Les Cahiers du GERAD ISSN: 0711–2440

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G–2017–14

March 2017

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On copula-based conditional quantile estimators

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March 2017

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G–2017–14

Abstract: Recently, two different copula-based approaches have been proposed to estimate the conditional quantile function of a variable Y with respect to a vector of covariates $\boldsymbol{X}$: the first estimator is related to quantile regression weighted by the conditional copula density, while the second estimator is based on the inverse of the conditional distribution function written in terms of margins and the copula. Using empirical processes, we show that even if the two estimators look quite different, they converge to the same limit. Also, we propose a bootstrap procedure for the limiting process in order to be able to construct uniform confidence bands around the conditional quantile function.

Keywords: Conditional quantile function, copula, quantile regression, bootstrap

Résumé : Récemment, deux approches différentes basées sur la fonction copule ont été proposées pour estimer la fonction des quantiles conditionnels d'une variable Y par rapport à un vecteur de covariables $\boldsymbol{X}$: le premier estimateur est lié au modèle de régression des quantiles pondéré par la densité de la copule conditionnelle, tandis que le second estimateur est basé sur l'inverse de la distribution conditionnelle écrite en termes des marges et de la copule. En s'appuyant sur la théorie des processus empiriques, nous montrons que les deux estimateurs, même s'ils semblent différents, convergent en fait vers la même limite. Nous proposons aussi une méthode de ré-échantillonage permettant la construction d'une bande de confiance uniforme autour de la fonction des quantiles conditionnels.

Mots clés : Quantile conditionnel, copule, régression des quantiles, bootstrap

1 Introduction

Copulas, or dependence functions, are very popular to model the dependence between variables, because one can remove the effect of marginal distributions, provided the latter are continuous. This is why dependence measures based on the copula are so robust, compared to the traditional Pearson correlation coefficient. Copulas also enter naturally when computing the conditional distribution function of a random variable Y given covariates $\boldsymbol{X} = (X_1, \ldots, X_d)$. See, e.g., Bouyé and Salmon (2002); Bouyé and Salmon (2009) when $d = 1$. This relation between the conditional distribution of Y given $\mathbf{X} = \mathbf{x}$ and the associated copula was used recently to propose conditional quantile estimators, as alternative to the quantile regression methods (Koenker and Bassett, 1978; Koenker et al., 1994; Koenker, 2005) or the parameter approach (Chavez-Demoulin and Davison, 2005; Neville et al., 2011; Nasri et al., 2013, 2016).

A first copula-based estimator of the conditional quantile was proposed by Noh et al. (2015) and is based on a weighted quantile regression method. The asymptotic limiting distribution was proved to be Gaussian. More recently, a much more intuitive estimator of the plug-in type was proposed in Kraus and Czado (2017); Nasri and Bouezmarni (2017), who compared the estimated MISE of various competitors, including the estimator proposed by Noh et al. (2015). From the simulations performed in Kraus and Czado (2017); Nasri and Bouezmarni (2017), it seems that the plug-in estimator performs better than the other copula-based estimator. However the asymptotic behavior of this estimator was not discussed.

In Section 2, we describe the estimators of Noh et al. (2015) and Kraus and Czado (2017) and we discussed their implementation. Another closely related parametric estimator proposed in Nasri and Bouezmarni (2017) is also discussed. In Section 3, we study the asymptotic limiting distribution of the estimators viewed as stochastic processes over $(0, 1)$ and we show that the two semi-parametric estimators have the same limiting distribution. We also propose a bootstrapping method for constructing uniform confidence bands for the conditional quantile functions.

2 Estimation of conditional quantiles

One way to model the dependence between a variable of interest Y and covariates $\mathbf{X}$ is to use dependence functions called copulas; see, e.g., Nelsen (1999). More precisely, suppose that $(Y_1, \mathbf{X}_1), \ldots, (Y_n, \mathbf{X}_n)$ are i.i.d. observations of $(Y, \mathbf{X})$ with (unconditional) continuous margins $(F_0, F_1, \ldots, F_d)$ and copula C with density c.

By definition, a copula is a joint distribution function of uniform random variables. According to Sklar (1959), since the margins are continuous, there exists a unique copula C so that the joint distribution function of $(Y, \mathbf{X})$ can be written in terms of the copula and the margins viz.

$$P(Y \le y, \mathbf{X} \le \mathbf{x}) = C\{F_0(y), \mathbf{F}(\mathbf{x})\}, \quad y \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^d, \tag{1}$$

where $\mathbf{F}(\mathbf{x}) = (F_1(x_1), \ldots, F_d(x_d))$. The copula C is the cdf of $(U, \mathbf{V})$, where $U = F_0(Y)$ and $\mathbf{V} = \mathbf{F}(\mathbf{X})$.

2.1 Copula-based conditional quantiles

Denote by $\mathcal{H}(y, \mathbf{x})$ the conditional distribution function of Y given $\boldsymbol{X} = \mathbf{x}$. The expression of the conditional distribution function $\mathcal{H}$ in terms of the copula function and the marginal distributions appeared explicitly first in Bouyé and Salmon (2002); Bouyé and Salmon (2009) in the case $d = 1$. However, it is easy to extend it to any $d \ge 1$, and one can easily show that

$$\mathcal{H}(y, \mathbf{x}) = \mathrm{P}(Y \le y | \mathbf{X} = \mathbf{x}) = \mathcal{C}\{F_0(y), \mathbf{F}(\mathbf{x})\}, \quad y \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^d, \tag{2}$$

where $\mathcal{C}(u, \mathbf{v})$ is the conditional distribution function of U given $\mathbf{V} \equiv \mathbf{F}(\mathbf{X}) = \mathbf{v} \equiv \mathbf{F}(\mathbf{x})$. In fact, according to Rémillard (2013, Proposition 8.6.2), for $u \in [0, 1]$ and $\mathbf{v} = (v_1, \ldots, v_d) \in (0, 1)^d$,

$$\mathcal{C}(u, \mathbf{v}) = \frac{\partial v_1 \cdots \partial v_d C\,(u, v_1, \ldots, v_d)}{\partial v_1 \cdots \partial v_d C\,(1, v_1, \ldots, v_d)},$$

and $\mathfrak{c}(u, \mathbf{v}) = \partial_u \mathcal{C}(u, \mathbf{v}) = c(u, \mathbf{v}) / \int_0^1 c(z, \mathbf{v}) dz$, so $\mathcal{C}(u, \mathbf{v}) = \int_0^u c(z, \mathbf{v}) dz / \int_0^1 c(z, \mathbf{v}) dz$.

