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Improved exact approaches for row layout problems with departments of equal length

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Abstract: Facility layout is a well-known operations research problem that arises in various applications. The multi-row layout is a challenging optimization problem where the task is to determine the optimal placement of one-dimensional departments on a given number of rows. This paper is concerned with multi-row facility layout problems in which all the departments have the same length. This is an important special case that includes most multi-row facility layout applications from the literature. We prove two theoretical results about the structure of optimal layouts, namely that only spaces of unit length are necessary to obtain an optimal solution, and that exact expressions exist for the minimum number of such spaces that need to be added so as to preserve at least one global optimal solution. Using these results we propose a binary linear optimization model and a binary semidefinite optimization model for the problem, neither of which uses continuous variables, which has a significant positive computational impact. Our computational experiments show that our specially tailored approaches can handle much larger instances than other exact methods applicable to this important problem class.

Keywords: Facilities planning and design, facility layout, mixed integer linear programming, semidefinite programming, bundle method

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Note: A short paper containing a brief outline of the ideas specialized to the double-row problem, without any technical details, appeared in the proceedings of the OR 2014.

1 Introduction

Facility layout is a well-known operations research problem that arises in various applications. The task is to determine an optimal placement of departments inside a plant according to a given objective function. This function usually reflects the transportation costs (for the material flow) as well as the construction cost of an associated material-handling system.

The exact solution of facility layout problems is generally extremely challenging even for relatively small instances, see e.g. [9]. For this reason it is common to restrict the shape of the layout or of the associated path system. For example, in the single-row facility layout problem (SRFLP), all the departments are placed in a single row, i.e., on one side of a straight path. In this case there always exists an optimal solution such that there are no spaces between neighboring departments. The SRFLP is well-studied. Currently instances of the SRFLP with up to 42 departments can be solved to optimality in reasonable time [34]. For further details on a variety of both exact methods and heuristics for the SRFLP we refer the reader to the surveys [8, 38, 40].

1.1 Multi-row facility layout and related problems

The multi-row facility layout problem (MRFLP) is an extension of the SRFLP in which the departments can be placed in two or more parallel rows. In contrast to the SRFLP, the optimal layouts for MRFLP may include spaces between neighboring departments in the same row or at the left margin of the rows.

Given d one-dimensional departments $\{1,\dots,d\} = [d]$ with given positive lengths $l_1,\dots,l_d$, pairwise non-negative weights w_{ij} indicating the (material) flow between each pair i,j of departments, and a set $\mathcal{R} := \{1,\dots,m\} = [m]$ of rows available for placing the departments, the objective of the MRFLP is to determine

1. an assignment $r\colon [d] \to \mathcal{R}$ of departments to rows, and
2. a function $p\colon [d] \to \mathbb{R}$ such that $|p(i) - p(j)| \geq \frac{1}{2}(l_i + l_j)$ if $r(i) = r(j)$, $i \neq j$, i.e., horizontal positions for the centers of the departments within each row without overlap,

so that the total weighted sum of the center-to-center distances between all pairs of departments is minimized. The MRFLP can thus be formulated as the following optimization problem:

$$\begin{aligned} \min_{r,p} \quad & \sum_{\substack{i,j\in[d]\\ i<j}} w_{ij}|p(i)-p(j)| \\ \text{s.t.} \quad & |p(i)-p(j)| \geq \tfrac{1}{2}(l_i+l_j), \quad i,j\in[d],\ r(i)=r(j),\ i\neq j. \end{aligned}$$

Note that the inter-row (vertical) distances between the departments are neglected in the above objective function, and that each department can be placed next to another department without any clearance restrictions. If $|\mathcal{R}| = 2$, then there are two rows of departments with a straight path between them. We denote this important special case as the double-row facility layout problem (DRFLP). Figure 1 shows an example of a layout with three rows and seven departments, where $r(1) = r(2) = r(3) = 1,\ r(4) = r(5) = 2,\ r(6) = r(7) = 3$ and $p(1) = p(4) = p(6) = 1,\ p(2) = 2,\ p(3) = p(5) = p(7) = 3$.

Various applications and extensions of the MRFLP have been studied, see, e.g., [25, 31, 46, 48, 53, 54]. Somewhat surprisingly, the development of exact approaches to the MRFLP has received limited attention in the literature. Heragu and Kusiak [30] proposed a nonlinear programming model and obtained locally optimal solutions to the SRFLP and the DRFLP. More recently, Chung and Tanchoco [18] (see also Zhang and Murray [52]) focused on the double-row problem and proposed a mixed integer linear programming (MILP) formulation that was tested in conjunction with several heuristics. They solved instances with up to 10 departments within 10 minutes. Amaral [2] proposed an improved MILP formulation that solves instances with up to 12 departments. Hungerländer and Anjos [36] put forward a semidefinite programming (SDP) approach for the general MRFLP that can solve instances with fewer than 12 departments to global optimality. Recently Fischer et al. [23] were able to solve DRFLP instances with up to 16 departments to optimality by iteratively using MILPs in an enumerative scheme.

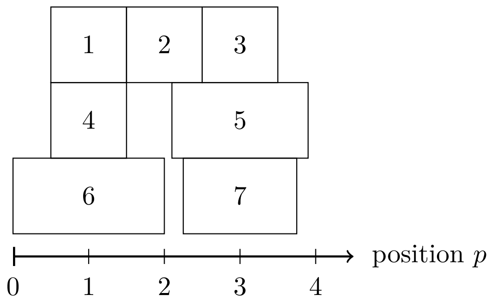

Figure 1: Illustration of a layout with three rows and seven departments.

Due to the challenging nature of the MRFLP, several simpler (but still $\mathcal{NP}$-hard) variants of the MRFLP have been considered in the literature. For example, in the space-free multi-row facility layout problem, spaces between the departments or at the left margin of the rows are forbidden. The special case $m = 2$ of space-free row layout is also known as the corridor allocation problem, and Hungerländer and Anjos [33] used an SDP approach that provides high-quality global bounds for space-free double-row instances with up to 15 departments and for space-free multi-row instances with up to 5 rows and 11 departments. Amaral [4] proposed a MILP formulation for the corridor allocation problem that is able to solve space-free instances with up to 13 departments. Recently Fischer et al. [23] were able to solve space-free double-row instances with up to 16 departments.

The parallel row ordering problem [5, 23, 33] is again a special case of the space-free MRFLP with the additional assumption that the assignment r of departments to rows is already given. Exact solutions for instances with up to 25 departments can be determined by a MILP model in [23]. This MILP is also called iteratively in the enumeration scheme of the currently best DRFLP solver [23]. An SDP approach [33] allows deriving good lower and upper bounds for instances with two to five rows and up to 100 departments.

1.2 Multi-row facility layout with departments of equal length

This paper is concerned with the special case of the MRFLP with departments of equal length, denoted (MREFLP), in which spaces are allowed, the row assignments are not given, and all department lengths are equal. The MREFLP is also known as the equidistant MRFLP, where we set w.l.o.g. $l_i = 1,\ i \in [d]$. The MREFLP can also be interpreted as an extension of the classical $\mathcal{NP}$-hard [24] (weighted) linear arrangement problem [20], where at most m nodes are assigned to one position. Hence the MREFLP is also $\mathcal{NP}$-hard.

The case of SRFLP with departments of equal length (SREFLP) has been studied before, and it turns out that the best models for the general SRFLP are also the best ones for the SREFLP [35]. This is not the case for the MREFLP, and we show in this paper that it is possible to exploit the additional problem structure for the development of tailored approaches.

Amaral [3] proposed an MILP formulation for the minimum duplex arrangement problem, which in our terminology is denoted as DRFLP with departments of equal length (DREFLP). His approach exploits the sparsity of the instances considered and is able to solve randomly generated instances with at most 10 departments (for dense instances) to 20 departments (for sparse instances). Amaral's MILP is closely related to models for the $\mathcal{NP}$-hard Quadratic Assignment Problem (QAP) that is known to be a particularly challenging combinatorial optimization problem in practice. The QAP asks for an assignment of n facilities to n locations that minimizes the sum of the distances between pairs of locations multiplied by the corresponding flows between pairs of facilities. For further details see, e.g., the survey paper [43] and the book [15].

Most of the earliest applications for the MRFLP were motivated by QAP problems where the locations were arranged on a regular grid, see [14, 21, 22, 26, 41, 44, 45, 49] among others. These QAPs are equivalent to MREFLPs, where the flows between pairs of facilities correspond to the connectivities between pairs of departments.

1.3 Research contributions

This paper is concerned with the MREFLP and its main contributions are:

- A proof that the MREFLP always has an optimal solution on the integer grid. This implies that it suffices to consider (multiple) spacing departments of unit length to obtain an optimal solution.
- Exact expressions for the minimum number of such spacing departments (as a function of the number of departments and of rows) such that from an optimal solution of the resulting "space-free" problem, it is possible to recover at least one global optimal solution for the MREFLP instance.
- An ILP model and an SDP model that are the first models for the MREFLP exploiting the fact that it can be modeled using only binary variables. This fact follows from the above theoretical results, and has a significant positive impact on the computational performance of the models. These models do not assign a row to each department but ensure that there are at most m departments at each (horizontal) integer position.
- Our computational experiments show that:
  - For the double-row case we increase the size of the largest instances solved to optimality from 16 departments (not extremely sparse) [23] to 25 departments.
  - For $3 \leq m \leq 5$ we increase the size of the largest instances solved to optimality from 8 departments [36] to 25 departments.
  - We achieve optimality gaps smaller than 1% for DREFLP and smaller than 4% for MREFLP, $m \in \{3, 4, 5\}$, for instances with up to 50 departments.

1.4 Outline

This paper is structured as follows. In Section 2 we state and prove our theoretical results on the structure of optimal MREFLP layouts. In Section 3 we focus on the double-row case; we present an ILP model for it in Subsection 3.1, an SDP model in Subsection 3.2, and these two models are extended to the multi-row case in Subsection 3.3. In Section 4 we describe a suitable combination of optimization methods to obtain both strong lower bounds and feasible MREFLP layouts using our proposed models. In Section 5 we report on a computational study of all relevant exact approaches for the MREFLP. Section 6 concludes the paper and proposes directions for future research.

2 The structure of optimal MREFLP layouts

The definitions of the MRFLP and the MREFLP allow the spaces between departments to be of arbitrary length. Thus, most optimization models use continuous variables to model the distances between departments. In this section we prove two theoretical results about the structure of optimal MREFLP layouts that allow to model the MREFLP with binary variables only.

In Subsection 2.1 we show that the MREFLP always has an optimal solution on the integer grid. The key insight here is that modeling the possible spaces between departments with *spacing departments* of unit length preserves at least one optimal solution.

In Subsection 2.2 we prove exact expressions for the minimum required number of such spacing departments, given the number of departments and of rows, to preserve at least one optimal solution.

These results are of interest because they reveal hitherto hidden structural properties of the MREFLP, and in turn these properties can be used to improve the practical performance of our models in Sections 3 and 3.3. These properties also allow improving other models applicable to the MREFLP [2, 18] by reducing the big-M value in these models.

2.1 A combinatorial property of MREFLP layouts on the integer grid

In this section we prove that the MREFLP always has an optimal solution on the integer grid. For a closely related result on general MRFLP layouts, we refer to [36, Theorem 2].

Theorem 1 *There is always an optimal solution to the MREFLP on the integer grid.*

Proof. Let an optimal solution of the MREFLP be given. We define an integer grid such that the centers of the leftmost departments are on a grid point. Next we divide the departments into two sets, a set S containing those departments with their centers already on the integer grid, and a set T containing the other departments. We assume, w.l.o.g., that the indices of the departments in S are all smaller than the indices of the departments in T, i.e., $i < j, \ \forall\, i \in S, \ j \in T$.

Observe that there exists $\varepsilon > 0$ sufficiently small so that we can move all the departments in T simultaneously, either to the left or to the right, by a distance ε. This holds because all departments have (the same) integer length, and because the departments in S are arranged on the integer grid. The change in the objective function from any such shift of the departments in T is given by

$$\delta = \sum_{i \in T} \left(\varepsilon \sum_{j \in S,\ j \dot{<} i} w_{ij} - \varepsilon \sum_{j \in S,\ i \dot{<} j} w_{ij} \right)$$

for a shift to the left, and by $-\delta$ for a shift to the right, where $i \dot{<} j$ means that the center of department j is to the right of the center of department i, and ε is chosen small enough such that no center of a department in T traverses a grid point. Due to the optimality of the given layout, δ has to be equal to zero because otherwise a shift either to the left ($\delta < 0$) or the right ($\delta > 0$) would improve the objective value. Hence the proposed shifting operation does not change the objective value.

Let us choose ε as the largest value such that the center of at least one department in T lies on a grid point after the shifting operation (to the left or right). If we apply this shifting to the given optimal solution, we can now move that department to the set S. Repeatedly applying this operation to the remaining departments allows us to arrange all departments on the integer grid in at most $n - 1$ steps without changing the objective value. □

Figure 2 depicts an MREFLP layout on the integer grid, where s denotes a spacing department and d_i denotes department i. For layouts satisfying the grid property, we say that department i lies in column j if the center of i is located at the j^{th} grid point. For example, department 5 lies in column 4 in Figure 2.

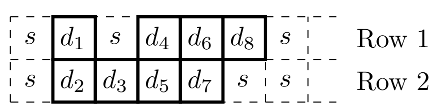

Figure 2: Illustration of the grid property of layouts.

We number the columns from 1 to d as for d departments obviously at most d columns are needed. By Theorem 1 we can represent an optimal solution of the MREFLP by an assignment $\alpha\colon [d] \to [d]$ of the d departments to d different columns with the interpretation

$$\alpha(i) = j, \quad \text{if department } i \text{ lies in column } j, \quad i, j \in [d], \tag{1}$$

where additionally at most m departments are assigned to each column $j \in [d]$, i.e.,

$$|\{i \in [d] \colon \alpha(i) = j\}| \leq m.$$

Note that the modeling approach in [3] builds directly on this assignment.

