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A comparison of formulations for a three-level lot sizing and replenishment problem with a distribution structure

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Abstract: We address a three-level lot sizing and replenishment problem with a distribution structure (3LSPD), which is an extension of the one-warehouse multi-retailer problem (OWMR). We consider one production plant that produces one type of item over a discrete and finite planning horizon. The items produced are used to replenish warehouses and then retailers using direct shipments. Each retailer is linked to a unique warehouse and there are no transfers between warehouses nor between retailers. We also assume that transportation is uncapacitated. However, we consider the possibility of imposing production capacity constraints at the production plant level. The objective is to minimize the sum of the fixed production and replenishment costs and of the unit variable inventory holding costs at all three levels. We compare 16 different MIP formulations to solve the problem. All of these formulations are adapted from existing MIP formulations found in the one-warehouse multi-retailer literature, but most formulations are new in the context of the 3LSPD. We run experiments on both balanced and unbalanced networks. In the balanced network each warehouse serves the same number of retailers whereas in the unbalanced network 20% of the warehouses serve 80% of the retailers. Our results indicate that the multi-commodity formulation is well suited for uncapacititated instances and that the echelon stock reformulations are better for capacitated instances. They also show that the richer formulations are not necessarily the best ones and that the unbalanced instances are harder to solve.

Keywords: Production planning and control, lot sizing, replenishment, mixed integer programming formulations, deterministic demand, one-warehouse multi-retailer problem, multi-level

Résumé : Nous étudions un problème intégré de planification de production et de transport sur trois niveaux avec une structure de distribution (3LSPD), problème qui est une extension du *one-warehouse multi-retailer problem* (OWMR). On considère une usine de production qui fabrique un type de produit sur un horizon de planification fini et discret. Les biens produits sont transportés de l'usine vers des centres de stockage puis ensuite vers des détaillants via des livraisons directes. Chaque détaillant est relié à un unique centre de stockage et les transferts de produits entre les centres de stockage ou entre les détaillants ne sont pas autorisés. Cependant, nous considérons la possibilité d'imposer des restrictions sur la capacité de production au niveau de l'usine de production. L'objectif est de minimiser la somme des coûts fixes de production et de commande et des coûts variables unitaires de stockage. On compare ici 16 formulations mixtes en nombres entiers différentes pour résoudre le problème. Toutes les formulations proposées sont des adaptations des formulations mixtes en nombres entiers rencontrées dans la littérature sur le problème *One-Warehouse Multi-Retailer*, et la plupart des formulations développées ici sont proposées pour la première fois dans le contexte du 3LSPD. Nous réalisons des expériences numériques tant sur un réseau équilibré que sur un réseau non équilibré. Dans le réseau équilibré chaque centre de stockage est responsable du même nombre de détaillants alors que dans le réseau non équilibré 20% des centres de stockage sont responsables de 80% des détaillants. Nos résultats indiquent que la formulation *multi-commodity* est la plus adaptée pour la résolution des instances sans contrainte de capacité alors que les formulation *echelon-stock* sont plus adaptées pour les formulations avec contraintes de capacité. Les résultats montrent aussi que les formulations les plus riches ne sont pas nécessairement les meilleures et que les instances ayant un réseau non équilibré sont les plus difficiles à résoudre.

Mots clés : Planification de production et transport, formulations mixtes en nombres entiers, demande déterministe, taille de lot, multi-niveau

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1 Introduction

Over the last decades, lot sizing problems have drawn the attention of many researchers, mainly because of their numerous applications in production, distribution and inventory management problems. Extensions of the basic lot sizing problem (LSP) are often encountered in the context of supply chain planning. Usually, the customers of a company, which have a certain demand, are located in a different area from the production plant where the items are actually produced and where lot sizing decisions are made. This leads to a replenishment problem where the company needs to determine when to replenish its customers so as to minimize the replenishment costs. Companies facing these two operational problems often make decisions in sequence. This leads, however, to solutions that can be far from the optimal solution of an integrated lot sizing and replenishment problem.

We address here an integrated three-level lot sizing and replenishment problem with a distribution structure (3LSPD). We consider a general manufacturing company that has one production plant (level zero), several warehouses (level one) and multiple retailers (level two) facing a dynamic and known demand for one item over a discrete and finite time horizon. The supply chain considered has a distribution structure: the warehouses are all linked to the single plant and all retailers are linked to exactly one warehouse. When we consider the demand of a particular retailer, the flow of goods in the supply chain network is hence as follows: an item is produced at the production plant, then sent to the warehouse linked to the retailer for storage and finally sent to the retailer to satisfy its demand. Figure 1 illustrates this flow of goods in a distribution network which consists of one production plant, three warehouses and three retailers linked to each warehouse. The objective of the problem is to determine the optimal timing and flows of goods between the different facilities while minimizing the operational and replenishment costs in the whole network (sum of the fixed setup and replenishment costs and unit inventory holding costs).

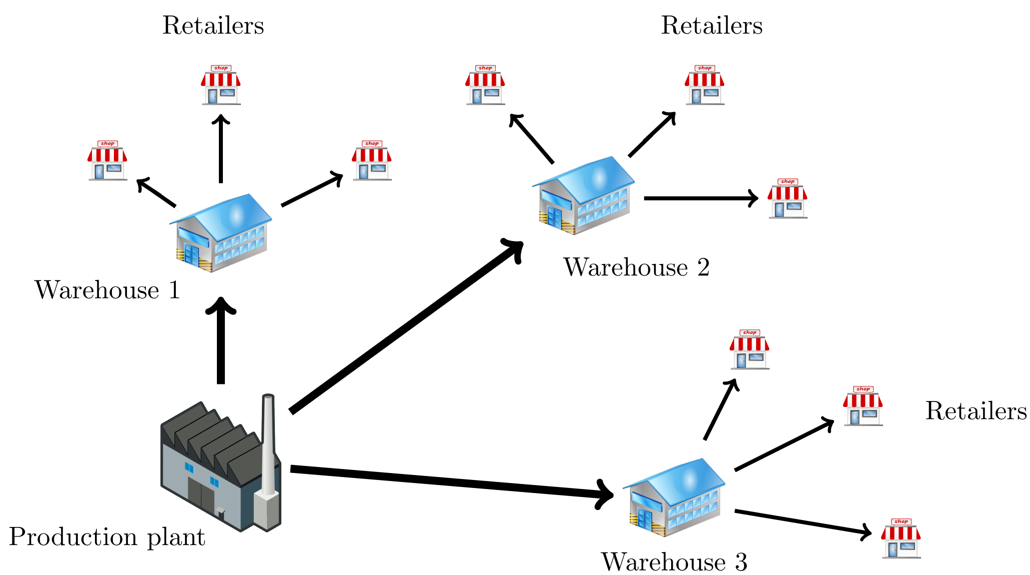

Figure 1: Graphical representation of the problem considered

More specifically, given the set T of time periods, we face an integrated problem where decisions are made at all facilities for each time period. The optimal solution of the problem will indicate, for each time period, the optimal quantities to be produced and to be ordered from their predecessor for the production plant and for the warehouses and retailers, respectively, so that the final demand at each retailer is satisfied. In this problem, the objective is to minimize the sum over all periods t of the fixed setup costs sc_t^p at the production plant, the fixed replenishment costs sc_t^w and sc_t^r of the warehouses and of the retailers, and the unit inventory holding costs hc_t^i of all facilities i. We do not include any unit production cost at the plant since the total production cost is a constant when all the demand is satisfied and when the unit production cost is constant over time. The same holds for the unit replenishment cost at the warehouses and retailers.

Transfers of goods between the warehouses and between the retailers are not allowed. Finally, we only consider uncapacitated direct shipments and do not incorporate any routing in the transportation decisions. Note that in a disaggregated context, the problem faced by any facility can be seen as the basic LSP. This basic LSP has attracted a lot of research since the seminal paper of Wagner and Whitin [39] who proposed a dynamic programming approach to solve the single item uncapacitated lot sizing problem (SI-ULSP). The reader is referred to Brahimi et al. [5] and to Pochet and Wolsey [34] for a review of the work done on the SI-ULSP and its extensions, and to Jans and Degraeve [19] for a review of industrial applications.

We consider both a capacitated and an uncapacitated version of the 3LSPD. In the capacitated version, the capacity constraints are imposed at the production plant level to limit the production quantities in each time period. There are no capacities on the flows between the facilities nor on the inventory level. Note that with the addition of the capacity constraints at the production plant level, the problem faced by the production plant can be seen as a basic capacitated lot sizing problem (CLSP). The reader is referred to Karimi et al. [21] for a review of models and algorithms used to solve the CLSP.

The motivation to work on MIP formulations for the 3LSPD is to extend the works of Solyalı and Süral [37] and Cunha and Melo [8] who compare several MIP formulations for the one-warehouse multi-retailer problem (OWMR). In the OWMR, a central warehouse replenishes several retailers that face a dynamic demand for one or several items over a discrete and finite time horizon. The objective of the problem is to jointly determine the optimal timing and quantities that are shipped between the warehouse and the retailers to minimize the sum of setup costs and inventory holding costs for the whole system. This problem has been shown to be *NP*-hard by Arkin et al. [2] and appears as a substructure in the production routing problem (PRP). Compared to the OWMR, the PRP also optimizes routing decisions to visit the different customers of the central warehouse. The reader is refered to Adulyasak et al. [1] for a detailed review of formulations and solution algorithms for the PRP.

Solyalı and Süral [37] compare four MIP formulations and Cunha and Melo [8] compare eight different MIP formulations for the OWMR. The 3LSPD can be considered as the generalization of the OWMR to three levels. Our aim in this work is to adapt these OWMR MIP formulations to the 3LSPD and to verify if the results obtained on the two-level OWMR still stand for the 3LSPD.

Our paper makes two main contributions. First, we fill a gap by adapting several MIP formulations that have been proposed in the context of the two-level OWMR (Solyalı and Süral [37], Cunha and Melo [8]) to the 3LSPD. To the best of our knowledge, this is the first attempt to provide strong formulations for the 3LSPD that could solve instances of large scale. We also give several properties about the relationships between the linear relaxations of these formulations. Second, we report the results of extensive numerical experiments using a general-purpose solver to assess the strengths and weaknesses of the different formulations. Indeed, we perform experiments for different structures of the main parameters (fixed or dynamic demand, fixed or dynamic setup costs) and for two distribution structures of the supply chain network. In one case we consider a balanced distribution network in which each warehouse is responsible for the same number of retailers. In the other case, we consider an unbalanced distribution network where 20% of the warehouses replenish 80% of the retailers. The results obtained highlight the importance of properly choosing a formulation depending on the characteristics of the problem.

The remainder of this paper is organized as follows. First, we survey the work related to our study in Section 2. We then present sixteen different MIP formulations for the problem in Section 3. These MIP formulations can be divided into three groups of formulations: the classical formulations, which use the standard MIP formulation of the basic LSP, the echelon stock based formulations, inspired from the echelon stock concept for the multi-level LSP, and the richer formulations, containing more information in the decision variables, inspired from the work on the polyhedral structure of the solutions of both the SI-ULSP and the two-level lot sizing problems. Section 4 presents computational results to determine the strengths and weaknesses of the different formulations that we propose, and to analyze the impact of the different parameters. This is followed by the conclusion in Section 4.3.

2 Literature review

We first review the literature on the OWMR in Section 2.1, followed by the literature on the three-level lot sizing problem in Section 2.2.

2.1 OWMR literature review

The 3LSPD studied in this paper is a generalization of the OWMR to three levels. Both problems have a distribution structure and there are similar production, inventory and replenishment decisions to be made at each time period to satisfy the demand of the retailers. The main difference is that the OWMR only considers two levels in its distribution network: the warehouse and the retailers.

Many formulations have been proposed for the OWMR. Federgruen and Tzur [14] propose the echelon stock formulation (ES), based on the echelon stock concept for multi-level lot sizing problems. Using the echelon stock concept, the traditional inventory decision variables are replaced by the echelon stock variables representing the total inventory of a component at a given facility and all of its descendents. Levi et al. [26] propose the transportation formulation inspired from the facility location literature. Melo and Wolsey [27] propose the multi-commodity formulation (MC) based on the distinction of each retailer-time period pair. Solyalı and Süral [37] compare four different MIP formulations for the OWMR: the shortest path formulation (SP), the transportation formulation (TP) and the echelon stock formulation and its strenghtened version (SES). The SES formulation is inspired from the ES formulation of Federgruen and Tzur [14] and uses transportation decision variables to strenghten the ES formulation. Solyalı and Süral [37] extend these formulations to the possibility of having a non-zero initial inventory. They also provide results concerning the LP bounds of each formulation and numerical experiments are performed with and without inital inventory. In the same vein, Cunha and Melo [8] consider eight different MIP formulations: the shortest path formulation (SP), the transportation formulation (TP), the strenghtened echelon stock formulation (SES), the Wagner-Whitin echelon stock based formulation (ESWW), the two-level lot sizing Wagner-Whitin based formulation (2LSWW) and its partial version (p2LSWW), the multi-commodity formulation (MC) and the dynamic programming formulation (DP). They compare the LP bounds of these formulations and show in particular that the DP formulation gives the best LP bound. They then perform numerous computational experiments with both dynamic and static unit transportation costs. Note that there also exists a classical MIP formulation for the OWMR which is the extension of the classical MIP formulation for the ULSP proposed by Zangwill [42].

Some work has also been done to develop families of valid inequalities for the OWMR to strengthen the MIP formulations given in Solyalı and Süral [37] and Cunha and Melo [8]. This is the case in Senoussi et al. [36]. Starting from a PRP and considering a warehouse that is really far from the retailers, they aggregate the retailers in different clusters to discard routing decisions and get a real OWMR with both fixed and unit transportation costs, and with transportation capacity. They propose six sets of valid inequalities: one to determine the maximum number of vehicles, one to break symmetries, one to have full trucks (based on the optimal properties of the solution), two which extend the (l, S) inequalities of the SI-ULSP proposed in Barany, Van Roy and Wolsey [3], and the last one to reduce the number of variables in their model. They conduct numerous experiments both with and without all the valid inequalities to see the impact of these valid inequalities. Melo [10] proposes another set of valid inequalities and also designs a separation algorithm to find the violated inequalities. This separation procedure is used in a cutting plane algorithm to perform experiments on a multi-item OWMR problem.

2.2 Three-level lot sizing problem

Because of the different nature of the decisions made at each facility and because of the three levels, one can find several supply chain structures in the literature on three-level lot sizing problems (3L-LSP). The following section only reviews the literature for which the supply chain structure is the same as in our problem: one production plant, several warehouses and several retailers. When not explicitely mentionned, the supply network structure considered in the papers reviewed in this section is a distribution structure as in our problem.

Only a few papers address a three-level lot sizing problem with a number of facilities per level which is the same as in our problem. The ones that we found all address extensions of the 3LSPD considered in this paper. Gebennini, Gamberini and Manzini [17] propose a heuristic to solve a problem where they consider safety stocks and allow backorders. The backorder in a particular period is the quantity of unmet demand for this time period. The basic model they propose is non-linear because of the safety stock cost but is linearized with an approximation of the objective function. There are also due dates for the deliveries to the customers. The authors design a procedure to solve the approximate problem.

Barbarosoglu and Ozgur [4] address the 3L-LSP where each retailer is linked to every warehouse. They thus do not have a distribution structure in their network but a general one instead. They also work in a just-in-time (JIT) environment. The JIT environment translates into a constraint that prevents retailers from keeping inventory. The model contains both fixed and unit transportation costs. The authors propose a transportation based MIP model and use Lagrangean relaxation to solve the problem. They relax the constraints linking the production and distribution components to obtain a production subproblem which can be decomposed into knapsack problems, and a distribution subproblem that can be easily solved for each item-customer pair. A customized procedure is then used to build feasible solutions from the solutions obtained in these two sub-problems.

Several extensions relate to applications for industrial cases. Kopanos, Puigjnaer and Georgiadis [23] address an industrial case in Greece in the food industry. They have a fixed cost per vehicle used for the deliveries between the facilities and there are several transportation modes available. They consider restrictions on the vehicles that can make the deliveries between facilities. They extend their MIP model to consider several production plants and use a general-purpose solver in both cases to solve their instances. Haq, Vrat and Kanda [18] also use a general-purpose solver to solve an industrial case of urea manufacturing. They propose a MIP model that contains transportation lead time and backlog but these features are discarded in the numerical experiments performed.

Heuristics have also been proposed to solve extensions of the 3LSPD applied to industrial cases. Lejeune [25] proposes to solve a problem with a fixed cost per truck used and unit transportation costs. The author also considers transportation capacities and time availability of the carriers. A combination of branch-and-bound (B&B) and variable neighborhood search (VNS) is used to solve the problem. In each node of the B&B tree, there are several neighborhoods where binary variables are split between fixed variables, variables to be fixed and free variables. The branching decisions are made depending on these sets. For each node there is also a limit on the children nodes. A computational experiment using data of a US chemical company indicates that this method outperforms CPLEX. In the same vein, Özdamar and Yazgaç [32] treat the case study of a detergent company in Turkey. They design an algorithm to approximately solve the problem. The authors consider transportation capacities and propose an aggregate and a disaggregate MIP model. The algorithm is based on an iterative hierarchical approach as well as on a rolling horizon.