Now, the associated conditional quantile function $Q(\alpha, \mathbf{x})$, $\alpha \in (0,1)$, is given by

$$Q(\alpha, \mathbf{x}) = \inf\{y \in \mathbb{R} : \mathcal{H}(y, \mathbf{x}) \geq \alpha\}. \tag{3}$$

Using (2), we get that Q depends only on the margins F_0, $\mathbf{F}$ and the copula C viz.

$$Q(\alpha, \mathbf{x}) = F_0^{-1} \left[\Gamma\{\alpha, \mathbf{F}(\mathbf{x})\} \right], \tag{4}$$

where $\Gamma(\alpha, \mathbf{v})$ is the quantile of order α of the distribution function $\mathcal{C}(u, \mathbf{v})$, $u \in [0,1]$, with $\mathbf{v} \in (0,1)^d$. Note that (4) is the basic equation for defining the plug-in estimator.

Next, using (2) and Koenker and Bassett (1978), one gets that $Q(\alpha, \mathbf{x})$ is also a solution of

$$\arg\min_a \mathrm{E}\left[\rho_\alpha (Y - a)\, \mathfrak{c}\{F_0(Y), \mathbf{F}(\mathbf{x})\} \right], \tag{5}$$

where $\rho_\alpha(y) = y\{\alpha - \mathbb{I}(y < 0)\} = (1-\alpha)|y|\mathbb{I}(y < 0) + \alpha y \mathbb{I}(y \geq 0)$, $y \in \mathbb{R}$, and $\mathbb{I}$ is the indicator function. The latter equation is used by Noh et al. (2015) to construct an estimator of $Q(\alpha, \mathbf{x})$.

2.2 Estimation of the copula and the margins

To estimate the conditional quantile using copulas, one needs to estimate the copula C associated with $(Y, \mathbf{X})$ or $(U, \mathbf{V})$, and the margins $F_0, \mathbf{F}$. First, one can assume that $Y_i = F_0^{-1}(U_i)$ and $X_{ij} = F_j^{-1}(V_{ij})$, where $(U_1, \mathbf{V}_1), \ldots, (U_n, \mathbf{V}_n)$ are i.i.d. observations from copula C.

2.2.1 Estimation of the copula

For sake of simplicity, we assume that the copula belongs to a parametric family $\{C_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \mathcal{O}\}$, so the estimation of the copula is given as $C_{\boldsymbol{\theta}_n}$, where $\boldsymbol{\theta}_n$ is a rank-based consistent estimator of the true parameter $\boldsymbol{\theta}_0$. Consequently, the quantile function $\Gamma(\alpha, \mathbf{v}) \equiv \Gamma_{\boldsymbol{\theta}}(\alpha, \mathbf{v})$ can be estimated by $\Gamma_{\boldsymbol{\theta}_n}(\alpha, \mathbf{v})$, $\alpha \in (0,1)$, $\mathbf{v} \in (0,1)^d$. The parametric family approach is also what Noh et al. (2015) and Kraus and Czado (2017) considered. In fact, in the case of several covariates, Kraus and Czado (2017) used a particular case of a parametric copula family, namely a D-vine model, which is a construction of a copula using a given set of parametric bivariate copula families. Note that instead of considering a parametric family of copulas, one could estimate the density of the copula non-parametrically, so that all the conditional quantile estimators discussed here could also be computed. However the convergence is slower and it often suffers from the curse of dimensionality (Bouezmarni et al., 2013; Bouezmarni and Rombouts, 2009, 2010; Janssen et al., 2016). The next step is to estimate the margins.

2.2.2 Estimation of the margins

Motivated by the IFM method, one could use parametric families to estimate each of the margins. This would make sense in several applications. For copula-based quantile estimators, this approach was suggested in Nasri and Bouezmarni (2017), where a parametric copula-based estimator was proposed. One can also consider non-parametric estimators, namely for any $y \in \mathbb{R}$ and any $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$F_{n0}(y) = \frac{1}{n+1} \sum_{i=1}^n \mathbb{I}(Y_i \leq y), \quad F_{nj}(x_j) = \frac{1}{n+1} \sum_{i=1}^n \mathbb{I}(X_{ij} \leq x_j), \quad j \in \{1, \ldots, d\}, \tag{6}$$

and set $\mathbf{F}_n(\mathbf{x}) = (F_{n1}(x_1), \ldots, F_{nd}(x_d))$. Further note that $F_{n0}(y) = D_n \circ F_0(y)$, where D_n is the empirical distribution distribution function of the U_i's and $\mathbf{F}_n(\mathbf{x}) = \mathbf{B}_n \circ \mathbf{F}(\mathbf{x})$, where $\mathbf{B}_n$ is the vector of empirical marginal distribution functions of $\mathbf{V}_1, \ldots, \mathbf{V}_n$. Noh et al. (2015) proposes a kernel-based estimator $\hat{F}_{n0}$ for F_0 such that $n^{1/2} \sup_y |\hat{F}_{n0}(y) - F_{n0}(y)| \stackrel{Pr}{\to} 0$ as $n \to \infty$. This was used in Kraus and Czado (2017). Even if $\hat{F}_{n0}$ is continuous, the precision of the estimation might not be better and there is always the question of the choice of the bandwidth. This is why we will use the estimators given by (6). For the rest of the section, let $\mathbf{x}$ be given and set $\mathbf{v} = \mathbf{F}(\mathbf{x})$. It then follows that $\mathbf{F}_n(\mathbf{x}) = \mathbf{B}_n(\mathbf{v})$. For sake of simplicity, $\mathbf{x}$ or $\mathbf{v}$ might be omitted. We present the copula-based estimators we will study.