For the remainder of the paper we restrict our analysis, w.l.o.g., to layouts satisfying the grid property. This restriction is clearly advantageous from both a theoretical and a practical point of view. Note that the grid property is automatically satisfied for the minimum duplex arrangement problem [3], the (weighted) linear arrangement problem [16, 17, 20, 39] and its extension, where two or more nodes can be assigned to the same position. By Theorem 1, these problems are all special cases of the `MREFLP`.

2.2 Exact expressions for the minimum number of spacing departments

We now consider the minimum number of spacing departments of length one, or simply spaces, that must be added to an instance of the `MREFLP` so that after solving the problem with the added spaces, we can recover at least one optimal solution for the original instance. Clearly this number is a function of the numbers of departments and the number of rows. Since for given cost coefficients we do not have a priori knowledge of the structure of optimal solutions, our function does not depend on the weights w_{ij} other than on their non-negativity.

We first state three additional assumptions that allow us to reduce the number of spaces required. We then prove that Lemma 1 ensures that at least one optimal layout is preserved under these assumptions, i.e., there always exists an optimal solution $\alpha^*\colon [d] \to [d]$ that satisfies these three assumptions. Using Lemma 1, we then prove Theorem 2 that gives exact expressions for the minimum number of spacing departments. We conclude this section with two examples whose optimal layouts contain many spaces and hence the results of Theorem 2 are tight.

Assumption 1 Columns that contain solely spaces can be deleted. Equivalently, if we number the columns from 1 to d there exists $k' \in [d]$ such that each column with index at most k' contains at least one department.

Assumption 2 If two nonempty neighboring columns together contain no more than m departments, then all corresponding departments can be assigned to the left column, and the right column can be deleted. Thus, with k' as in Assumption 1, we know that columns i and $i+1$ with $i \in [k'-1]$ contain at least $m+1$ departments.

Assumption 3 If $d > 2m$ and the first and third columns contain in total at most m departments, then all corresponding departments can be assigned to the third column, and the first column can be deleted. An analogous argument holds for columns $k'-2$ and k', with k' as in Assumption 1.

Figure 3 illustrates these assumptions: the left-hand side depicts a feasible layout and the right-hand side depicts the adaptation of that layout so that the respective assumption holds. Note that the adaptations cannot worsen the objective value of the layout.

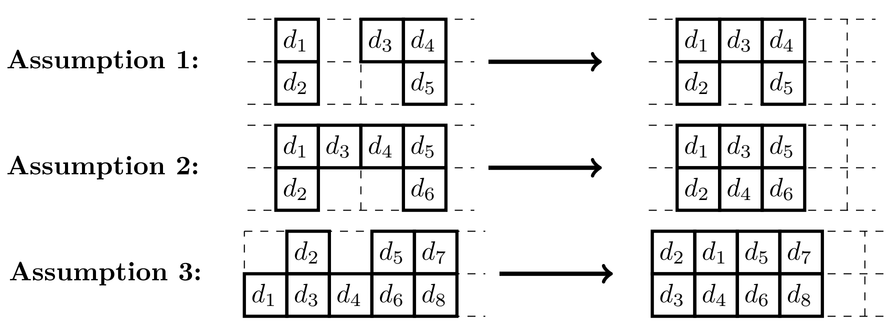

Figure 3: Illustration of Assumptions 1, 2, and 3.

Lemma 1 *Let $d, m \in \mathbb{N}$. Then there always exists an optimal solution $\alpha^*\colon [d] \to [d]$ of the `MREFLP` (satisfying the grid structure) that assigns each department $i \in [d]$ to a column $\alpha^*(i) \in [d]$ that satisfies the following properties:*

1. *There exists a* $k' \in [d]$ *such that* $|\{i \in [d]\colon \alpha^*(i) = l\}| \geq 1$ *for all* $l \in [d],\ l \leq k'$*, and* $|\{i \in [d]\colon \alpha^*(i) \geq k'+1\}| = 0$.
2. *If* $|\{i \in [d]\colon \alpha^*(i) = j\}| > 0$ *and* $|\{i \in [d]\colon \alpha^*(i) = j+1\}| > 0$ *for some* $j \in [d],\ j < d$*, then* $|\{i \in [d]\colon \alpha^*(i) = j\}| + |\{i \in [d]\colon \alpha^*(i) = j+1\}| \geq m+1$.
3. *Let* $d > 2m$*. Then* $|\{i \in [d]\colon \alpha^*(i) \geq k'+1\}| = 0$ *and* $|\{i \in [d]\colon \alpha^*(i) = k'\}| > 0$ *for some* $k' \in [d]$ *imply* $|\{i \in [d]\colon \alpha^*(i) = k'-2\}| + |\{i \in [d]\colon \alpha^*(i) = k'\}| \geq m+1$*. Furthermore,* $|\{i \in [d]\colon \alpha^*(i) = 1\}| + |\{i \in [d]\colon \alpha^*(i) = 3\}| \geq m+1$.

Proof. Let $d, m \in \mathbb{N}$ and α^* be an optimal solution of the `MREFLP` satisfying the grid structure.

1. If $|\{i \in [d]\colon \alpha^*(i) = j-1\}| = 0$ and $|\{i \in [d]\colon \alpha^*(i) = j\}| \geq 1$ for some $j \in [d]$, then the solution $\alpha'\colon [d] \to \mathbb{N}$ with $\alpha'(l) = \alpha^*(l)$ if $\alpha^*(l) < j$ and $\alpha'(l) = \alpha^*(l) - 1$ otherwise is also optimal for the `MREFLP` because the distances between departments are not increased. The repeated deletion of empty columns proves the statement.
2. Assume that $|\{i \in [d]\colon \alpha^*(i) = j\}| + |\{i \in [d]\colon \alpha^*(i) = j+1\}| \leq m$ for some $j \in [d], j < d$. Then $\alpha'\colon [d] \to \mathbb{N}$ with $\alpha'(l) = \alpha^*(l)$ if $\alpha^*(l) \leq j$ and $\alpha'(l) = \alpha^*(l) - 1$ otherwise is a feasible multi-row assignment and it is optimal because all distances are not increased (some are even decreased) and there are at most m departments in each row. Applying this approach repeatedly we get an optimal assignment $\bar{\alpha}$ such that $|\{i \in [d]\colon \bar{\alpha}(i) = j\}| > 0$ and $|\{i \in [d]\colon \bar{\alpha}(i) = j+1\}| > 0$ for some $j \in [d-1]$ imply $|\{i \in [d]\colon \bar{\alpha}(i) \in \{j, j+1\}\}| > m$.
3. Assume w.l.o.g. that there exists an optimal solution α^* of the `MREFLP` and $k' \in [d]$ such that $|\{i \in [d]\colon \alpha^*(i) = k'\}| > 1, |\{i \in [d]\colon \alpha^*(i) \geq k'+1\}| = 0$. By the previous statements we may assume $|\{i \in [d]\colon \alpha^*(i) = k'-1\}| > 0$ and —$\{i \in [d]\colon \alpha^*(i) \in \{k'-1, k'\}\}| > m$. If in addition $|\{i \in [d]\colon \alpha^*(i) \in \{k'-2, k\}\}| \leq m$, the solution $\alpha'\colon [d] \to \mathbb{N}$ with $\alpha'(l) = \alpha^*(l) - 2$ if $\alpha^*(l) = k'$ and $\alpha'(l) = \alpha^*(l)$ otherwise is also optimal because all distances between departments are not increased.

□

Theorem 2 *The minimum number of columns sufficient to preserve at least one optimal layout for an instance with* d *departments is*

1. *equal to* 1 *if* $d \leq m$*, and equal to* 2 *if* $m < d < \frac{3}{2}m + \frac{3}{2}$*;*
2. *equal to* $\lceil \frac{2d}{3} \rceil - 1$ *for the* `DREFLP` *with* $d \geq 9$*;*
3. *equal to* $\left\lfloor \frac{2d}{m+1} \right\rfloor$ *for the* `MREFLP` *with an odd number of rows* m*; and*
4. *equal to or at most* $2l+1$ *for the* `MREFLP` *with an even number of rows* m *and* $d \in \{\frac{m}{2} + 2 + (m+1)(l-1), \ldots, \frac{m}{2} + 1 + (m+1)l\}$ *for some* $l \in \mathbb{N}$.

Proof. We prove each of the four claims in turn:

- *Proof of 1:* Let $d, m \in \mathbb{N}$ be given. If $d \leq m$, it is clear that arranging all departments in one column leads to a cost of zero. Furthermore, as long as $m < d < \frac{3}{2}m + \frac{3}{2}$ there exists an arrangement such that only two columns are used because, w.l.o.g., we can assume that the first two columns contain $m+1$ departments and that the second column contains at most $\lceil \frac{m}{2} \rceil$ of these departments. Then the remaining departments could also be included in one of the first two columns, either all in the second column or also some of them in the first column.
- *Proof of 2:* Let $m = 2$, $d \geq 9$ and let α^* be an optimal solution of the `DREFLP` satisfying the grid structure and the properties given in Lemma 1. Then there exists a $k' \in [d]$ such that $|\{i \in [d]\colon \alpha^*(i) = l\}| \geq 1$ for all $l \in [d],\ l \leq k'$ and $|\{i \in [d]\colon \alpha^*(i) > k'\}| = 0$. (Note: $d \geq 9$ implies $k' \geq 5$.) By Lemma 1 the solution α^* satisfies $|\{i \in [d]\colon \alpha^*(i) \in \{j, j+1\}\}| \geq 3$ for all $j \in [d],\ j < k'$, as well as $|\{i \in [d]\colon \alpha^*(i) \in \{1,2,3\}\}| \geq 5$ and $|\{i \in [d]\colon \alpha^*(i) \in \{k'-2, k'-1, k'\}\}| \geq 5$. We consider two cases for k'. If $(k'-6) \bmod 2 \equiv 0$, then the first k' columns contain at least $10 + (k'-6)\frac{3}{2} = \frac{3}{2}k' + 1$ departments.

Otherwise, if $(k'-6) \bmod 2 \equiv 1$, then the first k' columns contain at least $5 + (k'-3)\frac{3}{2} = \frac{3}{2}k' + \frac{1}{2}$ departments. Now, assume that $k' \geq \left\lceil \frac{2d}{3} \right\rceil$. Then the first k' columns contain at least $\left\lceil \left(\frac{3}{2}\left\lceil\frac{2d}{3}\right\rceil + \frac{1}{2}\right) \right\rceil > d$ departments, a contradiction. The claim follows.

- *Proof of 3:* Let m be odd and $d > 2m$. Let α^* be an optimal solution of the MREFLP that satisfies the properties given in Lemma 1. By Lemma 1 there exists $k' \in [d]$ such that $|\{i \in [d] \colon \alpha^*(i) = l\}| \geq 1$ for all $l \in [d],\ l \leq k'$ and $|\{i \in [d] \colon \alpha^*(i) \geq k'+1\}| = 0$. Then we know by Lemma 1 that $|\{i \in [d] \colon \alpha^*(i) = j\}| + |\{i \in [d] \colon \alpha^*(i) = j+1\}| \geq m+1$ for all $j \in [d],\ j < k'$. Suppose now that $k' > \left\lfloor \frac{2d}{m+1} \right\rfloor$, then the k' columns contain at least $\frac{m+1}{2} \cdot k' \geq \frac{m+1}{2}(\left\lfloor \frac{2d}{m+1} \right\rfloor + 1) > d$ departments, a contradiction.
- *Proof of 4:* Let m be even and $d > 2m$. Let α^* be an optimal solution of the MREFLP that satisfies the properties given in Lemma 1. Assume $d \in \{\frac{m}{2}+2+(m+1)(l-1), \ldots, \frac{m}{2}+1+(m+1)l\}$ for some $l \in \mathbb{N}$. By Lemma 1 there exists a $k' \in [d]$ such that $|\{i \in [d] \colon \alpha^*(i) = m\}| \geq 1$ for all $m \in [d],\ m \leq k'$ and $|\{i \in [d] \colon \alpha^*(i) \geq k'+1\}| = 0$. Then we know by Lemma 1 that $|\{i \in [d] \colon \alpha^*(i) = j\}| + |\{i \in [d] \colon \alpha^*(i) = j+1\}| \geq m+1$ for all $j \in [d],\ j < k'$. Assume now that $k' \geq 2l+2$. Then the first k' columns contain at least $\frac{2l+2}{2}(m+1) = (m+1)l + m + 1 > (m+1)l + \frac{m}{2} + 1 \geq d$ departments, a contradiction.

□

Table 1 gives exact values of the minimum number of columns c for instances with 2, 3 and 4 rows and up to 16 departments. Note that $c \cdot m - d$ spaces are needed for the respective row-department combinations.

Table 1: Minimum number of columns required for $d \leq 16$ and $m \in \{2,3,4\}$.

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2 rows	1	1	2	2	3	4	4	5	5	6	7	7	8	9	9	10
3 rows	1	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
4 rows	1	1	1	1	2	2	2	3	3	4	4	4	5	5	5	6

We conclude this section with two examples for which the number of columns required as per Table 1 is tight.

1. Consider $m = 3$ rows, $d = 2l$ departments for some $l \in \mathbb{N}$, and weights $w_{i(i+1)} = 1, i = 1, 3, 5, \ldots, 2l-1$, and $w_{ij} = \varepsilon$ otherwise. For ε sufficiently small, the optimal solution contains exactly one space in each column. The left-hand side of Figure 4 illustrates the case $d = 10$. In this example the objective value is not worsened if we reduce the number of rows from 3 to 2.
2. For $m > 2$ even, the exact calculation of the bounds is complicated and might be slightly improved if d cannot be written as $\frac{m}{2}+1+(m+1)l$ for some $l \in \mathbb{N}$. Nevertheless no improvement of the (seemingly) large number of spaces is possible if we want to preserve an optimal solution. To see this consider a problem with four rows and 13 departments with $w_{12} = w_{13} = w_{45} = w_{67} = w_{68} = w_{9\,10} = w_{11\,12} = w_{11\,13} = 1$ and all other weights equal to a small $\varepsilon > 0$. Then all optimal solutions have a structure like the one visualized on the right-hand side of Figure 4. In this case $d = 13 = \frac{4}{2} + 1 + 5l$ with $l = 2$.

