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R. Salhab, J. Le Ny,
R. P. Malhamé

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GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

A dynamic ride-sourcing game with many drivers

Rabih Salhab
Jerome Le Ny
Roland P. Malhamé

GERAD & Department of Electrical Engineering, Polytechnique Montréal (Québec) Canada, H3T 1J4

rabih.salhab@polymtl.ca
jerome.le-ny@polymtl.ca
roland.malhame@polymtl.ca

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Abstract: We consider a dynamic game model of ride-sourcing, where a large number of private car owners provide rides to randomly appearing customers. Free drivers can travel around the city to improve their chance of being hired. They have access to the origin-destination statistics, to the current customer requests and to information about traffic congestion and the time- varying connectivity of the road network. We show how each driver can compute a best- response strategy to the anticipated behavior of the other drivers, which leads to an approximate Nash equilibrium in the limit of an infinite number of players. The outputs of our discrete-time model are the cars individual paths and the distribution of the free and busy drivers at each period. Finally, a numerical example illustrates three scenarios where a road closure in a city makes the cars desert an area. It shows how a financial incentive, namely increasing the ride fare in the deserted area, helps reestablish the ride service in that region.

Keywords: Ride-rourcing, mean field games

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1 Introduction

In its 2007 annual report, the United Nations Fund for Population Activities (UNFPA) announced that in 2008, half of the world's population (3.3 billion people) would be living in urban areas, a number that is expected to rise to almost 5 billion by 2030 [1]. Among the challenges associated with this growth is the development of efficient and sustainable transportation systems that can handle the increasing mobility needs of large city populations. Modernizing transportation systems requires providing incentives and designing regulations to satisfy different involved parties. This paper studies emerging "ride-sourcing" services, a term first coined by Rayle et al. [2] to refer to the process of "sourcing rides from a driver pool", which involve drivers, riders, a company (such as Uber and Lyft) providing the digital platform matching them, and the other users of the road network.

Our aim is to understand how a large population of *private car owners* providing rides for randomly appearing customers behaves on the population and individual levels. The drivers are assumed to be independent contractors working under the supervision of a ride-sourcing company [2]. The company provides its drivers with a (possibly partial) view of the customer demand, of the statistics of the customer requests' origin-destination (OD) pairs and of the drivers' distribution, while the drivers compete with each others to maximize their profits by trying to serve as many customers as possible, at the least cost. In contrast to the traditional taxicab sector, the drivers are within this market independent choice makers, i.e., they choose which area to serve, whether to accept or not to serve a customer or when to start working and when to stop. Although the ride-sourcing company has no direct control over its drivers, understanding their behavior in response to the information it sends them is crucial to design incentives that encourage the drivers to work during the rush hours, to drive towards a specific area, etc. For instance, by analyzing how the drivers react to the information about the demand and supply distribution in a city, the company can try to indirectly control their behavior by presenting this information in a certain way.

Our model considers a large population of private car owners circulating in a city to get matched with randomly appearing customers. It describes the behavior of an individual generic driver, i.e., its individual path, in response to the driver population's behavior and the statistics, assumed known, of the customer demand. Moreover, it anticipates the evolution of the free and busy drivers' distributions resulting from these individual behaviors.

The main contributions of this paper are as follows:

1. We introduce a game theoretic approach to model the ride-sourcing problem, which is a natural way of modeling the behavior of interacting individuals in a competitive market.
2. Our model describes the drivers' optimal paths and anticipates the macroscopic behavior of the population.
3. Our model takes into consideration the statistics of the customer requests, the traffic congestion and the state of the road network, including their temporal dynamics.

The study of the economic regulation of the taxicab industry started with the aggregate models [3, 4, 5] and later the equilibrium models [6, 7]. The main purpose of these models is to study the effect of different regulations on the entry to the taxicab market and on the taxi fare, and eventually suggest the optimal number of taxi licenses and the optimal taxi fare to achieve a supply-demand equilibrium. More recently, ride-sourcing companies have attracted much interest from researchers studying their impact on the taxicab sector, the overall transportation system and the environment [2]. Wang et al. [8] evaluate the influence of some pricing strategies on a taxi market that involves the following different parties: traditional taxi companies, ride-sourcing companies, the drivers and the customers. Zha et al. [9] analyze the social efficiency of ride-sourcing services under monopolist and duopolist scenarios. Moreover, they suggest some regulatory policies to increase the social welfare by controlling some parameters such as the percentage of the commission charged by the ride-sourcing company on each ride. Our model considers another aspect of the ride-sourcing problem. Namely, it addresses the question of how the private drivers should optimally circulate in a city to meet and serve randomly appearing customers, while taking into account the statistics of the requests' OD pairs and the time varying connectivity of the road network. It can also suggest strategies for the ride-sourcing companies to incentivize their drivers in a way that increases the overall system efficiency.