2.3 Weighted quantile regression estimator

Surprisingly, the natural plug-in estimator did not appeared first in the literature. In fact, Noh et al. (2015) proposed a copula-based model mixed with a quantile regression approach using (5) viz.

$$Q_{n,wqr}(\alpha, \mathbf{x}) = \arg\min_a \left[\sum_{i=1}^n \rho_\alpha (Y_i - a)\, \mathfrak{c}_{\boldsymbol{\theta}_n} \{F_{n0}(Y_i), \mathbf{F}_n(\mathbf{x})\} \right], \tag{7}$$

even if the solution is not necessarily unique. In fact they take $c_{\boldsymbol{\theta}_n}(u, \mathbf{v})$ instead of taking $\mathfrak{c}_{\boldsymbol{\theta}_n}(u, \mathbf{v})$ but it does not change anything. However, a unique way to define a solution to (7) is by using the empirical weighted distribution function H_n defined for any $y \in \mathbb{R}$ by

$$H_n(y) = \sum_{i=1}^n \mathbb{I}(Y_i \le y) w_{i,n} = G_n\{F_0(y)\}, \text{ with } G_n(u) = \sum_{i=1}^n \mathbb{I}(U_i \le u) w_{i,n},$$

where, for any $i \in \{1, \ldots, n\}$,

$$w_{i,n} = \frac{\mathfrak{c}_{\boldsymbol{\theta}_n}\{F_{n0}(Y_i), \mathbf{F}_n(\mathbf{x})\}}{\sum_{j=1}^n \mathfrak{c}_{\boldsymbol{\theta}_n}\{F_{n0}(Y_j), \mathbf{F}_n(\mathbf{x})\}} = \frac{c_{\boldsymbol{\theta}_n}\{F_{n0}(Y_i), \mathbf{F}_n(\mathbf{x})\}}{\sum_{j=1}^n c_{\boldsymbol{\theta}_n}\{F_{n0}(Y_j), \mathbf{F}_n(\mathbf{x})\}} = \frac{\mathfrak{c}_{\boldsymbol{\theta}_n}\{D_n(U_i), \mathbf{B}_n(\mathbf{v})\}}{\sum_{j=1}^n \mathfrak{c}_{\boldsymbol{\theta}_n}\{D_n(U_j), \mathbf{B}_n(\mathbf{v})\}}.$$

The estimator $Q_{n,wqr}(\alpha, \mathbf{x})$ is then defined as the quantile of level α of H_n, i.e.,

$$Q_{n,wqr}(\alpha, \mathbf{x}) = H_n^{-1}(\alpha) = F_0^{-1} \circ G_n^{-1}(\alpha), \quad \alpha \in (0,1). \tag{8}$$

If $\hat{a} = \arg\min_a \left[\sum_{i=1}^n \rho_\alpha (Y_i - a)\, \mathfrak{c}_{\boldsymbol{\theta}_n}\{F_{n0}(Y_i), \mathbf{F}_n(\mathbf{x})\}\right]$, then $H_n(\hat{a}) \ge \alpha \ge H_n(\hat{a}-)$. Hence $H_n^{-1}(\alpha)$ satisfies (7).[1] It is easy to show that H_n is a consistent estimator of the distribution function H given by $H(y) = \mathcal{H}(y, \mathbf{x}) = \mathcal{C}\{F_0(y), \mathbf{v}\}$, $y \in \mathbb{R}$. Also G_n is a consistent and asymptotically unbiased estimator of the distribution function G given by $G(u) = \mathcal{C}(u, \mathbf{v})$, $u \in [0,1]$.

2.4 Plug-in estimators

Expression (4) provides a natural way for estimating the conditional quantile. We already mentioned that we will estimate the margins by using the empirical distribution functions (6), but for sake of completeness, we describe both parametric and semi-parametric estimators of $Q(\alpha, \mathbf{x})$, as they are based on the following plug-in estimation $\mathcal{C}_{\boldsymbol{\theta}_n}\left\{\hat{F}_0(y), \hat{\mathbf{F}}(\mathbf{x})\right\}$ of $\mathcal{H}(y, \mathbf{x})$.

2.4.1 Parametric estimator

In the parametric approach, we assume that the marginal distributions F_0 and $\mathbf{F}$, belong to parametric families, denoted by $F_0(\cdot, \boldsymbol{\beta}_0)$ and $\mathbf{F}(\cdot, \boldsymbol{\beta})$ respectively. If $\boldsymbol{\beta}_{n0}$ and $\boldsymbol{\beta}_n$ are consistent estimators of $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}$, then for any $y \in \mathbb{R}$, $\check{H}_n(y) = \mathcal{C}_{\boldsymbol{\theta}_n}\{F_0(y, \boldsymbol{\beta}_{n0}), \mathbf{F}(\mathbf{x}, \boldsymbol{\beta}_n)\}$ is clearly a consistent estimator of $H(y) = \mathcal{H}(y, \mathbf{x})$, yielding the natural parametric estimator

$$Q_{n,p}(\alpha, \mathbf{x}) = \check{H}_n^{-1}(\alpha) = F_0^{-1}\left[\Gamma_{\boldsymbol{\theta}_n}\{\alpha, \mathbf{F}(\mathbf{x}, \boldsymbol{\beta}_n)\}, \boldsymbol{\beta}_{n0}\right], \quad \alpha \in (0,1). \tag{9}$$

Remark 1 *Two methods for estimating the parameters are developed in the literature. First, we can estimate simultaneously $\boldsymbol{\beta_0}$, $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ using the complete maximum likelihood, see, e.g. Shih and Louis (1995). However, this method requires intensive calculations and sometimes the numerical optimization problem is difficult to solve. Second, Xu (1996) and Joe (1997) proposed a two-step process, called inference function for margins (IFM), in order to estimate the marginal functions and copula parameters. This method consists in the first step in estimating the marginal functions parameters, followed in the second step by the estimation of the copula parameters through the pseudo-observations $\hat{U}_i = F_0(Y_i, \boldsymbol{\beta}_{n0})$ and $\hat{V}_{ij} = F_j(Y_i, \boldsymbol{\beta}_{nj})$, $j \in \{1, \ldots, d\}$, $i \in \{1, \ldots, n\}$. Most of the time, this is done by maximizing the likelihood density at these pseudo-observations. The IFM method is often used because it is easy to implement. Note that as discussed in Noh et al. (2013), if the estimation of the margins is incorrect, the estimation of the copula parameter $\boldsymbol{\theta}$ can then be biased.*

[1] If $H_n \circ H_n^{-1}(\alpha) = \alpha$, then every $\{y; H_n(y) = \alpha\} = \left[H_n^{-1}(\alpha), y_0\right]$, for some y_0, also satisfies (8). If $H_n \circ H_n^{-1}(\alpha) > \alpha$, then $H_n^{-1}(\alpha)$ is the unique solution of (8).

