0 \delta}{\beta (\delta^2 - v^2) (e^{vT} + e^{-vT})} = \frac{2\omega G_0 \delta}{k^2 (e^{vT} + e^{-vT})}.
$$

Since $\dot{G}(t)$ is increasing on $[0, T]$, $\dot{G}(t) \leq \dot{G}(T)$, $\forall t \in [0, T]$. Hence, the result. 

If the marginal salvage value is small enough, then the firm will reduce its advertising investment over time and, as a result, the goodwill decreases over time, and as does the market potential, given by $\alpha G(t)$. In turn, this brings down the price. As for the previous proposition, we note that if $s = 0$, then the condition in the proposition will be clearly always satisfied.

**Remark 2** We observe that $\hat{s} < \hat{s}$. Compute the difference

$$\hat{s} - \hat{s} = G_0 \left( \frac{k^2 \alpha^2 - 2\omega \beta s^2}{\beta ((\delta + v)e^{vT} + (\delta - v)e^{-vT}) k^2 (e^{vT} + e^{-vT})} \right) \left( e^{vT} + e^{-vT} \right) - 2\omega \delta \beta v (e^{vT} - e^{-vT}).$$
As the denominator is positive, the sign of the above expression is the same as the sign of the numerator. By (4), \(k^2\alpha^2 - 2\omega\beta\delta^2\) is negative and hence the result.

The behavior over time of the reputation stock is ambiguous. Indeed, differentiating \(G(t)\) with respect to time, we get
\[
\dot{G}(t) = \frac{\nu \left( k^2 s (e^{\nu(T+t)} + e^{\nu(T-t)}) - \omega G_0 \left( (\delta - v)e^{\nu t} + e^{\nu(2T-t)}(\delta + v) \right) \right)}{\omega (\delta + v)e^{2\nu T} - (\delta - v)},
\]
which shows that the sign of \(\dot{G}(t)\) depends on all the model’s parameter values. If \(s = 0\) (respectively, \(G_0 = 0\)), then \(\dot{G}(t)\) is negative (respectively, positive) for all \(t\).

### 4 Existence and uniqueness of the solution

Proposition 1 is stated assuming that a solution exists and is interior. This proposition provides a candidate, but it is not granted in our case that this candidate is indeed a solution, because the Hamiltonian in (8) is not concave. Indeed, the Hessian of the Hamiltonian function is given by
\[
\begin{bmatrix}
-2\beta & 0 & \alpha \\
0 & -\omega & 0 \\
\alpha & 0 & 0
\end{bmatrix},
\]
\[
H(p(t),a(t),G(t),\lambda(t)) = p(t) (\alpha G(t) - \beta p(t)) - \frac{\omega}{2} a^2(t) + \lambda(t) (ka(t) - \delta G(t)),
\]
which is clearly not negative semidefinite. Consequently, classical sufficient conditions of optimality cannot be invoked. We show in this section that our candidate is indeed a solution. To do so, consider the following optimal-control constrained problem (I):
\[
\max_{p(t),a(t)} \pi = \max_{p(t),a(t)} \int_0^T \left( p(t) (\alpha G(t) - \beta p(t)) - \frac{\omega}{2} a^2(t) \right) dt + s.G(T),
\]
\[
\dot{G}(t) = ka(t) - \delta G(t), \quad G(0) = G_0,
\]
\[
0 \leq a(t) \leq n, \quad 0 \leq p(t) \leq b,
\]
where \(n\) and \(b\) are positive real numbers.

**Proposition 4** Assume that
\[
\omega > \frac{k\alpha^2}{2\beta\delta} \left( \frac{G_0}{n} + \frac{k}{\delta} \right) + \frac{ks}{n},
\]
\[
b > \frac{\alpha}{2\beta} \left( G_0 + \frac{nk}{\delta} \right).
\]
Then, (5)–(7) is indeed the unique solution.

