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On the number of shortest paths in Cartesian product graphs and its robustness

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Abstract: In this paper, we establish the maximum number of basic shortest paths in Cartesian product graphs and bounds on the maximum number of the vertex-disjoint shortest paths and on this of the edge-disjoint shortest paths. To the best of our knowledge, the class of Cartesian product graphs has been intensively studied according to various invariants, except the maximum number of (basic, vertex-disjoint or edge-disjoint) shortest paths, whereas the latter invariants were investigated for other graph classes. The main contribution of this paper is to fill this gap. Moreover, we investigate the impact of a vertex or an edge removal on the maximum number of basic shortest paths in Cartesian product graphs.

Keywords: Cartesian product graph, basic shortest path, vertex-disjoint shortest path, edge-disjoint shortest path, edge removal impact, vertex removal impact

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1 Introduction

The Cartesian product of graphs was introduced by G. Sabidussi [24, 25] in 1957, and it has been studied since 1972 in the context of communication networks [7, 22]. Cartesian product graphs are well suited for network design and analysis, regarding scalability, performance, and fault-tolerance [23], due to the following properties. The Cartesian product $G \Box H$ of two connected graphs G and H provides a way of building a graph much larger than the first ones, while keeping relatively small diameter and maximum degree. That is, whereas the order of $G \Box H$ is given by the product of the orders of G and H, its diameter corresponds to the sum of the diameters of G and H, and its maximum degree corresponds to the sum of the maximum degrees of G and H. Besides this, the (edge) connectivity of $G \Box H$ is never less than the sum of the (edge) connectivities of G and H [24]. Thus, whatever the (edge) connectivities of two connected graphs G and H, $G \Box H$ will remain connected after the removal of any single edge or vertex. Such a removal can however impact the number of shortest paths in a Cartesian product graph but, to the best of our knowledge, this problem is not yet covered by the literature, even if other invariants [4, 5, 6, 8, 16, 19, 20, 21, 26, 27] are intensively studied for this class of graphs. The main contribution of this paper is to fill this gap.

Of course, in studying topological properties of interconnection networks, the number of shortest paths for fault-tolerance is frequently studied [9, 10, 11, 14, 18]. For instance, it is well known that there are $H(u \ XOR \ v)!$ distinct shortest paths between nodes u and v in the hypercube graph, where $H(.)$ is the Hamming weight function, i.e., the number of 1's in the binary instance, and XOR the exclusive-or operation. However, as mentioned before, none of them is dedicated to the Cartesian product graphs.

Naturally, in order to obtain networks with specific properties (for example, large number of vertices but small degree and diameter), some authors designed new networks or they generalized well known networks: for instance, the n-star graphs [1], the (n, k)-star graphs [12], the arrangement graphs [13], the higher dimensional hexagonal networks [14]. Moreover, other authors [2, 3] gave their proper definition of the product graphs. However, in this paper, we focus only on the class of Cartesian product graphs, as defined in Section 2.

This paper investigates in Section 3 the number of three kinds of shortest paths in Cartesian product graphs: the basic[1] shortest paths, the vertex-disjoint shortest paths and the edge-disjoint shortest paths. Moreover, we study the impact of a vertex or an edge removal on the first invariant. We provide the needed background of the Cartesian product of graphs in Section 2, and our results are summarized in Section 4. For an overview of Cartesian product graphs, we refer the reader to [16, 15, 17].

2 Background

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two connected graphs with the vertex set $V(G)$, respectively $V(H)$, and the edge set $E(G)$, respectively $E(H)$. An edge between vertices u and u' of G is denoted uu', similarly for the adjacency in H. In this paper, we assume that $|V(G)| \geq 2$ and $|V(H)| \geq 2$. By definition, the Cartesian product $G \Box H$ of these two graphs G and H is the following:

$$V(G \Box H) = \{(u, v) | u \in V(G) \text{ and } v \in V(H)\}$$

and

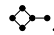
$$E(G \Box H) = \{(u, v)(u, v') | vv' \in E(H)\} \cup \{(u, v)(u', v) | uu' \in E(G)\}$$