2.4.2 Semiparametric estimator

Here, the marginal distributions are estimated using (6). To estimate the copula parameters using a rank-based estimator, one can use for example the pseudo-MLE method proposed by Genest et al. (1995) and Shih and Louis (1995). First, a consistent estimation $\tilde{H}_n$ of H is given by

$$\tilde{H}_n(y) = \mathcal{C}_{\boldsymbol{\theta}_n}\{F_{n0}(y), \mathbf{F}_n(\mathbf{x})\} = \tilde{G}_n \circ F_0(y), \quad y \in \mathbb{R}, \tag{10}$$

where $\tilde{G}_n(u) = \mathcal{C}_{\boldsymbol{\theta}_n}\{D_n(u), \mathbf{B}_n(\mathbf{v})\}$, which is a consistent estimate of $G(u) = \mathcal{C}(u, \mathbf{v})$, $u \in [0,1]$. As a result, the estimation of $Q(\alpha, \mathbf{x})$ is defined for any $\alpha \in (0,1)$ by

$$Q_{n,sp}(\alpha, \mathbf{x}) = \tilde{H}_n^{-1}(\alpha) = F_{n0}^{-1}\left[\Gamma_{\boldsymbol{\theta}_n}\{\alpha, \mathbf{F}_n(\mathbf{x})\}\right] = F_0^{-1} \circ \tilde{G}_n^{-1}(\alpha). \tag{11}$$

3 Asymptotic behavior of the copula-based estimators

In this section we find the asymptotic distribution of the conditional quantile functions for the proposed estimators, extending the results of Noh et al. (2015). As a result, we obtain that the plug-in estimator and the weighted quantile regression estimator converge to the same limit. We also propose a different bootstrap algorithm that can be used to construct uniform confidence bands about the conditional quantile function.

As before, $\mathbf{x}$ is fixed and $\mathbf{v} = \mathbf{F}(\mathbf{x})$. Recall that $H(y) = \mathcal{C}\{F_0(y), \mathbf{v}\} = G \circ F_0(y)$, for any $\in \mathbb{R}$. Throughout this section, it is assumed that the density $f_0 = F_0'$ exists and is positive everywhere. If the support is not $\mathbb{R}$, just transform Y accordingly. This way $F_0(y) \in (0,1)$ for any $y \in \mathbb{R}$. Also suppose that the density c of the $(d+1)$-dimensional copula C is positive on $(0,1)^{d+1}$. Then H is continuously differentiable and with density h satisfying $h(y) = f_0(y)\mathfrak{c}(u, \mathbf{v}) > 0$, for any $y \in \mathbb{R}$. Further write $Q(u) = H^{-1}(u)$ and $\Gamma(u) = G^{-1}(u)$, $u \in (0,1)$.

3.1 Convergence of the parametric estimator

In what follows, $\nabla_{\boldsymbol{\beta}_0} F_0(y, \boldsymbol{\beta}_0)$ is a p_0-dimensional column vector, $\nabla_{\boldsymbol{\beta}} \mathbf{F}$ is a $p \times d$ matrix, $\nabla_{\mathbf{v}} \mathcal{C}_{\boldsymbol{\theta}}(u, \mathbf{v})$ is a d-dimensional column vector, $\nabla_{\boldsymbol{\theta}} \mathcal{C}_{\boldsymbol{\theta}}(u, \mathbf{v}) = \dot{\mathcal{C}}_{\boldsymbol{\theta}}(u, \mathbf{v})$ is a q dimensional column vector which represent the partial derivatives with respect to $\boldsymbol{\beta}_0$, $\boldsymbol{\beta}$, $\mathbf{v}$ and $\boldsymbol{\theta}$ of F_0, $\mathbf{F}$, $\mathcal{C}_{\boldsymbol{\theta}}$ and $\mathcal{C}_{\boldsymbol{\theta}}$ respectively. Throughout this section, it is assumed that these derivatives are continuous. Also, we assume that $\mathfrak{c}_{\boldsymbol{\theta}}(u, \mathbf{v})$ is continuously differentiable with respect to $u \in (0,1)$.

Set $\mathcal{B}_{n0} = n^{1/2}(\boldsymbol{\beta}_{n0} - \boldsymbol{\beta}_0)$, $\mathcal{B}_n = n^{1/2}(\boldsymbol{\beta}_n - \boldsymbol{\beta})$, and $\boldsymbol{\Theta}_n = n^{1/2}(\boldsymbol{\theta}_n - \boldsymbol{\theta}_0)$. Finally, define $\check{\mathbb{H}}_n(y) = n^{1/2}\left\{\check{H}_n(y) - H(y)\right\}$ for any $y \in \mathbb{R}$, and $\mathbb{Q}_{n,p}(u) = n^{1/2}\left\{Q_{n,p}(u) - Q(u)\right\}$, $u \in (0,1)$. The proof of the following theorem, giving the asymptotic behavior of the parametric quantile process, follows readily from the Delta method (van der Vaart and Wellner, 1996). To simplify notations, set $g(u) = \mathfrak{c}(u, \mathbf{v}) = \mathfrak{c}_{\boldsymbol{\theta}_0}(u, \mathbf{v})$, $G(u) = \mathcal{C}_{\boldsymbol{\theta}_0}(u, \mathbf{v})$, $\dot{\mathcal{C}}(u, \mathbf{v}) = \nabla_{\boldsymbol{\theta}} \mathcal{C}_{\boldsymbol{\theta}}(u, \mathbf{v})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$ and $\nabla_{\mathbf{v}} \mathcal{C}_{\boldsymbol{\theta}_0}(u, \mathbf{v}) = \nabla_{\mathbf{v}} \mathcal{C}(u, \mathbf{v})$. Further set $F_0 = F_0(\cdot, \boldsymbol{\beta}_0)$ and $\mathbf{F} = \mathbf{F}(\cdot, \boldsymbol{\beta})$.

Theorem 1 *Assume that $(\mathcal{B}_{n0}, \mathcal{B}_n, \boldsymbol{\Theta}_n)$ converges in law to a centered Gaussian vector $(\mathcal{B}_0, \mathcal{B}, \boldsymbol{\Theta})$.*[2] *Then, as $n \to \infty$, $\check{\mathbb{H}}_n$ converges in $D(\mathbb{R})$*[3] *to a continuous centered Gaussian process $\check{\mathbb{H}}$, denoted $\check{\mathbb{H}}_n \rightsquigarrow \check{\mathbb{H}} = \check{\mathbb{G}} \circ F_0$, where*

$$\check{\mathbb{G}}(u) = \boldsymbol{\Theta}^\top \dot{\mathcal{C}}(u, \mathbf{v}) + \mathcal{B}^\top \nabla_{\boldsymbol{\beta}} \mathbf{F}\left\{\boldsymbol{F}^{-1}(\mathbf{v}), \boldsymbol{\beta}\right\} \nabla_{\mathbf{v}} \mathcal{C}(u, \mathbf{v}) + g(u) \mathcal{B}_0^\top \nabla_{\boldsymbol{\beta}_0} F_0\left\{F_0^{-1}(u), \boldsymbol{\beta}_0\right\}, \quad u \in [0,1].$$

Furthermore, $\mathbb{Q}_{n,p} \rightsquigarrow \mathbb{Q}_p$ in $D(0,1)$, where $\mathbb{Q}_p(u) = -\frac{\check{\mathbb{H}}\{Q(u)\}}{h\{Q(u)\}}$, $u \in (0,1)$. In particular, for any $[a,b] \subset (0,1)$, $n^{1/2} \sup_{u \in [a,b]} |Q_{n,p}(u) - Q(u)|$ converges in law to $\sup_{u \in [a,b]} \left|\frac{\check{\mathbb{H}}\{Q(u)\}}{h\{Q(u)\}}\right|$.

