

**An RLT approach for solving the
binary-constrained mixed linear
complementarity problem**

F. Djeumou Fomeni,
S.A. Gabriel, M.F. Anjos

G-2015-60

June 2015

Les textes publiés dans la série des rapports de recherche *Les Cahiers du GERAD* n'engagent que la responsabilité de leurs auteurs.

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2015.

The authors are exclusively responsible for the content of their research papers published in the series *Les Cahiers du GERAD*.

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

Legal deposit – Bibliothèque et Archives nationales du Québec, 2015.

An RLT approach for solving the binary-constrained mixed linear complementarity problem

Franklin Djeumou Fomeni^a

Steven A. Gabriel^b

Miguel F. Anjos^c

^a GERAD & Department of Mathematics and Industrial Engineering, Polytechnique Montréal, Montréal (Québec) Canada, H3C 3A7

^b Department of Mechanical Engineering, and Applied Mathematics & Statistics, and Scientific Computation Program, University of Maryland, College Park, MD 20742-3021, USA

^c Canada Research Chair in Discrete Nonlinear Optimization in Engineering, GERAD & Polytechnique Montréal, Montréal (Québec) Canada, H3C 3A7

franklin.djeumou-fomeni@polymtl.ca
sgabriel@umd.edu
anjios@stanfordalumni.org

June 2015

Les Cahiers du GERAD

G–2015–60

Copyright © 2015 GERAD

Abstract: It is well known that the mixed linear complementarity problem can be used to model the equilibrium in energy markets as well as a host of other engineering and economic problems. The *binary-constrained mixed linear complementarity problem* is a formulation of the mixed linear complementarity problem in which some variables are restricted to be binary. This paper presents a novel approach for solving the binary-constrained mixed linear complementarity problem. First we solve a series of linear programs that enables us to replace some of the complementarity constraints with linear equations. Then we solve an equivalent mixed integer linear program formulation of the original binary-constrained mixed linear complementarity problem (with a smaller number of complementarity constraints) to guarantee a solution to the problem. Our computational results on a wide range of test problems, including some engineering examples, demonstrate the usefulness and the effectiveness of this novel approach.

Key Words: Mixed linear complementarity problems, mixed integer linear program, reformulation linearization technique, electricity market, linear programs.

Acknowledgments: The work of F. Djeumou Fomeni was supported by a GERAD postdoctoral fellowship and by the Canada Research Chair in Discrete Nonlinear Optimization in Engineering at Polytechnique Montréal. Part of the work of S.A. Gabriel was done during a stay at GERAD as Trottier Senior Visiting Professor for 2014-2015, Institut de l'énergie Trottier, Polytechnique Montréal.

1 Introduction

The discretely-constrained mixed linear complementarity problem (DC-MLCP) is a formulation of the mixed linear complementarity problem (MLCP) in which some of the variables are restricted to be discrete. The DC-MLCP has many practical applications. For example, it includes a realistic model of market equilibrium in electric power markets and for grid based industries [17, 18, 31]. In most of these applications, the discrete variables are binary and they might for example represent the ON/OFF position of power generators. Therefore, this paper focuses on DC-MLCP with binary variables, we will call this problem the binary-constrained mixed linear complementarity problem (BC-MLCP).

More formally, the mathematical formulation of the BC-MLCP is described as follows. Given the vector $q = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} \in \mathbb{R}^n$ and the matrix $A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \in \mathbb{R}^{n \times n}$, find $\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $n_1 + n_2 = n$, such that:

$$0 \leq q^1 + \begin{pmatrix} A^{11} & A^{12} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \perp z^1 \geq 0 \quad (1a)$$

$$0 = q^2 + \begin{pmatrix} A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \quad z^2 \text{ free}, \quad (1b)$$

where some components of the variable $\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$ are constrained to be binary. z^1 and z^2 are called the complementarity and the free variables, respectively.

The DC-MLCP has not been studied widely in the literature. Gabriel *et al.* [17] proposed a mixed linear integer programming (MILP) approach in which the complementarity constraints, as well as the integrality of the discrete variables are relaxed. They showed that their approach was suitable to energy markets. Their approach has also been successfully used to solve the Nash-Cournot game with application to power markets [18], and to solve the electricity pool pricing [31].

As opposed to the DC-MLCP, the linear complementarity problem (LCP) has attracted a lot of interest due to its wide range of applications [10, 6, 7, 2, 38]. Several solution techniques have been proposed in the literature for the LCP [11, 24, 28]. Most of these techniques (Lemke's algorithm, Projected Gauss Seidel, etc.) are iterative algorithms and they make use of linear algebra to solve or approximately solve the LCP, see [25, 26, 5, 36, 37, 28]. Some of these methods are impressively successful when the matrix A has a particular structure. The classes of positive semi-definite (PSD) and P matrices are the most interesting of these classes. For the class of PSD matrices, the LCP always has a solution provided it is feasible i.e. the linear constraints are consistent, as shown by Cottle *et al.* [6].

It has been shown that solving an LCP is equivalent to finding a stationary point of a bilinear programming problem (BPP) [29]. In this case, the LCP has a solution if and only if there is a global solution to the corresponding BPP with optimal value equal to zero. However, the BPP is an \mathcal{NP} -hard problem. Therefore using global optimization techniques for the BPP in the case of the LCP will not make the problem less hard. Al-Khayyal [1], Júdice and Faustino [22], and Júdice [23] have solved the LCP using an equivalent quadratic programming formulation.

In this paper, we propose a new solution method for the BC-MLCP. Our method consists of three components. Firstly, we implement a bound refinement procedure to tighten the upper bounds of the complementarity variables. Secondly, we consider the RLT relaxation of the complementarity constraints and solve a series of LPs that enables us to replace some of these constraints with linear equations. We present theoretical results to show that we do not lose any point that satisfies the BC-MLCP constraints. Finally, we have a third component to our algorithm which ensures that it always finds a solution that satisfies all the constraints of the BC-MLCP. This is the approach proposed by Sherali *et al.* [34] for LCPs. It consists of solving an MILP that is equivalent to the BC-MLCP. We validate our proposed solution method on some 1400 test instances that include well known engineering applications.