We denote by $G^{\bar{v}}$ the induced subgraph of $G \Box H$ on the vertex set $\{(u, \bar{v}) \mid u \in G\}$ and say that $G^{\bar{v}}$ is the copy of G associated to the vertex $\bar{v} \in H$. Conversely, $H^{\bar{u}}$ denotes the induced subgraph on the vertex set $\{(\bar{u}, v) \mid v \in H\}$ and is called the copy of H associated to the vertex $\bar{u} \in G$. Obviously, $G^{\bar{v}}$ is isomorphic to G and $H^{\bar{u}}$ is isomorphic to H, and the different copies of G are connected only by edges in copies of H and vice-versa.

Let $d^G(u, u')$ denote the geodesic distance between vertices u and u' in G, and $d^H(v, v')$ be the one between v and v' in H. Similarly, $d^{G \Box H}((u, v), (u', v'))$ is the geodesic distance between vertices (u, v) and (u', v') in the Cartesian product $G \Box H$. Let $\delta^G(u)$ denote the degree of the vertex u in the graph G, similarly for $\delta^H(v)$.

[1] We say "basic" by opposition of vertex-disjoint and edge-disjoint.

3 Main results

The outcome of the paper is subdivided in three subsections: one for each kind of shortest paths. Through the three subsections, we illustrate the theory by two examples: the cube (see Figure 1) and the Cartesian product of a specific graph $\mathcal{G}$, namely , with itself (see Figure 2). Actually, the former is the Cartesian product of a cycle on 4 vertices and a path on 2 vertices, i.e., take two copies of the cycle on 4 vertices and join two corresponding vertices from different copies by an edge.

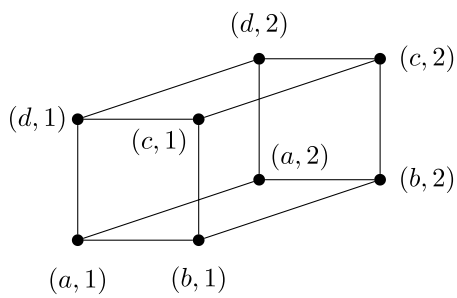


Figure 1: The cube is the Cartesian product of a cycle on vertices $\{a, b, c, d\}$ and a path on vertices $\{1, 2\}$.

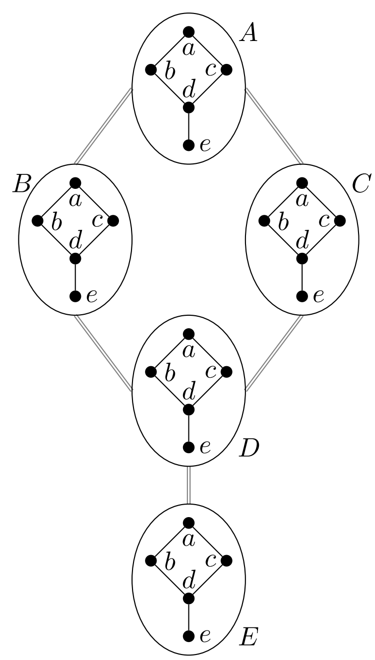


Figure 2: The Cartesian product $\mathcal{G}\square\mathcal{G}$ where edges between two copies of $\mathcal{G}$ are not explicitly drawn.

3.1 Maximum number of basic shortest paths

We denote by $\nu^s_G(u, u')$ the maximum number of basic shortest paths between u and u' in G, similarly in H and in $G\square H$. Following theorems establish the maximum number of basic shortest paths in a Cartesian product graph, and what happens after an edge removal or a vertex removal.