Our model is also related to robotic deployment problems [10, 11], the dynamic traveling repairman problem [12] and a class of dynamic vehicle routing problems [13]. These papers discuss how a group of vehicles or robots can be optimally deployed by centralized or distributed algorithms to serve randomly appearing events or customers in some region, e.g., to minimize the average customer's waiting time. Our model differs from this line of work in that it considers a competitive environment (non-cooperative game), which is suitable for modeling the ride-sourcing market. Moreover, it factors explicitly the time-varying nature of the environment dynamics in the driver's strategies, whereas deployment or vehicle routing problems generally assume a stationary environment.

We consider a discrete time dynamic game with a finite number of states, which involves a large number of weakly coupled players, that is, an isolated individual strategy has a negligible impact of the others' strategies, while the mass behavior of the population has a considerable influence on the individual strategies. To analyze our game involving a large population of players, we follow the Mean Field Games methodology (MFG), which was introduced in a series of papers by Huang et al. [14, 15, 16], and independently by Lions and Lasry [17, 18, 19]. Discrete time finite state MFGs were studied later [20, 21]. To solve the game, the MFG methodology starts by considering the limiting case of a continuum of players, which can be described by two coupled forward-backward difference equations. The backward equation characterizes a generic player's best response to the macroscopic distribution of players, while the forward equation propagates the distribution of the players under these best response strategies. Candidate sustainable macroscopic behaviors, if they exist, are then computed by a fixed point argument. The corresponding best response strategies, when applied to the finite population, typically constitute approximate Nash equilibria.

The mathematical model of the ride-sourcing game between drivers is presented in Section 2. In Section 3, we solve the game via the MFG methodology, compute the drivers' optimal strategies and show how to anticipate the population's macroscopic behavior. Section 4 discusses some numerical simulation results, while Section 5 presents our conclusion.

2 Mathematical model

We model our problem as a finite-state dynamic game in discrete time. Consider N players, which are private car owners circulating during a time interval $[0, T]$ in a city divided into l zones, in order to serve some randomly appearing customers. The drivers move on a time-varying graph G_t, with l nodes representing zones, which models the road network in the city and can take into account situations such as scheduled road closures. At discrete time $t \in \{0, \dots, T\}$ a car is characterized by its state, which takes values in the finite set $I = \{(i, d) \in \{1, \dots, l\} \times \{v, 1, \dots, l\}\}$, where i refers to the current location (zone number) of the car and d is its availability status. An availability status denoted v refers to a vacant car, whereas a value $d \geq 1$ refers to a busy car transporting a customer to a destination zone d. We denote by π^t_{id} the percentage of cars in zone i at time t with availability status d, we let $\pi^t := \{\pi^t_{id}\}_{i,d}$, and define $\pi^t_v := \left\{\pi^t_{i,v}\right\}_i$, the percentage of vacant cars in each zone. Let $\{c^t_i\}_{i,t}$ be the probabilities that a customer appears in zone i at time t, which are assumed known and fixed independently of the drivers' distribution.

The game is played as follows. First, at each time $t \in \{0, \dots, T-1\}$, a busy car with availability status $d \geq 1$ in the $i-$th zone drives its customer at time $t+1$ to the $j-$th zone with probability $Q^t_{ij}(d)$, and gets a reward of R^t_{ij}. The probabilities $Q^t_{ij}(d)$ are given and depend on the statistics of the preferred routes in the city at time t to go from i to d. Second, a vacant car $(d = v)$ in the i-th zone at time t is matched with a customer with probability denoted $\eta(\pi^t_{iv}, c^t_i)$, which is assumed to be a known function, increasing with c^t_i and decreasing with the fraction π^t_{iv} of vacant cars in zone i at t. For a matched car, the customer requests a ride to a destination $d \geq 1$ with probability D^t_{id}. The driver's availability status is therefore randomly updated to $d \geq 1$ with probability D^t_{id}, and he or she gets a "pick-up reward" $M^t_{id}(c^t_i, \pi^t)$, which in general is allowed to depend on the customer request probability c^i_t and at least on the percentage of vacant cars π^t_{iv} in zone i or on the distribution of vacant cars π^t_v, in order to be able to encourage free drivers to move to areas where service is insufficient, and hence reduce temporary imbalances between demand and supply. The distribution $D_i^t = (D_{i1}^t, \dots, D_{il}^t)$ depends on the statistics of the requests' OD pairs, which are assumed known. When matched, a driver moves in the next period to the $j-$th zone with probability $Q_{ij}^t(d)$ and gets a reward R_{ij}^t as above. Overall, the total reward that a driver gets for a ride from zone i to d includes a basic term $M_{id}^t(c_i^t, \pi^t)$ and a route dependent term given by the accumulation of the rewards R_{ij}^t along the path.