The rest of this paper is organized as follows. In Section 2, we present the different reformulations of the problem that the reader will encounter in this paper. We review the relevant literature in Section 3. The

description of our proposed solution approach is presented in Section 4 and the computational experience is discussed in Section 5. Finally some concluding remarks are given in Section 6.

Notation: Let $N = \{1, \dots, n_1, n_1 + 1, \dots, n_1 + n_2\} = \{1, \dots, n\}$, $B = \{i \in N : z_i \text{ is binary}\}$ and $N_1 = \{1, \dots, n_1\}$. In this paper, n_1 and n_2 designate the number of complementarity and free variables, respectively, and $n = n_1 + n_2$ is the total number of variables.

2 Problem formulations

In this section we present the different formulations used throughout this paper.

2.1 An equivalent reformulation of the BC-MLCP

In order to simplify the notation used for problem (1), we set $x = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$, and we let $A_i \in \mathbb{R}^{1 \times n}$ be the i^{th} row of the matrix A , and q_i the i^{th} element of the vector q . Then solving the BC-MLCP is equivalent to finding a point $x \in \mathbb{R}^n$ that satisfies the following system:

$$q_i + A_i x \geq 0 \quad \text{for } i = 1, \dots, n_1 \quad (2a)$$

$$(q_i + A_i x)x_i = 0 \quad \text{for } i = 1, \dots, n_1 \quad (2b)$$

$$q_i + A_i x = 0 \quad \text{for } i = n_1 + 1, \dots, n_1 + n_2 \quad (2c)$$

$$x_i \text{ free} \quad \text{for } i = n_1 + 1, \dots, n_1 + n_2 \quad (2d)$$

$$x_i \geq 0 \quad \text{for } i = 1, \dots, n_1 \quad (2e)$$

$$x_i \in \{0, 1\} \quad \text{for } i \in B \quad (2f)$$

Assumption 1 There exists a finite solution to the BC-MLCP i.e., all the continuous variables satisfy $-\infty < l_i \leq x_i \leq u_i < +\infty$ for $i \in N \setminus B$.

Gabriel et al. [17] give motivations for Assumption 1, and propose a method for finding suitable bounds.

Lemma 1 Suppose that Assumption 1 is in force such that $0 \leq z^1 \leq u^1$ with $u_i^1 > 0$ for all $i \in N_1 \setminus B$, and $l^2 \leq z^2 \leq u^2$ with $u_i^2 > l_i^2$ for all $i \in N_2 \setminus B$. Then, without loss of generality, $z_i \in [0, 1]$ for all $i \in N \setminus B$.

Proof. From the lemma's premise, there exists finite bounds such that $0 \leq z_i^1 \leq u_i^1$ for $i \in N_1 \setminus B$ and $l_i^2 \leq z_i^2 \leq u_i^2$ for $i \in N_2 \setminus B$. Thus, we can form a new variables $s_i^1 = \frac{z_i^1}{u_i^1}$ and $s_i^2 = \frac{z_i^2 - l_i^2}{u_i^2 - l_i^2}$ with $0 \leq s_i^1, s_i^2 \leq 1$. The original LCP (q, A) is given as

$$\begin{aligned} 0 &\leq q^1 + A^{11}z^1 + A^{12}z^2 \perp z^1 \geq 0 \\ 0 &= q^2 + A^{21}z^1 + A^{22}z^2, z^2 \text{ free} \end{aligned}$$

where z^1, z^2 are respectively, the vector of nonnegative and free variables and where $q = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}$, $A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}$, conformal with z^1, z^2 . Define the positive diagonal matrices $D^1 = \text{diag}(u^1)$ and $D^2 = \text{diag}(u^2 - l^2)$ whose diagonals are the u_i^1 and $u_i^2 - l_i^2$ values for z^1 and z^2 , respectively. Clearly, in light of the positive diagonals of D^1 , $s_i^1 = 0 \Leftrightarrow z_i^1 = 0$ and $s_i^1 > 0 \Leftrightarrow z_i^1 > 0$. Note that D^2 also has positive diagonals. We do have the following:

$$\begin{aligned} q^1 + A^{11}z^1 + A^{12}z^2 &= q^1 + A^{11}D^1s^1 + A^{12}D^2s^2 + A^{12}l^2 \\ &= (q^1 + A^{12}l^2) + A^{11}D^1s^1 + A^{12}D^2s^2 \\ &= \tilde{q}^1 + \tilde{A}^{11}s^1 + \tilde{A}^{12}s^2, \end{aligned}$$

where $\tilde{q}^1 = q^1 + A^{12}l^2$, $\tilde{A}^{11} = A^{11}D^1$ and $\tilde{A}^{12} = A^{12}D^2$.

