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Analytical valuation of compound options under regime switching dynamics

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Abstract: We propose an analytical formula for the evaluation of compound options when the underlying asset is described by a two-states Markov regime-switching log-normal model. One specific application of interest of such a formula is the pricing of principal protected callable notes with an early redemption feature. This approach provides practitioners with a Black-Scholes-type formula under a realistic assumption about market prices behavior.

Keywords: Pricing, compound options, regime switching, structured notes

Résumé: Nous proposons une formule analytique pour l'évaluation d'options composées lorsque la dynamique de l'actif sous-jacent est décrite par un modèle lognormal avec changements de régime markoviens. L'une des applications intéressantes d'une telle formule est l'évaluation de billets rachetables à capital protégé. Notre approche permet l'utilisation d'une formule de type Black-Scholes pour l'évaluation de tels produits sous une hypothèse réaliste de comportement des marchés.

Mots clés: Évaluation de produits dérivés, options composées, changement de régime, billets structurés

1 Introduction

A *compound option*, also named *split-free option*, is an option written on another option, meaning that it gives its owner the right to buy or sell another option, called the *underlying option*. According to the nature of the compound and the underlying option, there are four types of compound options: Call on Call, Call on Put, Put on Call and Put on Put. A compound option has two maturity dates (T_1, T_2) and two exercise prices (K_1, K_2) , where T_1 and K_1 denote respectively the maturity date and exercise price of the compound option while T_2 and K_2 denote the maturity date and exercise price of the underlying option, with $T_1 \leq T_2$.

Compound options are not traded in financial markets; however, many traded financial products can be expressed or approximated as a combination of compound options and other simpler instruments. For instance, it is the case of convertible bonds (Gong et al. 2006), American options (Geske and Johnson 1984), and more particularly some widespread types of principal protected callable notes. The first and most popular paper dedicated to the valuation of compound options is undoubtedly that of Geske (1979) where, assuming a geometric Brownian motion (GBM) for the dynamics of the underlying asset of the underlying option, the author derives a Black-Scholes type formula for the value of a Call on Call, using Fourier integration and some characteristics of the bivariate normal distribution. Another derivation based on the risk-neutral principle is suggested by Rubinstein (1991) and implemented in Lajeri-Chaherli (2002). In addition to providing an elegant proof, this last paper also gives an interesting interpretation of the different components that make up the formula. Based on the formula derived in Geske (1979), Geske and Johnson (1984) obtain an analytical approximation for the price of an American put option. This approximation method requires the infinite summation of multivariate normal integrals, which is not easy to implement and very time consuming. Selby and Hodges (1987) propose a technique that reduces the number of integrals to be evaluated for the implementation of methods that use multinomial distributions, as it is the case for the Geske formula. Agliardi and Agliardi (2003) provide a generalization of the Geske formula when the interest rate and the volatility are time-dependent. This generalization seems more realistic in practice, but the resulting formula is more complex to implement. Fouque and Han (2005) use a perturbation technique to derive an approximation of compound option prices under a two-factor stochastic volatility model. Their methodology can be summarized into two main steps; First the underlying option's value is approximated with a Black-Scholes term (with constant volatility) plus a perturbation term, and then a Taylor expansion of the compound option's payoff around the constant term of the underlying option's approximation is used to produce the risk-neutral discounted conditional expectation of the result. This method can be easily implemented in practice, but it requires calibration based on observed prices of compound options, which unfortunately are not exchanged in financial markets.

The main contribution of this paper is the development of an analytical formula for the evaluation of compound options when the underlying asset is described by a two-states Markov regime-switching log-normal (2RS) model. This model has been shown to capture several empirical properties of market returns, and often performs better than more complex econometric models when predicting the behavior of stocks or indexes. One specific application of interest of such a formula is the pricing of principal protected callable notes (PPCN) with an early redemption feature. Notes of this type are popular instruments that are widely used by financial institutions. In practice, the pricing and design of PPCN usually involves computationally costly Monte Carlo simulation procedures. The analytical formula proposed in this paper is an efficient alternative that can be easily implemented to evaluate portfolios containing PPCN or to determine the early redemption price of a given issue.