If a driver is not matched, then he or she moves to the $j-$th zone with a probability P_{ij}^t of his or her choice, and pays a cost

$$y_{ij}^t(P_{ij}^t; \pi^t) = a_{ij}(h_{ij}^t + \rho_{ij}^t(P_{ij}^t; \pi^t)) \tag{1}$$

$$\text{with } \rho_{ij}^t(P_{ij}^t; \pi_{iv}^t) := (\pi_{iv}^t(1 - \eta(\pi_{iv}^t, c_i^t)) + \epsilon)P_{ij}^t + \pi_{iv}^t \eta(\pi_{iv}^t, c_i^t) \sum_{d=1}^{l} D_{id}^t Q_{ij}^t(d) + \sum_{d=1}^{l} \pi_{id}^t Q_{ij}^t(d),$$

where $a_{ij} > 0$ is the traveled distance from zone i to j (in case $i \neq j$) or within zone i (in case $i = j$), $h_{ij}^t > 0$ models time-varying congestion due to cars that are not part of the game and $\rho_{ij}^t(P_{ij}^t; \pi^t)$ is the fraction of players moving from i to j at time t. For $\epsilon = 0$, this last term accounts for the traveled distance and the traffic congestion cost. The constant $\epsilon > 0$, assumed small, is included in this cost for a technical reason that will become clear later (see Remark 1 below). The transition probabilities P_{ij}^t are under the control of the drivers. We denote by $\mathcal{N}_i^t$, $1 \leq i \leq l$, the neighborhood of zone i at time t with respect to the graph G_t. Accordingly, the supports of the transition probabilities $Q_{ij}^t(d)$ and P_{ij}^t are included in the set $\mathcal{N}_i^t$. We illustrate the game dynamics in Figure 1.

Our model aims at capturing the competition between the drivers, who are trying to optimize their profit by remaining vacant for as little time as possible and by getting matched with customers requesting more rewarding trips. Since all drivers compete for the same customers, as captured by the matching probability η, a non-cooperative game ensues. By studying this game, we would like to predict macroscopic characteristics of the resulting transportation service, e.g., efficiency at moving customers, profitable strategies for the drivers and their expected profits, influence of the fare structure, etc. One apparent challenge in studying this game is due to the large number of players involved. However, since we assume a homogeneous population of drivers that are only weakly coupled through their density via the matching probability and rewards, the methodology of MFGs, which we introduce next, offers a promising avenue for analyzing our game.

3 Mean field equations

Following the mean field games methodology [20], we start by assuming a continuum of players whose distribution flow $\pi = \{\pi_{id}^t\}_{i,d,t}$ is then deterministic and assumed known for now. Later in this section, we explain how to compute this flow by using a fixed point argument capturing the fact that only certain behaviors are sustainable in a population of rational, profit-maximizing agents. With the distribution flow π known to all players, a generic driver faces an optimal control problem and chooses its transition probabilities P_{ij}^t to maximize its own reward by solving the following dynamic program [22]

$$V_{i,d}^t = \sum_{j=1, j\neq d}^{l} Q_{ij}^t(d) \left(R_{ij}^t + V_{j,d}^{t+1}\right) + Q_{id}^t(d) \left(R_{id}^t + V_{d,v}^{t+1}\right), \qquad 1 \leq d \leq l, \tag{2}$$

$$\begin{aligned} V_{i,v}^t = \eta(\pi_{iv}^t, c_i^t) \Bigg[\sum_{d=1}^{l} D_{id}^t \Bigg\{ & M_{id}^t(c^t, \pi^t) + \sum_{j=1, j\neq d}^{l} Q_{ij}^t(d) \\ & \times \left(R_{ij}^t + V_{j,d}^{t+1}\right) + Q_{id}^t(d) \left(R_{id}^t + V_{d,v}^{t+1}\right) \Bigg\} \Bigg] \\ & + \left(1 - \eta(\pi_{iv}^t, c_i^t)\right) \max_{P_i^t \in S_i^t} \sum_{j=1}^{l} P_{ij}^t \Big(- y_{ij}^t(P_{ij}^t) + V_{j,v}^{t+1} \Big). \end{aligned} \tag{3}$$