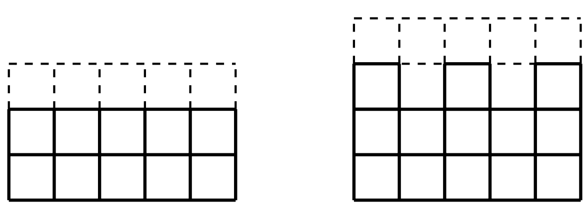

Figure 4: Worst-case examples for Theorem 2.

In summary Theorem 2 allows us to reduce the number of spaces, and hence of variables, in formulations such as the MILP model in [3] and the ILP and SDP models that we propose in the next section. Theorem 2 also helps to eliminate some of the symmetries in the problem, like, e.g., the position of empty columns.

3 New formulations for the DREFLP and MREFLP

In this section we present two new models for the DREFLP and their extensions to the MREFLP. Our models take only horizontal distances between the departments into account as they are assumed to be one-dimensional. This is a typical assumption in the row layout literature, especially for exact approaches that usually only discuss the double-row case because it is the most important case in practice.

For the MREFLP and the MRFLP, the only exact methods that are able to take vertical distances into account use an enumeration scheme to deal with each row assignment individually [23, 36]. For $d \geq 17$ these approaches have to consider a prohibitive number of $\mathcal{NP}$-hard subproblems to even obtain a lower bound and hence are not competitive for MREFLP instances of challenging size. Instead our suggested ILP and SDP models solely ensure that at most m departments are assigned to each column, which allows them to scale well for instances with up to 60 departments and up to 5 rows. However, our models cannot be used for applications where vertical distances matter.

In Subsection 3.1 we propose an ILP formulation for the DREFLP that uses betweenness variables together with variables modeling whether pairs of departments are assigned to the same column. In Subsection 3.2 an SDP formulation for DREFLP based on products of ordering variables is presented. To the best of our knowledge, Amaral [3] suggested the only other approach tailored specifically to the DREFLP. His MILP uses position variables and is loosely related to formulations for the QAP. In Subsection 3.3 we extend the two approaches to the MREFLP.

3.1 An ILP formulation for the DREFLP

Our first model is an ILP formulation for the DREFLP that extends the model proposed in [1] for the SRFLP. We use additional variables to model that two departments can be assigned to the same column. We also fill up the c columns with spaces, i.e., departments of length 1 and pairwise weights equal to 0 with all the departments (including other spaces). We collect all these spaces in a set S. To simplify notation we set the total number of departments (original ones plus spaces) to $n := 2c$ and the number of spaces is thus $s = n - d$. After the insertion of spaces we deal in fact with a space-free problem, and by Theorems 1 and 2 the optimal solution of the corresponding optimization problem is an optimal solution of the DREFLP.

Our model makes use of betweenness variables $b_{ijk} = b_{kji} \in \{0,1\},\ i,j,k \in [n],\ i < k,\ i \neq j \neq k$, and of column overlap variables $a_{ij} = a_{ji} \in \{0,1\}, i,j \in [n], i < j$. These two sets of binary variables have the following interpretations:

$$b_{ijk} = \begin{cases} 1, & \text{if department } j \text{ lies between departments } i \text{ and } k, \\ 0, & \text{otherwise;} \end{cases}$$

$$a_{ij} = \begin{cases} 1, & \text{if departments } i \text{ and } j \text{ are assigned to the same column,} \\ 0, & \text{otherwise.} \end{cases}$$

Our resulting ILP formulation of the DREFLP is

$$\min \sum_{i,j \in [n],\ i<j} \tfrac{w_{ij}}{2} \cdot \left(\sum_{k \in [n] \setminus \{i,j\}} b_{ikj} + 2(1 - a_{ij}) \right) \tag{2}$$

$$\text{s.t.} \quad a_{ij} + a_{ik} + a_{jk} + b_{ijk} + b_{ikj} + b_{jik} = 1, \qquad i,j,k \in [n],\ i<j<k, \tag{3}$$

$$\sum_{j \in [n] \setminus \{i\}} a_{ij} = 1, \qquad i \in [n], \tag{4}$$

$$\begin{aligned}
&b_{ihj} + b_{ihk} + b_{jhk} \le 2, && i,j,k,h \in [n],\ i<j<k \ne h,\ i \ne h \ne j, && (5)\\
&-b_{ihj} + b_{ihk} + b_{jhk} + b_{ikj} \ge 0, && i,j,k,h \in [n],\ i<j<k \ne h,\ i \ne h \ne j, && (6)\\
&+b_{ihj} - b_{ihk} + b_{jhk} + b_{ijk} \ge 0, && i,j,k,h \in [n],\ i<j<k \ne h,\ i \ne h \ne j, && (7)\\
&+b_{ihj} + b_{ihk} - b_{jhk} + b_{jik} \ge 0, && i,j,k,h \in [n],\ i<j<k \ne h,\ i \ne h \ne j, && (8)\\
&-b_{ihj} + b_{ihk} + b_{jhk} + a_{hk} \ge 0, && i,j,k,h \in [n],\ i<j<k \ne h,\ i \ne h \ne j, && (9)\\
&+b_{ihj} - b_{ihk} + b_{jhk} + a_{hj} \ge 0, && i,j,k,h \in [n],\ i<j<k \ne h,\ i \ne h \ne j, && (10)\\
&+b_{ihj} + b_{ihk} - b_{jhk} + a_{hi} \ge 0, && i,j,k,h \in [n],\ i<j<k \ne h,\ i \ne h \ne j, && (11)\\
&b_{ijk} \in \{0,1\}, && i,j,k \in [n],\ i<j,\ i \ne k \ne j, && (12)\\
&a_{ij} \in \{0,1\}, && i,j \in [n],\ i<j. && (13)
\end{aligned}$$

The objective function (2) counts all departments that lie between the departments i and j, because the distance between i and j equals the number of columns between them, regardless of which row they are in. This is equal to half of the number of departments between i and j, plus a term accounting for whether they lie in the same column or not. Specifically we divide the coefficient of the betweenness variables by 2 and count w_{ij} towards the cost if departments i and j do not lie in the same column.

Equations (3) ensure that for every choice of three different departments either these lie in three different columns and one of the betweenness variables equals 1, or two of the three departments lie in the same column and the associated overlap variable equals 1. Equations (4) ensure that each department $i \in [n]$ lies in the same column as exactly one other department. Inequalities (5) to (11) are extensions of the inequalities in [1] for the `SRFLP`: inequality (5) ensures that a department h cannot lie between each two of every choice of three departments $i,j,k \in [n] \setminus \{h\},\ i<j<k$, and inequalities (6)–(11) ensure that if department h lies between departments i and j, then h lies also between i,k, or between j,k, or in the same column as k (which also implies that k lies between i and j).

Due to the introduction of spaces, our model contains some symmetries that we break to improve its practical performance. The following constraints enforce an order of the s spaces such that space i is to the left of space j or in the same column as j iff $i<j,\ i,j \in S$:

$$\begin{aligned}
&a_{ij} = 0, && i,j \in S,\ i+2 \le j, && (14)\\
&b_{ijk} = 1, && i,j,k \in S,\ i+4 \le j+2 \le k, && (15)\\
&b_{ijk} = 0, && i,j,k \in S,\ i \ne k,\ (j > \max\{i,k\} \vee j < \min\{i,k\}). && (16)
\end{aligned}$$

A further improvement to the model is to include a variation on a class of inequalities for the `SRFLP` introduced by Amaral [1]. The precise statement of our proposed inequalities is given in Theorem 3. We note that taking $\beta = 4$ in Theorem 3 yields (6)–(11).

Theorem 3 *Let $\beta \in \mathbb{N}, \beta \ge 4$, be even and let $T \subseteq [n]$ with $|T| = \beta$. For a partition of T in $T_1, T_2, \{k\}$ such that $T = T_1 \dot\cup T_2 \dot\cup \{k\}, (T_1 \cap T_2 = \emptyset, k \notin T_1, k \notin T_2)$ and $|T_1| = \frac{\beta}{2}$ the following inequalities are valid for the DREFLP*

$$\sum_{\substack{p,q\in T_1,\\ p<q}} b_{pkq} + \sum_{\substack{p,q\in T_2,\\ p<q}} b_{pkq} - \sum_{\substack{p\in T_1,\\ q\in T_2}} b_{pkq} \le \sum_{p\in T_2} a_{kp}, \tag{17}$$

$$\sum_{\substack{p,q\in T_1,\\ p<q}} b_{pkq} + \sum_{\substack{p,q\in T_2,\\ p<q}} b_{pkq} - \sum_{\substack{p\in T_1,\\ q\in T_2}} b_{pkq} \le \sum_{\substack{p,q\in T_1,\ o\in T_2,\\ p<q}} b_{poq}. \tag{18}$$

Proof. Let $\beta \in \mathbb{N}, \beta \ge 4$, even and $T \subseteq [n]$ with $|T| = \beta$ be given. We consider a partition of T into $T_1, T_2, \{k\}$ such that $T = T_1 \dot\cup T_2 \dot\cup \{k\}$ and $|T_1| = \frac{\beta}{2}$ (so $|T_2| = \frac{\beta}{2} - 1$). In order to prove that inequalities (17) and (18) are valid for the `DREFLP` we consider a fixed double-row assignment $\alpha \colon [n] \to [\frac{n}{2}]$ that assigns each of the n departments (original and spaces) to one of the columns. We define $\sigma_1^1 := |\{i \in T_1 \colon \alpha(i) < \alpha(k)\}|$,

$\sigma_1^2 := |\{i \in T_2 \colon \alpha(i) < \alpha(k)\}|$, $\sigma_2^1 := |\{i \in T_1 \colon \alpha(i) > \alpha(k)\}|$, $\sigma_2^2 := |\{i \in T_2 \colon \alpha(i) > \alpha(k)\}|$, $\sigma_3^1 := |\{i \in T_1 \colon \alpha(i) = \alpha(k)\}|$, $\sigma_3^2 := |\{i \in T_2 \colon \alpha(i) = \alpha(k)\}|$. Then $\sigma_1^1 + \sigma_2^1 + \sigma_3^1 = \frac{\beta}{2}$, $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \frac{\beta}{2} - 1$ and $\sigma_3^1 + \sigma_3^2 \leq 1$. The left-hand side of (17) and (18) is equal to γ, where

$$\gamma := \sigma_1^1\sigma_2^1 + \sigma_1^2\sigma_2^2 - \sigma_1^1\sigma_2^2 - \sigma_2^1\sigma_1^2 = -(\sigma_1^1 - \sigma_1^2)^2 + \sigma_1^1 - \sigma_1^2 - \sigma_1^1\sigma_3^1 + \sigma_1^1\sigma_3^2 + \sigma_3^1\sigma_1^2 - \sigma_1^2\sigma_3^2.$$

The last equality follows by direct computations using $\sigma_2^1 = \frac{\beta}{2} - \sigma_1^1 - \sigma_3^1$ and $\sigma_2^2 = \frac{\beta}{2} - 1 - \sigma_1^2 - \sigma_3^2$. We consider three cases:

- $\sigma_3^1 = \sigma_3^2 = 0$: Then $\gamma = -(\sigma_1^1 - \sigma_1^2)^2 + \sigma_1^1 - \sigma_1^2 = -(\sigma_1^1 - \sigma_1^2)(\sigma_1^1 - \sigma_1^2 - 1) \leq 0$ and with $a_{ij} \geq 0,\ i,j \in [n],\ i < j,\ b_{ijk} \geq 0,\ i,j,k \in [n],\ i < k,\ |\{i,j,k\}| = 3$, the validity follows in this case.
- $\sigma_3^1 = 1, \sigma_3^2 = 0$: Then $\gamma = -(\sigma_1^1 - \sigma_1^2)^2 + \sigma_1^1 - \sigma_1^2 - \sigma_1^1 + \sigma_1^2 = -(\sigma_1^1 - \sigma_1^2)^2$ and with $a_{ij} \geq 0,\ i,j \in [n],\ i < j,$ $b_{ijk} \geq 0,\ i,j,k \in [n],\ i < k,\ |\{i,j,k\}| = 3$, the validity follows in this case.
- $\sigma_3^1 = 0, \sigma_3^2 = 1$: Then $\gamma = -(\sigma_1^1 - \sigma_1^2)^2 + \sigma_1^1 - \sigma_1^2 + \sigma_1^1 - \sigma_1^2 = -(\sigma_1^1 - \sigma_1^2)(\sigma_1^1 - \sigma_1^2 - 2)$. This term is positive if and only if $\sigma_1^1 - \sigma_1^2 = 1$ by the integrality of the σ_i^j.

So, it suffices to show that the right-hand sides of (17) and (18) are at least one if $\sigma_3^1 = 0,\ \sigma_3^2 = 1$ and $\sigma_1^1 - \sigma_1^2 = 1$. For (17) the term $\sigma_3^2 = 1$ implies the existence of an $o \in T_2$ that lies in the same column as k. Considering (18), $\sigma_3^2 = 1$ and $\sigma_1^1 - \sigma_1^2 = 1$ imply $\sigma_1^1 > 0,\ \sigma_2^1 > 0$ and so there exist $p, q \in T_1,\ p \neq q$, and $o \in T_2$ such that o lies between p, q. □

3.2 An SDP formulation for the DREFLP

Our second formulation for the DREFLP is based on a quadratic formulation using ordering variables that we rewrite using symmetric matrices. The matrix-based formulation is then relaxed into an SDP problem, and this SDP relaxation can be tightened using several classes of valid constraints. For more details on semidefinite programming we refer to the handbooks [6, 50].