[2]See, e.g. Joe (1997) for sufficient regularity conditions.

[3]Convergence in $D(I)$ means that for any close interval $[a,b] \subset I$, the process converges in law in the Skorokhod topology on $D([a,b])$. In particular, any continuous function of the process converges in distribution. See, e.g., Billingsley (1999).

3.2 Convergence of the semiparametric estimator

We now study the convergence of the conditional quantile process $\mathbb{Q}_{n,sp}(u) = n^{1/2}\{Q_{n,sp}(u) - Q(u)\}$, $u \in (0,1)$. Before stating the theorem, define $\mathbb{D}_n(u) = n^{1/2}\{D_n(u) - u\}$, and $\mathbb{B}_n(\mathbf{v}) = n^{1/2}(\mathbf{B}_n(\mathbf{v}) - \mathbf{v})$, $u \in [0,1]$, $\mathbf{v} \in (0,1)^d$. The proof of this theorem follows from the Delta method (van der Vaart and Wellner, 1996).

Theorem 2 *Assume* $(\mathbb{D}_n, \mathbb{B}_n, \boldsymbol{\Theta}_n)$ *converges in* $D\left([0,1]^{1+d} \times \mathbb{R}^q\right)$ *to centered Gaussian process* $(\mathbb{D}, \mathbb{B}, \boldsymbol{\Theta})$.[4] *Then, as* $n \to \infty$, $\tilde{\mathbb{G}}_n$ *converges in* $D([0,1])$ *to* $\tilde{\mathbb{G}} = \mathbb{H} + \mathbb{D}g$, *where*

$$\mathbb{H}(u) = \boldsymbol{\Theta}^\top \dot{\mathcal{C}}(u,\mathbf{v}) + \mathbb{B}(\mathbf{v})^\top \nabla_{\mathbf{v}} \mathcal{C}(u,\mathbf{v}), \quad u \in [0,1].$$

Furthermore, $\mathbb{Q}_{n,sp} \rightsquigarrow \mathbb{Q}_{sp}$ *in* $D(0,1)$, *where* $\mathbb{Q}_{sp}(u) = -\frac{\tilde{\mathbb{G}}\{\Gamma(u)\}}{h\{Q(u)\}}$, $u \in (0,1)$. *In particular, for any* $[a,b] \subset (0,1)$, $n^{1/2} \sup_{u\in[a,b]} |Q_{n,sp}(u) - Q(u)|$ *converges in law to* $\sup_{u\in[a,b]} \left|\dfrac{\tilde{\mathbb{G}}\{\Gamma(u)\}}{h\{Q(u)\}}\right|$.

A way to bootstrap the process $\tilde{\mathbb{G}}$ is given next.

Algorithm 1 (Bootstrapping $\tilde{\mathbb{G}}$) *First, estimate* $\boldsymbol{\theta}$ *using a regular rank-based estimator* $\boldsymbol{\theta}_n$ *of the form* $\boldsymbol{\theta}_n = \mathcal{T}_n(U_{1,n}, \mathbf{V}_{1,n}, \ldots, U_{n,n}, \mathbf{V}_{n,n})$ *in the sense of Genest and Rémillard (2008), and set* $\mathbf{v}_n = \mathbf{F}_n(x)$.

Then, for each $k \in \{1,\ldots,N\}$, *repeat the following steps:*

- *generate* $(U_i^\star, \mathbf{V}_i^\star) \sim C_{\boldsymbol{\theta}_n}$, $i \in \{1,\ldots,n\}$;
- *compute the associated empirical margins* $D_n^\star$, $\mathbf{F}_n^\star$;
- *calculate the pseudo-observations* $U_{i,n}^\star = D_n^\star(U_i^\star)$, $\mathbf{V}_{i,n}^\star = \mathbf{F}_n^\star(\mathbf{V}_i^\star)$, $i \in \{1,\ldots,n\}$;
- *estimate* $\boldsymbol{\theta}_n^\star = \mathcal{T}_n\left(U_{1,n}^\star, \mathbf{V}_{1,n}^\star, \ldots, U_{n,n}^\star, \mathbf{V}_{n,n}^\star\right)$;
- *define* $\tilde{\mathbb{G}}_n^{(k)}(u) = n^{1/2}\left[\mathcal{C}_{\boldsymbol{\theta}_n^\star}\{D_n^\star(u), \mathbf{B}_n^\star(\mathbf{v}_n)\} - \mathcal{C}_{\boldsymbol{\theta}_n}(u,\mathbf{v}_n)\right]$, $u \in [0,1]$.

The next theorem shows the consistency of the proposed bootstrap and is proven in Appendix A.

Theorem 3 *An* $n \to \infty$, $\tilde{\mathbb{G}}_n^{(1)}, \ldots, \tilde{\mathbb{G}}_n^{(N)}$ *converge to independent copies of* $\tilde{\mathbb{G}}$.

Remark 2 *Note that as shown in Genest and Rémillard (2008), most interesting estimators are regular. In particular, estimators of the class* $\mathcal{R}_1$: *this means that there exists a continuously differentiable function* J *so that* $E[J(U,\mathbf{V})] = 0$ *and* $\boldsymbol{\Theta}_n = n^{-1/2}\sum_{i=1}^n J\{D_n(U_i), B_n(\mathbf{V}_i)\} + o_P(1)$. *For example, pseudo-maximum likelihood estimators, as defined in Genest et al. (1995), belong to this class.*