**Proof.** We shall prove that (5)–(7) is the unique optimal solution. The proof proceeds in three steps.

**Step 1** We first show that there exists a solution to problem (I). Set \(F(G,a,p) = p(\alpha R - \beta p) - \frac{\omega}{2} a^2\), \(g(G(T)) = sG(T)\) and \(f(G,a,p) = ka - \delta G\). Set \(K = [0, n] \times [0, b]\). It is clear that \(G(t)\) is bounded, that \(F\) and \(g\) are continuous functions, and that \(f\) is Lipschitz in the state variable and continuous in the control variable, such that the set of admissible solutions is non-empty. Consider the set
\[
Q(G) = \{(v,z) / v \leq F(G,a,p), \quad z = f(G,a,p), \quad (a,p) \in K\}.
\]
It is convex since \(F\) is concave with respect to the control \((a,p)\) and \(f\) is linear with respect to the control. Thus, by Filippov (1959) existence theorem, there exists a solution to the problem (see also the survey in Cesari (1983)).
Step 2 Let us show that a solution of (I) is interior, i.e., it must satisfy $0 < p(t) < b$ and $0 < a(t) < n$ for all $t$.

The solution to problem (I) maximizes the following Hamiltonian at each date $t$:

$$H = p(t)(\alpha G(t) - \beta p(t)) - \frac{\omega}{2}a^2(t) + \lambda(t)(ka(t) - \delta G(t)) + \epsilon(t)p(t) + \eta(t)(b - p(t)) + \alpha(t)a(t) + \psi(t)(n - a(t)),$$

where $\epsilon(t), \eta(t), \alpha(t)$ and $\psi(t)$ are Lagrange multipliers. Therefore, the following first-order conditions must be satisfied:

**Optimality conditions:**

$$\frac{\partial H}{\partial p} = 0 \iff \alpha G(t) - 2\beta p(t) + \epsilon(t) - \eta(t) = 0,$$

$$\frac{\partial H}{\partial a} = 0 \iff -\omega a(t) + k\lambda(t) + \alpha(t) - \psi(t) = 0,$$

**Adjoint equation:**

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial G} = -\alpha p(t) + \lambda(t)\delta, \quad \lambda(T) = s,$$

**State equation:**

$$\dot{G}(t) = \frac{\partial H}{\partial \lambda} = ka(t) - \delta G(t), \quad G(0) = G_0,$$

**Complementarity conditions:**

$$p(t) \geq 0, \quad \epsilon(t) \geq 0, \quad \epsilon(t)p(t) = 0,$$

$$b - p(t) \geq 0, \quad \eta(t) \geq 0, \quad \eta(t)(b - p(t)) = 0,$$

$$a(t) \geq 0, \quad \alpha(t) \geq 0, \quad \alpha(t)a(t) = 0,$$

$$n - a(t) \geq 0, \quad \psi(t) \geq 0, \quad \psi(t)(n - a(t)) = 0.$$

Solving for $G(t)$, we have

$$G(t) = G_0e^{-\delta t} + k\int_0^t e^{\delta(u-t)}a(u)du > G_0e^{-\delta t} > 0. \quad (23)$$

Therefore, $G(t)$ is always positive.

Let us show that $0 < p(t) < b$, for all $t$. Assume that $p(t) = 0$. Then, $\eta(t) = 0$ and the first optimality condition becomes $\alpha G(t) + \epsilon(t) = 0$. Since $\epsilon(t) \geq 0$ and $G(t)$ is positive, we get a contradiction. Now, assume that $p(t) = b$ for some $t$. Then $\epsilon(t) = 0$, and from the first optimality condition, we have $p(t) = b = \frac{\alpha G(t) - \psi(t)}{2\delta}$. Since $\eta(t) \geq 0$, it follows that $b < \frac{\alpha G(t)}{2\delta}$. From (23) and the fact that $a(t) \leq n$, we have

$$b < \frac{\alpha}{2\beta} \left\{ G_0e^{-\delta t} + kn\int_0^t e^{\delta(u-t)}du \right\}, \quad (24)$$

$$= \frac{\alpha}{2\beta} \left\{ G_0e^{-\delta t} + \frac{kn}{\delta} (1 - e^{-\delta t}) \right\}, \quad (25)$$

$$< \frac{\alpha}{2\beta} \left( G_0 + \frac{kn}{\delta} \right). \quad (26)$$

But this contradicts assumption (22) in the statement of the proposition. We thus have $p(t) < b$ and $p(t) = \frac{\alpha G(t)}{2\beta}$. Now, we show that $a(t) > 0$ for all $t$. Assume by way of contradiction that there is some $t$ at which $a(t) = 0$. Then $\psi(t) = 0$, and the second optimality condition becomes $k\lambda(t) + \alpha(t) = 0$. Since $\alpha(t) \geq 0$, we therefore have $\lambda(t) = -\frac{\alpha(t)}{k} \leq 0$. It is immediate to see that $t \not\in T$. Now solving for $\lambda(t)$ and using
\( p(t) = \frac{\alpha G(t)}{\beta t} \), we get

\[
\lambda(t) = e^{\delta t} \left\{ \int_t^T e^{-\delta u} \frac{\alpha^2}{2\beta} G(u) du + e^{-\delta T} s \right\}.
\]  