Analogously, $0 = q^2 + A^{21}z^1 + A^{22}z^2 \Leftrightarrow 0 = \tilde{q}^2 + \tilde{A}^{21}s^1 + \tilde{A}^{22}s^2$, with $\tilde{q}^2 = q^2 + A^{22}l^2$, $\tilde{A}^{21} = A^{21}D^1$ and $\tilde{A}^{22} = A^{22}D^2$, so that the solution set of $\text{LCP}(q, A)$ maps one-to-one to the solution set for $\text{LCP}(\tilde{q}, \tilde{A})$, where $\tilde{q} = \begin{pmatrix} \tilde{q}^1 \\ \tilde{q}^2 \end{pmatrix}$, and $\tilde{A} = \begin{pmatrix} \tilde{A}^{11} & \tilde{A}^{12} \\ \tilde{A}^{21} & \tilde{A}^{22} \end{pmatrix}$. Thus, without loss of generality it can be assumed that $z_i \in [0, 1]$ for all $i \in N \setminus B$. \square

We can therefore define the following set \mathcal{F} , which is equivalent (in the sense that a solution x to any optimization problem over the set \mathcal{F} is a solution to the BC-MLCP) to the set of all the solutions of the BC-MLCP that satisfy Assumption 1.

$$\mathcal{F} = \left\{ x \in \mathbb{R}^n \left| \begin{array}{ll} q_i + A_i x \geq 0 & \text{for } i = 1, \dots, n_1 \\ (q_i + A_i x)x_i = 0 & \text{for } i = 1, \dots, n_1 \\ q_i + A_i x = 0 & \text{for } i = n_1 + 1, \dots, n_1 + n_2 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \setminus B \\ x_i \in \{0, 1\} & \text{for } i \in B \end{array} \right. \right\}.$$

Note that \mathcal{F} does not directly relate to an MLCP as it is not a square system. This is because the constraints $x_i \leq 1$ for all $i \in N \setminus B$ and $x_i \in \{0, 1\}$ for all $i \in B$ do not have associated dual variables. The rest of this paper focuses on finding a point $x \in \mathcal{F}$ that is a solution to the BC-MLCP.

2.2 The $\tilde{\mathcal{F}}$ relaxation of the set \mathcal{F}

Following the idea of the RLT relaxation [33], we introduce the variables $y_{ij} = x_i x_j$ for all $i, j \in N$ and relax the quadratic constraint $y_{ij} = x_i x_j$ with the linear inequalities $y_{ij} \geq 0$, $x_i - y_{ij} \geq 0$, $x_j - y_{ij} \geq 0$ and $y_{ij} - x_i - x_j + 1 \geq 0$ for all pairs i, j . These inequalities are also known as the McCormick inequalities [27]. We also relax the binary constraints as $0 \leq x_i \leq 1$ for $i \in B$.

The RLT relaxation of the set \mathcal{F} is given by the set

$$\tilde{\mathcal{F}} = \left\{ x \in \mathbb{R}^n \left| \begin{array}{ll} q_i + A_i x \geq 0 & \text{for } i = 1, \dots, n_1 \\ q_i x_i + \sum_{j=1}^n A_{ij} y_{ij} = 0 & \text{for } i = 1, \dots, n_1 \\ q_i + A_i x = 0 & \text{for } i = n_1 + 1, \dots, n_1 + n_2 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \\ y_{ij} \geq 0 & \text{for } i, j \in N \\ y_{ij} \leq x_i & \text{for } i, j \in N \\ y_{ij} \leq x_j & \text{for } i, j \in N \\ y_{ij} + 1 \geq x_i + x_j & \text{for } i, j \in N \end{array} \right. \right\}.$$

It is important to note that if both x_i and x_j are binary, then the McCormick inequalities are equivalent to $y_{ij} = x_i x_j$. Otherwise, they simply represent valid inequalities or linear over- and under-estimators of the quadratic equation $y_{ij} = x_i x_j$. This is the reason why $\tilde{\mathcal{F}}$ is a relaxation of the set \mathcal{F} . In addition, it is clear that for all $x \in \mathcal{F}$, one can find $y \in \mathbb{R}^{\binom{n}{2}}$ such that $(x, y) \in \tilde{\mathcal{F}}$ (take $y \in \mathbb{R}^{\binom{n}{2}}$ such that its components correspond to $y_{ij} = x_i x_j$).

2.3 Sub-problems related to $\tilde{\mathcal{F}}$

In this paper, we present an LP-based procedure that successively fixes the complementarity constraints $(q_i + A_i x)x_i = 0$ to either $q_i + A_i x = 0$ or $x_i = 0$, for $i \in N_1$. So, for a given index $i_0 \in N_1$, we define the

corresponding relaxed sets as:

$$\tilde{\mathcal{F}}_{i_0}^+ = \left\{ x \in \mathbb{R}^n \left| \begin{array}{ll} q_i + A_i x \geq 0 & \text{for } i = 1, \dots, n_1 \\ q_i x_i + \sum_{j=1}^n A_{ij} y_{ij} = 0 & \text{for } i = 1, \dots, n_1, i \neq i_0 \\ q_i + A_i x = 0 & \text{for } i = n_1 + 1, \dots, n_1 + n_2 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \\ y_{ij} \geq 0 & \text{for } i, j \in N \\ y_{ij} \leq x_i & \text{for } i, j \in N \\ y_{ij} \leq x_j & \text{for } i, j \in N \\ y_{ij} + 1 \geq x_i + x_j & \text{for } i, j \in N \\ q_{i_0} + A_{i_0} x = 0 & \end{array} \right. \right\}$$

and

$$\tilde{\mathcal{F}}_{i_0}^0 = \left\{ x \in \mathbb{R}^n \left| \begin{array}{ll} q_i + A_i x \geq 0 & \text{for } i = 1, \dots, n_1 \\ q_i x_i + \sum_{j=1}^n A_{ij} y_{ij} = 0 & \text{for } i = 1, \dots, n_1, i \neq i_0 \\ q_i + A_i x = 0 & \text{for } i = n_1 + 1, \dots, n_1 + n_2 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \\ y_{ij} \geq 0 & \text{for } i, j \in N \\ y_{ij} \leq x_i & \text{for } i, j \in N \\ y_{ij} \leq x_j & \text{for } i, j \in N \\ y_{ij} + 1 \geq x_i + x_j & \text{for } i, j \in N \\ x_{i_0} = 0 & \end{array} \right. \right\}.$$

$\tilde{\mathcal{F}}_{i_0}^+$ and $\tilde{\mathcal{F}}_{i_0}^0$ differ from $\tilde{\mathcal{F}}$ only in that the i_0^{th} equation of the form $q_i x_i + \sum_{j=1}^n A_{ij} y_{ij} = 0$ has been replaced by the equations $q_{i_0} + A_{i_0} x = 0$ and $x_{i_0} = 0$, respectively.