Note that in the works mentioned in this section, only three different types of MIP formulation have been used: Haq, Vrat and Kanda [18], Lejeune [25], Gebennini, Gamberini and Manzini [17] and Özdamar and Yazgaç [32] use a classical formulation, Barbarosoglu and Ozgur [4] use a combined classical and transportation formulation, and Kopanos, Puigjnaer and Georgiadis [23] use a transportation formulation. The classical formulation and the combined transportation and classical formulation will be presented in Section 3.1 while the transportation formulation will be given in Section 3.4.

3 Formulations

Let $G = (F, A)$ be a graph with F the set of nodes (facilities in our problem) and A the set of arcs. Let $P = \{p\} \subset F$ be the set containing the unique production plant, $W \subset F$ be the set of warehouses and $R \subset F$ be the set of retailers. Following the problem description in Section 1, we have $F = P \cup W \cup R$. Let $\delta(i)$ be the set of all direct successors of facility i and $\delta^w(r)$ be the warehouse linked to the retailer $r \in R$. Let d_t^r be the demand for retailer r in period t. The notion of the demand faced by any retailer is extended to the warehouses and to the production plant in the following fashion:

$$d_t^i = \begin{cases} \sum_{r\in R} d_t^r & \text{if } i = p \\ \sum_{r\in\delta(i)} d_t^r & \text{if } i \in W. \end{cases}$$

Using the notion of the demand faced by any facility, we introduce D_t^i, the total demand between period t and the end of the time horizon computed as $D_t^i = \sum_{k\geq t} d_k^i$. Similarly, we introduce, for any facility i, the demand between periods k and t as $d_{kt}^i = \sum_{k\leq l\leq t} d_l^i$. In the following sections, all the MIP formulations are presented in their capacitated version.

3.1 Classical formulations

We first present a simple MIP formulation that extends the basic MIP formulation for the ULSP as used by Pochet and Wolsey [34]. We call this formulation the classical formulation (C). This formulation is based on three sets of decisions variables: x_t^i represents the production quantities in period t if $i = p$ and the quantities ordered from the predecessor if $i \in W \cup R$, s_t^i is the inventory held at the end of period t in facility i, and y_t^i is a boolean setup variable taking value 1 iff $x_t^i > 0$. The formulation is as follows:

$$\text{Min} \sum_{t\in T}\left(\sum_{i\in F} sc_t^i y_t^i + \sum_{i\in F} hc_t^i s_t^i\right) \tag{1}$$

$$\begin{aligned}
\text{s.t. } x_t^i &\leq D_t^i y_t^i && \forall\, t\in T, i\in F && (2)\\
s_{t-1}^i + x_t^i &= \sum_{j\in\delta(i)} x_t^j + s_t^i && \forall\, t\in T, i\in P\cup W && (3)\\
s_{t-1}^r + x_t^r &= d_t^r + s_t^r && \forall\, t\in T, r\in R && (4)\\
x_t^p &\leq \min\{C_t, D_t^p\} y_t^p && \forall t\in T && (5)\\
x_t^i, s_t^i &\geq 0 && \forall\, t\in T, i\in F && (6)\\
y_t^i &\in \{0,1\} && \forall\, t\in T, i\in F. && (7)
\end{aligned}$$

The objective function minimizes the sum of the fixed setup and replenishment costs and of the unit inventory holding costs. Constraints (2) are the setup forcing constraints for all facilities. Constraints (3) are the inventory balance equations for the production plant and the warehouses whereas (4) are the inventory balance equations for the retailers. Constraints (5) are the capacity constraints at the production plant.

The classical formulation C can be improved by using some ideas coming from the ULSP literature. We observe that when we only consider the inventory balance equations (4) and the setup constraints (2) specifically for the retailers, we have a single item lot sizing structure for each retailer since the inventory balance equations (4) only incorporate the independent demand for each retailer. This means that we can use the existing reformulations of the ULSP for each of the retailers. These reformulations are not directly applicable to the warehouse or plant level, since at these levels the inventory balance constraints contain dependent demand in the form of decision variables related to the ordering decisions at the direct successors. We will propose three different alternative formulations to model the lot sizing structure at the retailer level.

First, we use the network reformulation proposed by Eppen and Martin [12] to change the decision variables linked to the retailers and rewrite the constraints where these variables appear. The reformulation proposed by Eppen and Martin [12] is based on the property of extreme flows in a network as applied by Zangwill [43] to the SI-ULSP. This property, also known as the zero inventory ordering property, states that if there is a positive entering stock at any period in the SI-ULSP, then the flow coming from production is equal to zero. Conversely, if the production is positive at any period, then the entering stock for this period is equal to zero. Although this property does not hold for the capacitated case, Eppen and Martin [12] show that their proposed reformulation is valid for the capacitated case. For any retailer $r \in R$, let z_{kt}^r be the proportion of d_{kt}^r that is ordered in period k. Let also $spc_{kt}^r = \sum_{k\leq u<t}\sum_{l=u+1}^{t} h_u^r d_l^r$ be the cost linked to the variable z_{kt}^r for any retailer i. The classical-network formulation (C-N) for the 3LSPD is as follows:

$$\text{Min} \sum_{t\in T}\left(\sum_{i\in F} sc_t^i y_t^i + \sum_{i\in P\cup W} hc_t^i s_t^i + \sum_{r\in R}\sum_{k\leq t} spc_{kt}^r z_{kt}^r\right) \tag{8}$$

$$\begin{aligned}
\text{s.t. } x_t^i &\leq D_t^i y_t^i && \forall\, t\in T, i\in P\cup W && (9)\\
\sum_{k=t:d_{tk}^r>0}^{|T|} z_{tk}^r &\leq y_t^r && \forall\, t\in T, r\in R && (10)\\
s_{t-1}^p + x_t^p &= \sum_{w\in\delta(p)} x_t^w + s_t^p && \forall\, t\in T && (11)\\
s_{t-1}^w + x_t^w &= \sum_{r\in\delta(w)}\sum_{k\geq t} z_{tk}^r d_{tk}^r + s_t^w && \forall\, t\in T, w\in W && (12)\\
\sum_{t=1}^{|T|} z_{1t}^r &= 1 && \forall\, r\in R && (13)\\
\sum_{l=1}^{t-1} z_{l,t-1}^r &= \sum_{k=t}^{|T|} z_{tk}^r && \forall\, t\geq 2, r\in R && (14)\\
x_t^p &\leq \min\{C_t, D_t^p\} y_t^p && \forall t\in T && (15)\\
z_{kt}^r &\geq 0 && \forall\, t\in T, k\leq t\in T, r\in R && (16)\\
x_t^i, s_t^i &\geq 0 && \forall\, t\in T, i\in P\cup W && (17)\\
y_t^i &\in \{0,1\} && \forall\, t\in T, i\in F. && (18)
\end{aligned}$$

Constraints (10) are the setup forcing constraints for the retailers. Constraints (12) are the inventory balance constraints for the warehouses. Constraints (13) are the initial flow constraints for each retailer and constraints (14) are the flow conservation constraints.

Second, one can use the transportation reformulation of the ULSP proposed by Krarup and Bilde [24] to give another formulation for the problem. For any retailer r, let ϕ_{kt}^r represent the quantity that is ordered in period $k \leq t$ and used to satisfy d_t^r. Let also $tc_{kt}^r = \sum_{k\leq u<t} h_u^r$ be the holding cost linked to the variable ϕ_{kt}^r. The classical-transportation formulation (C-TP) for the 3LSPD is as follows:

$$\text{Min} \sum_{t\in T}\left(\sum_{i\in F} sc_t^i y_t^i + \sum_{i\in P\cup W} hc_t^i s_t^i + \sum_{r\in R}\sum_{k\leq t} tc_{kt}^r \phi_{kt}^r\right) \tag{19}$$

$$\begin{aligned}
\text{s.t. } x_t^i &\leq D_t^i y_t^i && \forall\, t\in T, i\in P\cup W && (20)\\
\phi_{tk}^r &\leq d_k^r y_t^r && \forall\, t\in T, k\leq t\in T, r\in R && (21)\\
s_{t-1}^p + x_t^p &= \sum_{j\in\delta(p)} x_t^j + s_t^p && \forall\, t\in T && (22)\\
s_{t-1}^w + x_t^w &= \sum_{r\in\delta(w)}\sum_{k\geq t} \phi_{tk}^r + s_t^w && \forall\, t\in T, w\in W && (23)\\
\sum_{k=1}^{t} \phi_{kt}^r &= d_t^r && \forall\, t\in T, \forall\, r\in R && (24)\\
x_t^p &\leq \min\{C_t, D_t^p\} y_t^p && \forall t\in T && (25)\\
\phi_{kt}^r &\geq 0 && \forall\, t\in T, k\leq t\in T, r\in R && (26)\\
x_t^i, s_t^i &\geq 0 && \forall\, t\in T, i\in P\cup W && (27)\\
y_t^i &\in \{0,1\} && \forall\, t\in T, i\in F. && (28)
\end{aligned}$$

Constraints (21) are the setup forcing constraints for the retailers. Constraints (23) are the inventory balance constraints for the warehouses. Constraints (24) are the demand satisfaction constraints for each retailer.

Finally, one can also use the polyhedral results for the SI-ULSP to improve the classical formulation C at the retailer level. In particular, Barany et al. [3] propose the (l, S) valid inequalities that describe the polyhedron of solutions of the SI-ULSP. Besides, if the SI-ULSP has Wagner-Whitin costs (i.e., $pc_t + hc_t \geq pc_{t+1},\ \forall\ t \in T$, where pc_t is the unit production cost in period t), Pochet and Wolsey [34] propose the (l, S, WW) valid inequalities. When adapted to our problem, these (l, S, WW) inequalities are defined as follows:

$$s^r_{k-1} \geq \sum_{j=k}^{l} d^r_j \left(1 - \sum_{u=k}^{j} y^r_u\right) \quad \forall\ k \leq l \in T, r \in R. \tag{29}$$

These inequalities are always valid, even if the costs do not satisfy the Wagner-Whitin condition. However, in case the Wagner-Whitin cost condition holds, they are sufficient to describe the convex hull of the SI-ULSP. These inequalities are added to (1)–(7) to form the classical-lS formulation (C-LS).

3.2 Echelon stock formulations

Employing the idea of an echelon stock presented in Federgruen and Tzur [14], the 3LSPD can be decomposed into several independent SI-ULSPs. To do so, the inventory variables of the classical formulation C are replaced with echelon stock variables representing the total inventory at all descendents of a particular facility. We define the echelon stock I^i_t for facility i in period t as:

$$I^i_t = \begin{cases} s^i_t + \sum_{w \in W} s^w_t + \sum_{r \in R} s^r_t & \text{if } i = p \\ s^i_t + \sum_{r \in \delta(i)} s^r_t & \text{if } i \in W \\ s^i_t & \text{if } i \in R. \end{cases}$$

The echelon stock formulation (ES) is then as follows:

$$\text{Min} \sum_{t \in T} \left(\sum_{i \in F} sc^i_t y^i_t + \sum_{p \in P} hc^p_t I^p_t + \sum_{w \in W} (hc^w_t - hc^p_t) I^w_t + \sum_{r \in R} \left(hc^r_t - hc^{\delta_w(r)}_t \right) I^r_t \right) \tag{30}$$

$$\text{s.t. } I^i_{t-1} + x^i_t = d^i_t + I^i_t \quad \forall\ t \in T, i \in F \tag{31}$$

$$x^i_t \leq D^i_t y^i_t \quad \forall\ t \in T, i \in F \tag{32}$$

$$I^i_t \geq \sum_{j \in \delta(i)} I^j_t \quad \forall\ t \in T, i \in P \cup W \tag{33}$$

$$x^p_t \leq \min\{C_t, D^p_t\} y^p_t \quad \forall t \in T \tag{34}$$

$$x^i_t, I^i_t \geq 0 \quad \forall\ t \in T, i \in F \tag{35}$$

$$y^i_t \in \{0, 1\} \quad \forall\ t \in T, i \in F. \tag{36}$$

The objective function (30) is written in terms of echelon stock variables. Constraints (31) are the inventory balance constraints using the new echelon stock variables. Constraints (33) are the echelon stock constraints ensuring that the echelon stock at a specific facility is greater than the sum of the echelon stocks at all its direct successors. These constraints come from the non-negativity constraints (6) imposed on the stock variables in the classical formulation C. Note that with the introduction of the echelon stock variables, the problem has an uncapacitated lot sizing structure (in constraints (2) and (31)) with independent demand at each level. This means that we can now apply the known reformulation techniques for the ULSP (network, transportation and (l, S, WW) inequalities) at each level.

First, in the same spirit as in the C-N formulation, we can use a network reformulation on the ES formulation. We define Z^i_{kt} to be the proportion of d^i_{kt} that is produced in period k for $i = p$, and to be the proportion of d^i_{kt} that is ordered in period k for $i \in W \cup R$. The echelon stock network formulation (ES-N) is then as follows:

$$\text{Min} \sum_{t \in T} \left(\sum_{i \in F} sc^i_t y^i_t + \sum_{p \in P} hc^p_t I^p_t + \sum_{w \in W} (hc^w_t - hc^p_t) I^w_t + \sum_{r \in R} \left(hc^r_t - hc^{\delta_w(r)}_t \right) I^r_t \right) \tag{37}$$

$$\text{s.t.} \sum_{k=1}^{|T|} Z^i_{1k} = 1 \qquad \forall\, i \in F \tag{38}$$

$$\sum_{l=1}^{t-1} Z^i_{l,t-1} = \sum_{l=t}^{|T|} Z^i_{tl} \qquad \forall\, t \geq 2, i \in F \tag{39}$$

$$\sum_{k=t:d^i_{tk}}^{|T|} Z^i_{tk} \leq y^i_t \qquad \forall\, t \in T, i \in F \tag{40}$$

$$I^i_t = \left(\sum_{l=1}^{t} \sum_{k=l}^{|T|} d^i_{lk} Z^i_{lk} \right) - d^i_{1t} \qquad \forall\, t \in T, i \in F \tag{41}$$

$$I^i_t \geq \sum_{j \in \delta(i)} I^j_t \qquad \forall\, t \in T, i \in P \cup W \tag{42}$$

$$\sum_{k=t}^{|T|} Z^p_{tk} d^p_{tk} \leq \min\{C_t, D^p_t\} y^p_t \qquad \forall t \in T \tag{43}$$

$$I^i_t \geq 0 \qquad \forall\, t \in T, i \in F \tag{44}$$

$$Z^i_{tk} \geq 0 \qquad \forall\, t \in T, k \geq t \in T, i \in F \tag{45}$$

$$y^i_t \in \{0, 1\} \qquad \forall\, t \in T, i \in F. \tag{46}$$

Constraints (38) are the initial flow constraints for each facility and constraints (39) are the flow conservation constraints. Constraints (40) are the setup forcing constraints. Constraints (41) link the flow variables and the echelon stock variables. Constraints (43) are the capacity constraints at the production plant.

Then, in the same spirit as in the C-TP formulation, we can use a transportation reformulation on the ES formulation. We define X^i_{kt} to be the quantity that is produced in period k and used to satisfy d^i_t for $i = p$, and to be the quantity that is ordered in period k for $i \in W \cup R$ and used to satisfy d^i_t. The echelon stock transportation formulation (ES-TP) is then as follows:

$$\text{Min} \sum_{t \in T} \left(\sum_{i \in F} sc^i_t y^i_t + \sum_{p \in P} hc^p_t I^p_t + \sum_{w \in W} (hc^w_t - hc^p_t) I^w_t + \sum_{r \in R} \left(hc^r_t - hc^{\delta_w(r)}_t \right) I^r_t \right) \tag{47}$$

$$\text{s.t. } I^i_{t-1} + \sum_{k=t}^{|T|} X^i_{tk} = d^i_t + I^i_t \qquad \forall\, t \in T, i \in F \tag{48}$$

$$\sum_{k=1}^{t} X^i_{kt} = d^i_t \qquad \forall\, t \in T, i \in F \tag{49}$$

$$X^i_{tk} \leq d^i_k y^i_t \qquad \forall k \in T, t \leq k \in T, i \in F \tag{50}$$

$$I^i_t \geq \sum_{j \in \delta(i)} I^j_t \qquad \forall\, t \in T, i \in P \cup W \tag{51}$$

$$\sum_{k=t}^{|T|} X^p_{tk} \leq \min\{C_t, D^p_t\} y^p_t \qquad \forall t \in T \tag{52}$$

$$X^i_{kt} \geq 0 \qquad \forall\, k \leq t \in T, i \in F \tag{53}$$
$$I^i_t \geq 0 \qquad \forall\, t \in T, i \in F \tag{54}$$
$$y^i_t \in \{0,1\} \qquad \forall\, t \in T, i \in F. \tag{55}$$

Constraints (48) are the inventory balance constraints. These are included in order to correctly calculate the inventory levels. Constraints (49) are the demand satisfaction constraints. Constraints (50) are the setup forcing constraints. Constraints (52) are the capacity constraints at the production plant.

Finally, we can also add the (l, S, WW) valid inequalities in the context of the ES formulation. Using the echelon stock variables, these inequalities are given as follows:

$$I^i_{k-1} + \sum_{q=k}^{t} d^i_{qt} y^i_q \geq d^i_{kt} \qquad \forall k \leq t \in T, i \in F. \tag{56}$$

These inequalities are added to (30)–(36) to form the echelon stock-lS formulation (ES-LS).