We introduce the ordering variables $x_{ij},\ i,j \in [n],\ i \neq j$, where x_{ij} is 1 if department i lies left of department j, and -1 otherwise. We observed in Subsection 3.1 that the center-to-center distances between departments can be encoded using betweenness and column-overlap variables. Because we are willing to work with quadratic terms, we can express those two kinds of variables in terms of the ordering variables:

$$\begin{aligned} b_{ikj} &= \tfrac{1}{4}(x_{ik}x_{kj} + x_{jk}x_{ki} + x_{ik} + x_{kj} + x_{jk} + x_{ki}) + \frac{1}{2}, \quad i,j,k \in [n],\ i < j, \\ a_{ij} &= -\tfrac{1}{2}(x_{ij} + x_{ji}), \quad i,j \in [n],\ i < j. \end{aligned} \tag{19}$$

It directly follows that we can rewrite the objective function (2) as a linear-quadratic function of the ordering variables:

$$K + \sum_{\substack{i,j \in [n] \\ i<j}} \frac{w_{ij}}{8} \left(\sum_{\substack{k \in [n] \\ k \neq i,\ k \neq j}} (x_{ik}x_{kj} + x_{jk}x_{ki}) \right) + \sum_{\substack{i,j \in [n] \\ i<j}} \frac{w_{ij}}{4}(x_{ij} + x_{ji}), \tag{20}$$

where K is a constant defined as

$$K := \sum_{\substack{i,j \in [n] \\ i<j}} \frac{n \cdot w_{ij}}{4}. \tag{21}$$

Any feasible ordering of the departments has to satisfy the well-known 3-cycle inequalities

$$-1 \leq x_{ij} + x_{jk} - x_{ik} \leq 1, \qquad i,j,k \in [n],\ i \neq j \neq k,\ i \neq k. \tag{22}$$

that together with integrality conditions on the ordering variables suffice to describe feasible orderings, see e.g. [47, 51]. In the present context we need the following additional constraints

$$x_{ij} + x_{ji} \leq 0, \quad i,j \in [n],\ i < j, \tag{23}$$

that model the fact that either department i lies to the left of department j or department j lies to the left of department i or both departments are assigned to the same column.

Note from the definition of the ordering variables that if two departments i and j are placed in different columns then $x_{ij} + x_{ji}$ equals zero, while if they are assigned to the same column the sum is -2. In contexts where the departments cannot overlap, such as the SRFLP, this observation is often used to halve the number of variables in models using ordering variables by requiring that $x_{ij} + x_{ji} = 0$. While some overlap is allowed here, we ensure that exactly two departments are assigned to each column using the constraints

$$\sum_{j\in[n]\setminus\{i\}} (x_{ij} + x_{ji}) = -2, \quad i \in [n]. \tag{24}$$

Lemma 2 *Minimizing the objective function (20) over $x \in \{-1,1\}^{n(n-1)}$ and (22)–(24) solves the DREFLP.*

Proof. The constraints (22)–(24) together with the integrality conditions on x suffice to induce feasible double-row layouts and the definition of the objective function ensures that the distances between departments are computed correctly. □

Next we collect the ordering variables in a vector x and reformulate the DREFLP as a quadratic program in ordering variables.

We define $X := xx^\top$ and rewrite the quadratic objective function (20) in matrix notation to obtain:

$$\min \{\langle C_X, X\rangle + c_x^\top x + K \colon x \in \{-1,1\}^{n(n-1)} \text{ satisfies (22)–(24)}\}. \tag{DREFLP}$$

The cost matrix C_X and the cost vector c_x are deduced from (20):

$$\langle C_X, X\rangle = \sum_{\substack{i,j\in[n]\\ i<j}} \frac{w_{ij}}{8} \sum_{\substack{k\in[n]\\ i\neq k\neq j}} (x_{ik}x_{kj} + x_{jk}x_{ki}),$$

$$c_x^\top x = \sum_{\substack{i,j\in[n]\\ i<j}} \frac{w_{ij}}{4} (x_{ij} + x_{ji}).$$

We can further rewrite the above formulation as an SDP by relaxing the non-convex equation $X - xx^\top = 0$ to the positive semidefinite constraint $X - xx^\top \succcurlyeq 0$. Moreover, the main diagonal entries of X correspond to squared $\{-1,1\}$ variables, hence $\operatorname{diag}(X) = e$, where e denotes the vector of all ones. To simplify notation let us introduce

$$Z = Z(x,X) := \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}, \tag{25}$$

where $\dim(Z) = n(n-1)+1$. By the Schur complement theorem [7, Theorem 1.6], $X - xx^\top \succcurlyeq 0 \Leftrightarrow Z \succcurlyeq 0$. Hence any feasible layout is contained in the elliptope $\mathcal{E} := \{Z \colon \operatorname{diag}(Z) = e,\ Z \succcurlyeq 0\}$. In order to express constraints on x in terms of X, they have to be reformulated as quadratic conditions in x. A natural way to do this for the 3-cycle inequalities (22) is to express them as $|x_{ij} + x_{jk} - x_{ik}| = 1$ and square both sides [34]. Additionally using $x_{ij}^2 = 1$, we obtain

$$x_{ij,jk} - x_{ij,ik} - x_{ik,jk} = -1, \qquad i,j,k \in [n],\ i \neq j \neq k,\ i \neq k. \tag{26}$$

These conditions were first used for the SRFLP in [10].

Now we can formulate the DREFLP as a semidefinite optimization problem in binary variables.

Theorem 4 *The problem*

$$\min\left\{K + \langle C_Z, Z\rangle \colon Z \text{ satisfies } (26),\ Z \in \mathcal{E},\ x \in \{-1,1\}^{n(n-1)} \text{ satisfies } (23) \text{ and } (24)\right\}$$

where Z is given by (25), K is defined in (21) and the cost matrix C_Z is given by

$$C_Z := \begin{pmatrix} 0 & \frac{1}{2}c_x \\ \frac{1}{2}c_x & C_X \end{pmatrix},$$

is equivalent to the DREFLP*.*

Proof. Since $x_i^2 = 1,\ i \in \{1,\ldots,n(n-1)\}$, we have $\operatorname{diag}(X - xx^\top) = 0$, which together with $X - xx^\top \succcurlyeq 0$ shows that in fact $X = xx^\top$ is integral. Equations (26) ensure $|x_{ij} + x_{jk} - x_{ik}| = 1$, and constraints (23) and (24) together with the integrality of x suffice to induce feasible double-row layouts due to Lemma 2. Finally the definition of K and C_Z ensures that the distances between departments are computed correctly. □

Dropping the integrality condition on the first row and column of Z yields the basic semidefinite relaxation of the `DREFLP`:

$$\min\ \{K + \langle C_Z, Z\rangle \colon Z \text{ satisfies } (26),\ Z \in \mathcal{E},\ x \text{ satisfies } (23) \text{ and } (24)\}. \qquad (\texttt{SDP}_{\texttt{basic}})$$

We now consider possible ways to tighten the above relaxation. First we observe that adding Equations (3) from our ILP model does not improve $\texttt{SDP}_{\texttt{basic}}$.

Observation 1 *Equations (3) can be expressed as the sum of two equations of the form (26) using (19).*

We can add symmetry-breaking constraints arising from the addition of spaces (as already seen in Subsection 3.1):

$$x_{21} = -1, \qquad (27)$$

$$x_{ij} = 1, \qquad i,j \in S,\ i+2 \le j, \qquad (28)$$

$$x_{ij} = -1, \qquad i,j \in S,\ j < i, \qquad (29)$$

$$\begin{aligned} x_{i(i+1)}x_{ki} - x_{ki} - x_{i(i+1)} &= -1, \\ x_{i(i+1)}x_{k(i+1)} - x_{k(i+1)} - x_{i(i+1)} &= -1, \end{aligned} \qquad i \in S,\ i \ne n,\ k \in [d]. \qquad (30)$$

Constraint (27) breaks the symmetry of the overall arrangement. Constraints (28) ensure that two spaces i and j can be assigned to the same column only if $i+1 = j$. Constraints (29) guarantee that in all layouts considered the labels of the spaces increase from left to right. Finally, constraints (30) are related to Assumption 1 in Subsection 2.2: if two spaces $i,j \in S$ lie in the same column, then each department $k \in [d]$ has to lie left to them (see also Figure 3). Direct computations using (19) give the following result:

Observation 2 *The ILP symmetry-breaking Equations (14)–(16) can be derived from (28)–(30).*

Equations (28) and (29) allow us to reduce the size of the semidefinite problem for the computational experiments in Section 5. However, this requires all constraints containing the relevant variables to be transformed accordingly. While this is a straightforward exercise, it involves technical detail that does not provide further insight, so we omit the details of this transformation and of the resulting constraints. (For the same reason, we chose not to exploit (27) though this could be done in principle.)

Again because we allow quadratic terms, we can express the inequalities (23) as equations:

$$x_{ij}x_{ji} + x_{ij} + x_{ji} = -1, \qquad i,j \in [n],\ i < j. \qquad (31)$$

Equation (31) is valid because either $x_{ij} = x_{ji} = -1$ (both departments lie in the same column) or $x_{ij} + x_{ji} = 0$ and $x_{ij}x_{ji} = -1$ (they lie in different columns).

The theoretically smoothest way to deal with Equations (24) would be to use them to reduce the dimension of the problem by n (for details see [32, Proposition 4.4]). Unfortunately, this would make their practical implementation much more complicated. An alternative is to lift (24) into quadratic space via multiplication by an arbitrary ordering variable x_{lm}, $l,m \in [n]$, $l \neq m$, and the addition of the resulting linear-quadratic equations to the semidefinite relaxation:

$$\sum_{\substack{j\in[n]\\ j\neq i}} (x_{ij}x_{lm} + x_{ji}x_{lm}) = -2x_{lm}, \qquad i,l,m \in [n],\ l \neq m. \tag{32}$$

A well-known class of valid inequalities for our model is the triangle inequalities of the max-cut polytope, see e.g. [19]. Since Z is generated as the outer product of the vector $\begin{pmatrix}1 & x\end{pmatrix}^\top$ that has merely $\{-1,1\}$ entries in the (non-relaxed) SDP formulation, any matrix Z representing a feasible layout belongs to the metric polytope $\mathcal{M}$:

$$\mathcal{M} = \left\{ Z\colon \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} z_{ij} \\ z_{jk} \\ z_{ik} \end{pmatrix} \leq e,\ \ 1 \leq i < j < k \leq n(n-1)+1 \right\}, \tag{33}$$

which is defined through $\approx 4n^6$ facets.

In summary, we get the following tractable semidefinite relaxation of the DREFLP:

$$\min \left\{ K + \langle C_Z, Z\rangle \colon Z \in \mathcal{E} \cap \mathcal{M} \text{ satisfies } (26)\text{–}(32) \right\}. \tag{$\texttt{SDP}_{\texttt{full}}$}$$

All variables in Z with cost coefficient greater than zero appear in a 3-cycle equality (26) or in Equations (32) and thus are tightly constrained in the relaxation.

3.3 Extensions to the MREFLP

In this section we extend the models for double-row problems presented in Sections 3.1 and 3.2 to multi-row problems.

We again use Theorem 2 to reduce the MREFLP to a space-free version by introducing enough spacing departments. Let c be the minimum number of columns needed to preserve at least one of the original optimal solutions. Then our transformed problem has $n := cm$ departments, where $s = n - d$ are spaces.

3.3.1 Extension of the ILP formulation

We extend the ILP formulation for DREFLP proposed in Subsection 3.1 to the following formulation of the MREFLP:

$$\min \sum_{\substack{i,j\in[n],\\ i<j}} \tfrac{w_{ij}}{m} \cdot \left(\sum_{k\in[n]\setminus\{i,j\}} b_{ikj} + m(1-a_{ij}) \right) \tag{34}$$

s. t. (5)–(13)

$$a_{ij} + b_{ijk} + b_{ikj} + b_{jik} \leq 1, \qquad i,j,k \in [n],\ i<j<k, \tag{35}$$

$$a_{ik} + b_{ijk} + b_{ikj} + b_{jik} \leq 1, \qquad i,j,k \in [n],\ i<j<k, \tag{36}$$

$$a_{jk} + b_{ijk} + b_{ikj} + b_{jik} \leq 1, \qquad i,j,k \in [n],\ i<j<k, \tag{37}$$

$$a_{ij} + a_{jk} - a_{ik} \leq 1, \qquad i,j,k \in [n],\ i<k,\ i\neq j\neq k, \tag{38}$$

$$\sum_{j\in[n]\setminus\{i\}} a_{ij} = m-1, \qquad i \in [n], \tag{39}$$

$$\sum_{\substack{i,j,k\in[n],\\ i<k,\ j\neq k,\ j\neq i}} b_{ijk} = m^3\binom{c}{3}. \tag{40}$$

In the objective function (34), the distance between two departments $i, j \in [n], i \neq j$, in the same column is zero (all associated betweenness variables are zero). If i, j are not in the same row, the distance is at least one and we add the number of departments between i and j divided by m in the objective (so we derive the number of columns between i, j for that part).

Inequalities (35)–(37) ensure that three departments $i, j, k \in [n],\ i < j < k$, either lie next to each other or at least two of them are in the same column. Note that in the double-row case we used the strengthened version (3).

The inequalities (38) enforce the transitivity property that if departments i and j as well as j and k lie in the same column, then i and k also lie in the same column.

Equations (39) are the generalization of (4): each i lies in the same column as $m-1$ other departments (possibly including spaces).

Finally, we enforce exactly how many betweenness variables must equal 1 in a feasible solution. Let $c_1, c_2, c_3 \in \{1, \ldots, c\}$ be three different columns of a solution, then for each choice of one department from each of the three columns, we count 1 towards the left-hand side of (40).

3.3.2 Extension of the SDP formulation

The starting point for our semidefinite relaxation for the MREFLP is again a quadratic problem in ordering variables. We use the x-ordering variables and the 3-cycle inequalities (22) as well as (23). We change (24) to

$$\sum_{j\in[n]\setminus\{i\}} (x_{ij} + x_{ji}) = -2m + 2, \quad i \in [n], \tag{41}$$

and adjust the objective function (20) to

$$K^m + \sum_{\substack{i,j\in[n],\\ i<j}} \tfrac{w_{ij}}{4m} \sum_{k\in[n]\setminus\{i,j\}} (x_{ik}x_{kj} + x_{jk}x_{ki}) + \sum_{\substack{i,j\in[n],\\ i<j}} \tfrac{(m-1)w_{ij}}{2m}(x_{ij} + x_{ji}). \tag{42}$$

where $K^m = \frac{n}{2m}\sum_{i,j\in[n],i<j} w_{ij}$.

The following result for the MREFLP follows directly from Theorem 2.