3 Literature review

We refer the reader to [6, 5] for more details on LCPs. For the sake of brevity, we restrict our literature review to approaches that are relevant for a better understanding of this paper. Specifically, we focus our review on the RLT method proposed by Sherali et al. [34] for solving the LCP, and the bound refinement procedure of Sherali and Tuncbilek [35] which can be useful to strengthen the relaxation of \mathcal{F} .

3.1 The RLT approach for the LCP

Sherali et al. [34] proposed an enumerative algorithm that solves the LCP exactly. Their approach solves an equivalent MILP reformulation of the LCP based on the RLT relaxation. In fact they showed that the LCP, as formulated in (1) with $n_2 = 0$, is equivalent to the following MILP:

$$\min_{v, w} \quad q^T v + \sum_{i=1}^n \sum_{j=1}^n A_{ij} w_{ij} \quad (3a)$$

$$s.t. \quad (3b)$$

$$\sum_{j=1}^n A_{kj} w_{ij} + q_k v_i \geq 0 \quad \forall (i, k) \quad (3c)$$

$$\sum_{j=1}^n A_{kj} x_j + q_k \geq \sum_{j=1}^n A_{kj} w_{ij} + q_k v_i \quad \text{for } i, k \in N \quad (3d)$$

$$0 \leq w_{ij} \leq 1 \quad \text{for } i, j \in N \quad (3e)$$

$$w_{jj} = x_j \quad \text{for } j \in N \quad (3f)$$

$$v_j \in \{0, 1\} \quad \text{for } j \in N \quad (3g)$$

$$w_{ij} \geq 0 \quad \text{for } i, j \in N \quad (3h)$$

$$w_{ij} \leq x_i \quad \text{for } i, j \in N \quad (3i)$$

$$w_{ij} \leq v_j \quad \text{for } i, j \in N \quad (3j)$$

$$w_{ij} + 1 \geq x_i + v_j \quad \text{for } i, j \in N. \quad (3k)$$

Note that $v_i = 0$ if $x_i = 0$ and $v_i = 1$ if $x_i > 0$, and the variable w_{ij} represents the quadratic term $x_i v_j$ for all pair i, j . The paper [34] proposed a branch-and-bound method for solving problem (3). It is important to mention that such a branch-and-bound algorithm can converge even faster if some valid inequalities are added to tighten this RLT relaxation. Audet et al. [3] and Djeumou-Fomeni et al. [8] propose several families of cutting planes for this purpose.

The second component of the solution approach proposed in the current paper uses an RLT relaxation of the complementarity constraints. However, it differs from the method in [34] in the sense that it uses the traditional RLT relaxation ($y_{ij} = x_i x_j$) without introducing new binary variables. It also solves a series of LPs in order to replace some of the complementarity constraints with linear equations.

3.2 Bounds refinement

One way of strengthening the relaxation of the set \mathcal{F} is by tightening the bounds of each variable x_i . Given the nonconvex nature of the set \mathcal{F} , finding such bounds is non-trivial. A range reduction procedure was proposed for this purpose by Sherali and Tuncbilek [35]. This technique, which proved to be useful for solving nonconvex optimization problems (see [3]), works as follows. For $0 \leq x_i \leq 1$, one can obtain bounds \tilde{l}_i, \tilde{u}_i on variable x_i s.t. $0 \leq \tilde{l}_i \leq x_i \leq \tilde{u}_i \leq 1$ by minimizing (respectively maximizing) x_i subject to the set $\tilde{\mathcal{F}}$. Given the solutions (x^-, y^-) (respectively (x^+, y^+)) of such a minimization (respectively maximization) problem, the bounds \tilde{l}_i and \tilde{u}_i are given by $\tilde{l}_i := \max\{x_i^-, \sqrt{y_{ii}^-}\}$, $\tilde{u}_i := \min\{x_i^+, \sqrt{y_{ii}^+}\}$. This procedure can be repeated as long as the bound of one of the variables is improved.

4 Our solution approach for the BC-MLCP

In this section we describe our proposed solution approach for the BC-MLCP. This approach proceeds in three main steps. The first step consists of reducing the upper bounds of each complementarity variable using the bound refinement procedure of Sherali and Tuncbilek [35] as discussed in Subsection 3.2. In the second step, we solve a series of LPs which enables us to fix some of the complementarity constraints to linear equations. Finally, we solve an MILP that is equivalent to the original BC-MLCP with a reduced number of complementarity constraints. In Subsection 4.1, we show how some complementarity constraints can be replaced by linear equations, and in Subsection 4.2 we present our proposed solution method for the BC-MLCP.

4.1 Replacement of the complementarity constraints

In this section we show how we can replace some of the complementarity constraints by solving a series of LPs. One way of finding a point that belongs to the set \mathcal{F} is by solving any optimization problem over the set defined by \mathcal{F} . Any such problem will be an \mathcal{NP} -hard problem as it is a nonconvex mixed integer nonlinear program. However, we show that one can solve a finite series of LPs over the set $\tilde{\mathcal{F}}$ and use the results to replace some of the complementarity constraints with linear equations. In order to do so, we proceed as follows.

For $i_0 \in N_1$, we define the following LP : $\min_{(x,y) \in \tilde{\mathcal{F}}} x_{i_0}$. We can use the solution of this LP to replace the complementarity constraint $(q_{i_0} + A_{i_0} x)x_{i_0} = 0$ with a linear equation using the following result.