Theorem 1 *The maximum number of basic shortest paths in the Cartesian product graph $G\Box H$ between (u,v) and (u',v') is exactly*

$$\begin{aligned}\nu^s_{G\Box H}((u,v),(u',v')) &= \nu^s_G(u,u')\nu^s_H(v,v')\binom{d^G(u,u')+d^H(v,v')}{d^G(u,u')}\\ &= \nu^s_G(u,u')\nu^s_H(v,v')\binom{d^G(u,u')+d^H(v,v')}{d^H(v,v')}.\end{aligned}$$

Proof. Let $u, u_2, u_3, \ldots, u_k, u'$ be a sequence of vertices in a shortest path P_G in G and $v, v_2, v_3, \ldots, v_j, v'$ be a sequence of vertices in a shortest path P_H in H, where $k = d^G(u,u')$ and $j = d^H(v,v')$. Notice that by property of binomial coefficients, $\binom{d^G(u,u')+d^H(v,v')}{d^H(v,v')} = \binom{d^G(u,u')+d^H(v,v')}{d^G(u,u')}$. We prove that from these two paths P_G and P_H, we can build $\binom{d^G(u,u')+d^H(v,v')}{d^H(v,v')} = \binom{k+j}{j}$ paths from (u,v) to (u',v') in $G\Box H$. By property of the Cartesian product graph, we know that $d^{G\Box H}((u,v),(u',v')) = k+j$, i.e., the shortest paths from (u,v) to (u',v') contain $k+j$ edges. In $G\Box H$, these edges are alternatively from P_G in a copy of G and from P_H in a copy of H. For instance, if we first take all the edges from P_G and then from P_H, the corresponding path is: $(u,v)(u_2,v)(u_3,v)\ldots(u_k,v)(u',v)(u',v_2)(u',v_3)\ldots(u',v_j)(u',v')$, and in the other direction, if we first take all the edges from P_H and then from P_G, vertices $(u,v),(u,v_2),(u,v_3),\ldots,(u,v_j)$, (u,v'), $(u_2,v'),(u_3,v'),\ldots,(u_k,v'),(u',v')$ form a shortest path from (u,v) to (u',v'). Another shortest path is $(u,v)(u_2,v)(u_2,v_2)(u_2,v_3)\ldots(u_2,v_j)(u_3,v_j)\ldots(u_k,v_j)(u',v_j)(u',v')$. By definition of the Cartesian product graph, all shortest paths can be built by this way. Actually, all such shortest paths are well-defined by the paths P_G and P_H if and only if we know exactly which edge is in a copy of which graph: G or H. So, to count the maximum number of shortest paths, it is sufficient to count the number of manners to choose among the $k+j$ edges which ones will be in G, or, equivalently, which ones will be in H. In fact, there are $\binom{k+j}{j}$ such manners. All in all, by considering all shortest paths P_G between u and u' in G and all ones P_H between v and v' in H, we obtain $\nu^s_G(u,u')\nu^s_H(v,v')\binom{k+j}{j}$ shortest paths between (u,v) and (u',v') in $G\Box H$. □

Let's go back to our first example. If we count all possible basic shortest paths between vertices $(a,1)$ and $(c,2)$ in the cube (see Figure 1), we can construct this family by considering shortest paths in the cycle on 4 vertices and those in the path on 2 vertices, and by combining them. Indeed, in the cycle C_4 on vertices $\{a,b,c,d\}$ with the edge set $\{ab,bc,cd,ad\}$, the only 2 shortest paths between a and c are one going through the vertex b (noted by α) and another one going through the vertex d (noted by β). The length of both is 2. Naturally, there is only one direct shortest path in the path P_2 on 2 vertices. To construct shortest paths in the Cartesian product graph, it is sufficient to determine which path we use among $\{\alpha,\beta\}$ and when we switch from the first copy of C_4 to the second one (at the beginning at the position "0", in the middle at the position "1" or at the end at the position "2"). All basic shortest paths of the cube are illustrated by Figure 3. There are exactly six such paths:

$$\nu^s_{C_4}(a,c)\nu^s_{P_2}(1,2)\binom{d^{C_4}(a,c)+d^{P_2}(1,2)}{d^{C_4}(a,c)} = 2\times 1\times\binom{2+1}{2} = 2\times 1\times 3 = 6.$$