The paper is organized as follows: Section 2 recalls the valuation of compound options under the GBM model, and Section 3 presents our extension of the Geske formula to the 2RS model of Hardy (2001). We show in Section 4 how this formula can be used to price and/or design PPCN with an early redemption feature. Section 5 is a conclusion.

2 Compound options valuation under the GBM model

Denote by $\Phi(\cdot)$ and $\Phi_2(\cdot, \cdot; \rho)$ the cumulative distribution functions of respectively the univariate standard normal and the bivariate normal distribution with correlation matrix

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

where $\rho \in [-1, 1]$. Recall that the density ϕ_2 and cumulative distribution Φ_2 of a bivariate standard normal variable are given respectively by

$$\begin{aligned} \phi_2(x, y; \rho) &= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}} \\ \Phi_2(u, v; \rho) &= \int_{-u}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \Phi\left(\frac{v+\rho x}{\sqrt{1-\rho^2}}\right) dx. \end{aligned}$$

For any vector $\gamma \equiv (r, \sigma, K, T)$ of positive real numbers, define

$$\begin{aligned} d_\gamma(s) &= \frac{\log(s/K) + T(r + \sigma^2/2)}{\sigma\sqrt{T}} \\ d'_\gamma(s) &= d_\gamma(s) - \sigma\sqrt{T}. \end{aligned}$$

Denote by $C_{BM}(s; \gamma)$ the value of a European call option, under the GBM model, when the price of the underlying asset is s for a parameter vector $\gamma = (r, \sigma, K, T)$, where T is the time to maturity, K is the strike price, σ is the volatility of the underlying asset, and r is the risk-free rate. This value is given by the Black and Scholes (1973) formula, that is,

$$C_{BM}(s; \gamma) = s\Phi(d_\gamma(s)) - Ke^{-rT}\Phi(d'_\gamma(s)). \quad (1)$$

Within the GBM framework, Geske (1979) showed that the value at $t = 0$ of a Call on Call with contractual parameters $\xi \equiv (K_1, K_2, T_1, T_2)$ when the underlying asset price is s is given by

$$\begin{aligned} CC_{BM}(s; r, \sigma, \xi) &= s\Phi_2(d_{\gamma_1}(s^*), d_{\gamma_2}(s); \rho) \\ &\quad - K_2e^{-rT_2}\Phi_2(d'_{\gamma_1}(s^*), d'_{\gamma_2}(s); \rho) \\ &\quad - K_1e^{-rT_1}\Phi(d'_{\gamma_1}(s^*)) \end{aligned} \quad (2)$$

where

$$\begin{aligned} \gamma_1 &= (r, \sigma, K_1, T_1) \\ \gamma_2 &= (r, \sigma, K_2, T_2) \\ \rho &= \sqrt{\frac{T_1}{T_2}}, \end{aligned}$$

and where s^* is the underlying asset level such that the payoff of the compound option at T_1 is zero, that is, s^* is the value of s that solves the following equation:

$$C_{BM}(s; (r, \sigma, K_2, T_2 - T_1)) = K_1. \quad (3)$$

Formula (2) has an analogue for a Call on Put, Put on Call and a Put on Put, and also for options on stocks that pay dividends. Notice that the Geske formula, while analytic, is not in closed-form since it requires the numerical solution of implicit Equation (3). To solve (3), one can use for instance the Newton-Raphson algorithm, which is guaranteed to converge regardless of the starting point because of the monotonicity and convexity of the Black-Scholes function C_{BM} .

3 Compound options valuation under regime switching dynamics

In this section, we extend the Geske formula to the two-state Markov regime-switching lognormal (2RS) model of Hardy (2001). The choice of such dynamics is motivated essentially by the fact that this model is able to capture several empirical properties of long-term financial stock returns, such as heteroscedasticity, autocorrelation and fat tails (see Hamilton 1989 and Hardy 2001). For instance, Hardy (2001) finds that, compared to several econometric models, the 2RS model describes better the behavior of the TSE-300 and the S&P-500 indexes.