Corollary 1 *Minimizing (42) over $x \in \{0,1\}^{n(n-1)}$ and (22), (23), (41) solves the MREFLP.*

In analogy to the double-row case, we can rewrite the MREFLP in matrix notation as

$$\min\,\{\langle C_X^m, X\rangle + c_x^m x + K^m \colon x \in \{-1,1\}^{n(n-1)} \text{ satisfies (22),(23) and (41)}\}, \tag{MREFLP}$$

where $X := xx^\top$ and the cost matrix C_X^m and cost vector c_x^m are deduced from (42). Rewriting the above formulation along the lines of the double-row case gives

$$\min\left\{K^m + \langle C_Z^m, Z\rangle \colon Z \text{ satisfies (26)},\ Z \in \mathcal{E},\ x \in \{-1,1\}^{n(n-1)} \text{ satisfies (23) and (41)}\right\}$$

where Z is given by (25), K^m is defined right after (42), and the cost matrix C_Z^m is given by

$$C_Z^m := \begin{pmatrix} 0 & \frac{1}{2}c_x^m \\ \frac{1}{2}c_x^m & C_X^m \end{pmatrix}.$$

The basic semidefinite relaxation of the MREFLP reads

$$\min\{K^m + \langle C_Z^m, Z\rangle \colon Z \text{ satisfies (26)},\ Z \in \mathcal{E},\ x \text{ satisfies (23) and (41)}\}. \tag{$\mathtt{SDP}^m_{\mathtt{basic}}$}$$

To strengthen this relaxation we use

$$\sum_{j\in[n]\setminus\{i\}}(x_{ij}x_{kl}+x_{ji}x_{kl})=(-2m+2)x_{kl},\qquad i,k,l\in[n],\ k\neq l, \tag{43}$$

which can be derived by multiplying (41) for fixed i with an x-variable $x_{kl},\ k,l\in[n],\ k\neq l$. Furthermore we use (31) instead of (23).

Finally, we add constraints to break the symmetry of the spaces S:

$$x_{ij}=1,\qquad i,j\in S,\ i+m\leq j, \tag{44}$$

$$x_{ij}=-1,\qquad i,j\in S,\ j<i, \tag{45}$$

$$x_{ij}x_{ki}-x_{ki}-x_{ij}=-1,\qquad i,j\in S,\ j=i+m-1,\ k\in[d], \tag{46}$$

$$x_{ij}x_{kj}-x_{kj}-x_{ij}=-1,\qquad i,j\in S,\ j=i+m-1,\ k\in[d], \tag{47}$$

$$-x_{i(i+j)}x_{i(i+k)}+x_{i(i+k)}-x_{i(i+j)}=-1,\qquad i\in S,\ j,k\in\mathbb{N},\ k<j<m,\ i+j\leq n. \tag{48}$$

The constraints (44) and (45) ensure that two spaces $i,j\in S,\ i<j$, can lie in the same column only if $i+m>j$. If two spaces $i,(i+m-1)\in S$ lie in the same column each of the original departments $k\in[d]$ lies left to them, see (46)–(47). Furthermore, if two spaces $i,(i+j)\in S$ lie in the same column, then all spaces $i+1,\ldots,i+j-1$ also lie in this column by (48). Additionally, we can use (27) and the triangle inequalities described in (33).

In summary we obtain the following tractable semidefinite relaxation of the `MREFLP`:

$$\min\left\{K^m+\langle C_Z^m,Z\rangle\colon Z\in\mathcal{E}\cap\mathcal{M}\text{ satisfies (26), (27), (31) and (43)–(48)}\right\}. \tag{$\texttt{SDP}^m_{\texttt{full}}$}$$

4 Computational implementation

We now describe our implementation of exact approaches based on our two new formulations. In Subsection 4.1 we discuss how we solve the linear and semidefinite optimization problems to obtain lower bounds on the optimal solution. In Subsection 4.2 we describe heuristics for the semidefinite approach that yield feasible layouts and hence upper bounds.

4.1 Computing the lower bounds

For the models from Chung and Tanchoco [18] (we use the variant proposed in [52]), Amaral [2] and [3] we included all the constraints directly and used CPLEX [37] as an ILP solver. For our new ILP model (2)–(16), tests with CPLEX showed that one should not add all inequalities at once, but should separate inequalities (5)–(11). We separate (5)–(11) exactly and dynamically in a branch-and-cut approach. We decided to not additionally separate (17), (18) because handling (5)–(11) is already computationally challenging. The same separation procedure was applied in the multi-row case.

For the SDP approach, we solve our new SDP relaxation using a spectral bundle method [28, 29] in conjunction with primal cutting plane generation [27]. In general the optimal objective value v of the relaxation is not integer but then $\lceil v\rceil$ is a global lower bound for the layout instance (if all weights are integral). We also use this property in the linear approaches. For the application of a spectral bundle method in the solution of the max-cut and bisection problems, see [12, 27].

One of the main advantages of the spectral bundle method is the ability to exploit the sparsity of the SDP relaxation [27]. In the objective function all the entries $x_{ij}x_{kl}$ with $|\{i,j,k,l\}|=4$ are zero, and the support of Equations (26)–(31) is also small. However (32) and the triangle inequalities of the metric polytope $\mathcal{M}$ have a larger support. To keep the small support consisting of the first row and column and the entries $x_{ij}x_{kl}$ with $i,j,k,l\in[n],i\neq j,k\neq l,|\{i,j,k,l\}|\leq 3$, we restrict to inequalities (32) with $i\in\{l,m\}$, i.e., we only multiply (24) for $i\in[n]$ fixed with $x_{lm},l,m\in[n],l\neq m$, if $i\in\{l,m\}$. Moreover we do not include the triangle inequalities and instead add the odd-cycle inequalities [13] (transformed to the $-1/1$-setting) on the small support of the objective function, where the coefficient matrix is interpreted as the adjacency matrix of a graph. In our tests we used a separator by C. Helmberg that is a variant of that by M. JÃijnger. Note that if we work with the full support (and thus on a complete graph) and exactly separate the triangle inequalities, then there is no need for an additional odd-cycle separator because all odd-cycles with length at least five are not chordless and are therefore implied by the triangle inequalities [13].

As mentioned before, for the DREFLP equalities (28) and (29) (respectively equalities (44) and (45) for the MREFLP) are used to reduce the size of the semidefinite relaxations. In our implementation we add all the equations of ($\mathtt{SDP_{full}}$) (respectively ($\mathtt{SDP^{m}_{full}}$)) right from the beginning (except the ones with large support mentioned above), and then iteratively include the odd-cycle inequalities. After 50 (null or descent) steps of the spectral bundle method [28] we determine violated odd-cycle inequalities and restrict the separation to adding at most 100 of the most-violated constraints. To speed up the implementation we also delete constraints if they are no longer important, see, e.g., [12].

4.2 Computing the upper bounds

CPLEX provides upper bounds while solving ILP formulations. We describe here how we derive feasible layouts using SDP primal information.

Let $\begin{pmatrix}1 & \tilde{x}\end{pmatrix}$ denote the first row of the SDP matrix Z. Hence $\tilde{x}$ gives the values of the x-variables in the relaxation. Given a partial solution consisting of k completely filled columns, $k \in \{0, \dots, \frac{n}{m}\}$ (we arrange departments and spaces simultaneously), we determine and position the next column in a greedy manner. First, we determine for each subset T of the remaining departments and spaces with $|T| = m$ the sum $\tau_T = \sum_{i,j \in T, i \neq j} \tilde{x}_{ij}$. A small value of τ_T indicates that the m elements of T should be arranged in the same column. (Note that if all the departments in T lie in the same column, then $\sum_{i,j \in T, i \neq j} \tilde{x}_{ij} = -m(m-1)$.) Hence we choose the smallest τ_T and place the departments in T in the same column, that we denote by C.

Finally we decide on the position of the new column using again the information encoded in $\tilde{x}$. More precisely let $N \subset [n]$ denote the set of all departments and spaces that have already been assigned and set $l = \frac{|N|}{m}$. The function $\alpha_{part} \colon N \to \left[\frac{|N|}{m}\right]$ gives an assignment of the elements of N to the $\frac{|N|}{m}$ columns. Now we calculate for the departments in C

$$\gamma_p = \sum_{\substack{i \in C,\ j \in N \\ \alpha_{part}(j) < p}} \tilde{x}_{ji} + \sum_{\substack{i \in C,\ j \in N \\ \alpha_{part}(j) \geq p}} \tilde{x}_{ij}$$

for all possible positions $p \in [l+1]$. Finally we determine $\hat{p} = \operatorname{argmax}_{p \in [l+1]} \gamma_p$, update α_{part} by

$$\alpha_{part}(i) \leftarrow \begin{cases} \hat{p}, & i \in C, \\ \alpha_{part}(i), & i \in N, \alpha_{part}(i) < \hat{p}, \\ \alpha_{part}(i) + 1, & i \in N, \hat{p} \leq \alpha_{part}(i), \end{cases}$$

and set $N \leftarrow N \cup C$.

When all departments and spaces have been arranged, we try to improve the layout using a 3-OPT heuristic (see e.g. [42]) that searches for advantageous exchanges of two or three departments in a greedy manner. We also test if the solution can be improved by reallocation of any column or by exchanging two or three columns. Apart from the presented heuristic, we use an adapted version of this in order to save running time. This determines the departments that lie in the same row in an alternative way. For each new column we start with the pair $\{i, j\} \subset [n] \setminus N$ of the currently unassigned departments that minimizes $\tilde{x}_{ij} + \tilde{x}_{ji}$ and set $D = \{i, j\}$. Next we iteratively add the department $k \in [n] \setminus (N \cup D)$ to D that minimizes the sum $\sum_{l \in D} (\tilde{x}_{kl} + \tilde{x}_{lk})$ until $|D| = m$. Then we set $N \leftarrow N \cup D$. If every department has been assigned to a column, we finally determine the order of the columns in the same way as above. For $m = 2$ the two heuristics are exactly the same. Additionally we use the first heuristic for $m = 3$ and the second computationally cheaper one for $m \in \{3, 4, 5\}$.

5 Computational experiments

We present computational results for double-row instances from the literature as well as instances originally studied for the SREFLP with at least 10 departments in order to highlight the practical impact of our theoretical results. All experiments were conducted on an INTEL-Core-I7-4770 (4x 3400MHz, 8 MB Cache) with 32 GB RAM in single processor mode using openSUSE Linux 42.1. We test instances with $d \geq 10$ used for the SREFLP in [35], instances proposed for the DREFLP by Amaral [3] (denoted by A-d-{edge probability·100}) where the pairwise weights w_{ij} are either zero or one because of the underlying graph problem, and instances constructed by Hungerländer and Anjos [33] (denoted by E-d-{edge probability·100}). The instances together with information on the source and the density are available at http://www.miguelanjos.com/flplib.

In the following we compare the computational times and final gaps calculated by the following five approaches:

- TAN: The MILP model from [18] (corrected according to [52]) that can be used directly for solving the MRFLP with $m \geq 3$.
- AMA: The MILP model of Amaral [2] that cannot be easily extended to the MRFLP with $m \geq 3$.
- AMA2: The MILP model of Amaral [3] that can be easily extended to the MRFLP with $m \geq 3$.
- ILP: The ILP models in Section 3.1 for the DRFLP and Section 3.3.1 for the MRFLP.
- SDP: The SDP relaxations in Section 3.2 for the DRFLP and in Section 3.3.2 for the MRFLP.

We tested these approaches on all available benchmark instances from the literature with $d \geq 10$ departments. We considered $m \in \{2, 3, 4, 5\}$ rows, except for AMA that is only applicable for $m = 2$.

We improve the models TAN, AMA and AMA2 from the literature by setting the big-M term or the number of possible positions of the departments according to Theorem 2.

We do not test against the approaches in [36] and [23] that are both based on enumerating over all different row assignments and so for larger d even deriving strong lower bounds is out of scope due to the large number of challenging subproblems to be (approximately) solved.

We calculate the percentage gap between the best upper bound found (regardless of which method(s) derived it) and each lower bound:

$$\left(\frac{\text{upper bound}}{\text{lower bound}} - 1\right) \cdot 100\%.$$

Note that except for a simple start construction heuristic for SDP, each method determines its own upper bound during the solution process.

First we consider the results of the two approaches not specifically tailored to DREFLP and MREFLP, TAN and AMA, for $m = 2$ in Tables 2 and 3, one can see that both approaches are usually slower than our new approaches. AMA is in most cases the faster of the two and if the time limit is reached, the gaps are smaller. With TAN 24 instances and with AMA 32 instances out of 61 instances could be solved to optimality within the time limit of one hour for $m = 2$. For $m \in \{3, 4, 5\}$ we also tested TAN. In total 23 instances for $m = 3$, 21 for $m = 4$, and 22 for $m = 5$ could be solved to optimality within the time limit. Usually, the running times for $m \in \{3, 4, 5\}$ are larger than for $m = 2$; only when the number of departments is small in comparison to the number of rows do the running times partially decrease. One explanation for this behavior might be that the big-M value is rather small then. Both approaches from the literature have in common, that for the medium-sized instances ($d = 13, 14, 15$) usually optimality cannot be proved within the time limit of one hour and the gaps are relatively large. For this reason we do not test them on the larger instances with $d \geq 16$. In general, it seems that sparsity of the objective function helps in both approaches to reduce the running times or to derive good bounds. In particular, this can be seen on the instances A-13-10, A-13-20, A-14-10 and A-14-20 that are the only instances with $d \geq 13$ that were solved to optimality by both methods.

Next we look at the results for AMA2. Tables 2 and 3 show that for sparse instances, this approach is sometimes the best approach if $m = 2$. For $m \in \{3, 4, 5\}$ it is very often the best approach for small instances, especially for the sparse ones. If the instances could not be solved within the time limit, the gaps are usually rather high in comparison to SDP, but better than the ones of TAN and AMA. The running times decrease with increasing m. The main reason for this seems to be our improvement of the model presented in [3] according to Theorem 2 that significantly reduces the number of potential positions, and hence the number of variables. All instances with $d \leq 15$ could be solved to optimality for $m = 5$ within one hour. Due to this good performance we also tested AMA2 on the medium-sized instances with a time limit of one and of three hours. Table 4 shows that only sparse instances can be solved to optimality. But usually the gaps are extremely large and SDP (as well as ILP for $m = 2$) behaves much better. This is not surprising since the model is closely related to formulations for the quadratic assignment problem [43]. Comparing the results for different values of m, AMA2 also works better for larger m on the medium-sized instances. But even for $m = 5$ the gaps are greater than 10% for 13 of 21 instances after three hours (for $m = 2$ the gap of 20 of the 21 instances is greater than 39% after three hours).