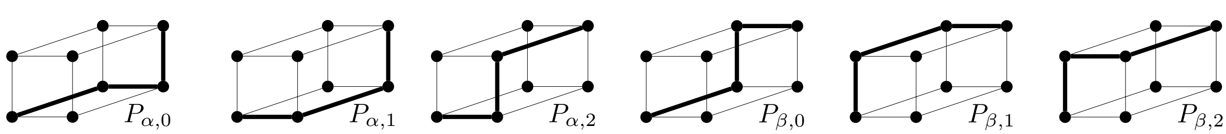


Figure 3: The cube and the only 6 basic shortest paths between $(a,1)$ and $(c,2)$, of length 3.

For the second example, Theorem 1 ensures that the maximum number of basic shortest paths between (E,a) and (A,e) is exactly

$$\begin{aligned}\nu^s_{\mathcal{G}\Box\mathcal{G}}((E,a),(A,e)) &= \nu^s_{\mathcal{G}}(E,A)\times\nu^s_{\mathcal{G}}(a,e)\binom{d^{\mathcal{G}}(E,A)+d^{\mathcal{G}}(a,e)}{d^{\mathcal{G}}(E,A)}\\ &= 2\times 2\times\binom{6}{3}\\ &= 80.\end{aligned}$$

The goal of this example is not to enumerate all basic shortest paths in $\mathcal{G}\Box\mathcal{G}$ but we will see that counting vertex-disjoint shortest paths, or edge-disjoint shortest paths, is not as easy as expected, due to this graph.

Theorem 2 *Let e be an arbitrary edge in $G\Box H$. Without loss of generality, we can suppose that e links vertices (u'',v'') and (u'',v'''). Then the number of shortest paths going through this edge from (u,v) to (u',v') in $G\Box H$ is exactly the following: in the case where the vertex u'' appears in at least one shortest path P_G in G between u and u', and where the edge $v''v'''$ appears in at least one shortest path P_H in H between v and v',*

$$\sum_{P_G:u''\in V(P_G)}\quad\sum_{P_H:v''v'''\in E(P_H)}\binom{d^G(u,u'')+d^H(v,v''v''',P_H)}{d^G(u,u'')}\binom{d^G(u'',u')+d^H(v',v''v''',P_H)}{d^G(u'',u')},$$

where $d^H(x,yz,P_H)=d^H(x,y)$ if y is the first vertex among $\{y,z\}$ encountered in P_H from x, $d^H(x,z)$ otherwise. In other cases, this number is 0.

Proof. Let P_G be a shortest path from u to u' in G and P_H be a shortest path from v to v' in H. Clearly if the vertex u'' does not appear in P_G or if the edge $v''v'''$ does not appear in P_H, no shortest path from (u,v) to (u',v') in $G\Box H$, built from P_G and P_H, goes through the edge e. So we may suppose that $u''\in V(P_G)$ and $v''v'''\in E(P_H)$. Without loss of generality, we can assume that v'' is the first vertex among $\{v'',v'''\}$ encountered in P_H from v. In that case, it is sufficient to count the number of shortest paths between (u,v) and (u'',v'') and the number of shortest paths between (u'',v''') and (u',v'), because a shortest path from (u,v) to (u',v'), going through e, begings with a shortest path from (u,v) to (u'',v''), goes through e and then finishes by a shortest path from (u'',v''') to (u',v'). Therefore, based on the previous result, we obtain

$$\sum_{P_G:u''\in V(P_G)}\quad\sum_{P_H:v''v'''\in E(P_H)}\binom{d^G(u,u'')+d^H(v,v'')}{d^G(u,u'')}\binom{d^G(u'',u')+d^H(v',v''')}{d^G(u'',u')}$$

shortest paths from (u,v) to (u',v'), going through e, which completes the proof. □

With a similar proof of the previous theorem, Theorem 3 counts the number of basic shortest paths going through a vertex.