Assume that the underlying asset price dynamics is described by the 2RS model of Hardy (2001). Accordingly, during a time interval $[m, m + 1]$ where the regime is $k \in \{H, L\}$, the log-return of the underlying asset follows a Gaussian distribution with mean and variance indexed on k , that is,

$$\ln \frac{S_{m+1}}{S_m} \sim \mathcal{N}(\mu_k, \sigma_k^2), \quad k \in \{H, L\},$$

where S_m denotes the price of the underlying asset at date m . Regime transitions are modeled by a discrete-stage Markov chain. We denote by p_{kl} the transition probability from regime k to regime l , $k, l \in \{H, L\}$. We assume that the time unit for expressing maturities corresponds to the time elapsed between two successive stages of the Markov chain. To specify a risk-neutral measure, one can follow Bollen (1998), who assumes that the additional risk brought by the possibility of regime changes is not priced in the market, so that the risk-neutral dynamics of assets is obtained by replacing μ_k with $r - \sigma_k^2/2$. The 2RS risk-neutral model is then fully characterized by the vector (r, ψ) where $\psi \equiv (\sigma_H, \sigma_L, p_{HL}, p_{LH})$.

Consider a European call option with exercise price K and maturity T , where the underlying asset price is modeled with a 2RS model as described above. Hardy (2001) shows that the price of such an option at $t = 0$ when the underlying asset price is s is given by¹

$$C_{RS}(s; r, \psi, K, T) = \sum_{i=0}^T C_{BM}(s; r, \bar{\sigma}_T(i), K, T) P_T(i) \quad (4)$$

where

$$\bar{\sigma}_T^2(i) = \frac{i\sigma_H^2 + (T-i)\sigma_L^2}{T}$$

and $P_T(i)$ is the probability of the underlying asset price process spending i time steps in Regime H during T time periods. Following Hardy (2001), denote by $Q_m(j|k)$ the probability at some stage t that the total sojourn (measured in time steps) in Regime H during time period $[t, t + m]$ be $j \in [0, m]$ if the prevailing regime in period $[t - 1, t)$ is $k \in \{H, L\}$, where $m = 1, \dots, T$. We then have:

$$Q_1(H|k) = p_{kH} \quad (5a)$$

$$Q_1(L|k) = p_{kL} \quad (5b)$$

$$Q_m(j|k) = p_{kH} Q_{m-1}(j-1|H) + p_{kL} Q_{m-1}(j|L), \quad 1 < m \leq T, \quad (5c)$$

and the probabilities $P_T(i)$ are given by

$$P_T(i) = \pi_H Q_T(i|H) + \pi_L Q_T(i|L) \quad (6)$$

where π_H and π_L describe the unconditional regime probability distribution for the process $\{S_t\}$, that is,

$$\pi_H = 1 - \pi_L = p_{LH} / (p_{HL} + p_{LH}).$$

Now consider a compound option where the underlying asset price is modeled with a 2RS model as described above. The following proposition characterizes the price at $t = 0$ of a compound Call on Call option in the regime switching model.

¹Notice that the Hardy formula does not depend on the current regime, which is assumed unobservable.

Proposition 1 *The value at $t = 0$ of a Call on Call with contractual parameters $\xi \equiv (K_1, K_2, T_1, T_2)$ under the 2RS model characterized by (r, ψ) , $\psi \equiv (\sigma_H, \sigma_L, p_{HL}, p_{LH})$ is given by*

$$CC_{RS}(s; r, \psi, \xi) = \sum_{i=0}^{T_1} A_i(s; r, \psi, \xi) P_{T_1}(i) \quad (7)$$

with

$$A_i(s; r, \psi, \xi) = -K_1 e^{-rT_1} \Phi(d'_{\gamma_1(i)}(s^*)) + \sum_{j=0}^{T_2-T_1} B_{ij}(s; r, \psi, \xi) P_{T_2-T_1}(j) \quad (8)$$

$$B_{ij}(s; r, \psi, \xi) = s \Phi_2(d_{\gamma_2(i+j)}(s), d_{\gamma^*(i)}(s); \rho_{ij}) - K_2 e^{-rT_2} \Phi_2(d'_{\gamma_2(i+j)}(s), d_{\gamma^*(i)}(s); \rho_{ij}) \quad (9)$$