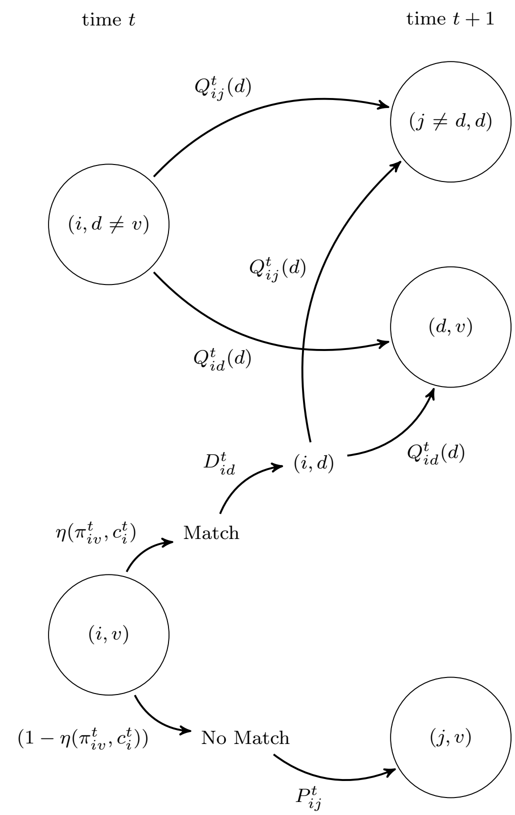

Figure 1: The dynamics of the ride-sourcing game.

for $1 \le i \le l$ and $V^T_{i,d} = 0$. Here, $V^t_{i,d}$ is the expected optimal utility from time t to T given the current state (i, d) of the generic agent, $P^t_i = (P^t_{i1}, \ldots, P^t_{il})$, and S^t_i is the set of probability distributions on $\{1, \ldots, l\}$ with supports included in $\mathcal{N}^t_i$. Note that the utility $V^t_{i,d}$ depends on the distribution flow π. The expected reward-to-go of a driver serving a customer at time t is given by (2). On the other hand, (3) computes the expected reward-to-go for a driver that is vacant at time t, which depends if he or she is matched at this time period or must continue looking for a customer.

From (3), the drivers' best responses $\bar{P}^t_i$ to the assumed given distribution flow π solve the following convex program:

$$\begin{aligned} &\max_{P^t_i \in \mathbb{R}^l} \sum_{j \in \mathcal{N}^t_i} P^t_{ij} \Big(- y^t_{ij}(P^t_{ij}) + V^{t+1}_{j,v} \Big) \\ &\text{s.t. } P^t_{ij} \ge 0, \sum_{j \in \mathcal{N}^t_i} P^t_{ij} = 1 \text{ and } \sum_{j \notin \mathcal{N}^t_i} P^t_{ij} = 0, \end{aligned} \tag{4}$$

where $y^t_{ij}(P^t_{ij})$ is defined in (1).

Theorem 1 *The convex program (4) has a unique solution:*

$$\begin{aligned} \bar{P}^t_{ij} &= \frac{\max\big(V^{t+1}_{j,v} - u^t_{ij}, 0\big)}{2a_{ij}(\pi^t_{iv}(1 - \eta(\pi^t_{iv}, c^t_i)) + \epsilon)}, && \textit{for } j \in \mathcal{N}^t_i, \qquad (5) \\ \bar{P}^t_{ij} &= 0, && \textit{for } j \notin \mathcal{N}^t_i, \end{aligned}$$

where

$$u_{ij}^t = a_{ij}\Big(h_{ij}^t + \pi_{iv}^t \eta(\pi_{iv}^t, c_i^t)\sum_{d=1}^{l} D_{id}^t Q_{ij}^t(d) + \sum_{d=1}^{l} \pi_{id}^t Q_{ij}^t(d)\Big) + \lambda_i^t,$$

and λ_i^t *is the unique solution of*

$$g_i^t(\lambda_i^t) \triangleq \sum_{j\in\mathcal{N}_i^t} \frac{\max\left(V_{j,v}^{t+1} - u_{ij}^t, 0\right)}{2a_{ij}(\pi_{iv}^t(1-\eta(\pi_{iv}^t, c_i^t)) + \epsilon)} = 1. \tag{6}$$