Theorem 3 *Let (u'',v'') be an arbitrary vertex in $G\Box H$. Then the number of shortest paths going through this vertex from (u,v) to (u',v') in $G\Box H$ is exactly*

$$\sum_{P_G:u''\in V(P_G)}\quad\sum_{P_H:v''\in V(P_H)}\binom{d^G(u,u'')+d^H(v,v'')}{d^H(v,v'')}\binom{d^G(u'',u')+d^H(v'',v')}{d^H(v'',v')},$$

if the vertex u'' (respectively, v'') appears in at least one shortest path P_G (respectively, P_H) in G (respectively, in H) between u and u' (respectively, between v and v'), 0 otherwise.

3.2 Maximum number of vertex-disjoint shortest paths

We denote by $\nu^v_G(u,u')$ the maximum number of vertex-disjoint shortest paths between u and u' in G, similarly in H and in $G\Box H$. Let $G^s_{(u,u')}$ be the subgraph of G induced by all basic shortest paths between u and u', similarly for $H^s_{(v,v')}$. Following theorem establishes lower and upper bounds on the maximum number of vertex-disjoint shortest paths in a Cartesian product graph.

Theorem 4 *The maximum number of vertex-disjoint shortest paths in the Cartesian product graph $G\Box H$ between (u,v) and (u',v') is bounded as follows:*

$$\alpha \geqslant \nu^v_{G\Box H}((u,v),(u',v')) \geqslant \nu^v_G(u,u') + \nu^v_H(v,v'),$$

where α is the minimum between $\delta^{G^s_{(u,u')}}(u) + \delta^{H^s_{(v,v')}}(v)$ and $\delta^{G^s_{(u,u')}}(u') + \delta^{H^s_{(v,v')}}(v')$.

Proof. Because all shortest paths are made from shortest paths in G and shortest paths in H, the number of neighbors of (u,v) (or edges adjacent to (u,v)) used in a shortest path from (u,v) to (u',v') in $G\Box H$ is at most the sum between the number of neighbors of u (or edges adjacent to u) used in a shortest path from u to u' in G and the number of neighbors of v (or edges adjacent to v) used in a shortest path from v to v' in H. Thus,

$$\nu^v_{G\Box H}((u,v),(u',v')) \leqslant \delta^{G^s_{(u,u')}}(u) + \delta^{H^s_{(v,v')}}(v).$$

The same argumentation holds for the other extremity (u',v') to obtain

$$\nu^v_{G\Box H}((u,v),(u',v')) \leqslant \min(\delta^{G^s_{(u,u')}}(u) + \delta^{H^s_{(v,v')}}(v), \delta^{G^s_{(u,u')}}(u') + \delta^{H^s_{(v,v')}}(v')).$$

We will describe $\nu^v_G(u,u') + \nu^v_H(v,v')$ shortest paths from (u,v) to (u',v'). Let $P^\star_G$ be a shortest path from u to u' in G, and let $P^\star_H$ be a shortest path from v to v' in H. For these two paths, we will consider the two shortest paths in $G\Box H$, by taking first all the edges from $P^\star_G$ and then from $P^\star_H$, and conversely, i.e., if $P^\star_G = uu^\star_2\ldots u^\star_k u'$ and $P^\star_H = vv^\star_2\ldots v^\star_j v'$, then one shortest path in $G\Box H$ is $(u,v)(u^\star_2,v)\ldots(u^\star_k,v)(u',v)(u',v^\star_2)\ldots(u',v^\star_j)(u',v')$ whereas the other one is $(u,v)(u,v^\star_2)\ldots(u,v^\star_j)(u,v')(u^\star_2,v')\ldots(u^\star_k,v')(u',v')$. Clearly these two paths are vertex-disjoint.