where

$$\begin{aligned} \gamma_l(i) &= (r, \bar{\sigma}_{T_l}(i), K_l, T_l), l \in \{1, 2\} \\ \gamma^*(i) &= (r, \bar{\sigma}_{T_1}(i), s^*, T_1) \\ \rho_{ij} &= \sqrt{\frac{T_1}{T_2} \frac{\bar{\sigma}_{T_1}(i)}{\bar{\sigma}_{T_2}(i+j)}} = \sqrt{\frac{T_1 \bar{\sigma}_{T_1}^2(i)}{T_1 \bar{\sigma}_{T_1}^2(i) + (T_2 - T_1) \bar{\sigma}_{T_2-T_1}^2(i)}} \end{aligned}$$

and s^* is the value of the underlying asset price such that the compound option is at the money at maturity, that is, s^* solves

$$C_{RS}(s; r, \psi, K_2, T_2 - T_1) = K_1, \quad (10)$$

where C_{RS} is given by Equation (4).

Proof. See Appendix. □

Formula (7) is easy to implement and can be easily extended to the case of Call on Put, Put on Call and Put on Put. The following algorithm gives a step by step implementation for the computation of a compound option in the 2RS model.

1. Set $s, r, \psi = (\sigma_H, \sigma_L, p_{HL}, p_{LH})$ and $\xi = (K_1, K_2, T_1, T_2)$.
2. Compute $Q_m(j|k)$ for $m = 0, \dots, T_2$, $j = 0, \dots, m$ and $k \in \{H, L\}$ using Equations (5a), (5b) and (5c); use Equation (6) to obtain the values $P_{T_1}(i)$ for $i = 0, \dots, T_1$ and $P_{T_2-T_1}(j)$ for $j = 0, \dots, T_2 - T_1$.
3. Define the function $C_{RS}(\cdot; r, \psi, K_2, T_2 - T_1)$ using (4) and find s^* that solves Equation (10).
4. Compute $B_{ij}(s; r, \psi, \xi)$, $i = 0, \dots, T_1$ and $j = 0, \dots, T_2 - T_1$ using Equation (9).
5. Compute $A_i(s; r, \psi, \xi)$, $i = 0, \dots, T_1$ using Equation 8.
6. Compute $CC_{RS}(s; r, \psi, \xi)$ using Equation 7.

For the 2RS model, the analytic formula (7) directly gives the value of a compound option as a function of the numerical solution of Equation (10), which can be readily obtained using, for instance, the Newton-Raphson algorithm. This formula can be very helpful to evaluate long-term contracts that can be expressed as a combination of compound options and simpler instruments. In the next section, we explore a specific application of compound options, that is, principal protected callable notes.

4 Principal protected callable notes

Principal protected callable notes (PPCN) are structured products with principal protection and call features. More specifically, an investor holding such a derivative will receive at maturity at least the principal amount initially invested, provided that the note has not been redeemed by the issuer, who has the right to do so at some predefined dates. Like most structured products, the potential return of PPCN is linked to the performance of a given risky asset (e.g. an index, a fund, a stock or a portfolio), called herein the *reference*

asset. In addition to their reference asset, PPCN are also characterized by their early-redemption date(s) and prices, which determine what is called the *adjusted potential return*. According to the number and the nature of the intermediate decisions and to the way that the potential return is linked to the reference asset's performance, one can distinguish several types of PPCN. In this section, we focus on PPCN with a single redemption date, where the issuer has the option of redeeming the contract at a pre-determined redemption price, while if the contract reaches maturity, the holder receives an amount linked to the total return of the reference asset. Examples of such contracts are the Canadian Blue Chip III Deposit Notes issued by the National Bank of Canada, the Callable Canadian Equity Deposit Notes issued by the Canadian Imperial Bank of Commerce, and the Callable Index-Linked Notes, linked to the Eurostoxx 50 Index and issued by HSBC Holdings plc.