Following the model proposed in Federgruen and Tzur [14], another change can be made to the echelon stock formulation ES. Indeed, one can alternatively write the echelon stock constraints (33) using the production variables of the ES, ES-N or ES-TP formulation, respectively:

$$\sum_{k=1}^{t} x^i_k \geq \sum_{j \in \delta(i)} \sum_{k=1}^{t} x^j_k \qquad \forall t \in T, i \in P \cup W. \tag{57}$$
$$\sum_{k=1}^{t} \sum_{l \geq k} d^i_{kl} Z^i_{kl} \geq \sum_{j \in \delta(i)} \sum_{k=1}^{t} \sum_{l \geq k} d^j_{kl} Z^j_{kl} \qquad \forall t \in T, i \in P \cup W. \tag{58}$$
$$\sum_{k=1}^{t} \sum_{l \geq k} X^i_{kl} \geq \sum_{j \in \delta(i)} \sum_{k=1}^{t} \sum_{l \geq k} X^j_{kl} \qquad \forall t \in T, i \in P \cup W. \tag{59}$$

If we substitute (33) by (57), (58) and (59) in formulations ES or ES-LS, ES-N and ES-TP, respectively, we obtain the echelon stock Federgruen formulations ES-F or ES-F-LS, ES-F-N and ES-F-TP, respectively.

3.3 Network formulation

The following formulation uses the network reformulation as proposed by Eppen and Martin [12] for the SI-ULSP to rewrite the variables and constraints of the problem. Such a reformulation has also been applied by Solyalı and Süral [37] and Cunha and Melo [8] for the OWMR. For any retailer r, let ψ^r_{klst} be the proportion of d^r_{st} that is produced by the production plant in period k, transported to the warehouse of retailer r in period l and to retailer r in period s. Let also nc^r_{klst} be the cost linked to the variable ψ^r_{klst}: $nc^r_{klst} = \sum_{j=k}^{l-1} hc^p_j d^r_{st} + \sum_{j=l}^{s-1} hc^{\delta_w(r)}_j d^r_{st} + \sum_{j=s}^{t-1} hc^r_j d^r_{j+1,t}$. The network formulation (N) is given as follows:

$$\text{Min} \sum_{t \in T} \left(\sum_{i \in F} sc^i_t y^i_t + \sum_{r \in R} \sum_{k=1}^{t} \sum_{l=k}^{t} \sum_{s=l}^{t} nc^r_{klst} \psi^r_{klst} \right) \tag{60}$$

$$\sum_{t=1}^{|T|} \psi^r_{111t} = 1 \qquad \forall\, r \in R \tag{61}$$
$$\sum_{k=1}^{t-1} \sum_{l=k}^{t-1} \sum_{s=l}^{t-1} \psi^i_{k,l,s,t-1} = \sum_{k=1}^{t} \sum_{l=k}^{t} \sum_{s=t}^{|T|} \psi^i_{klts} \qquad \forall\, t \geq 2, r \in R \tag{62}$$
$$\sum_{l=k}^{t} \sum_{s=l}^{t} \sum_{j=t:d^r_{sj}>0}^{|T|} \psi^r_{klsj} \leq y^p_k \qquad \forall\, t \in T, k \leq t \in T, r \in R \tag{63}$$

$$\sum_{k=1}^{l}\sum_{s=l}^{t}\sum_{j=t:d^r_{sj}>0}^{|T|}\psi^r_{klsj}\leq y_l^{\delta_w(r)} \qquad \forall\, t\in T, l\leq t\in T, r\in R \tag{64}$$

$$\sum_{k=1}^{s}\sum_{l=k}^{s}\sum_{j=t:d^r_{sj}>0}^{|T|}\psi^r_{klsj}\leq y_s^r \qquad \forall\, t\in T, s\leq t\in T, r\in R \tag{65}$$

$$\sum_{i\in R}\sum_{l=k}^{|T|}\sum_{s=l}^{|T|}\sum_{t=s}^{|T|}\psi^i_{klst}d^i_{st}\leq \min\{C_k, D^p_k\}y^p_k \qquad \forall k\in T \tag{66}$$

$$\psi^r_{klst}\geq 0 \qquad \forall\, k\leq l\leq s\leq t\in T, r\in R \tag{67}$$

$$y^i_t\in\{0,1\} \qquad \forall t\in T, i\in F. \tag{68}$$

Constraints (61) are the demand satisfaction constraints written as initial flow constraints. Constraints (62) are the flow conservation constraints. Constraints (63), (64) and (65) are the setup forcing constraints for the production plant, the warehouses and the retailers, respectively. Constraints (66) are the capacity constraints at the production plant.

3.4 Transportation formulation

In the following formulation, the interactions between the facilities are modeled based on the transportation formulation of Krarup and Bilde [24] for the SI-ULSP. For any retailer r, let θ^r_{klst} be the quantity that is produced by the production plant in period k, transported to the warehouse of retailer r in period l, transported to retailer r in period s and used to satisfy d^r_t. Let also H^r_{klst} be the cost linked to θ^r_{klst}: $H^r_{klst}=\sum_{j=k}^{l-1}hc^p_j+\sum_{j=l}^{s-1}hc^{\delta_w(r)}_j+\sum_{j=s}^{t-1}hc^r_j$. The transportation formulation (TP) is given as follows:

$$\text{Min}\sum_{t\in T}\left(\sum_{i\in F}sc^i_ty^i_t+\sum_{r\in R}\sum_{k=1}^{t}\sum_{l=k}^{t}\sum_{s=l}^{t}H^r_{klst}\theta^r_{klst}\right) \tag{69}$$

$$\text{s.t.}\ \sum_{k=1}^{t}\sum_{l=k}^{t}\sum_{s=l}^{t}\theta^r_{klst}=d^r_t \qquad \forall\, t\in T, r\in R \tag{70}$$

$$\sum_{l=k}^{t}\sum_{s=l}^{t}\theta^r_{klst}\leq d^r_ty^p_k \qquad \forall\, t\in T, k\leq t\in T, r\in R \tag{71}$$

$$\sum_{k=1}^{l}\sum_{s=l}^{t}\theta^r_{klst}\leq d^r_ty_l^{\delta_w(r)} \qquad \forall\, t\in T, l\leq t\in T, r\in R \tag{72}$$

$$\sum_{k=1}^{s}\sum_{l=k}^{s}\theta^r_{klst}\leq d^r_ty^r_s \qquad \forall\, t\in T, s\leq t\in T, r\in R \tag{73}$$

$$\sum_{i\in R}\sum_{l=k}^{|T|}\sum_{s=l}^{|T|}\sum_{t=s}^{|T|}\theta^i_{klst}\leq \min\{C_k, D^p_k\}y^p_k \qquad \forall k\in T \tag{74}$$

$$\theta^r_{klst}\geq 0 \qquad \forall\, k\leq l\leq s\leq t\in T, r\in R \tag{75}$$

$$y^i_t\in\{0,1\} \qquad \forall t\in T, i\in F. \tag{76}$$

Constraints (70) are the demand satisfaction constraints. Constraints (71), (72) and (73) are the setup forcing constraints for the production plant, the warehouses and the retailers, respectively. Constraints (74) are the capacity constraints at the production plant.

3.5 Multi-commodity formulation

The next formulation proposed is based on the distinction of each retailer-period pair (i.e., each d^r_t is viewed as a distinct commodity). For this formulation, for any retailer r, let w^{0r}_{kt} be the amount produced at the production plant in period k to satisfy d_t^r, let w_{kt}^{1r} be the amount transported from the production plant to the warehouse of retailer r in period k to satisfy d_t^r and let w_{kt}^{2r} be the amount transported from the warehouse of retailer r to retailer r in period k to satisfy d_t^r. Let also σ_{kt}^{0r} be the amount stocked at the production plant at the end of period k to satisfy d_t^r, let σ_{kt}^{1r} be the amount stocked at the warehouse of retailer r at the end of period k to satisfy d_t^r and let σ_{kt}^{2r} be the amount stocked at retailer r at the end of period k to satisfy d_t^r. In the following formulation, we denote by δ_{kt} the Kronecker delta which takes the value 1 if $k = t$ and 0 otherwise. The multi-commodity formulation (MC) is as follows:

$$\text{Min} \sum_{t\in T} \left(\sum_{i\in F} sc_t^i y_t^i + \sum_{r\in R}\sum_{k\le t} hc_k^p \sigma_{kt}^{0r} + \sum_{r\in R}\sum_{k\le t} hc_k^{\delta_w(r)} \sigma_{kt}^{1r} + \sum_{r\in R}\sum_{k\le t} hc_k^r \sigma_{kt}^{2r} \right) \tag{77}$$

$$\begin{aligned}
\sigma_{k-1,t}^{0r} + w_{kt}^{0r} &= w_{kt}^{1r} + \sigma_{kt}^{0r} && \forall\, t\in T, k\le t\in T, r\in R && (78)\\
\sigma_{k-1,t}^{1r} + w_{kt}^{1r} &= w_{kt}^{2r} + \sigma_{kt}^{1r} && \forall\, t\in T, k\le t\in T, r\in R && (79)\\
\sigma_{k-1,t}^{2r} + w_{kt}^{2r} &= \delta_{kt} d_t^r + (1-\delta_{kt})\sigma_{kt}^{2r} && \forall\, t\in T, k\le t\in T, r\in R && (80)\\
w_{kt}^{0r} &\le d_t^r y_k^p && \forall\, t\in T, k\le t\in T, r\in R && (81)\\
w_{kt}^{1r} &\le d_t^r y_k^{\delta_w(r)} && \forall\, t\in T, k\le t\in T, r\in R && (82)\\
w_{kt}^{2r} &\le d_t^r y_k^r && \forall\, t\in T, k\le t\in T, r\in R && (83)\\
\sum_{r\in R}\sum_{t=k}^{|T|} w_{kt}^{0r} &\le \min\{C_k, D_k^p\} y_k^p && \forall k\in T && (84)\\
w_{kt}^{0r}, w_{kt}^{1r}, w_{kt}^{2r}, \sigma_{kt}^{0r}, \sigma_{kt}^{1r}, \sigma_{kt}^{2r} &\ge 0 && \forall\, t\in T, k\le t\in T, r\in R && (85)\\
y_t^i &\in \{0;1\} && \forall t\in T, i\in F. && (86)
\end{aligned}$$

Constraints (78), (79) and (80) are the balance constraints for each commodity at the production plant, at the warehouses and at the retailers, respectively. Constraints (81), (82) and (83) are the setup forcing constraints for the production plant, the warehouses and the retailers, respectively. Constraints (84) are the capacity constraints at the production plant.

The last formulation combines the idea of an echelon stock presented in Federgruen and Tzur [14] and the MC formulation. It is called the multi-commodity echelon formulation (MCE). To get this formulation, the inventory variables of the MC formulation are replaced with multi-commodity echelon variables E_{kt}^{lr} representing the amount stocked at the end of period k at all predecessors of retailer r which are in level l or more, and which will be used to fulfill the specific demand d_t^r. We define the multi-commodity echelon variables E_{kt}^{lr} as:

$$E_{kt}^{lr} = \begin{cases} \sigma_{kt}^{0r} + \sigma_{kt}^{1r} + \sigma_{kt}^{2r} & \text{if } l = 0\\ \sigma_{kt}^{1r} + \sigma_{kt}^{2r} & \text{if } l = 1\\ \sigma_{kt}^{2r} & \text{if } l = 2. \end{cases}$$

The multi-commodity echelon formulation (MCE) is then as follows:

$$\text{Min} \sum_{t\in T} \left(\sum_{i\in F} sc_t^i y_t^i + \sum_{r\in R}\sum_{k\le t} hc_k^p E_{kt}^{0r} + \sum_{r\in R}\sum_{k\le t} \left(hc_k^{\delta_w(r)} - hc_k^p \right) E_{kt}^{1r} + \sum_{r\in R}\sum_{k\le t} \left(hc_k^r - hc_k^{\delta_w(r)} \right) E_{kt}^{2r} \right) \tag{87}$$

$$\text{s. t. } (81) - (86)$$

$$\begin{aligned}
E_{k-1,t}^{0r} + w_{kt}^{0r} &= \delta_{kt} d_t^r + (1-\delta_{kt}) E_{kt}^{0r} && \forall\, t\in T, k\le t\in T, r\in R && (88)\\
E_{k-1,t}^{1r} + w_{kt}^{1r} &= \delta_{kt} d_t^r + (1-\delta_{kt}) E_{kt}^{1r} && \forall\, t\in T, k\le t\in T, r\in R && (89)\\
E_{k-1,t}^{2r} + w_{kt}^{2r} &= \delta_{kt} d_t^r + (1-\delta_{kt}) E_{kt}^{2r} && \forall\, t\in T, k\le t\in T, r\in R && (90)\\
E_{kt}^{0r} &\ge E_{kt}^{1r} && \forall\, t\in T, k\le t\in T, r\in R && (91)\\
E_{kt}^{1r} &\ge E_{kt}^{2r} && \forall\, t\in T, k\le t\in T, r\in R && (92)\\
E_{kt}^{0r}, E_{kt}^{1r}, E_{kt}^{2r} &\ge 0 && \forall\, t\in T, k\le t\in T, r\in R. && (93)
\end{aligned}$$

Constraints (88), (89) and (90) are the balance constraints for each commodity at the production plant, at the warehouses and at the retailers respectively. Constraints (91) and (92) are the echelon constraints ensuring that the multi-echelon stock at a specific facility for a specific commodity is greater than or equal to the sum of the multi-echelon stocks at all its direct successors for the same commodity.

3.6 Summary

The formulations previously introduced are extensions of the MIP formulations proposed for the OWMR. For all the formulations presented, the adaptation of the original decision variables naturally leads to an increase in their number. For the N and TP formulations, this increase translates into an additionnal dimension with the new subscript k in the decision variables ψ^r_{klst} and θ^r_{klst} to reflect the third level. For all the other formulations, the increase in the number of decision variables is just the result of the increase in the number of facilities due to the added third level. Thus, the increase in the number of decision variables for the N and TP formulations is much higher than for the other formulations when going from a two-level LSP to a three-level LSP.

Table 1 gives a summary of the number of variables and constraints for each formulation previously introduced, and the paper from which the formulation has been adapted to our problem. Recall that these papers present a one-level or two-level problem whereas we consider a three-level problem. Note that, to the best of our knowledge, the ES-N, ES-F-N, ES-F-TP, ES-F-LS and MCE formulations we propose are completely new. In Table 1, one can see that the richer formulations, i.e., the ones that have more information in the decision variables, are the largest ones.

Table 1: Summary of the sizes of all formulations

Formulation	Variables	Constraints	Reference
C	$O(\|F\| \times \|T\|)$	$O(\|F\| \times \|T\|)$	Pochet and Wolsey [34]
C-N	$O(\|R\| \times \|T\|^2)$	$O(\|F\| \times \|T\|)$	Eppen and Martin [12]
C-TP	$O(\|R\| \times \|T\|^2)$	$O(\|R\| \times \|T\|^2)$	Krarup and Bilde [24]
C-LS	$O(\|R\| \times \|T\|^2)$	$O(\|R\| \times \|T\|^2)$	Pochet and Wolsey [34]
ES	$O(\|F\| \times \|T\|)$	$O(\|F\| \times \|T\|)$	Pochet and Wolsey [34]
ES-N	$O(\|F\| \times \|T\|^2)$	$O(\|F\| \times \|T\|^2)$	
ES-TP	$O(\|F\| \times \|T\|^2)$	$O(\|F\| \times \|T\|^2)$	Solyalı and Süral [37]
ES-LS	$O(\|F\| \times \|T\|)$	$O(\|F\| \times \|T\|^2)$	Melo and Wolsey [27]
ES-F	$O(\|F\| \times \|T\|)$	$O(\|F\| \times \|T\|)$	Federgruen and Tzur [14]
ES-F-N	$O(\|F\| \times \|T\|^2)$	$O(\|F\| \times \|T\|^2)$	
ES-F-TP	$O(\|F\| \times \|T\|^2)$	$O(\|F\| \times \|T\|^2)$	
ES-F-LS	$O(\|F\| \times \|T\|)$	$O(\|F\| \times \|T\|^2)$	
N	$O(\|R\| \times \|T\|^4)$	$O(\|R\| \times \|T\|^2)$	Solyalı and Süral [37]
TP	$O(\|R\| \times \|T\|^4)$	$O(\|R\| \times \|T\|^2)$	Levi et al. [26]
MC	$O(\|R\| \times \|T\|^2)$	$O(\|R\| \times \|T\|^2)$	Melo and Wolsey [27]
MCE	$O(\|R\| \times \|T\|^2)$	$O(\|R\| \times \|T\|^2)$	

3.7 Analysis of the LP relaxation of formulations

We explore the strength of the MIP formulations in terms of the objective function value of their LP relaxation, without considering the production capacity constraint (5). In the LP relaxations of the MIP formulations, we replace the binary requirements on the setup variables by the following constraints:

$$0 \leq y^i_t \leq 1 \qquad \forall\ i \in F, \forall\ t \in T. \tag{94}$$

We denote by z^X_{LP} the objective function value of the LP relaxation of formulation X. We denote by $F(X)$ the set of feasible solutions for formulation X. The following example is used to illustrate most of the strict dominance relations between the formulations. The strict dominance relation between formulations MC and N cannot be observed empirically on small instances such as the one presented hereafter. However, we have observed it for large instances, for example with $|R| = 200$ and $|T| = 30$.