For any other possible shortest path P_G from u to u' in G (notice that if such path exists, then $d^G(u,u') \geqslant 2$), we proceed in the following way: if $P_G = uu_2\ldots u_k u'$, then taking the first edge from P_G then following $P^\star_H$ except the last edge, finishing P_G and finally finishing $P^\star_H$, i.e., $(u,v)(u_2,v)(u_2,v^\star_2)\ldots(u_2,v^\star_j)(u_3,v^\star_j)\ldots(u_k,v^\star_j)(u',v^\star_j)(u',v')$ is a shortest path from (u,v) to (u',v'), which is vertex-disjoint from the other ones since it is also the case for paths in G. For any other possible shortest path P_H from v to v' in H, a similar argument holds, by symmetry of the Cartesian product graph. □

In our example of the cube, it is easy to see that the maximum number of vertex-disjoint shortest paths between $(a,1)$ and $(c,2)$ is $3 = \nu^v_{C_4}(a,c) + \nu^v_{P_2}(1,2)$. Moreover, an instance of family reaching this bound is $\{P_{\alpha,0}, P_{\alpha,2}, P_{\beta,1}\}$ from Figure 3.

Unfortunately, the lower bound in Theorem 4 is not tight. Indeed, if we strengthen it by $\nu^{s,2}_G(u,u') + \nu^{s,2}_H(v,v')$, where $\nu^{s,2}_G(u,u')$ is the maximum number of basic shortest paths between u and u' in G which are vertex-disjoint for the second vertices and ones before the last, similarly for $\nu^{s,2}_H(v,v')$, then our construction is still valid. However, even with this new lower bound, Theorem 4 remains loose. The second example is a good illustration of this phenomenon:

$$\nu^v_{\mathcal{G}\Box\mathcal{G}}((E,a),(A,e)) = 3 \neq \nu^v_{\mathcal{G}}(E,A) + \nu^v_{\mathcal{G}}(a,e) = \nu^{s,2}_{\mathcal{G}}(E,A) + \nu^{s,2}_{\mathcal{G}}(a,e) = 1+1.$$

because of the three vertex-disjoint shortest paths:

- $(E,a)(D,a)(B,a)(A,a)(A,b)(A,d)(A,e)$
- $(E,a)(E,b)(E,d)(E,e)(D,e)(B,e)(A,e)$
- $(E,a)(E,c)(D,c)(C,c)(C,d)(C,e)(A,e)$

Since the degree of a vertex in an induced subgraph is always upper bounded by the degree of this vertex in the original graph and since the degree of a vertex z in the subgraph induced by all basic shortest paths whose z is one extremity is always upper bouded by the maximum number of basic shortest paths, the upper bound in Theorem 4 can be loosened as mentioned in the following corollary.

Corollary 1 *The maximum number of vertex-disjoint shortest paths in the Cartesian product graph $G\Box H$ between (u,v) and (u',v') is upper bounded as follows:*

$$\nu^s_G(u,u') + \nu^s_H(v,v') \geqslant \nu^v_{G\Box H}((u,v),(u',v'))$$

and

$$\min(\delta^G(u) + \delta^H(v), \delta^G(u') + \delta^H(v')) \geqslant \nu^v_{G\Box H}((u,v),(u',v')).$$

This corollary implies that the maximum number of vertex-disjoint shortest paths in a Cartesian product graph is globally the sum between those in the original graphs, in comparison to the basic shortest path case.

3.3 Maximum number of edge-disjoint shortest paths

We denote by $\nu^e_G(u,u')$ the maximum number of edge-disjoint shortest paths between u and u' in G, similarly in H and in $G\Box H$. The theory about vertex-disjoint shortest paths in the previous subsection yields same results on the maximum number $\nu^e_{G\Box H}((u,v),(u',v'))$ of edge-disjoint shortest paths between (u,v) and (u',v') in $G\Box H$. Indeed, in this paper, we consider only simple[2] graphs and if $P = uu_2 \dots u_k u'$ and $P^\bullet = uu^\bullet_2 \dots u^\bullet_k u'$ are two edge-disjoint shortest paths from u to u' in G then u_2 must be different from $u^\bullet_2$ and u_k must be different from $u^\bullet_k$, which implies that paths in the collection $\mathcal{C}$ described in Theorem 4 are edge-disjoint if it is also the case for paths from u to u' in G and paths from v to v' in H.