More specifically, let T_2 denote the maturity of the contract, T_1 the early redemption date, K_1 the corresponding early redemption price, and D the initial investment of the note holder. The contract is then described by the following events:

- At issuance ($t = 0$), the issuer receives from the holder an amount D , which is linked to the value S_0 of the reference asset. Usually, this initial investment does not includes management fees eventually charged by the issuer.
- At the redemption date ($t = T_1$), the issuer has the option to call the PPCN for redemption at price K_1 .
- If it has not been redeemed earlier, the holder receives at maturity ($t = T_2$) the maximum between his initial investment D and the value S_{T_2} of thereference asset.

To the best of our knowledge, the valuation of PPCN is not directly addressed in the literature. In practice, issuers generally use Monte Carlo simulation to evaluate such products. In this section, we show that the valuation of PPCN with a single early-redemption date can be linked to the value of a compound option. Using the results of the previous section, we therefore obtain an analytical pricing formula for PPCN under both the GBM and the 2RS models. Notice that this approach can be generalized for PPCN with multiple early redemption dates, which can be expressed as a combination of multi-fold sequential compound options, but the resulting numerical scheme becomes rapidly inefficient.

In the sequel, we assume that the initial investment D is equal to the initial value of the reference asset. This assumption does not change the procedure, but it simplifies the notation. Let Θ denote a vector containing the parameters of the model representing the dynamics of the reference asset, including the risk-free rate r (for instance, $\Theta_{BM} = (r, \sigma)$ for the GBM model, while $\Theta_{RS} = (r, \sigma_H, \sigma_L, \rho_{HL}, \rho_{LH})$ for the 2RS model). Denote by $C(s; \Theta, K, T)$ the value at $S_0 = s$ of a European call option with maturity T and strike price K and by $CC(s; \Theta, K_1, K_2, T_1, T_2)$ the value at $S_0 = s$ of a Call on Call option with maturity dates (T_1, T_2) and strike prices (K_1, K_2) . The following proposition shows that PPCN with a single redemption date can be replicated by a long position on a European call option and a Treasury bond and a short position on a Call on Call.

Proposition 2 *When there is a single early-redemption date, the value at issuance of PPCN is given by*

$$V_0(D; \Theta, T_1, T_2, K_1) = \beta_0 \beta_1 D + C(D; \Theta, D, T_2) - CC(D; \Theta, K_1 - \beta_1 D, D, T_1, T_2),$$

where $\beta_0 = e^{-rT_1}$ and $\beta_1 = e^{-r(T_2 - T_1)}$.

Proof. If the note reaches maturity, its final value is given by

$$V_2 = \max \{D; S_{T_2}\} = D + (S_{T_2} - D)^+.$$

At the early redemption date, the value of the note is given by

$$\begin{aligned} V_1 &= \min \{K_1; \beta_1 \mathbb{E}_{T_1} [V_2]\} \\ &= \min \left\{ K_1; \beta_1 D + \beta_1 \mathbb{E}_{T_1} \left[(S_{T_2} - D)^+ \right] \right\} \\ &= \beta_1 D + \beta_1 \mathbb{E}_{T_1} \left[(S_{T_2} - D)^+ \right] - \left(\beta_1 \mathbb{E}_{T_1} \left[(S_{T_2} - D)^+ \right] - (K_1 - \beta_1 D) \right)^+. \end{aligned}$$

Finally, the value of the note at issuance where $S_0 = D$ is

$$\begin{aligned}
V_0(D; \Theta, T_1, T_2, K_1) &= \beta_0 \mathbb{E}_0 [V_1] \\
&= \beta_0 \beta_1 D + \beta_0 \mathbb{E}_0 \left[\beta_1 \mathbb{E}_{S_{T_1}} \left[(S_{T_2} - D)^+ \right] \right] \\
&\quad - \beta_0 \mathbb{E}_0 \left[\left(\beta_1 \mathbb{E}_{T_1} \left[(S_{T_2} - D)^+ \right] - (K_1 - \beta_1 D) \right)^+ \right] \\
&= \beta_0 \beta_1 D + C(D; \Theta, D, T_2) - CC(D; \Theta, K_1 - \beta_1 D, D, T_1, T_2),
\end{aligned}$$

where $\mathbb{E}_t[\cdot]$ is the expectation at date t conditional on S_t . \square

Proposition 2 shows that an analytical formula can be obtained for the value of PPCN with a single early-redemption date when analytical formulas are available for the price of a European call option and a Call on Call under the model assumptions for the dynamics of the reference asset. This is the case, for instance, where the reference asset evolution is described by a geometric Brownian motion or a two-state Markov regime-switching lognormal model. For the GBM model, functions C and CC are given by C_{BM} and CC_{BM} in Equations (1) and (2) respectively, and for the 2RS model, they are given by the Hardy formula (4) and Equation (7) respectively.