Theorem 1 *Let us assume that $\tilde{\mathcal{F}}$ is non-empty. For $i_0 \in N_1$, let $x_{i_0}^*$ be the optimal objective value of $\min_{(x,y) \in \tilde{\mathcal{F}}} x_{i_0}$. We have the following:*

1. *If $x_{i_0}^* \neq 0$, then all the feasible points of $\tilde{\mathcal{F}}$ are contained in the set defined by $\tilde{\mathcal{F}}_{i_0}^+$. Thus the constraint $(q_{i_0} + A_{i_0} x)x_{i_0} = 0$ can be replaced by $q_{i_0} + A_{i_0} x = 0$.*

2. If $x_{i_0}^* = 0$ and the problem $\min_{(x,y) \in \tilde{\mathcal{F}}_{i_0}^+} x_{i_0}$ is infeasible, then all the feasible points of $\tilde{\mathcal{F}}$ are contained in the set defined by $\tilde{\mathcal{F}}_{i_0}^0$. Thus the constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ can be replaced by $x_{i_0} = 0$.

Proof.

1. By definition the set $\tilde{\mathcal{F}}$ contains the set \mathcal{F} . In addition, $x_{i_0}^* \neq 0$ means that there is no feasible point $(x, y) \in \tilde{\mathcal{F}}$ that satisfies $x_{i_0} = 0$. Therefore, there is no such point in \mathcal{F} either, and the complementarity constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ can only be satisfied if $q_{i_0} + A_{i_0}x = 0$.
2. Let the problem $\min_{(x,y) \in \tilde{\mathcal{F}}_{i_0}^+} x_{i_0}$ be infeasible. This means that there is no feasible point $(x, y) \in \tilde{\mathcal{F}}$ that satisfies $q_{i_0} + A_{i_0}x = 0$. Therefore, there is no such point in \mathcal{F} either, and the complementarity constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ can only be satisfied if $x_{i_0} = 0$.

□

Remark: It is important to note that when $x_{i_0}^* = 0$ and the problem $\min_{(x,y) \in \tilde{\mathcal{F}}_{i_0}^+} x_{i_0}$ is feasible, no conclusion can be drawn i.e. we do not have any justifiable reason to replace the complementarity constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ with either $q_{i_0} + A_{i_0}x = 0$ or $x_{i_0} = 0$.

A result that follows from Theorem 1 and that can help limiting the number of LPs to be solved is described as follows. We consider the LP $\min_{(x,y) \in \tilde{\mathcal{F}}} x_{i_0}$, and let (x^*, y^*) be an optimal solution and $x_{i_0}^*$ its optimal objective value. We define ρ_i to be the *allowable increase* associated with the variable x_i for $i = 1, \dots, n$. By definition, the allowable increase of a variable for an LP defines the range within which if the coefficient of this variable in the objective function is increased, the optimal solution will be unchanged. We refer the reader to [4] for further details on the sensitivity analysis of linear programming. We have the following result.

Theorem 2 *Let us assume that $\tilde{\mathcal{F}}$ is non-empty. For $i_0 \in N_1$, let $x_{i_0}^*$ be the optimal objective value of $\min_{(x,y) \in \tilde{\mathcal{F}}} x_{i_0}$. Suppose that $x_{i_0}^* = 0$. For $i \neq i_0$, if $x_i^* \neq 0$ and $\rho_i \geq 1$, then we can replace the complementarity constraint $(q_i + A_i x)x_i = 0$ with the equation $q_i + A_i x = 0$.*

Proof. Let $i \neq i_0$ so that $x_i^* > 0$ and $\rho_i \geq 1$. Note that the coefficient of x_i in the current objective function is 0. From the definition of the allowable increase, $\rho_i \geq 1$ implies that (x^*, y^*) will still be an optimal solution of the LP $\min_{(x,y) \in \tilde{\mathcal{F}}} (x_{i_0} + x_i)$. Since $x_{i_0}^* = 0$ and $x_i^* > 0$, then for the same reasons as in Theorem 1, there is no point $(x, y) \in \tilde{\mathcal{F}}$ that satisfies $x_i = 0$. Consequently, the complementarity condition $(q_i + A_i x)x_i = 0$ can only be satisfied by $q_i + A_i x = 0$. □

The result of Theorem 1 enables us to design a procedure that successively replaces some of the complementarity constraints with linear equations. This procedure starts with $i_0 = 1$, then solves the LP $\min_{(x,y) \in \tilde{\mathcal{F}}} x_{i_0}$ and if any condition in Theorem 1 is satisfied, then the complementarity constraint $(q_{i_0} + A_{i_0}x)x_{i_0} = 0$ is replaced appropriately. This is then repeated for $i_0 = 2, \dots, n_1$, except for indexes $i \in N_1$ that already satisfy the condition of Theorem 2. The detailed step-by-step execution of this procedure is given by Algorithm 1. Our computational experiments show that this procedure alone is capable of solving some BC-MLCP instances.

4.2 On finding a solution of the BC-MLCP

In this subsection we describe the different steps of our proposed solution method for the BC-MLCP. Our approach proceeds in three main steps. First, we reduce the upper bounds of each complementarity variable. Then we fix some of the complementarity constraints to linear equations. Finally we solve an MILP that is equivalent to the original BC-MLCP with a reduced number of complementarity constraints. A more detailed description of each step is given as follows.