Proof. $\bar{P}_{ij}^t = 0$ for $j \notin \mathcal{N}_i^t$ follows from the constraints. In addition, the unique solution $\bar{P}_i^t$ of the convex program (4) satisfies the following KKT conditions [23, Section 5.5.3]:

$$\mu_j + \lambda_i^t = -2a_{ij}(\pi_{iv}^t(1-\eta(\pi_{iv}^t, c_i^t)) + \epsilon)\bar{P}_{ij}^t - a_{ij} \tag{7}$$

$$\Big(h_{ij}^t + \pi_{iv}^t \eta(\pi_{iv}^t, c_i^t)\sum_{d=1}^{l} D_{id}^t Q_{ij}^t(d) + \sum_{d=1}^{l} \pi_{id}^t Q_{ij}^t(d)\Big) + V_{j,v}^{t+1} \tag{8}$$

$$\bar{P}_{ij}^t \geq 0, \tag{9}$$

$$\sum_{j\in\mathcal{N}_i^t} \bar{P}_{ij}^t = 1 \tag{10}$$

$$\mu_j \geq 0, \tag{11}$$

$$\mu_j \bar{P}_{ij}^t = 0, \tag{12}$$

$\forall j \in \mathcal{N}_i^t$, for some $\lambda_i^t \in \mathbb{R}$ and $\mu_j \in \mathbb{R}$. By multiplying both sides of (8) by μ_j and noting (12), one can show (5). Now $g_i^t(\lambda_i^t) = 1$ follows from (5) and (10). It remains to prove that there exists actually a unique solution of $g_i^t(\lambda_i^t) = 1$. In fact, g_i^t is a continuous piecewise affine strictly decreasing function from $\mathbb{R}$ onto $[0,\infty)$, hence (6) has a unique solution λ_i^t. □

According to the policies (5), there exists at each step time t a set of threshold utilities $(u_{i1}^t, \ldots, u_{il}^t)$, such that if a free driver is at time t in zone i and is not matched, then he or she will not move at time $t+1$ to an "unprofitable" zone j with expected utility less then the threshold utility u_{ij}^t. We denote in the remaining of the paper the best responses $\bar{P}_{ij}^t$ defined in (5) by $\bar{P}_{ij}^t(V^{t+1})$ to express their dependence on the expected optimal utility $V^{t+1} = \{V_{i,d}^{t+1}\}_{(i,d)\in I}$.

Having computed the drivers' best responses to the assumed known distribution flow, we now turn to the problem of finding such flow. In fact, a flow of distributions is admissible if it can be reproduced by the continuum of players when they optimally respond to it. Hence, an admissible $\pi = \{\pi_{id}^t\}_{i,d,t}$ is a fixed point of the following mean field equations [20]:

$$V^t = \mathcal{G}_t\left(V^{t+1}, \pi^t\right), \qquad V^T = 0 \tag{13}$$

$$\pi^{t+1} = \mathcal{P}_t(V^{t+1}, \pi^t), \qquad \pi^0, \tag{14}$$

where $\left[\mathcal{G}_t(V^{t+1}, \pi^t)\right]_{i,d\neq v}$ is the right-hand side of (2), $\left[\mathcal{G}_t(V^{t+1}, \pi^t)\right]_{i,v}$ is the right-hand side of (3) where we replace the term $\max_{P_i^t\in S_i^t} \sum_{j=1}^{l} P_{ij}^t\Big(-y_{ij}^t(P_{ij}^t) + V_{j,v}^{t+1}\Big)$ by its value $\sum_{j=1}^{l} \bar{P}_{ij}^t(V^{t+1})\Big(-y_{ij}^t(\bar{P}_{ij}^t\left(V^{t+1}\right)) + V_{j,v}^{t+1}\Big)$, and

$$\left[\mathcal{P}_t(V^{t+1}, \pi^t)\right]_{d,d} = 0$$

$$\left[\mathcal{P}_t(V^{t+1}, \pi^t)\right]_{j\neq d, d\neq v} = \sum_{i=1}^{l}\left(\pi_{id}^t Q_{ij}^t(d) + \pi_{iv}^t \eta(\pi_{iv}^t, c_i^t) D_{id}^t Q_{ij}^t(d)\right)$$

$$\left[\mathcal{P}_t(V^{t+1}, \pi^t)\right]_{j,v} = \sum_{i=1}^{l}\left(\pi_{ij}^t Q_{ij}^t(j) + \pi_{iv}^t \eta(\pi_{iv}^t, c_i^t) D_{ij}^t Q_{ij}^t(j) + \pi_{iv}^t(1-\eta(\pi_{iv}^t, c_i^t))\bar{P}_{ij}^t(V^{t+1})\right).$$

The system of mean field equations (13)–(14) comprises a backward difference equation (13) coupled with a forward equation (14). The first computes the best response of a generic driver to the driver population distribution, and the second describes the evolution of the cars' distribution under these best response strategies.