ILP is the best approach for small and medium-sized instances with up to 20 departments in the case $m = 2$. All instances with $d \leq 18$ and $m = 2$ could be solved to optimality in less than 5 minutes. For larger m, the solution times are much higher than for $m = 2$ and the obtained lower bounds are rather weak. Furthermore, CPLEX has difficulties proving optimality: sometimes, although the lower bound equals the optimal solution value (found by a different method), the calculation goes on because of the higher upper bound, see e.g. instance A-20-10 in Table 4. One explanation for this behavior could be that Equations (3) for $m = 2$ are rather strong in comparison to inequalities (35)–(37) for $m \geq 3$. Furthermore, ILP and SDP both seem to suffer from the fact that for larger d, the number of spaces (additional departments) needed grows with m, see Tables 4 and 5 for large d.

Looking at the results of SDP, one can see that the lower bounds for MREFLP are often very strong, while the quality of the bounds is even a bit better for the case $m = 2$. Although we do not combine the lower bound computation with a bounding scheme, we are able to prove optimality for 76 instances with $m = 2$ and for 60/58/65 instances with $m = 3/4/5$ within a time limit of one hour. The largest instances solved to optimality have 25 departments for $m = 2$ and at least 20 departments for $m \in \{3, 4, 5\}$, see Tables 4 and 5. For the instances that cannot be solved exactly within the time limit, we achieve gaps of less than 2% or an absolute difference of at most 1 for 102/103/90/95 instances out of 108 when $m = 2/3/4/5$. Table 5 shows as well that the gaps get worse as the number of departments increases. Comparing SDP to ILP for $m = 2$ shows that for small instances, ILP is faster and allows to solve all instances to optimality. But for instances with $d \geq 21$ the bounds with SDP are much stronger whereas the gaps with ILP are often greater than 10%.

For ILP and SDP and $d \geq 16$, we further increased the time limit to three hours for $m = 2$ for both approaches, and for $m \in \{3, 4, 5\}$ for SDP only, see Tables 4 and 5. The increased time limit mainly helps SDP as the gaps can be reduced significantly. For ILP the improvements are small, especially for instances with $d \geq 30$. Here it seems that the solver has some problems handling the large number of constraints (usually a large number of constraints are violated during the solution process).

We also looked at the upper bounds derived with the various approaches. In most cases our SDP construction heuristic is the best and provides rather high-quality solutions. These good upper bounds make it possible to stop our SDP relaxation approach because we are close enough. Indeed, in all approaches we used the fact that in the instances from the literature all weights are integer and so the optimal solution value is also integer. If our SDP heuristic fails to determine an optimal solution, then the solution process might continue although theoretically the gap is closed, see e.g. instance A-20-60 and $m = 4$.

In summary, we conclude that:

- for $d \leq 19$ and $m = 2$ ILP is the best choice,
- for $d = 20, m = 2$ there is no clear winner, and
- for $m = 2, d \geq 21$ and $m \in \{3, 4, 5\}, d \geq 15$ SDP is the best choice.

For larger m and small $d \leq 14$, AMA2 is a good alternative. Moreover the SDP approach is well-suited for providing good lower bounds for large MREFLP instances in a reasonable time. Finally the upper bounds derived from the SDP fractional solutions are of high-quality.

6 Conclusions and future work

We considered the special case of multi-row layout problems in which all departments have the same length. We showed that only spaces of unit length are required when modeling the problem, and we stated and proved exact expressions for the minimum number of spaces that need to be added so as to preserve at least one optimal solution. These results show that the MREFLP can be modeled using only binary variables, which has a significant computational impact.

Using the results on the structure of optimal solutions, we proposed ILP and SDP models for the DREFLP and the general MREFLP. Our results show that the SDP approach dominates for medium-sized and large instances. For the double-row case we increased the largest instances solved to optimality from 16 departments [23] to 25 departments. When considering 3 to 5 rows, we increased the largest instances solved to optimality from 8 departments [36] to 25 departments. Furthermore we achieved optimality gaps smaller than 1% for instances with up to 45 departments.

One direction for future research is the use of the SDP relaxation within a branch-and-bound scheme in order to solve larger instances to optimality. This is worth exploring given the very good lower bounds provided by the SDP approach for instances with up to 60 departments and up to 5 rows.

Our approaches to the MREFLP can also be used for the development of new exact solution methods for the MRFLP. For example one might apply our approaches to compute tight lower bounds of MRFLP instances if the lengths of the departments of the given instance do not vary much. In this case we suggest to reduce all department lengths to multiples of the minimum length over all departments and then to compute an optimal solution, or at least a lower bound, for this modified instance containing only departments of equal length.

Table 2: Computation times (in (m)m:ss) and gaps (in percent) for instances of Amaral [3] with 10 to 14 departments.

	m=2	TAN		AMA		AMA2		ILP		SDP		m=3	TAN		AMA2		ILP		SDP	
	best	gap	time	gap	time	gap	time	gap	time	gap	time	best	gap	time	gap	time	gap	time	gap	time
A-10-10	0	0.00	00:00	0.00	0:00	0.00	0:00	0	0:00	0	0:00	1	0.00	0:01	0.00	0:00	0.00	0:53	0	0:04
A-10-20	1	0.00	00:01	0.00	0:00	0.00	0:00	0	0:00	0	0:00	3	0.00	0:10	0.00	0:00	0.00	1:33	0	0:60
A-10-30	1	0.00	00:09	0.00	0:01	0.00	0:00	0	0:00	0	0:00	5	0.00	0:25	0.00	0:00	0.00	0:18	0	0:28
A-10-40	10	0.00	03:03	0.00	1:39	0.00	0:04	0	0:00	0	0:07	20	0.00	7:04	0.00	0:02	5.26	TL	0	0:44
A-10-50	9	0.00	03:52	0.00	0:53	0.00	0:04	0	0:00	0	0:01	19	0.00	5:22	0.00	0:01	0.00	45:04	0	1:06
A-10-60	8	0.00	01:39	0.00	1:02	0.00	0:03	0	0:01	0	1:47	15	0.00	1:38	0.00	0:00	0.00	4:25	0	0:13
A-10-70	16	0.00	13:17	0.00	4:42	0.00	0:20	0	0:00	0	0:01	33	0.00	18:43	0.00	0:06	10.00	TL	0	0:37
A-10-80	21	0.00	40:50	0.00	0:44	0.00	0:45	0	0:00	0	0:02	44	7.32	TL	0.00	0:11	12.82	TL	0	2:48
A-10-90	21	0.00	44:26	0.00	0:40	0.00	0:48	0	0:00	0	0:02	44	4.76	TL	0.00	0:12	7.32	TL	0	2:07
A-11-10	0	0.00	00:00	0.00	0:00	0.00	0:00	0	0:01	0	0:01	0	0.00	0:00	0.00	0:00	0.00	1:27	0	0:04
A-11-20	5	0.00	05:12	0.00	2:10	0.00	0:03	0	0:06	0	0:14	11	0.00	8:40	0.00	0:01	0.00	TL	0	0:20
A-11-30	8	0.00	04:47	0.00	2:01	0.00	0:08	0	0:01	0	0:18	16	0.00	8:31	0.00	0:01	0.00	2:31	0	0:12
A-11-40	11	0.00	12:34	0.00	5:37	0.00	0:12	0	0:01	0	0:13	20	0.00	22:08	0.00	0:02	5.26	TL	0	4:53
A-11-50	18	24.39	TL	0.00	32:00	0.00	1:17	0	0:02	0	0:22	34	9.68	TL	0.00	0:06	9.68	TL	0	2:09
A-11-60	14	00.00	06:43	0.00	4:11	0.00	0:14	0	0:00	0	0:03	24	0.00	7:25	0.00	0:01	0.00	57:51	0	0:05
A-11-70	19	22.73	TL	0.00	24:54	0.00	1:34	0	0:02	0	0:56	35	9.38	TL	0.00	0:07	2.94	TL	0	0:46
A-11-80	27	21.31	TL	0.00	41:04	0.00	6:09	0	0:00	0	0:29	49	40.00	TL	0.00	0:16	8.89	TL	0	1:50
A-11-90	37	71.19	TL	0.00	6:47	0.00	58:32	0	0:06	0	4:01	66	78.38	TL	0.00	3:12	15.79	TL	0	2:20
A-12-10	0	00.00	00:00	0.00	0:00	0.00	0:00	0	0:00	0	0:01	1	0.00	0:01	0.00	0:00	0.00	5:03	0	0:42
A-12-20	4	00.00	01:19	0.00	0:28	0.00	0:00	0	0:01	0	0:02	7	0.00	2:29	0.00	0:00	0.00	58:21	0	0:40
A-12-30	4	00.00	05:33	0.00	1:09	0.00	0:07	0	0:01	0	0:13	8	0.00	2:26	0.00	0:01	0.00	25:00	0	0:41
A-12-40	15	19.35	TL	0.00	39:14	0.00	1:12	0	0:02	0	0:06	24	4.35	TL	0.00	0:12	9.09	TL	0	0:60
A-12-50	17	07.50	TL	0.00	24:37	0.00	0:37	0	0:01	0	0:16	27	0.00	53:05	0.00	0:08	3.85	TL	0	0:20
A-12-60	21	29.27	TL	8.16	TL	0.00	2:20	0	0:01	0	0:28	33	13.79	TL	0.00	0:13	3.12	TL	0	0:14
A-12-70	30	79.07	TL	35.09	TL	0.00	22:32	0	0:01	0	0:35	49	63.33	TL	0.00	1:48	11.36	TL	0	0:16
A-12-80	40	82.14	TL	64.52	TL	0.00	40:31	0	0:01	0	0:07	65	91.18	TL	0.00	3:46	22.64	TL	0	0:07
A-12-90	42	103.77	TL	0.00	55:11	0.00	30:04	0	0:01	0	0:14	70	89.19	TL	0.00	5:45	20.69	TL	0	0:32
A-13-10	1	00.00	00:01	0.00	0:01	0.00	0:00	0	0:02	0	0:02	1	0.00	0:01	0.00	0:00	0.00	1:13	0	0:21
A-13-20	9	00.00	15:22	0.00	22:05	0.00	1:01	0	0:18	0	1:57	15	0.00	23:54	0.00	0:03	7.14	TL	0	1:05
A-13-30	14	31.03	TL	5.56	TL	0.00	1:15	0	0:13	0	0:32	25	25.00	TL	0.00	0:09	8.70	TL	0	1:19
A-13-40	16	55.56	TL	27.27	TL	0.00	10:49	0	0:07	0	0:44	27	50.00	TL	0.00	0:35	0.00	51:29	0	1:16
A-13-50	26	106.06	TL	51.11	TL	0.00	27:36	0	0:24	0	2:12	44	76.00	TL	0.00	1:14	10.00	TL	0	2:07
A-13-60	26	94.44	TL	62.79	TL	0.00	16:46	0	0:02	0	0:57	46	91.67	TL	0.00	1:20	6.98	TL	0	2:16
A-13-70	41	144.19	TL	114.29	TL	17.98	TL	0	0:13	0	2:06	69	137.93	TL	0.00	12:36	15.00	TL	0	6:07
A-13-80	53	165.38	TL	72.50	TL	43.75	TL	0	0:02	0	2:06	90	210.34	TL	0.00	42:58	16.88	TL	0	2:07
A-13-90	58	173.21	TL	45.71	TL	54.55	TL	0	0:03	0	5:13	101	225.81	TL	10.99	TL	18.82	TL	0	20:35
A-14-10	1	0.00	00:07	0.00	00:08	0.00	0:00	0	0:17	0	0:11	3	0.00	0:07	0.00	0:00	0.00	TL	0	3:07
A-14-20	8	0.00	58:11	0.00	39:32	0.00	0:46	0	1:23	0	3:11	16	14.29	TL	0.00	0:09	14.29	TL	0	32:04
A-14-30	13	100.00	TL	63.64	TL	0.00	41:39	0	0:40	0	1:60	24	84.62	TL	0.00	1:16	14.29	TL	0	11:51
A-14-40	16	86.96	TL	38.71	TL	0.00	28:35	0	0:12	0	1:31	28	86.67	TL	0.00	1:46	7.69	TL	0	3:43
A-14-50	35	193.75	TL	147.37	TL	30.56	TL	0	0:39	0	2:26	63	231.58	TL	0.00	56:53	21.15	TL	0	39:53
A-14-60	37	160.53	TL	153.85	TL	28.57	TL	0	0:06	0	0:43	65	182.61	TL	0.00	20:31	10.17	TL	0	1:34
A-14-70	52	228.57	TL	187.50	TL	72.50	TL	0	4:42	0	4:27	92	283.33	TL	22.67	TL	19.48	TL	0	18:36
A-14-80	64	209.26	TL	169.35	TL	62.14	TL	0	2:53	0	2:01	111	362.50	TL	21.98	TL	23.33	TL	0	6:11
A-14-90	71	297.87	TL	128.05	TL	136.71	TL	0	1:18	0	1:46	125	420.83	TL	48.81	TL	20.19	TL	0	28:19