Algorithm 1 : Procedure to reduce the number of complementarity constraints

```

Initialize  $i_0 = 1$ 
while  $i_0 \leq n_1$  do
  Solve the LP  $\min_{(x,y) \in \tilde{\mathcal{F}}} x_{i_0}$ .
  if the above LP is infeasible, then
    Stop and exit (the BC-MLCP is infeasible).
  else
    Let  $(x^*, y^*)$  be its optimal solution.
    if  $(q_i + A_i x^*)x_i^* = 0$  for all  $i = 1, \dots, n_1$  and  $x_i^* \in \{0, 1\}$  for all  $i \in B$  then
      Stop and output  $x^*$  as a solution of the BC-MLCP.
    else
      if  $x_{i_0}^* > 0$  then
        Replace the constraint  $q_{i_0}x_{i_0} + \sum_{j=1}^n A_{i_0}y_{i_0,j} = 0$ 
        with the equation  $q_{i_0} + A_{i_0}x = 0$ .
        (We can further set  $x_{i_0} = 1$  if  $i_0 \in B$ .)
      else
        Solve the LP  $\min_{(x,y) \in \tilde{\mathcal{F}}_{i_0}^+} x_{i_0}$ .
        if this LP is infeasible then
          Set  $x_{i_0} = 0$ .
        end if
      end if
    end if
     $i_0 = i_0 + 1$ 
  end while

```

1. **Reducing the upper bounds of complementarity variables:** In order to reduce the upper bound of each complementarity variable, we implement the bound refinement procedure of Sherali and Tuncbilek [35] described in Subsection 3.2. So for each $i \in \{1, \dots, n_1\}$, we solve the LP $\max_{x \in \tilde{\mathcal{F}}} x_i$, and replace the upper bound of the variable x_i with the optimal objective value of this LP. This reduction process contributes to the success of Algorithm 1. For example, the improved upper bound on the variable x_i can cut-off the intersection between the hyperplane $q_i + A_i x = 0$ and the x_i axis. This makes it easier for Algorithm 1 to replace the complementarity constraint $(q_i + A_i x)x_i = 0$ with either $q_i + A_i x = 0$ or $x_i = 0$.
2. **Replacing the complementarity constraints:** At this stage, we implement the procedure described in Algorithm 1, which is the result of Theorem 1. It is important to note that fixing one complementarity constraint may affect the fixing of another complementarity constraint. This suggests that the order in which the variables are considered can affect the total number of complementarity constraints fixed. Therefore, in our implementation we have decided to repeat Algorithm 1 until it is no longer possible to fix any complementarity constraint.
 At the end of this step, we define the following sets: $N_1^+ = \{i \in N_1 : (q_i + A_i x)x_i = 0 \text{ is replaced by } q_i + A_i x = 0\}$ and $N_1^0 = \{i \in N_1 : (q_i + A_i x)x_i = 0 \text{ is replaced by } x_i = 0\}$.
3. **Solving the final MILP reformulation of the BC-MLCP:** The procedure of replacing the complementarity constraints may end with either $N_1^+ \cup N_1^0 = N_1$ or $N_1^+ \cup N_1^0 \subsetneq N_1$.
 If $N_1^+ \cup N_1^0 = N_1$ then all the complementarity constraints have been fixed and therefore the set \mathcal{F} is equivalent to

$$\mathcal{F}^1 = \left\{ x \in \mathbb{R}^n \left| \begin{array}{ll} q_i + A_i x = 0 & \text{for } i \in N_1^+ \\ x_i = 0 & \text{for } i \in N_1^0 \\ q_i + A_i x = 0 & \text{for } i \in N \setminus N_1 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \setminus B \\ x_i \in \{0, 1\} & \text{for } i \in B \end{array} \right. \right\}.$$

Hence, we find a solution to the BC-MLCP by solving an MILP over the set \mathcal{F}^1 with an arbitrary objective function.

If $N_1^+ \cup N_1^0 \subsetneq N_1$ i.e., only some of the complementarity constraints have been fixed and therefore the set \mathcal{F} is equivalent to

$$\mathcal{F}^2 = \left\{ x \in \mathbb{R}^n \left| \begin{array}{ll} q_i + A_i x \geq 0 & \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0) \\ (q_i + A_i x)x_i = 0 & \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0) \\ q_i + A_i x = 0 & \text{for } i \in N_1^+ \\ x_i = 0 & \text{for } i \in N_1^0 \\ q_i + A_i x = 0 & \text{for } i \in N \setminus N_1 \\ 0 \leq x_i \leq 1 & \text{for } i \in N \setminus B \\ x_i \in \{0, 1\} & \text{for } i \in B \end{array} \right. \right\}.$$

Note that finding a solution in the set \mathcal{F}^2 is again equivalent to finding a solution of a BC-MLCP problem, but with a smaller set of complementarity variables. In our approach, we follow the idea of Sherali et al. [34] and define an MILP in the form of (3), which is equivalent to finding a solution in the set \mathcal{F}^2 . This MILP is defined by:

$$\min_{x,v,w} \quad q^T v + \sum_{i \in N_1 \setminus (N_1^+ \cup N_1^0)} \sum_{j \in N} A_{ij} w_{ij} \quad (4a)$$

$$s.t. \quad (4b)$$

$$\sum_{j=1}^n A_{kj} w_{ij} + q_k v_i \geq 0 \quad \forall i, k \in N_1 \setminus (N_1^+ \cup N_1^0) \quad (4c)$$

$$\sum_{j=1}^n A_{kj} x_j + q_k \geq \sum_{j=1}^n A_{kj} w_{ij} + q_k v_i \quad \text{for } i, k \in N_1 \setminus (N_1^+ \cup N_1^0) \quad (4d)$$

$$q_i + A_i x = 0 \quad \text{for } i \in N_1^+ \quad (4e)$$

$$x_i = 0 \quad \text{for } i \in N_1^0 \quad (4f)$$

$$q_i + A_i x = 0 \quad \text{for } i \in N \setminus N_1 \quad (4g)$$

$$x_j \in \{0, 1\} \quad \text{for } j \in B \quad (4h)$$

$$0 \leq w_{ij} \leq 1 \quad \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N \quad (4i)$$

$$w_{jj} = x_j \quad \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0) \quad (4j)$$

$$v_i \in \{0, 1\} \quad \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0) \quad (4k)$$

$$w_{ij} \geq 0 \quad \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N \quad (4l)$$

$$w_{ij} \leq x_j \quad \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N \quad (4m)$$

$$w_{ij} \leq v_i \quad \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N \quad (4n)$$

$$w_{ij} + 1 \geq x_j + v_i \quad \text{for } i \in N_1 \setminus (N_1^+ \cup N_1^0), j \in N. \quad (4o)$$

5 Computational experiments

We report computational results on a variety of test problems that include pure LCP, MLCP and BC-MLCP instances. Some of these problem instances are well known engineering applications. We have used problems that are important from both the theoretical and the practical point of view. The instances are described in Subsection 5.1 and the results are presented in Subsection 5.2.