Example 1. Consider an instance of the 3LSPD with $T = 4$, $|W| = 2$ and $|R| = 4$. Each warehouse is responsible for two retailers. The first warehouse is responsible for the first two retailers and the second warehouse is responsible for the other two. We have, for any $t \in T$, $hc_t^p = 30$, $hc_t^{w_1} = 50$, $hc_t^{w_2} = 60$, $hc_t^{r_1} = 10$, $hc_t^{r_2} = 20$, $hc_t^{r_3} = 100$, $hc_t^{r_4} = 10$, $sc_t^p = 100$, $sc_t^{w_1} = 500$, $sc_t^{w_2} = 600$, $sc_t^{r_1} = 100$, $sc_t^{r_2} = 200$, $sc_t^{r_3} = 300$, $sc_t^{r_4} = 50$ and $d^{r_1} = (10, 20, 15, 10)$, $d^{r_2} = (5, 30, 10, 10)$, $d^{r_3} = (45, 20, 20, 10)$, $d^{r_4} = (10, 20, 15, 20)$. For this instance, the optimal LP solutions values for six of the formulations are $z_{LP}^{C} = 3903.56$, $z_{LP}^{C-N} = 4813.46$, $z_{LP}^{ES-LS} = 6017.25$, $z_{LP}^{ES-N} = 6096.343$, $z_{LP}^{MC} = 6750.00$ and $z_{LP}^{N} = 6750.00$.

Proposition 1

$$\begin{aligned} z_{LP}^{C} = z_{LP}^{ES} = z_{LP}^{ES-F} \le z_{LP}^{C-LS} \le z_{LP}^{C-TP} = z_{LP}^{C-N} \le z_{LP}^{ES-LS} = z_{LP}^{ES-F-LS} \le z_{LP}^{ES-N} = z_{LP}^{ES-TP} \\ = z_{LP}^{ES-F-N} = z_{LP}^{ES-F-TP} \le z_{LP}^{TP} = z_{LP}^{MC} = z_{LP}^{ME} \le z_{LP}^{N} \end{aligned} \tag{95}$$

In case the unit inventory holding costs are increasing when we go deeper in the supply chain (i.e., if $hc_t^p \le hc_t^{\delta_w(r)} \le hc_t^r$ for $p \in P$ and any $r \in R$), Proposition 1 can be slightly improved.

Proposition 2 *If, for $p \in P$ and for any $r \in R$, we have $hc_t^p \le hc_t^{\delta_w(r)} \le hc_t^r$, then:*

$$z_{LP}^{C-LS} = z_{LP}^{C-N} \tag{96}$$

Proposition 3 $z_{LP}^{C} \le z_{LP}^{C-N}$

Proof. This result follows from the fact that the network reformulation used describes the convex hull of the solutions satisfying (2), (6) and (7) for $i \in R$ and (4). □

Proposition 4 $z_{LP}^{C-LS} \le z_{LP}^{C-TP}$

Proof. In our case, since the production variables for the retailers also appear in constraints (3), we may not have an exact Wagner-Whitin cost structure for the retailers. Therefore, inequalities (29) may not be sufficient to describe the convex hull of the solution space for the retailers, whereas the network reformulation at the retailer level does. Indeed, suppose that in the LP optimal solution one constraint (3) has a non-zero dual variable. If we dualize this constraint in the objective function, the Wagner-Whitin cost structure may be violated, and the (l, S, WW) valid inequalities do not describe the convex hull of the solution space for the retailers part anymore. □

Proposition 5 $z_{LP}^{C-N} = z_{LP}^{C-TP}$

Proof. This results follows from the fact that at the retailer level both the network and transportation reformulations exactly describe the convex hull of the solutions satisfying (2), (6) and (7) for $i \in R$ and (4) as stated in Pochet and Wolsey [34]. □

Proposition 6 *If, for $p \in P$ and for any $r \in R$, we have $hc_t^p \le hc_t^{\delta_w(r)} \le hc_t^r$, then:*

$$z_{LP}^{C-LS} = z_{LP}^{C-N} \tag{97}$$

Proof. To prove this equality, it is sufficient to prove that we still have Wagner-Whitin costs for the retailers despite the fact that the production variables x^r also appear in constraint (3) for $w = \delta_w(r)$. Therefore, as proved in Pochet and Wolsey [34], the (l, S, WW) inequalities are sufficient to describe the convex hull of solutions satisfying (2), (6) and (7) for $i \in R$ and (4). Let us dualise constraints (2) (for $i \in P \cup W$) with positive dual variables μ_t^i and constraints (3) with dual variables ν_t^i. For any retailer r and any time period t, the new production cost is $\nu_t^{\delta_w(r)}$ while the holding cost remains the same. We denote by $pc_t'^r$ this new production cost. In this modified objective function, we have Wagner-Whitin costs for one particular retailer r iff:

$$pc_t'^r + hc_t^r \ge pc_{t+1}'^r \qquad \forall t \in T$$

$$\Leftrightarrow \nu_t^{\delta_w(r)} + hc_t^r \geq \nu_{t+1}^{\delta_w(r)} \qquad \forall t \in T$$
$$\Leftrightarrow \nu_t^{\delta_w(r)} - \nu_{t+1}^{\delta_w(r)} + hc_t^r \geq 0 \qquad \forall t \in T.$$

Furthermore, in the dual of problem C, at optimality, the constraint linked to the stock variables of the warehouse $\delta_w(r)$ linked to retailer r is:

$$\nu_{t+1}^{\delta_w(r)} - \nu_t^{\delta_w(r)} \leq hc_t^{\delta_w(r)} \qquad \forall t \in T. \tag{98}$$

Therefore:

$$\nu_t^{\delta_w(r)} - \nu_{t+1}^{\delta_w(r)} + hc_t^r \geq hc_t^r - hc_t^{\delta_w(r)} \qquad \forall t \in T. \tag{99}$$

As, by hypothesis, we have $hc_t^r \geq hc_t^{\delta_w(r)}$, the Wagner-Whitin cost structure still holds for any retailer. This concludes the proof. □

Proposition 7 $z_{LP}^{ES} = z_{LP}^{C}$

Proof. The proof consists in showing that any solution to the linear relaxation of ES can be converted into a solution to the linear relaxation of C with the same total cost, and that the reverse is also true. We will thus prove that $F(C) = F(ES)$. By the construction indicated in Section 3.2, we directly have $F(C) \subseteq F(ES)$. We will now prove that $F(ES) \subseteq F(C)$. Let us take a feasible solution $(x, I, y) \in F(ES)$ and construct a feasible solution $(x, s, y) \in F(C)$ with the same objective function value. We construct $(x, s, y) \in F(C)$ as follows:

$$s_t^i = \begin{cases} I_t^i & \text{if } i \in R \\ I_t^i - \sum_{r \in \delta(i)} I_t^r & \text{if } i \in W \\ I_t^i - \sum_{w \in W} I_t^w & \text{if } i = p. \end{cases} \tag{100}$$

We furthermore directly map the x and y variables. We now verify that all constraints hold.

1. Constraints (4). For any $i \in R$ and any $t \in T$, constraints (4) hold directly because they are equivalent to constraints (31).
2. Constraints (3). Let us take $i \in W$. We have, according to (31), for any $t \in T$:

$$I_{t-1}^i + x_t^i = d_t^i + I_t^i$$
$$\Leftrightarrow I_{t-1}^i + x_t^i = \sum_{r \in \delta(i)} d_t^r + I_t^i$$
$$\Leftrightarrow I_{t-1}^i + x_t^i = \sum_{r \in \delta(i)} \left(I_{t-1}^r + x_t^r - I_t^r\right) + I_t^i$$
$$\Leftrightarrow I_{t-1}^i - \sum_{r \in \delta(i)} I_{t-1}^r + x_t^i = \sum_{r \in \delta(i)} x_t^r + I_t^i - \sum_{r \in \delta(i)} I_t^i$$
$$\Leftrightarrow s_{t-1}^i + x_t^i = \sum_{r \in \delta(i)} x_t^r + s_t^i.$$

   Thus, constraints (3) hold for any warehouse as well. Using a similar approach, one can prove that constraints (3) hold for the production plant. Therefore, constraints (3) hold.
3. Constraints (2). These constraints directly hold since the production and setup variables used in formulations C and ES are the same.
4. Constraint (6). Due to constraints (33), constraints (6) directly hold.
5. Objective function value. A straightforward substitution of the I variables for the s variables in the objective function of formulation ES directly gives the objective function expression of formulation C. This concludes the proof.

□

Proposition 8 $z_{LP}^{C-LS} \leq z_{LP}^{ES-LS}$

Proof. Using a similar approach as in the proof of Proposition 7, and using the fact that the (l, S, WW) valid inequalities (29) are a subset of the (l, S, WW) valid inequalities (56), we directly have that $F(ES-LS) \subseteq F(C-LS)$. Indeed, the (l, S, WW) valid inequalities (29) are defined for all retailers r whereas the (l, S, WW) valid inequalities (56) are defined for any facility i. □

Proposition 9 $z_{LP}^{ES-LS} \leq z_{LP}^{ES-N}$

Proof. The proof consists in showing that ES-N gives a stronger reformulation than ES-LS. The result follows from the fact that ES-N uses a network reformulation (38) and (39) that gives the convex hull of the set (2), (31), (35), (36). On the contrary, the (l, S, WW) valid inequalities (56) only give an approximation of the convex hull of this set. Indeed, suppose there exist one $i \in P \cup W$ such that, in the optimal LP solution of ES-LS, we have one constraint (33) whose dual variable is strictly positive. If we dualize this constraint with its dual value in the objective function, we may destroy the Wagner-Whitin cost structure for the subproblem linked to the facility i and therefore, the (l, S, WW) valid inequalities (56) only give an approximation of the convex hull of the SI-ULSP linked to facility i. This concludes the proof. □

Proposition 10 $z_{LP}^{ES-N} = z_{LP}^{ES-TP}$

Proof. This results follows from the fact that at the retailer level both the network and transportation reformulations exactly describe the convex hull of the solutions satisfying (31), (35), (2) and (36) (see Pochet and Wolsey [34]). □

Proposition 11 $z_{LP}^{ES} = z_{LP}^{ES-F}$

Proof. To prove this result, we just need to prove that the echelon constraints (33) and (57) are equivalent, since except for the echelon constraints, formulations ES and ES-F have exactly the same objective function and constraints.

1. (33) $\Rightarrow$ (57). Let $(x, I, y) \in F(ES)$ be a feasible solution for formulation ES. One has, thanks to (31), $I_t^i = \sum_{u=1}^{t} x_u^i - d_{1t}^i$. Therefore, for any $i \in F$ and any $t \in T$, one has

$$\begin{aligned} I_t^i &\geq \sum_{j \in \delta(i)} I_t^j \\ \Leftrightarrow \sum_{u=1}^{t} x_u^i - d_{1t}^i &\geq \sum_{j \in \delta(i)} \left(\sum_{u=1}^{t} x_u^j - d_{1t}^j \right) \\ \Leftrightarrow \sum_{u=1}^{t} x_u^i &\geq \sum_{j \in \delta(i)} \sum_{u=1}^{t} x_u^j \end{aligned}$$

since $d_{1t}^i = \sum_{j \in \delta(i)} d_{1t}^j$.

2. (57) $\Rightarrow$ (33). Let $(x, I, y) \in F(ES-F)$ be a feasible solution for formulation ES-F. One has, for any $i \in F$ and any $t \in T$, $\sum_{k=1}^{t} x_k^i = \sum_{k=1}^{t} \left(d_k^i + I_k^i - I_{k-1}^i \right) = I_t^i - I_0^i + \sum_{k=1}^{t} d_k^i = d_k^i + I_t^i$, since $I_0^i = 0$. Therefore,

$$\begin{aligned} \sum_{k=1}^{t} x_k^i &\geq \sum_{j \in \delta(i)} \sum_{k=1}^{t} x_k^j \\ \Leftrightarrow \sum_{k=1}^{t} d_k^i + I_t^i &\geq \sum_{j \in \delta(i)} \sum_{k=1}^{t} d_k^j + \sum_{j \in \delta(i)} I_t^j \\ \Leftrightarrow I_t^i &\geq \sum_{j \in \delta(i)} I_t^j \end{aligned}$$

since $\sum_{j \in \delta(i)} \sum_{k=1}^{t} d_k^j = \sum_{k=1}^{t} d_k^i$. □

Using similar arguments, one can prove Propositions 12, 13 and 14.

Proposition 12 $z_{LP}^{ES-N} = z_{LP}^{ES-F-N}$

Proposition 13 $z_{LP}^{ES-TP} = z_{LP}^{ES-F-TP}$

Proposition 14 $z_{LP}^{ES-LS} = z_{LP}^{ES-F-LS}$

Proposition 15 $z_{LP}^{ES-F-TP} \leq z_{LP}^{TP}$

Proof. Let $z_{LP}^{TP}(\theta, y)$ and $z_{LP}^{ES-F-TP}(X, I, y)$ be the LP relaxation objective function values of $(\theta, y) \in F(TP)$ and $(X, I, y) \in F(ES-F-TP)$, respectively. To prove the result, we prove that $F(TP) \subseteq F(ES-F-TP)$. The counter example presented at the beginning of the section shows that the strict equality does not hold in some cases. Let us take a feasible solution $(\theta, y) \in F(TP)$ and construct a feasible solution $(X, I, y) \in F(ES-F-TP)$ with the same objective function value. We construct $(X, I, y) \in F(ES-F-TP)$ as follows:

$$X_{qt}^p = \sum_{i \in R} \sum_{r=q}^{t} \sum_{s=r}^{t} \theta_{qrst}^i \qquad \forall q \leq t \in T \tag{101}$$

$$X_{rt}^w = \sum_{i \in \delta(w)} \sum_{q=1}^{r} \sum_{s=r}^{t} \theta_{qrst}^i \qquad \forall w \in W, r \leq t \in T \tag{102}$$

$$X_{st}^i = \sum_{q=1}^{s} \sum_{r=q}^{s} \theta_{qrst}^i \qquad \forall i \in R, s \leq t \in T \tag{103}$$

$$I_t^i = \sum_{u=1}^{t} \sum_{k=u}^{|T|} X_{uk}^i - d_{1t}^i \qquad \forall i \in F, t \in T. \tag{104}$$

We directly map the y variables since they are the same in both formulations. We show hereafter that (X, I, y) constructed using (101)–(104) belongs to $F(ES-F-TP)$.

1. Constraints (48). For any $i \in F$ and any $t \in T$, by construction one has

$$\begin{aligned} I_t^i - I_{t-1}^i &= \sum_{u=1}^{t} \sum_{k=u}^{|T|} X_{uk}^i - d_{1t}^i - \sum_{u=1}^{t-1} \sum_{k=u}^{|T|} X_{uk}^i + d_{1,t-1}^i \\ &= \sum_{k=t}^{|T|} X_{tk}^i - d_t^i. \end{aligned}$$

   Therefore, constraints (48) hold.

2. Constraints (49). For the production plant p and for any $t \in T$, one has

$$\begin{aligned} \sum_{q=1}^{t} X_{qt}^p &= \sum_{i \in R} \sum_{q=1}^{t} \sum_{r=q}^{t} \sum_{s=r}^{t} \theta_{qrst}^i \\ &= \sum_{i \in R} d_t^i \text{ by (70)} \\ &= d_t^p. \end{aligned}$$

   Using a similar approach one can prove that constraints (49) also hold for any warehouse, any retailer and any time period. Therefore, constraints (49) hold.

3. Constraints (50). If we sum up constraints (71) over all $i \in R$ for any $k \leq t \in T$, we have $\sum_{i \in R} \sum_{l=k}^{t} \sum_{s=l}^{t} \theta_{klst}^i \leq \sum_{i \in R} d_t^i y_k^p$. Besides, $\sum_{i \in R} \sum_{l=k}^{t} \sum_{s=l}^{t} \theta_{klst}^i = X_{kt}^p$ by construction and $\sum_{i \in R} d_t^i y_k^p = y_k^p \sum_{i \in R} d_t^i = d_t^p y_k^p$. Therefore, $X_{kt}^p \leq d_t^p y_k^p$ and constraints (50) hold for the production plant and any $k \leq t \in T$. Using a similar approach, one can prove that the constraints also hold for the warehouses and the retailers. Thus, constraints (50) hold.

4. Constraints (59). For the production plant (i.e., $i = p$) and for any $t \in T$, one has

$$\sum_{q=1}^{t}\sum_{l=q}^{|T|} X_{ql}^{p} \geq \sum_{w\in W}\sum_{k=1}^{t}\sum_{l=k}^{|T|} X_{kl}^{w} \tag{105}$$

$$\Leftrightarrow \sum_{q=1}^{t}\sum_{l=q}^{|T|}\sum_{i\in R}\sum_{k=q}^{l}\sum_{s=k}^{l}\theta_{qksl}^{i} \geq \sum_{i\in R}\sum_{k=1}^{t}\sum_{l=k}^{|T|}\sum_{q=1}^{k}\sum_{s=k}^{l}\theta_{qksl}^{i} \tag{106}$$

$$\Leftrightarrow \sum_{i\in R}\sum_{q=1}^{t}\sum_{k=q}^{|T|}\sum_{l=k}^{|T|}\sum_{s=k}^{l}\theta_{qksl}^{i} \geq \sum_{i\in R}\sum_{q=1}^{t}\sum_{k=q}^{t}\sum_{l=k}^{|T|}\sum_{s=k}^{l}\theta_{qksl}^{i} \tag{107}$$

$$\Leftrightarrow \sum_{i\in R}\sum_{q=1}^{t}\sum_{k=q}^{t}\sum_{l=k}^{|T|}\sum_{s=k}^{l}\theta_{qksl}^{i} + \sum_{i\in R}\sum_{q=1}^{t}\sum_{k=t+1}^{|T|}\sum_{l=k}^{|T|}\sum_{s=k}^{l}\theta_{qksl}^{i} \geq \sum_{i\in R}\sum_{q=1}^{t}\sum_{k=q}^{t}\sum_{l=k}^{|T|}\sum_{s=k}^{l}\theta_{qksl}^{i} \tag{108}$$

$$\Leftrightarrow \sum_{i\in R}\sum_{q=1}^{t}\sum_{k=t+1}^{|T|}\sum_{l=k}^{|T|}\sum_{s=k}^{l}\theta_{qksl}^{i} \geq 0. \tag{109}$$

As $\theta \geq 0$, (109) holds. Thus, thanks to the different equivalences, (105) holds and so do constraints (59) for the production plant and any $t \in T$. Using a similar approach one can prove that constraints (59) hold for the warehouses and any $t \in T$. Therefore, constraints (59) hold.