Assumption 1 *We assume that the reward $M^t_{id}(c^t, \pi^t)$ and the matching probability $\eta(\pi^t_{iv}, c^t_i)$ are continuous w.r.t. distribution flow π.*

Theorem 2 *Under Assumption 1, there exists an admissible distribution flow, i.e., a π that satisfies (13)–(14).*

Proof. We define S the convex compact set of distribution flows which is equal to the set K^{T+1} where K is the simplex $\{x \in \mathbb{R}^{l(l+1)} | x_k \geq 0, \text{ and } \sum_{k=1}^{l(l+1)} x_k = 1\}$. The difference equation (13) defines a map f_1 from S into D, the set of functions from $I \times \{0, \dots, T\}$ into $\mathbb{R}$, such that $f_1(\pi)(i, d, t) = V^t_{i,d}$, where $\{V^t_{i,d}\}_{t,i,d}$ is the unique solution of (13) for the given π. Similarly, (14) defines a map f_2 from D into S. A distribution flow $\pi \in S$ satisfies (13)–(14) if and only if it is a fixed point of $f_2 \circ f_1$, which is a function from the convex compact set S into itself. In the following, we show that λ^t_i defined in Theorem 1 is continuous with respect to V^{t+1}, which implies that the best responses $\bar{P}^t_{ij}(V^{t+1})$ defined in (5) and $f_2 \circ f_1$ are continuous. The result then follows from Brouwer's fixed point theorem [24, Section V.9].

The function g^t_i defined in (6) is continuous piecewise linear, with card$(\mathcal{N}^t_i)$ break points $\Big\{b_j \triangleq V^{t+1}_{j,v} - a_{ij}\Big(h^t_{ij} + \pi^t_{iv}\eta(\pi^t_{iv}, c^t_i) \sum_{d=1}^l D^t_{id} Q^t_{ij}(d) + \sum_{d=1}^l \pi^t_{id} Q^t_{ij}(d)\Big)\Big\}_{j \in \mathcal{N}^t_i}$, and gradients at $-\infty$ and $+\infty$ independent of V^{t+1}. Therefore, it is sufficient to show that the break points $(b_j, g^t_i(b_i))$ change continuously with V^{t+1}. Fix V^{t+1}. We assume at first that there exists $b_j \neq b_k$, and we fix $0 < \delta < 1/4 \min\{|b_j - b_k|, \text{ s.t. } b_j - b_k \neq 0\}$. Consider now another $\bar{V}^{t+1}$ such that $\max_j |\bar{V}^{t+1}_{j,v} - V^{t+1}_{j,v}| < \delta$. We denote by $\bar{g}^t_i$ the corresponding function defined in (6) and $(\bar{b}_i, \bar{g}^t_i(\bar{b}_i))$ the corresponding break points. Hence, $\max_j |\bar{b}_j - b_j| < \delta$. Fix a zone $j \in \mathcal{N}^t_i$. Let F be the set of zones $s \in \mathcal{N}^t_i$ such that b_s equal to b_j, E the set of zones s such that $b_s > b_j$. In view of the maximum value of δ,

$$|g^t_i(b_j) - \bar{g}^t_i(\bar{b}_j)| \leq \left| \sum_{s \in E} \frac{1}{2a_{is}\epsilon} \left((b_s - b_j) - (\bar{b}_s - \bar{b}_j)\right) \right| + \sum_{s \in F} \frac{1}{2a_{is}\epsilon} |\bar{b}_s - \bar{b}_j|.$$

By the definition of F, for $s \in F$, $|\bar{b}_s - \bar{b}_j| \leq |\bar{b}_s - b_s| + |b_j - \bar{b}_j|$. Hence, $|g^t_i(b_j) - \bar{g}^t_i(\bar{b}_j)| \leq 4\delta \sum_{j=1}^l 1/2a_{ij}\epsilon$.