Theorem 5 *The maximum number of edge-disjoint shortest paths in the Cartesian product graph $G\Box H$ between vertices (u,v) and (u',v') is bounded as follows:*

$$\alpha \geqslant \nu^e_{G\Box H}((u,v),(u',v')) \geqslant \nu^e_G(u,u') + \nu^e_H(v,v'),$$

where α is the minimum between $\delta^{G^s_{(u,u')}}(u) + \delta^{H^s_{(v,v')}}(v)$ and $\delta^{G^s_{(u,u')}}(u') + \delta^{H^s_{(v,v')}}(v')$.

As expected for our examples, results on the maximum number of edge-disjoint shortest paths are the same as those on the maximum number of vertex-disjoint shortest paths. Naturally, Corollary 1 can be adapted to the edge-disjoint case.

Corollary 2 *The maximum number of edge-disjoint shortest paths in the Cartesian product graph $G\Box H$ between (u,v) and (u',v') is upper bounded as follows:*

$$\nu^s_G(u,u') + \nu^s_H(v,v') \geqslant \nu^e_{G\Box H}((u,v),(u',v'))$$

and

$$\min(\delta^G(u) + \delta^H(v), \delta^G(u') + \delta^H(v')) \geqslant \nu^e_{G\Box H}((u,v),(u',v')).$$

4 Conclusions

In this paper, we found relations on the maximum number of basic, vertex-disjoint and edge-disjoint shortest paths in Cartesian product graph in terms of those from the original graphs. For basic shortest paths, the maximum number in the Cartesian product graph is the product of the maximum numbers in the original graphs multiplied by a binomial coefficient depending on distances in the original graphs, i.e., for every $u,u' \in V(G)$ and $v,v' \in V(H)$,

$$\nu^s_{G\Box H}((u,v),(u',v')) = \nu^s_G(u,u')\nu^s_H(v,v')\binom{d^G(u,u') + d^H(v,v')}{d^G(u,u')}.$$

Besides, we established the exact number of shortest paths going through a fixed vertex or a fixed edge, measuring the impact of a vertex removal or an edge removal.

[2]The word "simple" means here "without loops or multiple edges".

However, for vertex-disjoint and edge-disjoint shortest paths, the results are quite different from the basic case. Instead of a product, the maximum number of such paths in the Cartesian product graph is roughly the sum of the corresponding maximum numbers in the original graphs, i.e., for every $u, u' \in V(G)$ and $v, v' \in V(H)$,

$$\nu^s_G(u,u') + \nu^s_H(v,v') \geqslant \nu^v_{G\Box H}((u,v),(u',v')) \geqslant \nu^v_G(u,u') + \nu^v_H(v,v')$$

and

$$\nu^s_G(u,u') + \nu^s_H(v,v') \geqslant \nu^e_{G\Box H}((u,v),(u',v')) \geqslant \nu^e_G(u,u') + \nu^e_H(v,v').$$

Computing the maximum number of vertex-disjoint/edge-disjoint shortest paths in the Cartesian product graphs is not as easy as we can expect since a family of vertex-disjoint/edge-disjoint shortest paths can be built from families of shortest paths which are not necessary vertex-disjoint/edge-disjoint, as illustrated by our second example. Consequently, the two following open questions remain.

Open question 1 *Is it possible to compute precisely $\nu^v_{G\Box H}((u,v),(u',v'))$ in terms of $\nu^v_G(u,u')$ and $\nu^v_H(v,v')$? Similar question holds for $\nu^e_{G\Box H}((u,v),(u',v'))$ in terms of $\nu^e_G(u,u')$ and $\nu^e_H(v,v')$.*

Open question 2 *Is it possible to describe all cases for which $\nu^v_{G\Box H}((u,v),(u',v'))$ is decreasing by 1 after a vertex removal / an edge removal? Same question for $\nu^e_{G\Box H}((u,v),(u',v'))$.*

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