5. Objective function value. For the TP formulation, the holding cost linked to the production plant is

$$\sum_{i\in R}\sum_{q=1}^{|T|}\sum_{r=q}^{|T|}\sum_{t=r}^{|T|}\sum_{k=t}^{|T|}\sum_{l=q}^{r-1} hc_l^p \theta_{qrtk}^{i} = \sum_{i\in R}\sum_{q=1}^{|T|}\sum_{r=q}^{|T|}\sum_{t=r}^{|T|}\sum_{k=t}^{|T|}\sum_{l=q}^{k-1} hc_l^p \theta_{qrtk}^{i} - \sum_{i\in R}\sum_{q=1}^{|T|}\sum_{r=q}^{|T|}\sum_{t=r}^{|T|}\sum_{k=t}^{|T|}\sum_{l=r}^{k-1} hc_l^p \theta_{qrtk}^{i}$$

$$= \sum_{i\in R}\sum_{q=1}^{|T|}\sum_{k=q}^{|T|}\sum_{r=q}^{k}\sum_{t=r}^{k}\sum_{l=q}^{k-1} hc_l^p \theta_{qrtk}^{i} - \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{k=t}^{|T|}\sum_{q=1}^{t}\sum_{r=q}^{t}\sum_{l=r}^{k-1} hc_l^p \theta_{qrtk}^{i} \tag{110}$$

$$= \sum_{q=1}^{|T|}\sum_{k=q}^{|T|}\left(\sum_{l=q}^{k-1} hc_l^p\right)\sum_{i\in R}\sum_{r=q}^{k}\sum_{t=r}^{k}\theta_{qrtk}^{i} - \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{k=t}^{|T|}\left(\sum_{l=r}^{k-1} hc_l^p\right)\sum_{q=1}^{t}\sum_{r=q}^{t}\theta_{qrtk}^{i} \tag{111}$$

$$= \sum_{q=1}^{|T|}\sum_{k=q}^{|T|}\left(\sum_{l=q}^{k-1} hc_l^p\right)\sum_{i\in R}\sum_{r=q}^{k}\sum_{t=r}^{k}\theta_{qrtk}^{i} - \sum_{r=1}^{|T|}\sum_{k=r}^{|T|}\left(\sum_{l=r}^{k-1} hc_l^p\right)\sum_{i\in R}\sum_{q=1}^{r}\sum_{t=r}^{k}\theta_{qrtk}^{i} \tag{112}$$

$$= \sum_{q=1}^{|T|}\sum_{k=q}^{|T|}\left(\sum_{l=q}^{k-1} hc_l^p\right) X_{qk}^{p} - \sum_{r=1}^{|T|}\sum_{k=r}^{|T|}\left(\sum_{l=r}^{k-1} hc_l^p\right)\sum_{w\in W} X_{rk}^{w}. \tag{113}$$

Expression (113) is exactly the holding cost linked to the production plant in the objective function expression (47) of formulation ES-F-TP, when writen in terms of X variables. Using a similar approach, one can map the holding costs of the warehouses and the retailers in the two formulations. Besides, the setup costs for all facilities directly map since the setup variables are the same. Therefore, the objective function expression of the TP and ES-F-TP formulations are the same. This concludes the proof.

□

Proposition 16 $z_{LP}^{MC} = z_{LP}^{TP}$

Proof. The proof consists in showing that any solution to the linear relaxation of MC can be converted into a solution to the linear relaxation of TP with the same objective function value, and that the reverse is also true. We will thus prove that $F(MC) = F(TP)$. We first prove that $F(TP) \subseteq F(MC)$. Let us take a feasible solution $(\theta, y) \in F(TP)$ and construct a feasible one $(\sigma, w, y) \in F(MC)$ with the same objective function value. We construct $(\sigma, w, y) \in F(MC)$ as follows:

$$w_{qt}^{0i} = \sum_{r=q}^{t} \sum_{s=r}^{t} \theta_{qrst}^{i} \qquad \forall i \in R, q \leq t \in T \tag{114}$$

$$w_{rt}^{1i} = \sum_{q=1}^{r} \sum_{s=r}^{t} \theta_{qrst}^{i} \qquad \forall i \in R, r \leq t \in T \tag{115}$$

$$w_{st}^{2i} = \sum_{q=1}^{s} \sum_{r=q}^{s} \theta_{qrst}^{i} \qquad \forall i \in R, s \leq t \in T \tag{116}$$

$$\sigma_{kt}^{0i} = \sum_{u=1}^{k} \left(w_{ut}^{0i} - w_{ut}^{1i}\right) \qquad \forall i \in R, k \leq t \in T \tag{117}$$

$$\sigma_{kt}^{1i} = \sum_{u=1}^{k} \left(w_{ut}^{1i} - w_{ut}^{2i}\right) \qquad \forall i \in R, k \leq t \in T \tag{118}$$

$$\sigma_{kt}^{2i} = \begin{cases} \sum_{u=1}^{k} w_{ut}^{2i} & \forall i \in R, k < t \in T \\ 0 & \text{otherwise.} \end{cases} \tag{119}$$

We directly map the y variables since they are the same in both formulations. We show hereafter that (σ, w, y) constructed using (114)–(119) belongs to $F(MC)$.

1. Constraints (78)–(80). Note that these constraints can be written just in terms of the w variables if we eliminate the stock variables. We thus get the following constraints instead of (78)–(80):

$$\sum_{j=1}^{k} w_{jt}^{0i} \geq \sum_{j=1}^{k} w_{jt}^{1i} \qquad \forall i \in R,\ \forall k \leq t \in T \tag{120}$$

$$\sum_{j=1}^{k} w_{jt}^{1i} \geq \sum_{j=1}^{k} w_{jt}^{2i} \qquad \forall i \in R,\ \forall k \leq t \in T \tag{121}$$

$$\sum_{k=1}^{t} w_{kt}^{2i} = d_t^i \qquad \forall i \in R,\ \forall t \in T. \tag{122}$$

   For any $i \in R$ and any $k \leq t \in T$, we have:

$$\sum_{j=1}^{k} w_{jt}^{0i} = \sum_{j=1}^{k} \sum_{r=j}^{t} \sum_{s=r}^{t} \theta_{jrst}^{i} \geq \sum_{j=1}^{k} \sum_{r=j}^{k} \sum_{s=r}^{t} \theta_{jrst}^{i}$$

   since $\theta \geq 0$ and since $t \leq k$. Besides, we have:

$$\begin{aligned} \sum_{j=1}^{k} \sum_{r=j}^{k} \sum_{s=r}^{t} \theta_{jrst}^{i} &= \sum_{q=1}^{k} \sum_{r=q}^{k} \sum_{s=r}^{t} \theta_{qrst}^{i} \\ &= \sum_{r=1}^{k} \sum_{q=1}^{r} \sum_{s=r}^{t} \theta_{qrst}^{i} \\ &= \sum_{r=1}^{k} w_{rt}^{1i}. \end{aligned}$$

   Therefore, $\sum_{j=1}^{k} w_{jt}^{0i} \geq \sum_{r=1}^{k} w_{rt}^{1i}$ and constraints (120) hold. Using a similar approach, one can prove that constraints (121) and (122) also hold.

2. Constraints (81)–(83). These constraints hold directly by substituing (114)–(116) in constraints (71)–(73).
3. Objective function value. In the TP formulation, the holding cost linked to the production plant is given by

$$\sum_{i\in R}\sum_{t=1}^{|T|}\sum_{q=1}^{t}\sum_{r=q}^{t}\sum_{s=r}^{t}\sum_{j=q}^{r-1} hc_j^p \theta_{qrst}^i$$
$$= \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{q=1}^{t}\sum_{r=q}^{t}\sum_{s=r}^{t}\sum_{j=q}^{t} hc_j^p \theta_{qrst}^i - \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{q=1}^{t}\sum_{r=q}^{t}\sum_{s=r}^{t}\sum_{j=r}^{t} hc_j^p \theta_{qrst}^i$$
$$= \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{j=1}^{t}\sum_{q=1}^{j}\sum_{r=q}^{t}\sum_{s=r}^{t} hc_j^p \theta_{qrst}^i - \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{j=1}^{t}\sum_{r=1}^{j}\sum_{q=1}^{r}\sum_{s=r}^{t} hc_j^p \theta_{qrst}^i$$
$$= \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{j=1}^{t}\sum_{q=1}^{j} hc_j^p \left(\sum_{r=q}^{t}\sum_{s=r}^{t} \theta_{qrst}^i\right) - \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{j=1}^{t}\sum_{r=1}^{j} hc_j^p \left(\sum_{q=1}^{r}\sum_{s=r}^{t} \theta_{qrst}^i\right)$$
$$= \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{j=1}^{t}\sum_{u=1}^{j} hc_j^p \left(w_{ut}^{0i} - w_{ut}^{1i}\right),$$

which is exactly the holding cost linked to the production plant in the objective function of the MC formulation if the inventory variables of the production plant level are replaced by the production variables using (117). Using a similar approach, one can map the holding costs linked to the warehouses and retailers and obtain the same expression as in the MC formulation. Besides, the setup costs directly match between the two formulations. Therefore, the objective function expression of the TP and MC formulations are the same. This concludes the proof showing that $F(TP) \subseteq F(MC)$.

We now prove that $F(MC) \subseteq F(TP)$. Let us consider, for each retailer i and each period t, a network with three layers representing the three levels of our distribution structure. In each layer, the nodes (l, t_1) represent each time period $t_1 \le t$ at each level l and there are arcs going from one node to the node representing the next period. Figure 2 illustrates this network for a particular retailer i and with $t = 4$. In Figure 2, the node S represents the source node and we have displayed the variables linked to each arc in the network.

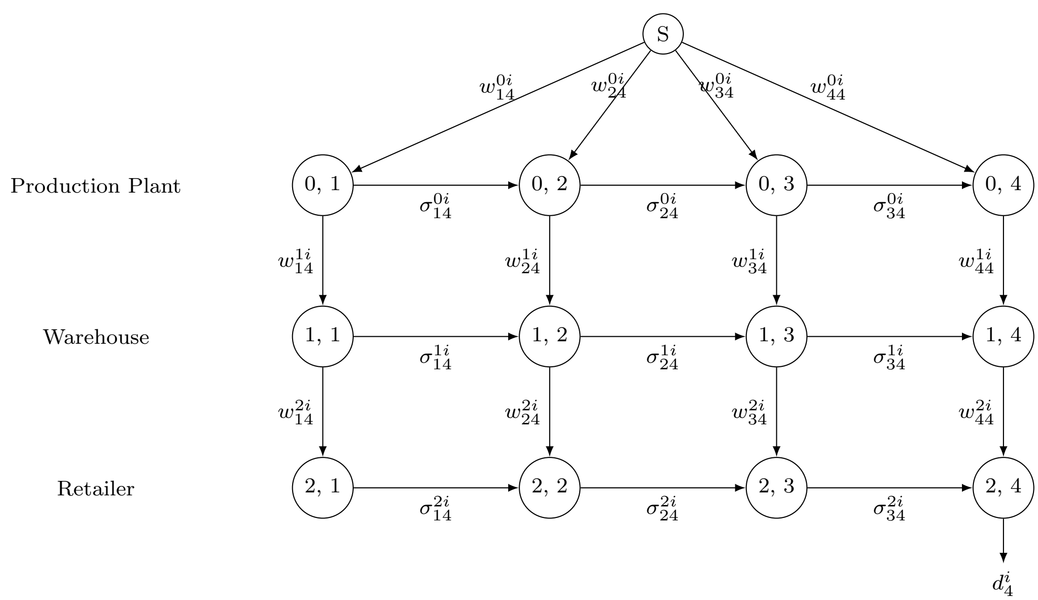

Figure 2: Graphical representation of the network used for the flow decomposition for $t = 4$

The key idea is the same as in Cunha and Melo [8]. Indeed, here also we see that variables $w^{0i}_{kt}, w^{1i}_{kt}, w^{2i}_{kt}$, $\sigma^{0i}_{kt}, \sigma^{1i}_{kt}$ and σ^{2i}_{kt} describe a feasible flow of d^i_t units of demand arriving in node $(2,t)$ in the network for retailer i. Besides, we have $\sum_{k=1}^{t} w^{2i}_{kt} = d^i_t$. Based on the flow decomposition theorem of Ford and Fulkerson [15], any feasible flow in a network can be decomposed into paths and cycles. In our general distribution network, there is no directed cycle, which means that the feasible flow can be decomposed into paths only. For any feasible flow, the decomposition into paths θ^i can be done in such a way that (114)–(116) are satisfied. Indeed, the subscripts k of the set of w^{li}_{kt} variables along a path directly translates into the subscripts q, r and s of the θ^i_{qrst} variables. This results comes fom the fact that, in the MC formulation, the flow of goods between facilities is depicted by the w variables while the flow of goods between facilities in the TP formulation is obtained through the subscripts of the θ variables. An example of such a decomposition is given in Figures 3 and 4. In Figure 3, the flow between facilities is shown in terms of w^0, w^1 and w^2 variables, and the inventory at a facility is shown in terms of σ^0, σ^1 and σ^3 variables. This flow is decomposed into paths θ in Figure 4. Note that the w variables may belong to several paths.

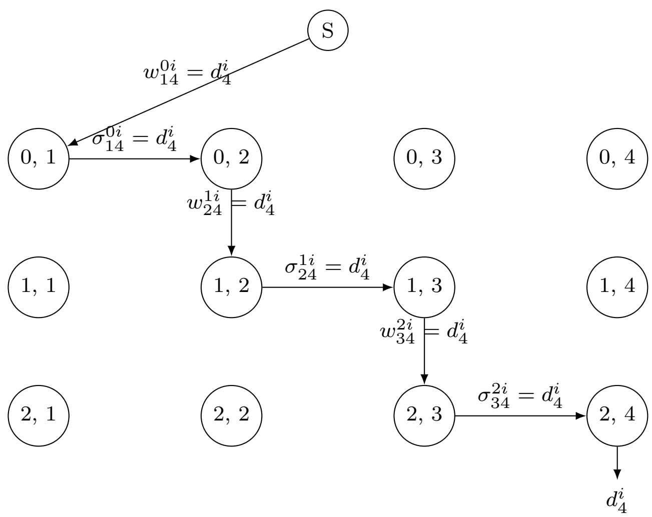

Figure 3: Graphical representation of the flow decomposition in terms of w^0, w^1 and w^2 variables, for $t = 4$

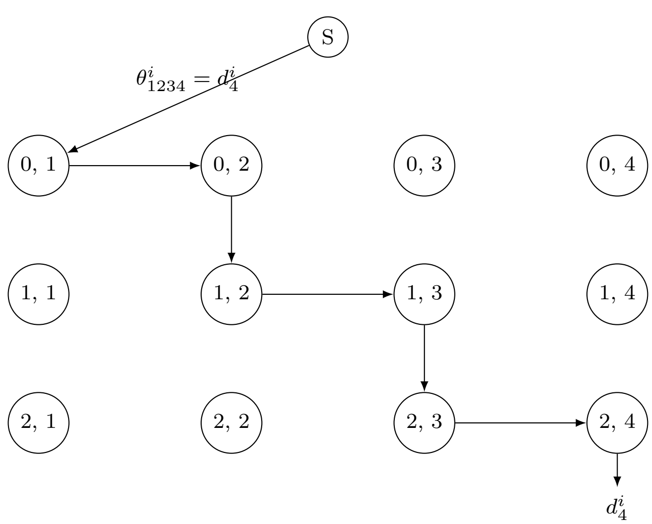

Figure 4: Graphical representation of the flow decomposition in paths θ^i for $t = 4$

We now need to prove that the constraints of the formulation TP are satisfied with the variables θ built as in the example previously.

1. Constraints (70). For any $i \in R$ and any $t \in T$, we have $\sum_{k=1}^{t} w_{kt}^{2i} = d_t^i$. Therefore, by immediate subsitution $\sum_{q=1}^{t}\sum_{r=q}^{t}\sum_{s=r}^{t}\theta_{qrst}^{i} = \sum_{s=1}^{t}\sum_{q=1}^{s}\sum_{r=q}^{s}\theta_{qrst}^{i} = \sum_{s=1}^{t} w_{st}^{2i} = d_t^i$. Therefore, constraints (70) hold.
2. Constraints (71)–(73). Substituting the w variables in (81)–(83) using (114)–(116) results directly in (71)–(73).
3. Objective function value. As stated previously, the formulations TP and MC have the same objective function value if (114)–(116) are used. This concludes the proof.