Otherwise, suppose $b_j = b_k$ for all $j, k \in \mathcal{N}^t_i$. Fix $\delta > 0$, and $\bar{V}^{t+1}$ such that $\max_j |\bar{V}^{t+1}_{j,v} - V^{t+1}_{j,v}| < \delta$. Without loss of generality, we assume that $\mathcal{N}^t_i = \{1, \dots, k\}$ and that $\bar{b}_j$ are ordered as follows: $\bar{b}_1 \leq \dots \leq \bar{b}_k$. We have

$$|\bar{g}^t_i(\bar{b}_j) - g^t_i(b_j)| \leq |\bar{g}^t_i(\bar{b}_1) - g^t_i(b_1)| = \sum_{j=2}^k \frac{\bar{b}_j - \bar{b}_k}{2a_{ij}(\pi^t_{iv}(1 - \eta(\pi^t_{iv}, c^t_i)) + \epsilon)} \leq 2\delta \sum_{j=1}^l 1/2a_{ij}\epsilon.$$

Therefore, the break points change continuously with V^{t+1}. $\square$

Remark 1 *If $\epsilon = \pi^t_{iv} = 0$, then one of the $\bar{P}^t_{ij}$ is equal to one and the others are zero. In this case, the function f_1 defined in the proof of Theorem 2 is no longer continuous. Hence, an $\epsilon > 0$ is required to guaranty the existence of a fixed point π.*

As discussed in the proof of Theorem 2, a flow of distributions $\{\pi^t_{id}\}_{t,i,d}$ satisfying (13)–(14) is a fixed point of the continuous map $f_2 \circ f_1$. One can try to compute $\{\pi^t_{id}\}_{t,i,d}$ using the fixed point iteration method, i.e., by defining the sequence $\pi_k = f_2 \circ f_1(\pi_{k-1})$, with $(\pi_0)^t_{id} = \pi^0_{id}$ for all t, i and d. If this method converges, the limit is a fixed point flow. The best responses to this flow are then computed by (5).

4 Illustrative scenario

We consider in this section a scenario for a group of private car owners providing rides in a city divided into $l = 22$ zones under the supervision of a ride-sourcing company. The zones' layout and numbers are illustrated in Figure 2. The game lasts for $T = 15$ periods of time. At time $t = 0$, all the cars are vacant, where 30% of them are in zone 21, 20% in zone 15, 20% in zone 6, 20% in zone 20 and 10% in zone 19. The cars can move one step per period vertically or horizontally. For example, a car in zone 1 can only move to the zones $1, 2, \dots, 8$ (i.e. $\mathcal{N}_1^t = \{1, 2, \dots, 8\}$), while a car in zone 9 can only move to the zones $2, 9, 10, 15$ and 16. The transition probabilities $Q_{ij}^t(d)$ are defined as follows:

- If d is one of the neighbors of i then $Q_{id}^t(d) = 1$.
- If the destination is the center of the city ($d = 1$), then a busy driver in zone i moves with probability 0.8 to the zone below i, with probability zero to the zone above i, and with equal probabilities to the rest of i's neighbors.
- If i and j belong to the same ring, for example $\{9, \dots, 15\}$, then a driver in zone i moves with probability 0.8 to the closest neighbor $j \in \mathcal{N}_i^t$ to d, such that j belongs to the same ring as i, and with equal probabilities to the rest of the neighbors.
- If i and d belong to different rings, then a car in zone i moves with probability 0.8 to the zone above (resp. below) i if $d > i$ (resp. $d < i$), and with equal probabilities to the rest of the neighbors.

We assume that most of the ride requests are local, that is, when a driver is matched with a customer at zone i, then the requested destination d belongs to the neighborhood of i with probability 0.8 and to the rest of the zones with probability 0.2. The customers appear with probability 0.5 in zone 22, probability 0.3 in zone 14 and probability 0.2 in zone 17. The matching probability is taken as $\eta(\pi_{iv}^t, c_i^t) = c_i^t \exp(-\sqrt{\pi_{iv}^t})$. Next, we consider 3 scenarios.

4.1 First scenario: No scheduled road closure.

In this scenario, we assume that no road closure is scheduled. The ride-sourcing company fixes a reward $R_{ij}^t = 10$ with a "pick-up reward" $M_{id}^t(c^t, \pi^t)$ equal to zero. Figure 3 shows the evolution of cars' distribution in the city. The cars are essentially concentrated around the zones of appearance of the customers, namely zones 22, 14 and 17. Figure 2 shows the path of a driver who started his trip from zone 20 and kept driving until being matched at $t = 7$ in zone 22 (red color refers to busy car) with a customer who requested a ride to zone 16. At $t = 11$, he is matched with another customer in zone 22 who requested a ride to zone 21.