5.1 Test instances

5.1.1 LCP benchmark examples

We carried out the first set of tests on LCP examples from the literature. These are LCP instances that are known to be hard to solve. They are defined as follows, where $e = (1, 1, \dots, 1)$.

- LCP 1. This test instance is taken from [9], where the vector $q = -e$ and the matrix A_1 is defined as in (5).
- LCP 2. This test instance is taken from [19], where the vector $q = -e$ and the matrix A_2 is defined as in (5).

$$A_1 = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 2 & 5 & 6 & \dots & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 10 & \dots & 4n-3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4 & -1 & 0 & \dots & 0 \\ -1 & 4 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 4 \end{pmatrix}. \quad (5)$$

- LCP 3. This test instance is taken from [29], where the vector $q = -e$ and the matrix A_3 is defined as in (6).

$$A_3 = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (6)$$

- LCP 4. This test instance is also taken from [29]. For this instance, the matrix A_4 is the same as the matrix A_3 defined in (6) but the vector q is now given $q = (1, 1, \dots, 1, 0)$. This problem is known not to have any strictly interior point.
- LCP 5. This test instance is taken from [19], where the vector $q = -e$ and the matrix A_5 is defined by $A_5 = \text{diag}(1/n, 2/n, \dots, 1)$.

5.1.2 An energy network equilibrium problem

This test example is the energy network equilibrium problem with multiple players described in Gabriel et al. [17]. This problem features four energy price-taking producers denoted A, B, C, D with the first two located at node 1 and the latter two at node 2. The production levels are denoted as q_n^p , where node $n \in \{1, 2\}$ and producer $p \in \{A, B, C, D\}$. Similarly, the sales levels are denoted as s_n^p . Lastly, at node 1, the two producers A and B have the additional option of sending energy to node 2 and f_{12}^A, f_{12}^B represents the associated amounts of flow. (Note that the producers at node 2 are not allowed to ship their product to node 1.) Each producer solves an optimization problem for which the KKT conditions are necessary and sufficient. The KKT conditions of each producer are given by:

- Producers $X = A, B$ at node 1.

$$0 \leq -\pi_1 + \delta_1^X \perp s_1^X \geq 0 \quad (7a)$$

$$0 \leq \gamma_1^X + \lambda_1^X - \delta_1^X \perp q_1^X \geq 0 \quad (7b)$$

$$0 \leq -\pi_2 + (\tau_{12}^{Reg} + \tau_{12}) + \delta_1^X \perp f_{12}^X \geq 0 \quad (7c)$$

$$0 \leq \bar{q}_1^X - q_1^X \perp \lambda_1^X \geq 0 \quad (7d)$$

$$0 = s_1^X - q_1^X + f_{12}^X, \quad \delta_1^X \text{ free.} \quad (7e)$$

- Producers $X = C, D$ at node 2.

$$0 \leq -\pi_2 + \delta_2^X \perp s_2^X \geq 0 \quad (8a)$$

$$0 \leq \gamma_2^X + \lambda_2^X - \delta_2^X \perp q_2^X \geq 0 \quad (8b)$$

$$0 \leq \bar{q}_2^X - q_2^X \perp \lambda_2^X \geq 0 \quad (8c)$$

$$0 = s_2^X - q_2^X, \quad \delta_1^X \text{ free.} \quad (8d)$$

In addition to the KKT conditions of the four producers, there are market-clearing conditions that force supply to equal demand:

$$0 = [s_1^A + s_1^B] - (a_1 - b_1\pi_1), \quad \pi_1 \text{ free,} \quad (9a)$$

$$0 = [s_2^C + s_2^D + f_{12}^A + f_{12}^B] - (a_2 - b_2\pi_2), \quad \pi_2 \text{ free.} \quad (9b)$$

There is also a network system operator (NSO) who makes sure that the network is running smoothly. This operator solves an optimization problem for which the KKT conditions are stated as follows:

$$0 \leq -\tau_{12}^{Reg} - \tau_{12} + \gamma^{NSO} + \epsilon_{12} \perp g_{12} \geq 0 \quad (10a)$$

$$0 \leq \bar{g}_{12} - g_{12} \perp \epsilon_{12} \geq 0, \quad (10b)$$

$$0 = s_2^X - q_2^X, \quad \delta_1^X \text{ free.} \quad (10c)$$

Finally, the following market-clearing condition is added to determine the congestion tariff:

$$0 = g_{12} - [f_{12}^A + f_{12}^B], \quad \tau_{12} \text{ free.} \quad (11)$$

The MLCP model of this problem is obtained by combining equations (7)–(11).