When issuing PPCN, a financial institution faces the problem of fixing the early-redemption price so that the value of the contract is equal to its nominal initial investment. Alternatively, one may want to determine the fair value of the initial investment corresponding to a given early-redemption price. In both cases, this corresponds to solving the fixed-point problem

$$V_0(D; \Theta, T_1, T_2, K_1) = D. \quad (11)$$

The Newton-Raphson algorithm can be used to solve (11) for either K_1 or D .

5 Conclusion

In this paper, an analytical formula is derived for compound options under a regime-switching lognormal model. This formula can be used for any product that possesses compound option components. As an illustration, we use it in a specific application to derive a formula for the price of a callable structured product with one early-redemption date. The resulting pricing algorithm is much more efficient than Monte Carlo simulation, and allows to easily price PPCN or determine the fair value of their early-redemption price.

Appendix

Proof of Proposition 1

Proof. At date T_1 , if the underlying asset price is s , the value of the Call on Call is equal to

$$CC_{RS}(s; r, \psi, K_1, K_2, 0, T_2 - T_1) = [C_{RS}(s; r, \psi, K_2, T_2 - T_1) - K_1]^+.$$

Denote by s^* the value of s solving (10). Since function C_{RS} is increasing in s , we have

$$CC_{RS}(s; r, \psi, K_1, K_2, 0, T_2 - T_1) = (C_{RS}(s; r, \psi, K_2, T_2 - T_1) - K_1) \mathbb{I}_{s \geq s^*}$$

where \mathbb{I}_X denotes the indicator of the condition X . The value of the compound option at $t = 0$ is therefore

$$CC_{RS}(s; r, \psi, \xi) = e^{-rT_1} \mathbb{E}_s \left[(C_{RS}(S_{T_1}; r, \psi, K_2, T_2 - T_1) - K_1) \mathbb{I}_{S_{T_1} \geq s^*} \right] \quad (12)$$

where $\mathbb{E}_s[\cdot]$ denotes the expectation, conditional on $S_0 = s$.

On the other hand, as shown by Hardy (2001), conditional on i , the number of time steps spent in Regime H during the time interval $[0, T_1)$, the asset price S_{T_1} at date T_1 satisfies

$$S_{T_1} = S_0 \exp \left(\left(r - \frac{\bar{\sigma}_{T_1}^2(i)}{2} \right) T_1 + \bar{\sigma}_{T_1}(i) \sqrt{T_1} \mathbb{Z} \right) \quad (13)$$

where \mathbb{Z} is a standard normal random variable. Thereby, to compute the conditional expectation in Equation (12), we condition on the number of time units spent in Regime H during time interval $[0, T_1)$, yielding

$$\begin{aligned} CC_{RS}(s; r, \psi, \xi) &= e^{-rT_1} \sum_{i=0}^{T_1} \mathbb{E}_{s,i} \left[(C_{RS}(S_{T_1}; r, \psi, K_2, T_2 - T_1) - K_1) \mathbb{1}_{S_{T_1} \geq s^*} \right] P_{T_1}(i) \\ &= \sum_{i=0}^{T_1} A_i(s; r, \psi, \xi) P_{T_1}(i) \end{aligned}$$

with

$$A_i(s; r, \psi, \xi) \equiv e^{-rT_1} \mathbb{E}_{s,i} \left[(C_{RS}(S_{T_1}; r, \psi, K_2, T_2 - T_1) - K_1) \mathbb{1}_{S_{T_1} \geq s^*} \right]$$