3.2.1 Construction of the uniform $100(1-\alpha)\%$ confidence band for Q

To construct the uniform confidence band on $[a,b] \subset (0,1)$, we generate N processes $\tilde{\mathbb{G}}^{(k)}$, $k \in \{1,\ldots,N\}$ and they are evaluated at $u \in \mathcal{A} = \{a + j(b-a)/m; j = 0,\ldots,m\}$, where m is fixed but large enough (say $m = 1000$). The density f_0 is estimated with a Gaussian kernel estimator f_{n0}, so $\mathfrak{h}(u) = h \circ Q(u)$ is estimated by $\mathfrak{h}_n(u) = f_{n0} \circ Q_{n,sp}(u)\mathfrak{c}_{\boldsymbol{\theta}_n}(u,\mathbf{v}_n)$, when $\mathbf{v}_n = \mathbf{F}_n(\mathbf{x})$. One then compute

$$b_{k,n} = \max_{u\in\mathcal{A}} \left|\tilde{\mathbb{G}}^{(k)}(u)\right| / \mathfrak{h}_n(u), \quad k \in \{1,\ldots,N\},$$

and let $b_n(\alpha)$ be the associated quantile of order $1-\alpha$. The uniform confidence band about Q is given by $Q_{n,sp}(u) \pm n^{-1/2} b_n(\alpha)$, $u \in [a,b]$. A 95% confidence interval about a single point $Q(u)$ is given by $Q_{n,sp}(u) \pm n^{-1/2} 1.96\hat{\sigma}/\mathfrak{h}_n(u)$ where $\hat{\sigma}^2$ is the sample variance of the values $\tilde{\mathbb{G}}^{(k)}(u)$, $k \in \{1,\ldots,N\}$.

[4] This assumption is satisfied for most well-behaved rank-based estimator of θ. See, e.g., Genest and Rémillard (2008).

3.3 Convergence of the weighted quantile regression estimator

In this section, for $u \in (0,1)$ we study the convergence of the conditional quantile process $\mathbb{Q}_{n,wqr}(u) = n^{1/2}\left\{Q_{n,wqr}(u) - Q(u)\right\}$. It extends the results in Noh et al. (2015), where only the convergence at a single value was proven. In order to formulate the result, we need to define another sequence of stochastic processes, namely

$$\mathring{\mathbb{G}}_n(u) = n^{-1/2}\sum_{i=1}^n \left\{\mathbb{I}(U_i \le u) g(U_i) - G(u)\right\}, \quad u \in [0,1].$$

It follows from the theory of stochastic processes (van der Vaart and Wellner, 1996) that $(\mathbb{D}_n, \mathbb{B}_n, \mathring{\mathbb{G}}_n)$ converges in $D\left([0,1]^{2+d}\right)$ to centered Gaussian processes $(\mathbb{D}, \mathbb{B}, \mathring{\mathbb{G}})$. The proof of the following theorem is given in Section B. It shows that the two estimators have the same asymptotic distribution.

Theorem 4 *Assume that $(\mathbb{D}_n, \mathbb{B}_n, \mathring{\mathbb{G}}_n, \mathbf{\Theta}_n)$ converges in $D\left([0,1]^{2+d} \times \mathbb{R}^q\right)$ to centered Gaussian processes $(\mathbb{D}, \mathbb{B}, \mathring{\mathbb{G}}, \mathbf{\Theta})$. Then, as $n \to \infty$, $\mathbb{G}_n$ converges in $D([0,1])$ to $\mathbb{G} = \tilde{\mathbb{G}}$. Furthermore, $\mathbb{Q}_{n,wgr} \rightsquigarrow \mathbb{Q}_{wqr}$ in $D(0,1)$, where $\mathbb{Q}_{wqr}(u) = -\frac{\mathbb{G}\{\Gamma(u)\}}{h\{Q(u)\}}$, $u \in (0,1)$. In particular, for any $[a,b] \subset (0,1)$, $n^{1/2} \sup_{u\in[a,b]} |Q_{n,wqr}(u) - Q(u)|$ converges in law to $\sup_{u\in[a,b]} \left|\frac{\mathbb{G}\{\Gamma(u)\}}{h\{Q(u)\}}\right|$.*

Remark 3 *Using our notations, the bootstrap algorithm proposed in Noh et al. (2015) yields values $Q^{(k)}_{n,wqr}$, $k \in \{1,\ldots,N\}$, so that $\mathbb{Q}^{(k)}_{n,wqr} = n^{1/2}\left\{Q^{(k)}_{n,wqr} - Q\right\}$ converges to $\mathbb{Q}^{(k)}_{wqr} + \mathbb{Q}_{wqr}$, where $\mathbb{Q}^{(k)}_{wqr}$ is an independent copy of $\mathbb{Q}_{wqr}$. It then follows that their algorithm works for estimating the asymptotic variance σ^2_α, in the sense that what they call $\hat{\sigma}^2_{boot}$ satisfies $\hat{\sigma}^2_{boot} \approx \frac{\sigma^2_\alpha}{n}$ if n and N are large. However, their procedure is slower than the one we propose since we do not need to compute $Y^\star_i = F^{-1}_{n0}(U^\star_i)$ and $\mathbf{X}^\star_i = \mathbf{F}^{-1}_n(\mathbf{V}^\star_i)$, $i \in \{1,\ldots,n\}$. Also computing $\tilde{H}_n$ is faster than computing H_n.*

4 Conclusion

We have shown that two seemingly different estimators for the conditional quantile function have in fact the same limit. However, the plug-in estimator is easier and faster to implement, in addition to being more accurate for small samples, as shown by simulations in Kraus and Czado (2017); Nasri and Bouezmarni (2017). Therefore, this is the one we recommend.

A Proof of Theorem 3

Proof. Using Genest and Rémillard (2008) and Theorem 2, we get that

$$(\mathbb{D}^\star_n, \mathbb{B}_n, \mathbb{B}^\star_n, \mathbf{\Theta}_n, \mathbf{\Theta}^\star_n) \rightsquigarrow \left(\mathbb{D}^\perp, \mathbb{B}, \mathbb{B}^\perp, \mathbf{\Theta}, \mathbf{\Theta} + \mathbf{\Theta}^\perp\right),$$

where $\left(\mathbb{D}^\perp, \mathbb{B}^\perp, \mathbf{\Theta}^\perp\right)$ is an independent copy of $(\mathbb{D}, \mathbb{B}, \mathbf{\Theta})$. Hence, since $n^{1/2}\{\mathbf{B}^\star_n(\mathbf{v}_n) - \mathbf{v}\} = \mathbb{B}^\star_n(\mathbf{v}_n) + \mathbb{B}_n(\mathbf{v})$, it follows from the Delta Method and Theorem 2 that