	m=4	TAN		AMA2		ILP		SDP		m=5	TAN		AMA2		ILP		SDP	
	best	gap	time	gap	time	gap	time	gap	time	best	gap	time	gap	time	gap	time	gap	time
A-10-10	0	0.00	0:00	0.00	0:00	0.00	1:17	0.00	0:02	0	0.00	0:00	0	0:00	0.00	0:42	0	0:02
A-10-20	1	0.00	0:00	0.00	0:00	0.00	0:02	0.00	0:03	1	0.00	0:01	0	0:00	0.00	0:14	0	0:03
A-10-30	3	0.00	0:32	0.00	0:00	0.00	0:52	0.00	0:10	1	0.00	0:00	0	0:00	0.00	0:00	0	0:01
A-10-40	13	0.00	5:00	0.00	0:00	0.00	3:02	0.00	0:03	10	0.00	7:35	0	0:00	0.00	0:18	0	0:06
A-10-50	13	0.00	3:38	0.00	0:00	0.00	1:19	0.00	0:14	9	0.00	7:46	0	0:00	0.00	0:07	0	0:05
A-10-60	11	0.00	3:51	0.00	0:00	0.00	7:37	0.00	0:21	8	0.00	3:10	0	0:00	0.00	0:00	0	0:06
A-10-70	24	0.00	31:40	0.00	0:01	4.35	TL	0.00	0:27	16	0.00	14:32	0	0:00	0.00	0:10	0	0:04
A-10-80	32	14.29	TL	0.00	0:03	10.34	TL	0.00	1:18	21	0.00	27:49	0	0:00	0.00	0:00	0	0:02
A-10-90	32	10.34	TL	0.00	0:03	10.34	TL	0.00	0:54	21	0.00	29:03	0	0:00	0.00	0:00	0	0:02
A-11-10	0	0.00	0:00	0.00	0:00	0.00	8:34	0.00	0:04	0	0.00	0:00	0	0:00	0.00	1:10	0	0:02
A-11-20	8	0.00	12:19	0.00	0:00	0.00	39:12	0.00	0:38	5	0.00	5:10	0	0:00	0.00	0:01	0	0:02
A-11-30	12	0.00	22:09	0.00	0:00	0.00	27:35	0.00	0:21	8	0.00	20:10	0	0:00	0.00	0:03	0	0:03
A-11-40	14	0.00	40:21	0.00	0:00	0.00	48:12	0.00	0:21	11	10.00	TL	0	0:00	0.00	1:39	0	0:39
A-11-50	24	20.00	TL	0.00	0:01	9.09	TL	0.00	0:36	18	28.57	TL	0	0:00	0.00	9:11	0	1:37
A-11-60	18	0.00	23:36	0.00	0:00	0.00	4:31	0.00	0:17	14	0.00	49:40	0	0:00	0.00	0:38	0	0:22
A-11-70	26	30.00	TL	0.00	0:01	13.04	TL	0.00	2:03	19	26.67	TL	0	0:00	0.00	3:46	0	0:30
A-11-80	36	50.00	TL	0.00	0:04	09.09	TL	0.00	1:50	27	58.82	TL	0	0:01	0.00	2:60	0	1:32
A-11-90	48	84.62	TL	0.00	0:27	20.00	TL	0.00	2:15	37	105.56	TL	0	0:03	8.82	TL	0	8:22
A-12-10	0	0.00	0:00	0.00	0:00	0.00	0:21	0.00	0:03	0	0.00	0:00	0	0:00	0.00	20:16	0	0:12
A-12-20	5	0.00	5:59	0.00	0:00	0.00	0:26	0.00	0:13	4	0.00	33:39	0	0:00	0.00	8:06	0	0:56
A-12-30	5	0.00	1:08	0.00	0:00	0.00	4:30	0.00	0:12	4	0.00	8:38	0	0:00	0.00	TL	0	0:37
A-12-40	16	0.00	33:14	0.00	0:01	0.00	1:36	0.00	0:04	15	66.67	TL	0	0:01	15.38	TL	0	11:58
A-12-50	20	25.00	TL	0.00	0:01	0.00	TL	0.00	0:39	17	54.55	TL	0	0:01	13.33	TL	0	4:12
A-12-60	24	26.32	TL	0.00	0:02	0.00	10:27	0.00	0:18	21	75.00	TL	0	0:01	16.67	TL	0	10:14
A-12-70	34	70.00	TL	0.00	0:05	0.00	5:48	0.00	0:01	30	130.77	TL	0	0:04	15.38	TL	0	9:19
A-12-80	47	88.00	TL	0.00	0:17	9.30	TL	0.00	0:07	40	150.00	TL	0	0:11	14.29	TL	0	8:55
A-12-90	50	100.00	TL	0.00	0:12	11.11	TL	0.00	0:10	42	147.06	TL	0	0:09	13.51	TL	0	10:25
A-13-10	1	0.00	0:01	0.00	0:00	0.00	29:18	0.00	0:36	1	00.00	0:04	0	0:00	0.00	21:27	0	0:44
A-13-20	11	0.00	58:29	0.00	0:01	10.00	TL	0.00	2:06	9	28.57	TL	0	0:00	12.50	TL	0	5:44
A-13-30	19	46.15	TL	0.00	0:03	18.75	TL	0.00	8:21	14	55.56	TL	0	0:00	7.69	TL	0	0:42
A-13-40	20	66.67	TL	0.00	0:06	11.11	TL	0.00	2:45	16	100.00	TL	0	0:01	14.29	TL	0	2:40
A-13-50	32	88.24	TL	0.00	0:14	14.29	TL	0.00	2:37	26	116.67	TL	0	0:04	23.81	TL	0	4:22
A-13-60	33	94.12	TL	0.00	0:12	10.00	TL	0.00	2:33	26	116.67	TL	0	0:03	13.04	TL	0	3:13
A-13-70	50	177.78	TL	0.00	1:13	19.05	TL	0.00	6:01	41	215.38	TL	0	0:18	20.59	TL	0	33:39
A-13-80	66	153.85	TL	0.00	4:40	20.00	TL	0.00	4:60	53	194.44	TL	0	0:43	15.22	TL	0	6:20
A-13-90	74	236.36	TL	0.00	8:20	27.59	TL	0.00	22:32	58	205.26	TL	0	0:55	16.00	TL	0	4:40
A-14-10	1	0.00	0:02	0.00	0:00	0.00	TL	0.00	1:32	1	0.00	0:14	0	0:00	0.00	20:57	0	0:17
A-14-20	11	22.22	TL	0.00	0:03	10.00	TL	0.00	8:35	8	33.33	TL	0	0:00	0.00	47:49	0	0:40
A-14-30	18	100.00	TL	0.00	0:20	38.46	TL	5.88	TL	13	85.71	TL	0	0:01	8.33	TL	0	1:41
A-14-40	21	110.00	TL	0.00	0:31	31.25	TL	0.00	24:54	16	100.00	TL	0	0:01	14.29	TL	0	5:25
A-14-50	47	235.71	TL	0.00	14:16	38.24	TL	2.17	TL	35	191.67	TL	0	0:09	20.69	TL	0	6:17
A-14-60	49	250.00	TL	0.00	6:23	32.43	TL	0.00	20:41	37	184.62	TL	0	0:08	8.82	TL	0	1:50
A-14-70	68	325.00	TL	0.00	45:13	30.77	TL	0.00	32:37	52	271.43	TL	0	0:39	13.04	TL	0	3:32
A-14-80	83	361.11	TL	12.16	TL	36.07	TL	1.22	TL	64	433.33	TL	0	1:13	16.36	TL	0	5:19
A-14-90	93	342.86	TL	20.78	TL	38.81	TL	1.09	TL	71	446.15	TL	0	5:01	18.33	TL	0	6:37

*A time "TL" indicates that an optimal solution could not be determined within the time limit of one hour.

Table 3: Computation times (in (m)m:ss) and gaps (in percent) for instances with 10 to 14 departments.

$m = 2$

		TAN		AMA		AMA2		ILP		SDP	
	best	gap	time	gap	time	gap	time	gap	time	gap	time
E-10-100	427	0.00	22:07	0.00	10:57	0.00	1:43	0	0:01	0	0:59
E-10-50	191	0.00	4:24	0.00	1:06	0.00	0:06	0	0:02	0	3:35
E-11-100	539	35.77	TL	11.36	TL	0.00	14:22	0	0:01	0	1:48
N-15	1064	247.71	TL	166.00	TL	61.46	TL	0	0:04	0	4:02
O-10	670	0.00	12:17	0.00	6:20	0.00	0:26	0	0:00	0	0:09
O-15	2556	266.19	TL	256.98	TL	158.44	TL	0	0:04	0	4:55
S-12	2167	116.27	TL	92.28	TL	17.90	TL	0	0:00	0	0:54
S-13	2940	177.88	TL	127.20	TL	79.71	TL	0	0:15	0	7:48
S-14	3608	221.00	TL	217.33	TL	124.66	TL	0	0:60	0	33:55
S-15	4466	289.70	TL	304.90	TL	180.00	TL	0	0:24	0	50:22
Y-10	1697	3.60	TL	0.00	21:02	0.00	1:54	0	0:00	0	0:21
Y-11	2008	53.99	TL	34.31	TL	0.00	52:24	0	0:00	0	2:10
Y-12	2342	124.54	TL	82.68	TL	18.76	TL	0	0:00	0	0:44
Y-13	2730	177.16	TL	152.31	TL	55.64	TL	0	0:01	0	11:13
Y-14	3164	229.93	TL	180.50	TL	158.50	TL	0	0:06	0	7:18
Y-15	3676	296.55	TL	278.58	TL	180.40	TL	0	0:05	0	51:48

$m = 3$

		TAN		AMA2		ILP		SDP	
	best	gap	time	gap	time	gap	time	gap	time
E-10-100	277	0.00	27:34	0.00	0:16	12.60	TL	0.00	0:57
E-10-50	114	0.00	01:35	0.00	0:01	0.00	12:50	0.00	0:15
E-11-100	351	51.29	TL	0.00	0:49	10.73	TL	0.00	4:04
N-15	668	307.32	TL	0.00	33:08	6.37	TL	0.00	7:11
O-10	450	0.00	25:26	0.00	0:10	12.22	TL	0.00	0:59
O-15	1660	492.86	TL	39.73	TL	12.93	TL	0.00	28:25
S-12	1404	113.05	TL	0.00	21:23	16.32	TL	0.07	49:53
S-13	1938	224.08	TL	20.22	TL	18.68	TL	0.16	TL
S-14	2408	357.79	TL	42.99	TL	21.43	TL	0.08	TL
S-15	2883	513.40	TL	90.80	TL	14.31	TL	0.00	21:44
Y-10	1140	9.83	TL	0.00	0:33	19.25	TL	0.00	6:30
Y-11	1314	71.09	TL	0.00	2:33	8.60	TL	0.00	4:36
Y-12	1510	128.44	TL	0.00	18:34	13.79	TL	0.00	2:59
Y-13	1798	226.32	TL	19.87	TL	17.98	TL	0.00	26:48
Y-14	2110	432.83	TL	46.32	TL	20.99	TL	0.00	42:54
Y-15	2357	526.86	TL	73.31	TL	12.45	TL	0.00	3:41

$m = 4$

		TAN		AMA2		ILP		SDP	
	best	gap	time	gap	time	gap	time	gap	time
E-10-100	209	8.85	TL	0.00	0:04	13.59	TL	0.00	7:52
E-10-50	89	0.00	4:36	0.00	0:00	7.23	TL	0.00	1:46
E-11-100	256	56.10	TL	0.00	0:08	15.84	TL	0.00	22:13
N-15	500	267.65	TL	0.00	10:59	30.21	TL	0.81	TL
O-10	334	0.00	52:05	0.00	0:02	6.71	TL	0.00	1:24
O-15	1250	392.13	TL	17.92	TL	35.14	TL	0.73	TL
S-12	995	108.16	TL	0.00	0:45	12.56	TL	0.00	4:08
S-13	1413	242.96	TL	0.00	14:33	25.27	TL	0.00	19:19
S-14	1794	340.79	TL	24.24	TL	45.03	TL	1.36	TL
S-15	2180	393.21	TL	41.94	TL	46.80	TL	1.02	TL
Y-10	845	26.31	TL	0.00	0:07	19.52	TL	0.00	10:42
Y-11	947	71.87	TL	0.00	0:17	18.37	TL	0.00	3:04
Y-12	1070	115.29	TL	0.00	0:35	12.39	TL	0.00	0:41
Y-13	1314	265.00	TL	0.00	15:41	24.31	TL	0.00	21:59
Y-14	1574	333.61	TL	29.23	TL	46.55	TL	1.35	TL
Y-15	1782	395.00	TL	50.51	TL	47.15	TL	0.96	TL

$m = 5$

		TAN		AMA2		ILP		SDP	
	best	gap	time	gap	time	gap	time	gap	time
E-10-100	133	0.00	16:49	0	0:00	0.00	0:00	0.00	0:11
E-10-50	59	0.00	02:59	0	0:00	0.00	0:00	0.00	0:14
E-11-100	191	076.85	TL	0	0:01	0.00	7:54	0.00	5:00
N-15	382	274.51	TL	0	1:01	18.27	TL	0.79	TL
O-10	222	0.00	25:53	0	0:00	0.00	0:01	0.00	0:54
O-15	914	603.08	TL	0	6:36	17.03	TL	0.00	5:41
S-12	841	148.82	TL	0	0:29	18.79	TL	0.00	16:01
S-13	1132	208.45	TL	0	1:52	21.98	TL	0.00	50:06
S-14	1369	508.44	TL	0	5:31	20.19	TL	0.00	18:33
S-15	1612	588.89	TL	0	28:26	21.11	TL	0.31	TL
Y-10	530	0.00	32:35	0	0:00	0.00	0:00	0.00	0:36
Y-11	724	95.15	TL	0	0:02	0.00	26:56	0.00	8:24
Y-12	908	157.95	TL	0	0:35	18.85	TL	0.00	15:43
Y-13	1048	217.58	TL	0	1:46	22.43	TL	0.00	29:40
Y-14	1201	469.19	TL	0	7:04	21.44	TL	0.00	58:23
Y-15	1322	567.68	TL	0	54:42	22.18	TL	0.30	TL

*A time "TL" indicates that an optimal solution could not be determined within the time limit of one hour.

Table 4: Computation times (in h:mm:ss or (m)m:ss) and gaps (in percent) for medium-sized instances solved with SDP and AMA2 as well as with ILP in the case $m = 2$.