where $\mathbb{E}_{s,i}[\cdot]$ denotes the expectation, conditional on $S_0 = s$ and on the number of time units spent in Regime H being i . Let $\gamma^*(i) = (r, \bar{\sigma}_{T_1}(i), s^*, T_1)$. Using

$$S_{T_1} \geq s^* \iff \mathbb{Z} \geq -d'_{\gamma^*(i)}(s),$$

we then have

$$\begin{aligned} \mathbb{E}_{s,i} \left[(C_{RS}(S_{T_1}; r, \psi, K_2, T_2 - T_1) - K_1) \mathbb{1}_{S_{T_1} \geq s^*} \right] \\ = \mathbb{E}_{s,i} \left[C_{RS}(S_{T_1}; r, \psi, K_2, T_2 - T_1) \mathbb{1}_{S_{T_1} \geq s^*} \right] - K_1 \mathbb{E}_{s,i} \left[\mathbb{1}_{S_{T_1} \geq s^*} \right] \\ = \mathbb{E}_{s,i} \left[C_{RS}(S_{T_1}; r, \psi, K_2, T_2 - T_1) \mathbb{1}_{S_{T_1} \geq s^*} \right] - K_1 \Phi \left(d'_{\gamma^*(i)}(s) \right). \end{aligned}$$

Now, using $\gamma^o(j) \equiv (r, \bar{\sigma}_{T_2-T_1}(j), K_2, T_2 - T_1)$, Equation (4) yields

$$C_{RS}(S_{T_1}; r, \psi, K_2, T_2 - T_1) = \sum_{j=0}^{T_2-T_1} C_{BM}(S_{T_1}; \gamma^o(j)) P_{T_2-T_1}(j)$$

and we can write

$$\begin{aligned} A_i(s; r, \psi, \xi) &= -K_1 e^{-rT_1} \Phi \left(d'_{\gamma^*(i)}(s) \right) \\ &\quad + e^{-rT_1} \mathbb{E}_{s,i} \left[C_{RS}(S_{T_1}; r, \psi, K_2, T_2 - T_1) \mathbb{1}_{S_{T_1} \geq s^*} \right] \\ &= -K_1 e^{-rT_1} \Phi \left(d'_{\gamma^*(i)}(s) \right) \\ &\quad + e^{-rT_1} \mathbb{E}_{s,i} \sum_{j=0}^{T_2-T_1} \left[C_{BM}(S_{T_1}; \gamma^o(j)) \mathbb{1}_{S_{T_1} \geq s^*} \right] P_{T_2-T_1}(j) \\ &= -K_1 e^{-rT_1} \Phi \left(d'_{\gamma^*(i)}(s) \right) + \sum_{j=0}^{T_2-T_1} B_{ij}(s; r, \psi, \xi) P_{T_2-T_1}(j) \end{aligned}$$

with

$$B_{ij}(s; r, \psi, \xi) \equiv e^{-rT_1} \mathbb{E}_{s,i} \left[C_{BM}(S_{T_1}; \gamma^o(j)) \mathbb{1}_{S_{T_1} \geq s^*} \right].$$

Using Equation (1) yields

$$\begin{aligned} B_{ij}(s; r, \psi, \xi) &= e^{-rT_1} \mathbb{E}_{s,i} [C_{BM}(S_{T_1}; \gamma^o(j)) \mathbb{I}_{S_{T_1} \geq s^*}] \\ &= e^{-rT_1} \mathbb{E}_{s,i} \left[\left(S_{T_1} \Phi(d_{\gamma^o(j)}(S_{T_1})) - K_2 e^{-r(T_2-T_1)} \Phi(d'_{\gamma^o(j)}(S_{T_1})) \right) \mathbb{I}_{S_{T_1} \geq s^*} \right] \\ &= e^{-rT_1} \mathbb{E}_{s,i} [S_{T_1} \Phi(d_{\gamma^o(j)}(S_{T_1})) \mathbb{I}_{S_{T_1} \geq s^*}] - K_2 e^{-rT_2} \mathbb{E}_{s,i} \left[\Phi(d'_{\gamma^o(j)}(S_{T_1})) \mathbb{I}_{S_{T_1} \geq s^*} \right]. \end{aligned}$$