4.2 Second scenario: Road Closure with uniform pricing strategy

In this scenario, the passage $16 - 9 - 2$ is closed from the left and right, that is, a car in zone $i \in \{22, 15, 8, 17, 10, 3\}$ cannot move to zone $j \in \{16, 9, 2\}$. With the same pricing strategy as in scenario one, the cars desert zone 17. This is due to the fact that with the closure of the passage $16 - 9 - 2$, the drivers initially in the neighborhoods of zones 22 and 14 prefer serving these two areas than driving for long distances to zone 17 to get the same rewards M_{id}^t and R_{ij}^t. Figure 4 shows that the fraction of cars in zone 17 drops from 5% to 1.8% after the road closure.

4.3 Third scenario: Road Closure with nonuniform pricing strategy

To encourage some cars to serve zone 17, the company declares a "pick-up reward" $M_{17d}^t(c^t, \pi^t)$ in this zone equal to 1.2. Figure 4 illustrates how this financial incentive helps reestablishing the service in zone 17. In fact, the fraction of cars in this zone increases from 1.8% to 5%.

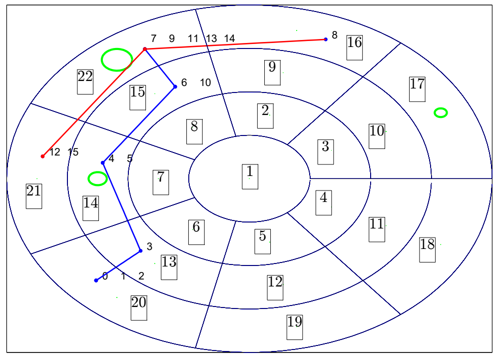

Figure 2: City layout and sample path of a car, where the red points (resp. blue points) indicate that the car is busy (resp. vacant). The trajectory is explained in Section 4.1. The numbers above the points refer to the time, while those in the boxes are the zones' numbers.

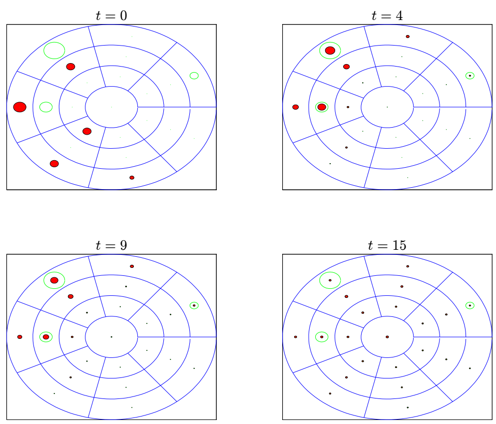

Figure 3: First Scenario. The evolution of the cars' distribution at time $t = 0, 4, 9, 15$, where the radius of a red ball in zone i is proportional to the fraction of cars in this zone. The probability of appearance of a customer is proportional to the radius of the green balls.

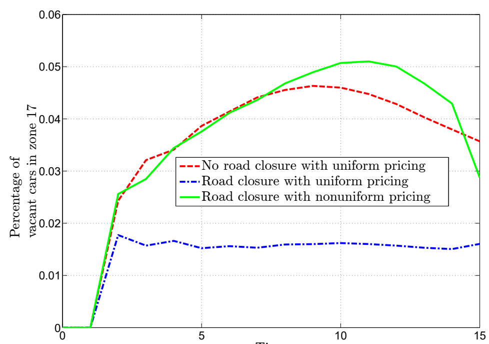

Figure 4: Comparison of the fraction of cars in zone 17 before and after the road closure with uniform and nonuniform pricing strategies.

5 Conclusion

We consider in this paper a ride-sourcing dynamic game, where a large number of private car owners are optimally circulating in a city to serve some randomly appearing customers, while taking into consideration the statistics of the requests' OD pairs, the time varying configuration of the road network and the traffic congestion. We develop via the MFG methodology a set of Nash strategies. Moreover, we anticipate the distributions of the free and busy drivers in the city. For future work, it is of interest to extend the interaction between drivers and customers in two ways. On the one hand, one can assume that the drivers don't know the probability of appearance of the customers, and learn it while playing the game. On the other hand, the customers can also be considered as players in the game, which would allow us to take into consideration the quality of service, captured by the clients' waiting time for example. Another extension of the current model would be to also allow the drivers to freely enter and leave the game.

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