(27)

It follows that \( \lambda(t) > 0 \), which is a contradiction.

Let us now show that \( a(t) < n \) for all \( t \). Assume by way of contradiction that there is a date \( t \) at which \( a(t) = n \). Then \( \alpha(t) = 0 \), and we must have \(-\omega n + k\lambda(t) - \psi(t) = 0\). Thus, \( k\lambda(t) = \omega n + \psi(t) \), or \( \lambda(t) \geq \frac{\omega n}{k} \). Now from (23), we have

\[
G(t) = G_0 e^{-\delta t} + k \int_0^t e^{\delta(u-t)} a(u) du,
\]  

(28)

\[
\leq G_0 e^{-\delta t} + kn \int_0^t e^{\delta(u-t)} du
\]  

(29)

\[
= G_0 e^{-\delta t} + \frac{kn}{\delta} (1 - e^{-\delta t})
\]  

(30)

\[
\leq G_0 + \frac{kn}{\delta}.
\]  

(31)

Using (27), we get

\[
\frac{\omega n}{k} \leq \lambda(t) \leq e^{\delta t} \left\{ \int_t^T e^{-\delta u} \frac{\alpha^2}{2\beta} (G_0 + \frac{kn}{\delta}) du + e^{-\delta T} s \right\}
\]  

(32)

\[
= \frac{\alpha^2}{2\beta} (G_0 + \frac{kn}{\delta})(1 - e^{\delta(t-T)}) + e^{\delta(t-T)} s
\]  

(33)

\[
< \frac{\alpha^2}{2\beta} (G_0 + \frac{kn}{\delta}) + s.
\]  

(34)

But this last inequation contradicts assumption (21) made in the statement of the proposition.

**Step 3** Now we can check that (5)–(7) is the unique solution of the first-order conditions of Problem (I) (where \( \epsilon(t) = \eta(t) = \alpha(t) = \psi(t) = 0 \)). Since the optimal solution of Problem (I) exists, it is this unique solution. Hence the result.

\[\square\]

Although the proof of the above proposition is long, its interpretation is straightforward. The firm’s instantaneous profit is linearly increasing in the brand’s reputation. To avoid having an infinite value for \( G \), which would be the result of very large investments in advertising, condition (21) requires that the advertising cost to be large enough. Further, if the upper bound (22) on the instantaneous price is large enough, then it is never optimal to charge this upper bound (because demand would otherwise be too low). Naturally, \( G_0 \) must also not be too high; otherwise, it would be profitable to increase reputation in order to obtain higher profits. Notice, however, that the higher the bound on instantaneous advertising, the lower \( \omega \) can be, but the higher \( b \) must be. This is rather intuitive. The higher is \( n \), the higher the reputation can be, and the higher the price can be set. And, to avoid being constrained, \( b \) must also be higher. Likewise, it is less necessary that \( \omega \) be high to avoid choosing \( a(t) = n \). Finally, also notice that the higher is the efficiency of advertising, that is, the higher is \( k \), then the higher must be the cost \( \omega \) of advertising be to get an interior solution. The influence of the other parameters, \( s, \delta \), can be understood in the same way.

## 5 Concluding remarks

In this paper, we provided sufficiency conditions to show the existence and uniqueness of an interior solution of a dynamic pricing and advertising non-concave problem. The control model considered is of finite horizon and uses the Nerlove-Arrow dynamics. Our results show that the pricing trajectory follows the goodwill, and that the advertising trajectory is time-varying and monotone. The behavior over time of the control
and state trajectories depends on the parameter values. Interesting extensions of our work include the case where the price also influences the dynamics and the case where horizontal and/or vertical strategic interactions are present.

References