Now, using (13) and replacing $d_{\gamma^o(j)}(S_{T_1})$ by its value, we get

$$\begin{aligned} & \mathbb{E}_{s,i} [S_{T_1} \Phi(d_{\gamma^o(j)}(S_{T_1})) \mathbb{I}_{S_{T_1} \geq s^*}] \\ &= s e^{T_1(r - \frac{1}{2}\bar{\sigma}_{T_1}^2(i))} \mathbb{E}_s \left[e^{\bar{\sigma}_{T_1}(i)\sqrt{T_1}Z} \Phi \left(\frac{\log(S_{T_1}/K_2) + (T_2 - T_1)(r + \frac{1}{2}\bar{\sigma}_{T_2-T_1}^2(j))}{\bar{\sigma}_{T_2-T_1}(j)\sqrt{(T_2 - T_1)}} \right) \mathbb{I}_{Z \geq -d'_{\gamma^*(i)}(s)} \right] \\ &= s e^{T_1(r - \frac{1}{2}\bar{\sigma}_{T_1}^2(i))} \mathbb{E}_s \left[e^{\bar{\sigma}_{T_1}(i)\sqrt{T_1}Z} \Phi \left(\frac{\log(s/K_2) + \left(r - \frac{\bar{\sigma}_{T_1}^2(i)}{2}\right)T_1}{\bar{\sigma}_{T_2-T_1}(j)\sqrt{(T_2 - T_1)}} + \frac{\bar{\sigma}_{T_1}(i)\sqrt{T_1}Z + (T_2 - T_1)(r + \frac{1}{2}\bar{\sigma}_{T_2-T_1}^2(j))}{\bar{\sigma}_{T_2-T_1}(j)\sqrt{(T_2 - T_1)}} \right) \mathbb{I}_{Z \geq -d'_{\gamma^*(i)}(s)} \right] \\ &= s e^{T_1(r - \frac{1}{2}\bar{\sigma}_{T_1}^2(i))} \mathbb{E}_s \left[e^{\bar{\sigma}_{T_1}(i)\sqrt{T_1}Z} \Phi \left(\frac{d_{(r, \bar{\sigma}_{T_2}(i+j), K_2, T_2)}(s) + \rho_{ij}(Z - \bar{\sigma}_{T_1}(i))}{\sqrt{1 - \rho_{ij}^2}} \right) \mathbb{I}_{Z \geq -d'_{\gamma^*(i)}(s)} \right] \\ &= s e^{rT_1} \int_{-d'_{\gamma^*(i)}(s)}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - \bar{\sigma}_{T_1}(i))^2} \Phi \left(\frac{d_{\gamma_2(i+j)}(s) + \rho_{i,j}(z - \bar{\sigma}_{T_1}(i))}{\sqrt{1 - \rho_{ij}^2}} \right) dz \\ &= s e^{rT_1} \int_{-d'_{\gamma^*(i)}(s) + \bar{\sigma}_{T_1}(i)}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} N \left(\frac{d_{\gamma_2(i+j)}(s) + \rho_{i,j}z}{\sqrt{1 - \rho_{ij}^2}} \right) dz \\ &= s e^{rT_1} \Phi_2(d_{\gamma_2(i+j)}(s), d_{\gamma^*(i)}(s); \rho_{ij}). \end{aligned}$$

The same procedure can be used to show that:

$$\mathbb{E}_{s,i} \left[\Phi(d'_{\gamma^o(j)}(S_{T_1})) \mathbb{I}_{S_{T_1} \geq s^*} \right] = \Phi_2(d'_{\gamma_2(i+j)}(s), d_{\gamma^*(i)}(s); \rho_{ij})$$

and we obtain finally

$$B_{ij}(s; r, \psi, \xi) = s \Phi_2(d_{\gamma_2(i+j)}(s), d_{\gamma^*(i)}(s); \rho_{ij}) - K_2 e^{-rT_2} \Phi_2(d'_{\gamma_2(i+j)}(s), d_{\gamma^*(i)}(s); \rho_{ij}).$$

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