$$\begin{aligned}
n^{1/2}\left[\mathcal{C}_{\boldsymbol{\theta}^\star_n}\left\{D^\star_n(u), \mathbf{B}^\star_n(\mathbf{v}_n)\right\} - G(u)\right] &= \dot{\mathcal{C}}(u,v)^\top \mathbf{\Theta}^\star_n + \nabla_{\mathbf{v}}\mathcal{C}(u,\mathbf{v})\{\mathbb{B}^\star_n(\mathbf{v}_n) + \mathbb{B}_n(\mathbf{v})\} \\
&\quad + g(u)\mathbb{D}^\star_n(u) + o_P(1) \\
&\rightsquigarrow \dot{\mathcal{C}}(u,v)^\top\left(\mathbf{\Theta}^\perp + \mathbf{\Theta}\right) + \nabla_{\mathbf{v}}\mathcal{C}(u,\mathbf{v})\{\mathbb{B}^\perp(\mathbf{v}) + \mathbb{B}(\mathbf{v})\} + g(u)\mathbb{D}^\perp(u) \\
&= \dot{\mathcal{C}}(u,v)^\top \mathbf{\Theta}^\perp + \nabla_{\mathbf{v}}\mathcal{C}(u,\mathbf{v})\mathbb{B}^\perp(\mathbf{v}) + g(u)\mathbb{D}^\perp(u) + \mathbb{H}(u) \\
&= \tilde{\mathbb{G}}^\perp(u) + \mathbb{H}(u),
\end{aligned}$$

where $\tilde{\mathbb{G}}^\perp$ is an independent copy of $\tilde{\mathbb{G}}$, while $n^{1/2}\left\{\mathcal{C}_{\boldsymbol{\theta}_n}(u,\mathbf{v}_n) - G(u)\right\} \rightsquigarrow \mathbb{H}$. As a result, $\tilde{\mathbb{G}}^{(1)}_n, \ldots, \tilde{\mathbb{G}}^{(N)}_n$ converge to independent copies of $\tilde{\mathbb{G}}$. □

B Proof of Theorem 4

Set $\dot{\mathfrak{c}}(u,\mathbf{v}) = \nabla_{\boldsymbol\theta}\mathfrak{c}_{\boldsymbol\theta}(u,\mathbf{v})|_{\boldsymbol\theta=\boldsymbol\theta_0}$, $\nabla_{\mathbf{v}}\mathfrak{c}_{\boldsymbol\theta_0}(u,\mathbf{v}) = \nabla_{\mathbf{v}}\mathfrak{c}(u,\mathbf{v})$. It suffices to prove the convergence of $\mathbb{G}_n(u) = \sqrt{n}(G_n(u) - G(u))$. We can write

$$G_n(u) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}(U_i \le u)\mathfrak{c}_{\boldsymbol\theta_n}\{D_n(U_i), \mathbf{B}_n(\mathbf{v})\}/s_n,$$

where $s_n = \frac{1}{n}\sum_{i=1}^n \mathfrak{c}_{\boldsymbol\theta_n}\{D_n(U_i), \mathbf{B}_n(\mathbf{v})\}$.

Next, set $r_n(u) = \mathfrak{c}_{\boldsymbol\theta_n}\{D_n(u), \mathbb{B}_n(\mathbf{v})\} - g(u) - \frac{\left\{\boldsymbol\Theta_n^\top \dot{\mathfrak{c}}(u,\mathbf{v}) + g'(u)\mathbb{D}_n(u) + \nabla_{\mathbf{v}}\mathfrak{c}(u,\mathbf{v})^\top \mathbb{B}_n(\mathbf{v})\right\}}{n^{1/2}}$, $u \in [0,1]$. By hypothesis, as $n\to\infty$, $n^{1/2}\sup_{u\in[0,1]}|r_n(u)|$ converges in probability to 0. It then follows that

$$\begin{aligned}
\mathbb{G}_n(u) &= \frac{1}{n^{1/2}s_n}\sum_{i=1}^n \mathbb{I}(U_i\le u)\{\mathfrak{c}_{\boldsymbol\theta_n}\{D_n(U_i),\mathbf{B}_n(\mathbf{v})\} - g(U_i)\} + \mathring{\mathbb{G}}_n(u)/s_n - G(u)n^{1/2}(s_n-1)/s_n \\
&= \{\mathbb{L}_n(u) + \mathring{\mathbb{G}}_n(u) - G(u)\mathbb{L}_n(1) - G(u)\mathring{\mathbb{G}}_n(1)\}/s_n,
\end{aligned}$$

where $\mathbb{L}_n(u) = n^{-1/2}\sum_{i=1}^n \mathbb{I}(U_i\le u)\{\mathfrak{c}_{\boldsymbol\theta_n}\{D_n(U_i),\mathbf{B}_n(\mathbf{v})\} - g(U_i)\}$. Now,

$$\begin{aligned}
\mathbb{L}_n(u) &= \boldsymbol\Theta_n^\top\left\{\frac{1}{n}\sum_{i=1}\mathbb{I}(U_i\le u)\dot{\mathfrak{c}}(U_i,\mathbf{v})\right\} + \frac{1}{n}\sum_{i=1}^n \mathbb{I}(U_i\le u)\mathbb{D}_n(U_i)g'(U_i) \\
&\quad + \mathbb{B}_n(\mathbf{v})^\top\left\{\frac{1}{n}\sum_{i=1}^n \mathbb{I}(U_i\le u)\nabla_{\mathbf{v}}\mathfrak{c}(U_i,\mathbf{v})\right\} + o_P(1) \\
&= \boldsymbol\Theta_n^\top\dot{\mathcal{C}}(u,\mathbf{v}) + \int_0^u \mathbb{D}_n(z)g'(z)dz + \mathbb{B}_n(\mathbf{v})^\top\nabla_{\mathbf{v}}\mathcal{C}(u,\mathbf{v}) + o_P(1).
\end{aligned}$$

Next, assuming that $ug(u)\to 0$ as $u\to 0$, we have

$$\begin{aligned}
\int_0^u \mathbb{D}_n(z)g'(z)dz &= n^{-1/2}\sum_{i=1}^n\int_0^u g'(z)\{\mathbb{I}(U_i\le z) - z\}dz \\
&= n^{-1/2}\sum_{i=1}^n \mathbb{I}(U_i\le u)\{g(u) - g(U_i)\} - n^{1/2}\{ug(u) - G(u)\} \\
&= g(u)\mathbb{D}_n(u) - \mathring{\mathbb{G}}_n(u).
\end{aligned}$$

As a result, $\mathbb{L}_n \rightsquigarrow \mathbb{H} + g\mathbb{D} - \mathring{\mathbb{G}} = \tilde{\mathbb{G}} - \mathring{\mathbb{G}}$. so, $\mathbb{G}_n \rightsquigarrow \mathbb{G} = \tilde{\mathbb{G}}$ in $D([0,1])$. □