□

Proposition 17 $z_{LP}^{MC} = z_{LP}^{ME}$

Proof. The proof consists in showing that any solution to the LP relaxation of MC can be converted into a solution to the LP relaxation of ME and that the reverse is also true. We will thus prove that $F(MC) = F(ME)$. By the construction indicated in Section 3.5, we directly have $F(MC) \subseteq F(ME)$. We will now prove that $F(ME) \subseteq F(MC)$. Let us take a feasible solution $(w, E, y) \in F(ME)$ and construct a feasible solution $(w, \sigma, y) \in F(MC)$ with the same objective function value. We construct $(w, \sigma, y) \in F(MC)$ as follows: $\sigma_{kt}^{2i} = E_{kt}^{2i}$, $\sigma_{kt}^{1i} = E_{kt}^{1i} - E_{kt}^{2i}$ and $\sigma_{kt}^{0i} = E_{kt}^{0i} - E_{kt}^{1i}$. We furthermore directly map the w and y variables. We now verify that all constraints hold.

1. Balance constraints (78)–(80). For any $i \in R$ and any $k \leq t \in T$, constraints (80) hold directly because of constraint (90) and of the equality $\sigma_{kt}^{2i} = E_{kt}^{2i}$. For any $t \in T$, any $k \leq t$ and any $i \in R$, one can substract constraint (90) from (89) to obtain:

$$\begin{aligned} E_{k-1,t}^{1i} - E_{k-1,t}^{2i} + w_{kt}^{1i} - w_{kt}^{2i} &= (1-\delta_{kt})(E_{kt}^{1i} - E_{kt}^{2i}) \\ \Leftrightarrow \sigma_{k-1,t}^{1i} + w_{kt}^{1i} &= w_{kt}^{2i} + (1-\delta_{kt})\sigma_{kt}^{1i}. \end{aligned}$$

   In the previous calculations, we have used the fact that constraints (89) and (90) hold for any retailer i and any $k \leq t \in T$. If $k < t$, we directly have (79) since $\delta_{kt} = 0$. If $k = t$, it is obvious that any optimal solution will have $\sigma_{tt}^{li} = 0$ since it represents the inventory on hand at the end of period t to satisfy the demand of the current period. Thus, constraints (79) hold. Using a similar approach, one can prove that constraints (78) hold.
2. Constraints (81)–(84). These constraints directly hold since the production and setup variables used in formulations MC and ME are the same.
3. Constraint (85). Due to constraints (91) and (92), constraints (85) directly hold.
4. Objective function value. A straightforward substitution of the E variables for the σ variables in the objective function of formulation ME directly gives the objective function expression of formulation MC. This concludes the proof. □

Proposition 18 $z_{LP}^{TP} \leq z_{LP}^{N}$

Proof. Let $z_{LP}^{TP}(\theta, y)$ and $z_{LP}^{N}(\psi, y)$ be the LP relaxation objective function values of $(\theta, y) \in F(TP)$ and $(\psi, y) \in F(N)$, respectively. To prove the result, we prove that $F(N) \subseteq F(TP)$. The counter example presented at the beginning of the section shows that the strict inequality holds in some cases. Let us take a feasible solution $(\psi, y) \in F(N)$ and construct a feasible solution $(\theta, y) \in F(TP)$ with the same objective function value. We construct $(\theta, y) \in F(TP)$ as follows:

$$\theta_{klst}^{i} = \sum_{j=t}^{|T|} d_t^i \psi_{klsj}^{i} \qquad \forall k \leq l \leq s \leq t \in T. \tag{123}$$

We directly map the y variables since they are the same in both formulations. We show hereafter that (θ, y) constructed using (123) belongs to $F(TP)$.

1. Constraints (70). If, for any $i \in R$ and any demand point $k \in T$, we sum up constraints (62) over t $(2 \le t \le k)$ with constraint (61), one gets

$$\sum_{t=1}^{|T|}\psi^i_{111t} + \sum_{t=2}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=t}^{|T|}\psi^i_{plts} - \sum_{t=2}^{k}\sum_{p=1}^{t-1}\sum_{l=p}^{t-1}\sum_{s=l}^{t-1}\psi^i_{p,l,s,t-1} = 1$$

$$\Leftrightarrow \sum_{t=1}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=t}^{|T|}\psi^i_{plts} - \sum_{t=2}^{k}\sum_{p=1}^{t-1}\sum_{l=p}^{t-1}\sum_{s=l}^{t-1}\psi^i_{p,l,s,t-1} = 1$$

$$\Leftrightarrow \sum_{t=1}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=k}^{|T|}\psi^i_{plts} + \sum_{t=1}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=t}^{k-1}\psi^i_{plts} - \sum_{t=2}^{k}\sum_{p=1}^{t-1}\sum_{l=p}^{t-1}\sum_{s=l}^{t-1}\psi^i_{p,l,s,t-1} = 1$$

$$\Leftrightarrow \sum_{t=1}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=k}^{|T|}\psi^i_{plts} + \sum_{t=1}^{k-1}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=t}^{k-1}\psi^i_{plts} - \sum_{t=1}^{k-1}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=l}^{t}\psi^i_{plst} = 1$$

$$\Leftrightarrow \sum_{t=1}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=k}^{|T|}\psi^i_{plts} + \sum_{t=1}^{k-1}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=t}^{k-1}\psi^i_{plts} - \sum_{s=1}^{k-1}\sum_{p=1}^{s}\sum_{l=p}^{s}\sum_{t=l}^{s}\psi^i_{plts} = 1$$

$$\Leftrightarrow \sum_{t=1}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=k}^{|T|}\psi^i_{plts} + \sum_{t=1}^{k-1}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=t}^{k-1}\psi^i_{plts} - \sum_{t=1}^{k-1}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=t}^{k-1}\psi^i_{plts} = 1$$

$$\Leftrightarrow \sum_{t=1}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=k}^{|T|}\psi^i_{plts} = 1$$

$$\Rightarrow \sum_{t=1}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\sum_{s=k}^{|T|}d^i_k\psi^i_{plts} = d^i_k$$

$$\Leftrightarrow \sum_{t=1}^{k}\sum_{p=1}^{t}\sum_{l=p}^{t}\theta^i_{pltk} = d^i_k$$

$$\Leftrightarrow \sum_{p=1}^{k}\sum_{l=p}^{k}\sum_{t=l}^{k}\theta^i_{pltk} = d^i_k$$

Thus, constraints (70) hold. Note that the single implication that appears in the previous calculations comes from the special case where $d^i_k = 0$ for some i and some k.

2. Constraints (71). For any $i \in R$ and any $t \le q \in T$, let us define a parameter a^i_{tq} as follows:

$$a^i_{tq} = \begin{cases} 1 & \text{if } D^i_{tq} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

With this parameter, (123) can be written as $\theta^i_{klst} = \sum_{j=t}^{|T|} a^i_{sj} d^i_t \psi^i_{klsj}$ since $a^i_{sj} = 1$ for $s \le t \le j \in T$ if $d^i_t > 0$, otherwise θ^i_{klst} becomes zero. For any retailer i and for any $k \le t \in T$, (63) gives

$$\sum_{l=k}^{t}\sum_{s=l}^{t}\sum_{j=t:d^i_{sj}>0}^{|T|}\psi^i_{klsj} \le y^p_k$$

$$\Leftrightarrow \sum_{l=k}^{t}\sum_{s=l}^{t}\sum_{j=t}^{|T|}a^i_{sj}\psi^i_{klsj} \le y^p_k$$

$$\Leftrightarrow \sum_{l=k}^{t}\sum_{s=l}^{t}\frac{\theta^i_{klst}}{d^i_t} \le y^p_k$$

$$\Leftrightarrow \sum_{l=k}^{t}\sum_{s=l}^{t}\theta^i_{klst} \le d^i_t y^p_k.$$

Therefore, constraints (71) hold. In the same vain, one can prove that constraints (72) and (73) also hold.

3. Objective function value. The holding cost linked to the production plant in formulation N is given by

$$\begin{aligned}
&\sum_{i\in R}\sum_{t=1}^{|T|}\sum_{k=1}^{t}\sum_{l=k}^{t}\sum_{s=l}^{t}\sum_{j=k}^{l-1} hc_j^p d_{st}^i \psi_{klst}^i \\
&= \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{k=1}^{t}\sum_{l=k}^{t}\sum_{s=l}^{t}\sum_{j=k}^{l-1}\sum_{u=s}^{t} hc_j^p d_u^i \psi_{klst}^i \\
&= \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{k=1}^{t}\sum_{l=k}^{t}\sum_{s=l}^{t}\sum_{u=s}^{t}\sum_{j=k}^{l-1} hc_j^p d_u^i \psi_{klst}^i \\
&= \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{k=1}^{t}\sum_{l=k}^{t}\sum_{s=l}^{t}\sum_{u=s}^{t} d_u^i \psi_{klst}^i \sum_{j=k}^{l-1} hc_j^p \\
&= \sum_{i\in R}\sum_{k=1}^{|T|}\sum_{l=k}^{|T|}\sum_{s=l}^{|T|}\sum_{u=s}^{|T|}\sum_{t=u}^{|T|} \left(\sum_{j=k}^{l-1} hc_j^p\right) d_u^i \psi_{klst}^i \\
&= \sum_{i\in R}\sum_{k=1}^{|T|}\sum_{l=k}^{|T|}\sum_{s=l}^{|T|}\sum_{u=s}^{|T|} \left(\sum_{j=k}^{l-1} hc_j^p\right) \theta_{klsu}^i \\
&= \sum_{i\in R}\sum_{k=1}^{|T|}\sum_{l=k}^{|T|}\sum_{s=l}^{|T|}\sum_{u=s}^{|T|}\sum_{j=k}^{l-1} hc_j^p \theta_{klsu}^i \\
&= \sum_{i\in R}\sum_{k=1}^{|T|}\sum_{l=k}^{|T|}\sum_{s=l}^{|T|}\sum_{t=s}^{|T|}\sum_{j=k}^{l-1} hc_j^p \theta_{klst}^i \\
&= \sum_{i\in R}\sum_{t=1}^{|T|}\sum_{k=1}^{t}\sum_{l=k}^{t}\sum_{s=l}^{t}\sum_{j=k}^{l-1} hc_j^p \theta_{klst}^i,
\end{aligned}$$

which is exactly the holding cost linked to the production plant in the objective function of formulation TP. Using a similar approach, one can prove the equivalence between formulations N an TP for the holding costs at the warehouse and retailer level. Besides, the setup costs in the two formulations are already identical. Therefore, the objective function expression of the TP and N formulations are the same. This concludes the proof.

□

4 Numerical experiments

In order to assess the strengths and weaknesses of the different formulations, we conducted computational experiments based on the instances used in Solyalı and Süral [37]. In their experiments, Solyalı and Süral [37] set the number of retailers $|R|$ equal to 50, 100 or 150, and the length of the time horizon $|T|$ is equal to 15 or 30. The demand at the retailers is generated both in a static and dynamic way from U$[5, 100]$. The fixed costs at all levels are also generated both in a static and in a dynamic way. For the warehouse, the fixed costs are generated from U$[1500, 4500]$. For the retailers, the fixed costs are generated from U$[5, 100]$. All the demands and fixed costs are generated as integer values. The unit inventory holding costs are static and are set to 0.5 for the warehouse. For the retailers, the unit inventory holding costs are also static and are generated from U$[0.5, 1]$. The holding costs take continuous values. The authors generated 10 random instances for each combination of settings, resulting in a total of 240 instances.

As we have one more level than in Solyalı and Süral [37], we adapted these instances. In our instances, the number of retailers $|R|$ is set equal to 50, 100 or 200. The number of warehouses $|W|$ is set equal to 5, 10, 15 or 20. We used two different horizon lengths: $|T| = 15$ and 30. The demand at the retailers is generated both in a static and dynamic way from U[5, 100]. In the case of a static demand, we have $d_t^r = d^r \; \forall \; t \in T, r \in R$. The fixed costs at all levels are also generated in a static and in a dynamic way. For the production plant, the fixed costs are generated from U[30000, 45000]. For the warehouses, the fixed costs are generated from U[1500, 4500]. For the retailers, the fixed costs are generated from U[5, 100]. All the demands and fixed costs are generated as integer values. The unit inventory holding costs are static and are set to 0.25 for the production plant and 0.5 for the warehouses. For the retailers, the unit inventory holding costs are generated from U[0.5, 1]. The holding costs take continuous values. For each combination of settings, we generate five different instances leading to 480 different instances to be solved for each formulation.

In order to test our formulations, we additionnally define two structures for the distribution network represented in Figure 1. In the first structure, we consider a balanced network where each warehouse has the same number of retailers, except when the number of retailers is not a multiple of the number of warehouses. In the second structure, we consider an unbalanced network where 80% of the retailers are assigned to 20% of the warehouses. For each pair $(|W|, |R|)$, Tables 2 and 3 give the number of retailers assigned to each warehouse for the balanced and unbalanced networks, respectively. Each structure is tested on the 480 instances we generated.

Table 2: Assignment of the retailers to the warehouses for the balanced network

Number of warehouses	Number of retailers 50	 100	 200
5	$10 \; \forall w \in W$	$20 \; \forall w \in W$	$40 \; \forall w \in W$
10	$5 \; \forall w \in W$	$10 \; \forall w \in W$	$20 \; \forall w \in W$
15	3 if $w \in [\![1, 10]\!]$ 4 if $w \in [\![11, 15]\!]$	6 if $w \in [\![1, 5]\!]$ 7 if $w \in [\![6, 15]\!]$	14 if $w \in [\![1, 10]\!]$ 12 if $w \in [\![12, 15]\!]$
20	3 if $w \in [\![1, 10]\!]$ 2 if $w \in [\![11, 20]\!]$	$5 \; \forall w \in W$	$10 \; \forall w \in W$

Table 3: Assignment of the retailers to the warehouses for the unbalanced network

Number of warehouses	Number of retailers 50	 100	 200
5	40 if $w = 1$ 3 if $w \in [\![2, 3]\!]$ 2 if $w \in [\![4, 5]\!]$	80 if $w = 1$ 5 if $w =\in [\![2, 5]\!]$	160 if $w = 1$ 10 if $w =\in [\![2, 5]\!]$
10	17 if $w \in [\![1, 2]\!]$ 2 if $w \in [\![3, 10]\!]$	38 if $w \in [\![1, 2]\!]$ 3 if $w \in [\![3, 10]\!]$	80 if $w \in [\![1, 2]\!]$ 5 if $w \in [\![3, 10]\!]$
15	9 if $w \in [\![1, 2]\!]$ 8 if $w = 3$ 2 if $w \in [\![4, 15]\!]$	25 if $w \in [\![1, 2]\!]$ 26 if $w = 3$ 2 if $w \in [\![4, 15]\!]$	54 if $w \in [\![1, 2]\!]$ 56 if $w = 3$ 3 if $w \in [\![4, 15]\!]$
20	5 if $w \in [\![1, 2]\!]$ 4 if $w \in [\![3, 4]\!]$ 2 if $w \in [\![5, 20]\!]$	17 if $w \in [\![1, 4]\!]$ 2 if $w \in [\![5, 20]\!]$	38 if $w \in [\![1, 4]\!]$ 3 if $w \in [\![5, 20]\!]$

For the experiments, we used the CPLEX 12.6.1.0 C++ library and turned off CPLEX's parallel mode. We set the CPLEX MIP tolerance parameter to 10^{-6}. All the other CPLEX parameters are set to their default value. The computation time limit imposed to solve each MIP instance is 6 hours.

We compare the formulations with respect to different indicators:

- number of instances for which the MIP is solved to optimality;
- CPU time (s) taken to solve the LP relaxation;
- CPU time (s) taken to solve the MIP;
- objective function value of the LP relaxation;
- objective function value of the MIP optimal solution when available, cost of the best solution found otherwise;
- number of nodes in the branch-and-cut tree;
- integrality gap (%);
- optimality gap (%).

For a particular instance, if we denote by z^X_{LP} the objective function value of the LP relaxation with formulation X and by z^* the optimal objective function value of this instance when available (or the best objective function value obtained among all formulations for this instance otherwise), the integrality gap is computed as $(z^* - z^X_{LP})\ /\ z^*$. The optimality gap is the gap between the best solution found and the best lower bound given by CPLEX at the end of the CPU time limit. Detailed results can be found in the appendices of this report.

In the following sections, results will be reported in two tables. The first table illustrates the aggregated results obtained for $|T| = 15$ while the second table displays the aggregated results obtained for $|T| = 30$. In each table, each row represents the results obtained for a particular formulation while each column refers to the different indicators previously defined. In the tables, MIP-opt denotes the number of MIP optimal solutions obtained (out of 240 instances in each table); LP-CPU and MIP-CPU represent the CPU time taken to solve the LP and MIP instances, respectively; LP-cost and MIP-cost represent the cost of the LP and MIP optimal solutions (or best solution found at the end of the time limit for the MIP solutions), respectively; I-gap gives the integrality gap and O-gap indicates the optimality gap. In Sections 4.1 and 4.2 we will report the results for the uncapacitated and capacitated instances, respectively. In Section 4.3, we will perform an analysis of the influence of the parameters in our experiments.

4.1 Uncapacitated instances

We first report the results for the balanced network in Section 4.1.1, followed by the unbalanced network in Section 4.1.2. For the uncapacitated instances, we performed our experiments on a 3.07 GHz Intel Xeon processor with only one thread. For these instances, CPLEX was able to find a feasible MIP solution for all uncapacitated instances with a balanced network and with an unbalanced network. The LP relaxation values are calculated separately. Note that we do not impose any time limit to solve the LP relaxations.

4.1.1 Balanced network

In the balanced network, each warehouse is responsible for approximately the same number of retailers (see Table 2).