$m = 2$

		AMA2 1h		AMA2 3h		ILP 1h		ILP 3h		SDP 1h		SDP 3h	
instance	best ub	gap	time	gap	time	gap	time	gap	time	gap	time	gap	time
A-20-10	12	0.00	4:29			0.00	TL	0	TL	9.09	TL	0.00	2:28:32
A-20-20	73	170.37	TL	108.57	TL	0.00	TL	0	1:25:41	0.00	17:53		
A-20-30	111	258.06	TL	164.29	TL	0.00	26:39			0.00	10:00		
A-20-40	149	210.42	TL	148.33	TL	0.00	20:50			0.00	19:36		
A-20-50	249	591.67	TL	352.73	TL	0.00	57:37			0.00	28:21		
A-20-60	345	618.75	TL	618.75	TL	0.00	21:46			0.00	20:30		
A-20-70	385	474.63	TL	352.94	TL	0.00	13:12			0.00	9:17		
A-20-80	434	600.00	TL	600.00	TL	0.00	38:29			0.00	15:28		
A-20-90	521	740.32	TL	689.39	TL	0.00	TL	0	2:38:36	0.19	47:03		
N-16a	1496	119.68	TL	89.37	TL	0.00	0:18			0.00	4:47		
N-16b	1168	63.59	TL	39.71	TL	0.00	0:17			0.00	2:45		
N-17	1678	250.31	TL	180.60	TL	0.00	0:59			0.00	12:45		
N-18	1970	200.30	TL	149.05	TL	0.00	1:55			0.00	25:31		
N-20	2782	990.98	TL	522.37	TL	0.00	5:20			0.00	17:43		
O-20	6414	1055.68	TL	728.68	TL	0.00	9:41			0.00	30:34		
S-16	5446	219.04	TL	154.13	TL	0.00	0:20			0.00	8:28		
S-17	6577	412.23	TL	277.34	TL	0.00	1:18			0.03	TL	0.00	1:31:08
S-18	7788	477.74	TL	411.02	TL	0.00	3:53			0.08	TL	0.01	TL
S-19	9343	643.87	TL	488.72	TL	0.00	31:18			0.39	TL	0.34	TL
S-20	10841	930.51	TL	562.25	TL	0.01	TL	0	2:00:20	0.06	40:00		
Y-20	6046	1200.22	TL	581.62	TL	0.00	20:04			0.05	TL	0.00	1:20:46

$m = 3$

		AMA2 1h		AMA2 3h		SDP 1h		SDP 3h	
instance	best ub	gap	time	gap	time	gap	time	gap	time
A-20-10	7	0.00	0:29			16.67	TL	16.67	TL
A-20-20	49	44.12	TL	28.95	TL	2.08	TL	2.08	TL
A-20-30	74	54.17	TL	39.62	TL	1.37	TL	0.00	1:21:36
A-20-40	98	122.73	TL	92.16	TL	0.00	42:15		
A-20-50	166	127.40	TL	102.44	TL	0.61	TL	0.61	TL
A-20-60	229	252.31	TL	172.62	TL	0.44	TL	0.00	1:29:48
A-20-70	258	309.52	TL	226.58	TL	1.18	TL	0.39	TL
A-20-80	290	308.45	TL	215.22	TL	1.05	TL	0.69	TL
A-20-90	347	298.85	TL	230.48	TL	0.58	TL	0.00	1:22:25
N-16a	1002	48.01	TL	29.12	TL	0.80	TL	0.00	1:10:20
N-16b	792	14.78	TL	0.00	2:05:12	1.15	TL	0.76	TL
N-17	1114	66.77	TL	41.73	TL	0.00	58:58		
N-18	1292	125.09	TL	92.84	TL	0.31	TL	0.00	1:34:43
N-20	1856	235.02	TL	159.58	TL	1.31	TL	0.49	TL
O-20	4284	537.50	TL	335.81	TL	0.59	TL	0.45	TL
S-16	3638	134.86	TL	105.65	TL	0.89	TL	0.30	TL
S-17	4354	202.57	TL	148.66	TL	0.23	TL	0.02	TL
S-18	5110	307.17	TL	226.94	TL	0.10	TL	0.00	1:41:13
S-19	6190	431.79	TL	288.82	TL	0.98	TL	0.52	TL
S-20	7227	542.40	TL	392.64	TL	0.85	TL	0.54	TL
Y-20	4033	540.16	TL	399.13	TL	0.55	TL	0.30	TL

$m = 4$

		AMA2 1h		AMA2 3h		SDP 1h		SDP 3h	
instance	best ub	gap	time	gap	time	gap	time	gap	time
A-20-10	4	0.00	0:02			33.33	TL	0.00	1:25:38
A-20-20	34	0.00	13:31			0.00	43:35		
A-20-30	54	22.73	TL	8.00	TL	1.89	TL	0.00	2:24:31
A-20-40	73	52.08	TL	37.74	TL	1.39	TL	1.39	TL
A-20-50	122	90.62	TL	64.86	TL	0.83	TL	0.83	TL
A-20-60	167	131.94	TL	96.47	TL	0.00	TL	0.00	1:01:46
A-20-70	187	139.74	TL	101.08	TL	0.54	TL	0.00	1:28:58
A-20-80	211	197.18	TL	134.44	TL	0.48	TL	0.00	1:24:21
A-20-90	252	300.00	TL	215.00	TL	0.00	22:02		
N-16a	706	0.00	42:20			0.00	6:31		
N-16b	570	0.00	11:27			0.71	TL	0.00	1:50:17
N-17	808	23.36	TL	0.00	2:43:27	0.00	14:03		
N-18	972	41.28	TL	23.19	TL	0.00	34:11		
N-20	1360	115.19	TL	81.33	TL	1.42	TL	0.97	TL
O-20	3118	320.22	TL	224.45	TL	0.42	TL	0.32	TL
S-16	2600	54.30	TL	38.52	TL	0.00	26:32		
S-17	3225	108.20	TL	82.10	TL	1.07	TL	0.84	TL
S-18	3892	175.25	TL	131.39	TL	1.35	TL	0.36	TL
S-19	4599	285.50	TL	194.43	TL	1.66	TL	0.79	TL
S-20	5260	308.07	TL	215.54	TL	0.32	TL	0.17	TL
Y-20	2934	260.89	TL	179.16	TL	0.24	TL	0.17	TL

$m = 5$

		AMA2 1h		AMA2 3h		SDP 1h		SDP 3h	
instance	best ub	gap	time	gap	time	gap	time	gap	time
A-20-10	3	0.00	0:00			0.00	13:11		
A-20-20	27	0.00	1:56			0.00	42:55		
A-20-30	42	0.00	4:51			2.44	TL	0.00	1:03:11
A-20-40	58	7.41	TL	0.00	1:21:14	1.75	TL	1.75	TL
A-20-50	96	33.33	TL	18.52	TL	1.05	TL	1.05	TL
A-20-60	132	53.49	TL	40.43	TL	0.76	TL	0.76	TL
A-20-70	146	65.91	TL	46.00	TL	0.00	34:26		
A-20-80	165	81.32	TL	55.66	TL	0.61	TL	0.00	1:21:09
A-20-90	197	105.21	TL	79.09	TL	0.00	8:13		
N-16a	584	0.00	7:21			1.57	TL	0.00	2:02:19
N-16b	462	0.00	1:41			1.76	TL	1.09	TL
N-17	662	0.00	17:52			1.69	TL	0.30	TL
N-18	772	11.08	TL	0.00	1:26:27	1.45	TL	0.00	1:47:02
N-20	1068	37.98	TL	23.61	TL	1.33	TL	0.47	TL
O-20	2444	100.66	TL	75.07	TL	0.70	TL	0.53	TL
S-16	2094	24.13	TL	12.10	TL	0.48	TL	0.00	1:24:53
S-17	2577	46.25	TL	28.02	TL	0.98	TL	0.27	TL
S-18	3083	76.88	TL	53.77	TL	2.12	TL	0.29	TL
S-19	3612	111.23	TL	82.89	TL	1.06	TL	0.56	TL
S-20	4105	118.93	TL	91.73	TL	0.00	34:22		
Y-20	2282	117.54	TL	88.28	TL	0.00	9:31		

*A time "TL" indicates that an optimal solution could not be determined within the time limit of one or three hours, respectively.

Table 5: Computation times (in h:mm:ss or mm:ss) and gaps (in percent) for large instances solved with SDP and with ILP for $m = 2$.

	$m=2$									$m=3$					$m=4$					$m=5$				
		ILP 1h		ILP 3h		SDP 1h		SDP 3h			SDP 1h		SDP 3h			SDP 1h		SDP 3h			SDP 1h		SDP 3h	
instance	best ub	gap	time	gap	time	gap	time	gap	time	best ub	gap	time	gap	time	best ub	gap	time	gap	time	best ub	gap	time	gap	time
A-25-10	41	20.59	TL	0.00	TL	0.00	34:36			27	3.85	TL	3.85	TL	20	17.65	TL	11.11	TL	15	7.14	TL	0.00	1:42:25
A-25-20	110	64.18	TL	34.15	TL	0.92	TL	0.00	1:29:15	75	7.14	TL	4.17	TL	55	10.00	TL	5.77	TL	40	2.56	TL	0.00	1:06:29
A-25-30	222	55.24	TL	21.31	TL	0.45	TL	0.00	1:24:44	146	0.69	TL	0.00	1:42:06	110	3.77	TL	1.85	TL	87	3.57	TL	1.16	TL
A-25-40	400	34.23	TL	22.70	TL	0.50	TL	0.25	TL	265	1.15	TL	0.76	TL	198	3.66	TL	2.06	TL	156	3.31	TL	2.63	TL
A-25-50	511	22.84	TL	14.32	TL	0.39	TL	0.39	TL	340	1.49	TL	0.89	TL	254	3.25	TL	2.42	TL	196	1.55	TL	0.51	TL
A-25-60	549	19.87	TL	12.50	TL	0.73	TL	0.37	TL	364	1.39	TL	0.83	TL	271	2.65	TL	1.88	TL	212	1.92	TL	1.44	TL
A-25-70	660	59.04	TL	5.60	TL	0.46	TL	0.30	TL	441	1.85	TL	1.38	TL	325	1.88	TL	1.25	TL	255	1.59	TL	1.19	TL
A-25-80	910	1.68	TL	0.78	TL	0.44	TL	0.33	TL	604	0.83	TL	0.50	TL	450	1.58	TL	1.35	TL	350	0.29	TL	0.00	2:45:03
A-25-90	1084	0.65	TL	0.00	TL	0.37	TL	0.28	TL	721	0.98	TL	0.70	TL	537	1.51	TL	1.13	TL	417	0.00	58:58		
N-21	2512	0.00	51:32			0.08	TL	0.00	1:18:51	1664	1.59	TL	1.16	TL	1248	2.89	TL	1.79	TL	972	1.78	TL	0.62	TL
N-22	3064	0.23	TL	0.16	TL	0.56	TL	0.20	TL	2034	1.60	TL	0.35	TL	1530	2.89	TL	2.00	TL	1188	1.19	TL	0.00	2:03:13
N-24	4120	3.75	TL	0.59	TL	0.81	TL	0.54	TL	2712	0.97	TL	0.59	TL	2022	2.59	TL	1.66	TL	1624	3.24	TL	2.07	TL
N-25	4604	10.43	TL	1.10	TL	0.70	TL	0.00	2:03:36	3062	1.16	TL	0.69	TL	2286	2.93	TL	1.83	TL	1796	2.16	TL	1.35	TL
N-30	8230	21.60	TL	21.60	TL	0.48	TL	0.23	TL	5442	1.34	TL	0.68	TL	4086	3.36	TL	1.95	TL	3232	3.06	TL	1.99	TL
S-21	12431	0.00	26:51			0.44	TL	0.24	TL	8144	0.11	TL	0.00	2:52:34	6136	1.86	TL	1.44	TL	4849	2.43	TL	1.83	TL
S-22	14208	0.04	TL	0.04	TL	0.04	TL	0.02	2:09:55	9484	1.18	TL	0.85	TL	7082	1.72	TL	0.88	TL	5623	2.03	TL	0.92	TL
S-23	16521	0.40	TL	0.07	TL	0.51	TL	0.42	TL	10974	1.07	TL	0.71	TL	8159	1.22	TL	0.80	TL	6523	1.99	TL	1.10	TL
S-24	18658	0.38	TL	0.04	TL	0.06	TL	0.05	TL	12349	0.32	TL	0.19	TL	9147	0.41	TL	0.29	TL	7342	1.61	TL	0.96	TL
S-25	21172	1.37	TL	0.31	TL	0.68	TL	0.40	TL	14070	1.04	TL	0.85	TL	10487	1.81	TL	1.44	TL	8149	0.30	TL	0.00	2:26:56
Y-25	10170	1.86	TL	1.24	TL	0.39	TL	0.37	TL	6761	1.00	TL	0.87	TL	5050	2.04	TL	1.61	TL	3930	0.69	TL	0.33	TL
Y-30	13790	3.67	TL	2.86	TL	0.17	TL	0.14	TL	9133	0.54	TL	0.12	TL	6889	2.18	TL	1.62	TL	5390	0.92	TL	0.62	TL
Y-35	19087	27.02	TL	27.02	TL	0.48	TL	0.26	TL	12705	1.28	TL	0.69	TL	9492	2.02	TL	1.32	TL	7504	1.28	TL	0.67	TL
Y-40	23739	31.37	TL	31.37	TL	0.55	TL	0.38	TL	15825	1.82	TL	1.18	TL	11801	2.86	TL	1.26	TL	9381	2.45	TL	1.22	TL
Y-45	31442	35.78	TL	35.78	TL	1.08	TL	0.66	TL	20887	2.74	TL	0.68	TL	15664	5.19	TL	2.33	TL	12434	4.21	TL	1.70	TL
Y-50	41517	39.56	TL	39.56	TL	1.94	TL	0.62	TL	27695	3.46	TL	1.29	TL	20760	7.73	TL	3.24	TL	16483	5.80	TL	2.46	TL
Y-60	55986	46.83	TL	46.83	TL	8.26	TL	1.51	TL	37304	15.44	TL	3.18	TL	27913	14.92	TL	5.47	TL	22291	15.34	TL	5.66	TL

*A time "TL" indicates that an optimal solution could not be determined within the time limit of one or three hours, respectively.

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