C Supplementary material on copula families

C.1 Conditional quantile function for common Archimedean copula families

Elliptical copulas are simply copulas associated with elliptical distributions through Sklar's representation (1). Since a copula is invariant by monotone increasing transformations, an elliptical copula is typically associated with a $(d+1)$-dimensional random vector $\boldsymbol{Z}$ having representation $\boldsymbol{Z} = \mathcal{R}^{1/2}\boldsymbol{A}\boldsymbol{S}$, where $\mathcal{R}$ is a positive random variable independent of $\boldsymbol{S}$, which is uniformly distributed of the unit sphere of $\mathbb{R}^{d+1}$, and $\boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{R}$, where $\boldsymbol{R} = \begin{pmatrix} 1 & \boldsymbol\beta^\top \\ \boldsymbol\beta & \boldsymbol\Sigma \end{pmatrix}$ is an invertible correlation matrix. In fact, $\boldsymbol{R}$ is the correlation matrix of $\boldsymbol{Z}$ iff $E(\mathcal{R}) < \infty$. However, in general, $\boldsymbol{Z}$ is never observed. Fortunately, there is a relationship between $\boldsymbol{R}$ and the matrix $\mathcal{T}$ of Kendall's tau associated with the copula (Fang et al., 2002) making it possible to estimate $\boldsymbol{R}$: for any $j,k \in \{1,\ldots,d+1\}$, $\tau_{jk} = \frac{2}{\pi}\arcsin(\boldsymbol{R}_{jk})$. For details on the estimation, see, e.g. Rémillard (2013, Section 8.7.2.1). We now present the quantile functions for the two most popular elliptical copulas: the Gaussian copula and the Student copula. The details are given in Appendix D.

Gaussian copula with parameter $\boldsymbol{R}$ Denote by Φ the distribution function of a standard Gaussian variate. Then, setting $\sigma^2 = 1 - \boldsymbol{\beta}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta}$, we have, for any $\alpha, u \in (0,1)$, and any $\mathbf{v} = (v_1, \ldots, v_d) \in (0,1)^d$,

$$\Gamma(\alpha, \mathbf{v}) = \Phi\left\{\sigma\Phi^{-1}(\alpha) + \sum_{k=1}^{d} (\boldsymbol{\Sigma}^{-1}\boldsymbol{\beta})_k \Phi^{-1}(v_k)\right\}. \tag{12}$$

Student copula with ν degrees of freedom and parameter $\boldsymbol{R}$ Let t_μ be the distribution function of a Student random variable with $\mu > 0$ degrees of freedom. Then, setting $\tilde{\sigma}^2 = \left(1 - \boldsymbol{\beta}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta}\right)\left(1 + \frac{\mathbf{w}^\top \boldsymbol{\Sigma}^{-1}\mathbf{w}}{\nu}\right)$, we have that for any $\alpha, u \in (0,1)$, and any $\mathbf{v} = (v_1, \ldots, v_d) \in (0,1)^d$,

$$\Gamma(\alpha, \mathbf{v}) = t_\nu\left\{\tilde{\sigma} t_{\nu+d}^{-1}(\alpha) + \sum_{k=1}^{d} (\boldsymbol{\Sigma}^{-1}\boldsymbol{\beta})_k t_\nu^{-1}(v_k)\right\}. \tag{13}$$

D Computations for the Gaussian and Student copulas

Let $\mathbf{Z} = (Z_0, \mathbf{W}) \sim N_{d+1}(0, \boldsymbol{R})$, with $\boldsymbol{R} = \begin{pmatrix} 1 & \boldsymbol{\beta}^\top \\ \boldsymbol{\beta} & \boldsymbol{\Sigma} \end{pmatrix}$. Then $\epsilon = Z_0 - \boldsymbol{\beta}^\top \boldsymbol{\Sigma}^{-1}\mathbf{W} \sim N(), \sigma^2)$ is independent of $\mathbf{W}$, and $\sigma^2 = 1 - \boldsymbol{\beta}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta}$. Thus the conditional distribution of Z_0 given $\mathbf{W} = \mathbf{w}$ is $N(\boldsymbol{\beta}^\top \boldsymbol{\Sigma}^{-1}\mathbf{w}, \sigma^2)$. As a result, $\mathcal{C}\{\Phi(z_0), \Phi(w_1), \ldots, \Phi(w_d)\} = \Phi\left(\frac{z_0 - \boldsymbol{\beta}^\top \boldsymbol{\Sigma}^{-1}\mathbf{w}}{\sigma}\right)$, yielding formula (12).

Similarly, if $\mathbf{Z} = (Z_0, \mathbf{W})$ has a Student distribution with $\nu > 0$ degrees of freedom and parameter $\boldsymbol{R} = \begin{pmatrix} 1 & \boldsymbol{\beta}^\top \\ \boldsymbol{\beta} & \boldsymbol{\Sigma} \end{pmatrix}$, denoted $\mathbf{Z} \sim T_{d+1}(\nu, \boldsymbol{R})$, then its density is proportional to $\left(1 + \frac{\mathbf{z}^\top \boldsymbol{R}^{-1}\mathbf{z}}{\nu}\right)^{-\frac{(\nu+d+1)}{2}}$, $\mathbf{z} = (z_0, \mathbf{w}) \in \mathbb{R}^{d+1}$. Thus, the conditional distribution of Z_0 given $\mathbf{W} = \mathbf{w}$ is the same as $\boldsymbol{\beta}^\top \boldsymbol{\Sigma}^{-1}\mathbf{w} + \tilde{\sigma}\mathcal{Z}$, where $\tilde{\sigma}^2 = \left(1 - \boldsymbol{\beta}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta}\right)\left(1 + \frac{\mathbf{w}^\top \boldsymbol{\Sigma}^{-1}\mathbf{w}}{\nu}\right)$ and $\mathcal{Z} \sim T_1(\tilde{\nu})$, with $\tilde{\nu} = \nu + d$ degrees of freedom (Simard and Rémillard, 2015, Appendix B). It follows from (2) that

$$\mathcal{C}\{t_\nu(z_0), t_\nu(w_1), \ldots, t_\nu(w_d)\} = t_{\tilde{\nu}}\left(\frac{z_0 - \boldsymbol{\beta}^\top \boldsymbol{\Sigma}^{-1}\mathbf{w}}{\tilde{\sigma}}\right).$$

Formula (13) is then easily obtained.

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