Tables 4 and 5 illustrate the performance of the different MIP formulations for $|T| = 15$ and $|T| = 30$, respectively. In Table 4, which presents the results for the small instances, one can see that the formulations MC, MCE, N and TP obtain the best performance in general, with all MIP optimal solutions found, the lowest MIP-CPU and a value of the LP relaxation which is very close to the optimal MIP cost. Yet, the LP relaxation for these three formulations is not the same as the MIP optimal cost, as witnessed by the small but positive values for the I-gap. Besides, the MC formulation has the lowest MIP-CPU time among all formulations. However, the CPU time needed to solve the LP relaxation of these formulations is much higher than with the other formulations. The high performance of these formulations is also expected because of the rich information which is contained in the decision variables used for each formulation.

Table 4: Performance of the formulations for the balanced network - 1h time limit, $|T| = 15$

Formulation	LP-cost	LP-CPU	MIP-cost	MIP-CPU	Nodes	MIP-opt	I-gap	O-gap
C	186156	0.03	327484	8291.35	71832.2	157	40.94	2.94
C-N	225136	0.2	327315	10023.89	202204.1	141	29.89	3.12
C-TP	225136	0.13	327247	8567.71	29431	158	29.89	2.78
C-LS	225136	0.19	327501	8337.91	14132.7	158	29.89	3.26
ES	186156	0.02	326906	600.53	29725.7	240	40.94	0
ES-N	320903	0.47	326906	117.47	4253.7	240	1.62	0
ES-TP	320903	1.69	326906	176.9	2652.1	240	1.62	0
ES-LS	320897	1.51	326906	297.58	1760.4	240	1.62	0
ES-F	186156	0.03	326906	875.35	29628	238	40.94	0
ES-F-N	320903	0.7	326906	120.92	3401.4	240	1.62	0
ES-F-TP	320903	1.3	326906	214.16	3673.9	240	1.62	0
ES-F-LS	320897	1.12	326906	208.61	3110.4	240	1.62	0
MC	326832	26.45	326906	35.51	0.7	240	0.02	0
MCE	326832	37.8	326906	40.44	0.7	240	0.02	0
N	326887	121.27	326906	74.25	0.3	240	0	0
T	326832	80.21	326906	81.67	0.8	240	0.02	0

Table 5: Performance of the formulations for the uncapacitated balanced network - $|T| = 30$

Formulation	LP-cost	LP-CPU	MIP-cost	MIP-CPU	Nodes	MIP-opt	I-gap	O-gap
C	240367	0.07	664638	21600.07	35356.6	0	60.86	24.58
C-N	338679	0.83	705070	21600.19	19020.5	0	46.62	30.58
C-TP	338679	0.62	780467	21600.2	3018	0	46.62	29.57
C-LS	338679	2	771246	21516.86	4602.1	0	46.62	28.39
ES	240367	0.05	645908	15252.18	91231.2	84	60.86	4.03
ES-N	624974	6.77	643306	6069.55	53462.9	186	2.77	0.09
ES-TP	624974	30.3	643714	7744.16	14847.4	175	2.77	0.64
ES-LS	624935	4.09	644312	9034.64	4785.6	160	2.77	0.78
ES-F	240367	0.14	644747	14404.42	24790.8	90	60.86	2
ES-F-N	624974	11.14	643863	6270.86	32940.9	181	2.77	0.1
ES-F-TP	624974	26.5	643385	8174.27	23707.9	173	2.77	0.4
ES-F-LS	624935	5.04	643843	9930.78	17481.5	155	2.77	0.76
MC	642779	826.09	643303	1021.77	5.1	240	0.08	0
MCE	642779	996.56	643303	1276.72	5.2	240	0.08	0
N	643057	27969.13	1068367	9209.08	0.9	188	0.04	16.86
T	642779	1901.78	693483	5773.58	2.4	211	0.08	3.62

For the small instances, the classical formulations obtain the worst results, mainly because of a poor LP relaxation as shown by the integrality gap reported in Table 4. The echelon stock based formulations can be divided into two groups with formulations ES and ES-F on one side, and formulations ES-N, ES-TP, ES-LS, ES-F-N, ES-F-TP and ES-F-LS on the other side. The last six formulations are much stronger than the first two formulations, as indicated by the integrality gap reported in Table 4. Formulations ES-N, ES-TP, ES-LS, ES-F-N, ES-F-TP and ES-F-LS were able to solve all instances, which is not the case for the ES-F formulations. This better performance of formulations ES-N, ES-TP, ES-LS, ES-F-N, ES-F-TP and ES-F-LS is easily explained by the use of a reformulation of the uncapacitated lot sizing structure found in the ES formulation, and the resulting improved LP bound.

The classical based formulations have in general a much higher number of nodes in the branch-and-cut tree than the other formulations, which is a consequence of the weak LP relaxation bound. The same remarks hold for the formulations ES and ES-F. For the MC, MCE, N and TP formulations, the number of nodes is really small, less than 1 on average, showing the high performance of the LP relaxation. Concerning the O-gap, the classical based formulations have a gap of approximately 3% while the other formulations have an average gap that is less than 0.0003%. This illustrates once again the weakness of the classical based formulations. Note that for the N and TP formulations, the LP-CPU is higher than the MIP-CPU because of the efficiency of the heuristic used by CPLEX at the root node before going in the branch-and-cut tree.

Finally, one can see in Table 4 that despite the reformulation used at the retailer level or the valid inequalities added, the C-N, C-TP and C-LS formulations do not succeed in closing a lot of the integrality gap, which remains high around 30%. This contrasts with the same reformulations or valid inequalities added in the ES formulation but at all levels instead of just at the retailer level. Indeed, the I-gap for the ES-N, ES-TP, ES-LS, ES-F-N, ES-F-TP and ES-F-LS formulations is low, around 1.6%. This indicates that the combination of the reformulation and the echelon stock structure is very efficient if we compare the performance of the ES-N, ES-TP, ES-LS, ES-F-N, ES-F-TP and ES-F-LS formulations to the one of the classical formulations.

Table 5 reports the performance of each formulation for the large instances, with $|T| = 30$. The poor performance of the classical formulations is even more apparent for these large instances. Yet, the LP relaxations are still easily solved to optimality but have a low value compared to the true MIP optimal cost. The performance of the richer formulations N, TP, MC and MCE is also more contrasted than for the small instances. The number of instances solved to optimality for the N formulation is much lower than for the three other rich formulations. This can be explained by the inability of the N formulation to solve the LP relaxation of the instances in a short time. One can see a similar behavior, but to a lesser extent, for the TP formulation. This difficulty for the formulations N and TP to even solve the LP relaxations of many large instances can be explained by the huge number of variables used in the models when $|T| = 30$, which is a major drawback of these two formulations. This practical drawback is the price one has to pay for the strong LP relaxation given by these two formulations, as stated by the theoretical results presented in Section 3.7. Finally, the MC formulation still provides the best performances for these large instances, both in terms of CPU time to solve the MIP instances and in terms of number of optimal solutions found within the time limit.

In light of the results provided in Tables 4 and 5, we can draw the following conclusions about the performance of our formulations on an uncapacitated balanced network:

- the classical formulations are the poorest, mainly because of a bad LP relaxation and providing a stronger reformulation only at the retailer level does not lead to better results at the MIP level;
- applying the echelon stock reformulation to the classical formulation does not have any impact on the LP relaxation value (as we also theoretically proved), but results nevertheless show a substantial improvement in CPU time, optimality gap and number of instances solved to optimality. The conjecture is that because the echelon stock reformulation exposes the single item lot sizing structure at the three different levels, CPLEX is able to derive better cuts;
- the echelon stock reformulation can still be improved by explicitly using one of the lot sizing reformulations at each level, i.e., using formulations ES-N, ES-TP, ES-LS, ES-F-N, ES-F-TP and ES-F-LS, with ES-N generally having the best performance among these six formulations;
- when comparing the various echelon stock reformulations with the traditional echelon stock constraints (33) to their counterpart using the constraint proposed in Federgruen and Tzur (57), we observe individual differences, but overall no general tendencies appear and the formulations provide fairly similar results;
- the N and TP formulations have difficulty to solve the LP relaxations of some instances because of the huge size of the model resulting in an overall substantially weaker performance compared to the best formulation;
- the MC formulation performs the best for the balanced network;
- the results we obtained here are in line with the ones obtained by Solyalı and Süral [37] and Cunha and Melo [8] for the OWMR.

4.1.2 Unbalanced network

We performed the same experiments as in Section 4.1.1 but considering an unbalanced distribution network. In the unbalanced network, 20% of the warehouses are responsible for 80% of the retailers (see Table 3).

Tables 6 and 7 illustrate the performance of our formulations for the small and large instances, respectively. In Table 6, one can see that, compared to Table 4 and except for the classical formulations, there is an increase in CPU time to solve the instances as MIPs. This increase ranges between 0.16% and 78.5% for the ES formulation and for the ES-F-LS formulation, respectively. As far as the classical based formulations are concerned, they have a better performance on the unbalanced network, compared to the balanced network, in

Table 6: Performance of the formulations for the uncapacitated unbalanced network - $|T| = 15$

Formulation	LP-cost	LP-CPU	MIP-cost	MIP-CPU	Nodes	MIP-opt	I-gap	O-gap
C	177633	0.02	310925	5668.75	21108.3	197	40.78	0.66
C-N	217549	0.18	310882	5267.93	119965	197	28.84	0.28
C-TP	217549	0.12	310892	5285.29	19746.6	197	28.84	0.68
C-LS	217549	0.18	311225	7311.02	4038	169	28.84	2.61
ES	177633	0.02	310871	601.5	14711.3	239	40.78	0
ES-N	300182	0.55	310871	182.27	4045.6	240	2.99	0
ES-TP	300182	1.68	310871	262.02	3084	240	2.99	0
ES-LS	300178	1.6	310871	413.61	2917.2	240	2.99	0
ES-F	177633	0.04	310871	1165.83	17758	240	40.78	0
ES-F-N	300182	0.89	310871	186.75	3731.2	240	2.99	0
ES-F-TP	300182	1.92	310871	303.41	3814.9	240	2.99	0
ES-F-LS	300178	1.24	310871	372.34	3508.8	240	2.99	0
MC	310750	20.33	310871	39.88	1.6	240	0.03	0
MCE	310750	41.06	310871	48.25	1.6	240	0.03	0
N	310832	125.33	310871	112.39	1	240	0.01	0
T	310750	58.37	310871	93.97	2.7	240	0.03	0

Table 7: Performance of the formulations for the uncapacitated unbalanced network - $|T| = 30$

Formulation	LP-cost	LP-CPU	MIP-cost	MIP-CPU	Nodes	MIP-opt	I-gap	O-gap
C	231785	0.06	624878	21103.84	36751.6	10	60.35	19.39
C-N	330865	0.73	627028	21600.18	25819	0	45.2	26.26
C-TP	330865	0.54	642133	21600.15	4824.3	0	45.2	23.54
C-LS	330865	1.81	647389	21600.2	3378.2	0	45.2	23.9
ES	231785	0.05	613737	14747.68	26616.9	91	60.35	4.89
ES-N	583375	8	610963	8690.4	30168.7	164	4.14	0.37
ES-TP	583375	29.78	611589	10271.05	15351.3	149	4.14	1.36
ES-LS	583349	6.53	612763	12294.87	7218.3	128	4.14	1.63
ES-F	231785	0.17	613421	14546.53	14335	90	60.35	2.99
ES-F-N	583375	20.82	611004	9512	24099.6	157	4.14	0.45
ES-F-TP	583375	45.58	611424	10509.58	18437.7	147	4.14	1.11
ES-F-LS	583349	10.48	612275	11766.99	10525.3	130	4.14	1.63
MC	610109	460.85	610908	1363.92	19.2	240	0.1	0
MCE	610109	994.48	610908	1476.23	18.7	239	0.1	0
N	610542	11473.94	828581	8018.58	3.7	204	0.04	9.05
T	610109	1700.49	705844	6356.45	14.7	201	0.1	5.55

terms of CPU time used to solve the MIP instances, number of MIP optimal solutions found and integrality and optimality gap. Note, however, that the improvements for the integrality gap is very limited compared to the other improvements. Despite these improvements, the performance of the classical formulations is still far from the performance of the other formulations, highlighting once again the weakness of the classical formulations. Apart from the two points mentioned here, all the other conclusions drawn in Section 4.1.1 for the small instances with a balanced network still hold for an unbalanced structure of the supply network.

In Table 7, one can see that there are once again small improvements for the classical formulations compared to instances solved on a balanced network. For the other formulations, the performance is worse than in the case of a balanced network. This difficulty is in particular reflected in the number of optimal MIP solutions found, which decreases by a number ranging from 0 for the MC formulation and up to 32 for the ES-LS formulation. This indicates that the unbalanced instances are harder to solve than the balanced instances. This difficulty can be explained by the fact that, in the network, the warehouses that are responsible for many retailers represent a much larger MIP to solve. Compared to the balanced instances, we have thus several big distribution channels to cope with, which makes the instances harder to solve. Note, however, that formulations C, ES and N were able to find more optimal MIP solutions for the unbalanced instances.

In light of the results provided in Tables 6 and 7, we can draw the following conclusions about the performance of our formulations on an unbalanced network:

- the unbalanced instances are generally harder to solve than the balanced instances;
- the C based formulations, the N and the ES formulations have a better performance on the unbalanced instances than on the balanced ones in terms of number of instances solved to optimality;
- the other formulations have a worse performance on the unbalanced instances compared to the balanced ones;
- the N and TP formulations have a large O-gap for many large instances;
- the MC formulation is the best suited for the unbalanced instances since it is able to solve all instances to optimality with the lowest CPU time.

4.2 Capacitated instances

For the capacitated instances, we set the production capacity as a given factor C of the average total demand. The production capacity imposed is thus $C_t = C \sum_{i\in R} \sum_{t\in T} d_t^i \;/\; |T|$. We additionally consider three different values for the capacity factor $C : C \in \{2, 1.75, 1.5\}$. We performed these experiments on a 6.67 GHz Intel Xeon X5650 Westmere processor with one thread. Because of the bad performance of the classical based formulations and of the formulations ES and ES-F in the previous section, and based on preliminary results, we decided not to run experiments using these formulations. Note that for the capacitated instances we impose a time limit of 6 hours even to solve the LP instances.

The results of this section will be reported in tables having the same columns as the tables in Section 4.1 plus two additional columns indicating the number of LP optimal solutions found within the time limit and the number of instances for which a MIP solution was found, in columns LP-opt and MIP-sol, respectively. For the columns LP-cost and I-gap, we only report the average cost and integrality gap obtained, respectively, over instances for which all formulations have both solved the LP relaxation to optimality and have found a MIP solution within the time limit. In the same vein, for the columns MIP-cost, Nodes and O-gap, we only report the average MIP cost, number of nodes and optimality gap obtained, respectively, over instances for which all formulations have found a MIP solution within the time limit. We first report the results for the balanced network in Section 4.2.1, followed by the unbalanced network in Section 4.2.2.

4.2.1 Balanced network

Tables 8–13 illustrate the performance of the different MIP formulations for the different values of the time horizon and capacity level. When comparing the results with those obtained for the uncapacitated instances on the balanced network, we can see that the results are completely different. Indeed, the richer formulations have more trouble achieving a good performance in terms of CPU time, MIP cost, number of MIP optimal solutions found and optimality gap. On the contrary, the echelon stock formulations have a better performance than the richer formulations on these indicators. This difference in performance is even more pronounced when the capacity level gets tighter. This indicates that the capacity constraint has a major impact on the performance of the formulations. Despite the properties related to the strength of their LP relaxation, the richer formulations seem to be less adequate to handle capacitated instances.

We also see that the MC formulation does not perform the best for the capacitated instances on the balanced network. The best performance, in terms of MIP-CPU time, number of optimal solutions found and optimality gap, is obtained by one of the echelon stock formulations, depending on the capacity level. Within the richer formulations, our newly introduced MCE formulation performs the best on average. Note also that the addition of the capacity constraint makes the problem harder, as stated by the increase in CPU time to solve both the MIP and LP instances. This difficulty is also apparent by observing that the number of MIP solutions found is not equal to the number of instances present in the data set used for the experiments.

Table 8: Performance of the formulations for the capacitated balanced network - $|T| = 15, C = 2.0$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	514798	1.06	240	510641	9517.05	141103	161	207	4.69	0.06
EST	514798	2.39	240	510677	10367.67	38086.4	151	240	4.69	0.1
ESLS	514786	1.71	240	510675	9030.44	37633.7	164	233	4.7	0.1
ES-F-N	514798	1.14	240	510641	9914.74	119257.7	156	220	4.69	0.05
ES-F-T	514798	1.48	240	510694	10892.41	59856.5	144	237	4.69	0.1
ES-F-LS	514786	1.5	240	510730	10732.17	50980.4	146	237	4.7	0.13
MC	519979	185.39	240	511242	16582.34	9397.9	92	240	3.76	1.02
MCE	519979	192.17	240	511042	14987.63	6900.8	113	240	3.76	0.77
N	520090	173.84	240	511024	17180.75	4452.9	89	240	3.74	0.92
T	519979	398.16	240	511809	18104.03	3372.2	76	240	3.76	1.14

Table 9: Performance of the formulations for the capacitated balanced network - $|T| = 30, C = 2.0$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	942317	18.72	240	904162	21544.09	107672.3	2	240	1.83	1
EST	942317	24.31	240	904226	21576.36	25025.4	1	240	1.83	1.16
ESLS	942276	7.91	240	904223	21538.3	26318.8	2	231	1.83	1.07
ES-F-N	942317	15.15	240	904163	21587.23	55846.7	1	240	1.83	1.05
ES-F-T	942317	26.26	240	904256	21600.23	31181.1	0	230	1.83	1.11
ES-F-LS	942276	8.89	240	904321	21591.14	21633.9	1	240	1.83	1.36
MC	950697	2626.04	240	905226	21600.5	1371.2	0	240	1	1.63
MCE	950697	2522.05	240	904873	21602.85	1543.1	0	240	1	1.55
N	950883	7689.24	191	904841	21705.65	89.8	0	141	0.98	14.78
T	950697	8999.21	193	906231	21866.83	61.9	0	191	1	22.66

Table 10: Performance of the formulations for the capacitated balanced network - $|T| = 15, C = 1.75$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	559141	1.18	240	572268	11059.05	118612.4	142	231	3.88	0.31
EST	559141	2.48	240	572377	15230.97	47566.4	80	161	3.88	0.32
ESLS	559131	1.88	240	572395	14929.48	43995.6	83	159	3.88	0.32
ES-F-N	559141	1.16	240	572279	14882.05	110598.3	81	150	3.88	0.28
ES-F-T	559141	1.58	240	572384	13894.41	59972.3	106	218	3.88	0.35
ES-F-LS	559131	1.66	240	572430	14001.68	44803.5	100	182	3.88	0.37
MC	564010	144.67	240	573036	17253.53	15058.5	83	240	3.06	0.9
MCE	564010	153.53	240	572939	17060.25	12080.5	83	240	3.06	0.8
N	564113	113.09	240	573023	17532.16	10689.8	77	240	3.04	0.72
T	564010	283.63	240	573307	18183.12	10585.5	67	240	3.06	1.06

Table 11: Performance of the formulations for the capacitated balanced network - $|T| = 30, C = 1.75$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	1039555	17.06	240	1008623	21600.1	41473.5	0	234	4.04	3.42
EST	1039555	21.94	240	1008814	21562.91	27470.6	2	236	4.04	0.98
ESLS	1039519	8.75	240	1008846	20913.82	34887.1	13	128	4.04	0.64
ES-F-N	1039555	16.97	240	1008642	21600.35	28611.6	0	240	4.04	3.5
ES-F-T	1039555	26.38	240	1008828	21600.22	39757.8	0	220	4.04	0.61
ES-F-LS	1039519	10.13	240	1008907	21582.47	22652.9	1	236	4.04	3.06
MC	1047761	2460.44	240	1009976	21602.48	1250.3	0	240	3.31	4.06
MCE	1047761	2196.98	240	1009805	21473.28	2074.7	6	239	3.31	2.75
N	1047959	6634.88	202	1009953	21673.02	54.8	0	131	3.29	17.99
T	1047761	7936.58	203	1010453	21737.07	44.7	0	210	3.31	19.81

Finally, note that in Tables 9, 11 and 13, for formulations N, TP, MC and MCE, the values obtained for O-gap is higher than the values obtained for I-gap. Since the I-gap is calculated relative to the optimal or best solution found among all formulations this indicates that these formulations have a good LP relaxation but are unable to provide a MIPsolution with a low objective function value.

Table 12: Performance of the formulations for the capacitated balanced network - $|T| = 15, C = 1.5$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	621721	1.7	240	583124	13304.05	184360.4	116	198	1.3	0.15
EST	621721	2.71	240	583221	14763.63	74318	99	210	1.3	0.22
ESLS	621714	3.1	240	583280	13068.93	52541.1	112	190	1.3	0.22
ES-F-N	621721	1.34	240	583119	13388.01	189937.4	112	193	1.3	0.15
ES-F-T	621721	1.8	240	583187	14895.36	98617	96	207	1.3	0.2
ES-F-LS	621714	2.09	240	583231	14321.8	65407.7	101	205	1.3	0.21
MC	626314	136.62	240	583888	17125.63	19017.5	75	238	0.58	0.23
MCE	626314	156.63	240	583528	16940.93	20167.1	78	234	0.58	0.21
N	626403	124.95	240	583690	18349.35	12262	62	212	0.57	0.24
T	626314	360.09	240	584047	18712.05	11320.1	55	227	0.58	0.32

Table 13: Performance of the formulations for the capacitated balanced network - $|T| = 30, C = 1.5$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	1174947	18.96	240	935711	21561.77	66408.5	1	194	1.59	1.43
EST	1174947	25.97	240	935867	21587.74	25253.1	1	169	1.59	1.48
ESLS	1174914	10.5	240	935960	21500.33	31446.1	2	193	1.59	1.36
ES-F-N	1174947	19.29	240	935702	21596.85	72474.6	1	236	1.59	1.43
ES-F-T	1174947	30.31	240	935812	21600.2	38381.7	0	234	1.59	1.43
ES-F-LS	1174914	11.96	240	935882	21516.6	28078.8	0	236	1.59	1.58
MC	1183634	2257.01	240	936937	21600.38	1858.3	0	240	0.89	3.5
MCE	1183634	1947.3	240	936360	21600.3	1762.5	0	240	0.89	3
N	1183835	5867.03	218	936620	21700.5	201.2	0	181	0.87	9.56
T	1183634	6773.7	222	937192	21781.93	175.3	0	213	0.89	9.52

4.2.2 Unbalanced network

Tables 14–19 illustrate the performance of the different MIP formulations on the unbalanced instances for the different values of the time horizon and capacity level. If we compare the results with those obtained for the uncapacitated instances on the unbalanced network, we can see similar differences as the ones observed in Section 4.2.1. The richer formulations also have more trouble obtaining a good performance than on the uncapacitated instances, and actually have a worse performance than the echelon stock formulations on numerous performance indicators. These differences are even clearer for the unbalanced instances, especially for the number of best solutions found, which is generally much higher for the echelon stock formulations. Within the richer formulations, the MCE formulation still has the best performance on average. Note finally that, compared to the balanced structure, the unbalanced structure of the supply network combined with the production capacity restriction results in general in better values for the number of MIP solutions found and for the number of MIP optimal solutions found.

Table 14: Performance of the formulations for the capacitated unbalanced network - $|T| = 15, C = 2.0$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	482906	1	240	504744	1263.92	28050.1	238	238	5.14	0
EST	482906	1.96	240	504744	2106.83	22230.3	236	240	5.14	0
ESLS	482904	1.28	240	504744	1531.26	13069.1	239	240	5.14	0
ES-F-N	482906	1.37	240	504744	971.1	22544.2	239	239	5.14	0
ES-F-T	482906	2.07	240	504744	1644.95	18161.5	238	240	5.14	0
ES-F-LS	482904	1.43	240	504744	1913.39	16423.3	238	240	5.14	0
MC	489860	142.82	240	504920	12934.8	6092.6	130	239	3.81	0.42
MCE	489860	129.69	240	504851	11098.95	6894.4	155	237	3.81	0.24
N	489920	118.19	240	504904	13804.04	5405.5	144	239	3.8	0.31
T	489860	302.27	240	505029	13626.37	4472.6	142	240	3.81	0.46

Table 15: Performance of the formulations for the capacitated unbalanced network - $|T| = 30, C = 2.0$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	907791	13.45	240	944359	19921.34	58469	27	239	2.35	1.13
EST	907791	16.8	240	944361	20266.64	28715.4	22	240	2.35	1.39
ESLS	907775	9.8	240	944359	20163.36	20849	24	219	2.35	1.39
ES-F-N	907791	20.31	240	944359	19939.28	54412.1	26	240	2.35	1.13
ES-F-T	907791	40.57	240	944359	20053.67	36504	27	240	2.35	1.25
ES-F-LS	907775	14.12	240	944359	20118.8	23147.8	23	238	2.35	1.35
MC	918996	2226.9	240	944688	21481.4	1455.2	3	240	1.19	2.36
MCE	918996	1806.59	240	944560	21522.96	1864.2	4	240	1.19	2.26
N	919144	6767.62	200	944659	21672.53	100.5	0	155	1.18	15.51
T	918996	8348.99	203	944892	21705.5	115.8	0	194	1.19	13.87

Table 16: Performance of the formulations for the capacitated unbalanced network - $|T| = 15, C = 1.75$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	526754	1.06	240	549514	2224.54	45994.8	238	240	4.26	0
EST	526754	2.01	240	549524	3193.44	28369.3	227	240	4.26	0.01
ESLS	526751	1.46	240	549516	2433.6	25883.5	236	239	4.26	0
ES-F-N	526754	1.38	240	549514	1786.29	39305.9	240	240	4.26	0
ES-F-T	526754	2.11	240	549521	2851.11	28501.2	231	240	4.26	0.01
ES-F-LS	526751	1.57	240	549536	3660.7	27184.3	225	240	4.26	0.02
MC	533035	125.55	240	549761	14359.29	10031.3	113	240	3.15	0.56
MCE	533035	122.02	240	549700	12873.73	11765.1	141	240	3.15	0.41
N	533092	106.24	240	549853	15213.96	7213.3	119	240	3.14	0.56
T	533035	391.23	240	550351	16143.75	5282.3	96	240	3.15	0.95

Table 17: Performance of the formulations for the capacitated unbalanced network - $|T| = 30, C = 1.75$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	997363	17.08	240	820837	21465.26	31462.3	2	240	4.44	3.59
EST	997363	18.61	240	820851	20596.13	24642	18	240	4.44	1.16
ESLS	997346	10.65	240	820839	20652.97	27727.7	16	236	4.44	1.2
ES-F-N	997363	21.66	240	820837	21569.18	25981.1	2	240	4.44	3.58
ES-F-T	997363	40.57	240	820847	20649.26	27377.4	17	240	4.44	0.9
ES-F-LS	997346	14.9	240	820870	21016.96	17628.5	11	239	4.44	2.67
MC	1008358	2052.43	240	821206	21600.27	1539	0	239	3.42	4.63
MCE	1008358	1608.27	240	821114	21228.93	2128.1	10	240	3.42	2.93
N	1003245	6127.87	212	821343	21691.07	96.8	0	178	3.92	14.76
T	1008358	7664.68	208	822087	21787.94	110.8	0	201	3.42	13.79

Table 18: Performance of the formulations for the capacitated unbalanced network - $|T| = 15, C = 1.5$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	587569	1.19	240	577246	5628.17	129453	203	222	1.6	0
EST	587569	2.03	240	577254	8291.62	82475.7	174	204	1.6	0.02
ESLS	587566	1.54	240	577264	7252.8	102829.8	185	215	1.6	0.02
ES-F-N	587569	1.47	240	577251	5830.11	107024.6	202	227	1.6	0
ES-F-T	587569	2.05	240	577255	6830.86	91531.8	195	225	1.6	0.01
ES-F-LS	587566	1.62	240	577268	7764.41	72522.2	184	216	1.6	0.02
MC	593073	130.26	240	577413	14334.42	26345.4	110	234	0.7	0.09
MCE	593073	134.1	240	577386	14168.51	34757.1	114	236	0.7	0.08
N	593130	105.81	240	577546	17307.97	14994.4	77	238	0.69	0.14
T	593073	321.95	240	577558	17069.16	15284	81	239	0.7	0.14

In light of the results provided in Tables 8–19, we can draw the following conclusions about the performance of our formulations on capacitated instances:

- the capacitated instances are harder to solve than the uncapacitated instances;
- the richer formulations have a relative worse performance than on uncapacitated instances compared to the echelon stock formulations;
- the echelon stock formulations are better than the richer formulations;
- within the richer formulations, the MCE formulation has the best performances.

Table 19: Performance of the formulations for the capacitated unbalanced network - $|T| = 30, C = 1.5$

Formulation	LP-cost	LP-CPU	LP-sol	MIP-cost	MIP-CPU	Nodes	MIP-opt	MIP-sol	I-gap	O-gap
ESN	1120686	18.64	240	940086	20794.78	42369.6	14	239	1.81	1.42
EST	1120686	21.21	240	940100	20907.78	28587.9	10	236	1.81	1.54
ESLS	1120671	12.15	240	940115	20799.68	23392.2	13	159	1.81	1.45
ES-F-N	1120686	23.88	240	940094	20896.34	40821.4	13	239	1.81	1.39
ES-F-T	1120686	40.68	240	940101	20726.33	35502.7	13	236	1.81	1.28
ES-F-LS	1120671	16.63	240	940122	20910.24	24093	11	234	1.81	1.4
MC	1131290	1982.83	240	940358	21526.09	2193.1	2	240	0.91	2.58
MCE	1131290	1472.36	240	940314	21551.76	3249	1	240	0.91	2.2
N	1131441	5397.68	218	940574	21810.5	155.1	0	203	0.89	13.01
T	1117718	6687.1	217	940594	21838.89	130.3	0	182	1.87	13.54

4.3 Influence of the parameters

Table 20 reports the performance of the MC formulation for all experiments with a balanced uncapacitated network and with $|T| = 30$. The first two columns indicate the parameter that varies and the respective values taken by the parameter. Since most of the following conclusions also apply for the other formulations and for the experiments with an unbalanced network, we only report here the results for the MC formulation with a balanced network. The analyses that are specific to this formulation are discussed at the end of this section. All the other results are available in the appendices of this report.

In Table 20, one can see that when $|R|$ increases, the problems gets harder and the CPU time taken to solve both the LP and MIP instances increases. On the contrary, when $|W|$ increases, the CPU time taken to solve the MIP instances decreases. Indeed, with the same number of retailers, if the number of warehouses increases, the supply network has a smaller number of channels linked to each warehouse. This leads to a smaller problem per warehouse and makes the global problem easier to solve, thus reducing the CPU time and the number of nodes. The integrality gap is also lower but less significantly.

Table 20 indicates that for the MC formulation, generally the instances with dynamic fixed costs are much easier to solve compared to the instances with a static fixed cost. We further note that the dynamic demand case is generally slightly easier to solve than the static demand case.

Finally, the detailed results provided in the appendices of this report illustrate the fact that the impact of the setting of the parameters (static or dynamic demand, static or dynamic fixed cost), depends on the kind of formulation used. For the classical based formulations, apart for the very small instances where $|R| = 50$ and $|T| = 15$, the instances with a dynamic fixed cost are harder to solve, thus requiring a higher CPU time. For the ES-N, ES-TP and ES-LS formulations, the instances with a dynamic fixed cost are also harder to solve. On the contrary, for the N, TP and MC formulations, the instances with a static fixed cost are harder to solve in terms of CPU time required. For the ES and ES-F formulations, there is no clear impact of the setting of the parameters on the CPU time required to solve the instances. Note however that this result does not question the higher global performance of the MC formulation stated in the previous sections.

Table 20: Performances of the MC formulation for the uncapacitated balanced network - $|T| = 30$

Parameter	Value	LP-cost	LP-CPU	MIP-cost	MIP-CPU	Nodes	MIP-opt	I-gap	O-gap
$\|R\|$	50	423630	60.36	423765	88.07	1.9	80	0.04	0
	100	609655	414.54	610096	643.15	4.5	80	0.08	0
	200	895053	2003.37	896048	2334.08	8.8	80	0.12	0
$\|W\|$	5	540034	587.33	541416	1451.26	14.8	60	0.23	0
	10	621960	912.23	622489	1095.86	3.5	60	0.07	0
	15	678045	1023.23	678196	827.76	1.5	60	0.02	0
	20	731078	781.58	731111	712.18	0.5	60	0	0
Costs	SF	658846	1040.45	659632	1508.85	8.6	120	0.12	0
	DF	626713	611.74	626974	534.68	1.6	120	0.04	0
Demand	SD	644294	840.35	644921	1077.51	6.4	120	0.1	0
	DD	641265	811.84	641685	966.03	3.7	120	0.06	0

Conclusions and future research

We have extended eleven MIP formulations proposed in the context of the OWMR and have applied them to the 3LSPD. We also introduced the ES-N, ES-F-N, ES-F-TP, ES-F-LS and MCE formulations that had not been tested before in the context of the OWMR. For our numerical experiments, we have considered two network structures (a balanced one and an unbalanced one) and have assessed the performance of the formulations proposed using several indicators. We have also considered the possibility of having production capacities at the plant level. The results indicate that, for the uncapacitated case, the unbalanced instances are harder to solve than the balanced instances and lead to a worse performance of all formulations, except for the classical formulations. On the contrary, for the capacitated case, the unbalanced instances give better values for our different performance indicators compared to the balanced instances. The classical formulations are much weaker than the other formulations and do not suit our problem, mainly because of a very weak LP relaxation. On the contrary, the MC formulation obtains the best performance on the uncapacitated instances and is able to solve all instances for both network structures. This result is similar to the conclusion of Cunha and Melo [8] for the OWMR. The other formulations obtain results that are not entirely satisfactory for the uncapacitated instances. In particular, for the rich formulations TP and N, the non-satisfactory performances on the large instances, in terms of number of MIP optimal solutions found and CPU time, are due to the huge size of the model. As a consequence, it is already very time-consuming to solve the LP relaxation of these formulations. When we impose capacity restrictions for production at the plant level, the performance of the formulations are reversed: the rich formulations have a worse performance and the echelon formulations have the best performance. Within the rich formulations, for the capacitated instances, our newly introduced MCE formulation generally has the best performance.

In future research, we want to introduce transportation capacities to limit the flows between all facilities. We will then use the results of our study and the possible substructures induced by transportation capacities to chose the best formulation possible to solve the problem, either heuristically or using decomposition methods.

Appendices

See the online document for Tables 21–204.

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