5.1.3 A two-node energy system

This example, taken from [18], presents a Nash-Cournot game between two energy producers. Each producer decides on their production level $q_p, p \in \{1, 2\}$ to maximize their profit function $[a - b(q_1 + q_2)]q_p - (\beta_p q_p^2 + \rho_p q_p)$ subject to their production capacity constraints $s_p q_{min} \leq q_p \leq s_p q_{max}$, where s_p is a binary variable that is 1 if player p decides to produce and 0 if not. Here the binary variable s_p might for example relate to the on/off status for a power generation unit. If on, then the minimum and maximum production quantities are in force. If off, then both the upper and lower bounds are equal to zero. The BC-MLCP model associated with this example is defined as follows.

$$0 \leq -2q_p(b + \beta_p) + bq_{-p} - (a - \rho_p) + (\lambda_p^{max} - \lambda_p^{min}) \perp q_p \geq 0, \quad (12a)$$

$$0 \leq s_p q_{max} - q_p \perp \lambda_p^{max} \geq 0, \quad (12b)$$

$$0 \leq -s_p q_{min} + q_p \perp \lambda_p^{min} \geq 0, \quad (12c)$$

$$0 \leq 1 - s_p \perp \delta_p \geq 0, \quad (12d)$$

$$s_p \in \{0, 1\}, \quad (12e)$$

for all $p \in \{1, 2\}$. The parameters are $a = 9, b = 1, \beta_1 = \beta_2 = 1, \rho_1 = 1, \rho_2 = 3, q_{min} = 1.5$ and $q_{max} = 4$.

5.1.4 Random BC-MLCP instances

We randomly generated BC-MLCP test instances as follows. First we generate the matrix A at random with all the entries following a uniform distribution in the interval $[-20, 20]$. Then we generate a vector x at random with entries following a uniform distribution in the interval $[0, 20]$ for the complementarity variables and $[-20, 20]$ for the free variables. We choose a percentage ($\alpha\%$) of these entries and change their values to either 0 or 1. This represents the percentage of binary variables present in the problem. We have done three sets of experiments with different values of α . A first set with $\alpha \in \{10, 20, 30, \dots, 100\}$, a second one

with $\alpha \in \{20, 40, 60, 80\}$, and a third one with $\alpha = 60$. We also set a fraction ($\rho\%$) of the variables to be the complementarity variables. We have also experimented with various values of $\rho \in \{50, 70, 90\}$. If all the non-binary entries of x are non-zero, we force some of them to be zero. The vector q is then calculated in the following manner. For $i \in \{1, \dots, n_1\}$, if $x_i \neq 0$ then $q_i = -A_i x$, if $x_i = 0$ then $q_i = -A_i x + \epsilon$, with ϵ being uniformly drawn between $[-5, 5]$, and finally for $i \in \{n_1 + 1, \dots, n_1 + n_2\}$, $q_i = -A_i x$. This procedure is standard for generating random instances of linear programs with equilibrium constraints (LPEC).

5.1.5 A market-clearing problem as BC-MLCP

In order to test our algorithm on a more practical BC-MLCP instances, we consider the power market-clearing problem presented in Section 3 of [17]. This is a market-clearing problem of a multi-period power network, which features 6 (production/consumption) nodes, 8 producers and 4 demand blocks. The model corresponds to a multi-period auction in which producers submit production offers consisting of energy blocks and their corresponding selling prices as well as start-up and shut-down costs. In addition, consumers submit consumption bids consisting of energy blocks and their corresponding buying prices. This model clears the market by maximizing the social welfare which is computed using the producers offers and the consumers bids (i.e., the “declared” social welfare). We refer the reader to [17] for a full description of this problem and the definition of its parameters. The variables of this problems are:

- P_{tib}^G : the power produced by block b of unit i in period t .
- P_{tjk}^D : the power consumed by block k of demand j in period t .
- μ_{tib}^{Gmax} : the Lagrangian multiplier associated with the bound of the production for block b of unit i in period t .
- μ_{tjk}^{Dmax} : the Lagrangian multiplier associated with the bound of the demand for block k of unit j in period t .
- μ_{ti}^{\min} : the Lagrangian multiplier associated with the minimum power output of unit i in period t .
- ν_{tnm}^{\min} and ν_{tnm}^{\max} : the Lagrangian multipliers associated with the limits of the transmission capacity of the line relating nodes n and m in period t .
- δ_{tn} : the voltage angle of node n in period t .
- ξ_{tn}^{\max} : the Lagrangian multiplier associated with δ_{tn} .
- C_{ti}^U and C_{ti}^D : are respectively the non-negative start-up and shut-down cost of unit i in period t . Their corresponding Lagrangian multipliers are η_{ti}^U and η_{ti}^D , respectively.
- v_{ti} : the binary variable describing the on/off status of unit i in period t . Its corresponding Lagrangian multiplier is β_{ti}^{\max} .
- λ_{tn} : the Lagrangian multiplier associated with the equation that enforces the power balance at node n in period t .
- $\xi_{t(n=1)}^1$: the Lagrangian multiplier associated with the equation that imposes the reference node.

5.2 Results

In this subsection, we present our computational results. All our routines were coded in the C programming language and compiled with *gcc 4.6* [20]. All the results were obtained by running our code on a laptop with processor at 1.98 GHz and 4 GB of RAM. All the linear programs and mixed linear integer programs were solved with the simplex and MIP solvers of CPLEX version 12.6 [21]. All the complementarity constraints of our computational experiment are met with an accuracy of 10^{-15} .

5.2.1 LCP benchmark examples

For the purely LCP instances, we report our results in Table 1. For each class of LCP problem LCP 1, LCP 2, LCP 3, LCP 4 and LCP 5, we generate 5 instances of size $\{1000, 1500, 2000, 2500, 3000\}$. We report in Column 3, the number of iterations processed by Algorithm 1 for each instance. In Column 4, we report the total time needed to solve the LCP instances. Column 5 tells whether the solution found satisfies all

the complementarity constraints or not, while Column 6 tells whether or not the branch-and-bound phase of our algorithm was needed to solve the instances. We can see that for all of these problems, the procedure described in Algorithm 1 finds a solution that satisfies all the complementarity constraints. It can also be seen that apart from the class LCP 1, which needed three iteration into the first run of Algorithm 1, all the other classes of problems only needed 1 iteration to solve the problem.

Table 1: Results for the LCP test problems

Inst.	n	# iter. Alg. 1	time (s)	comp.	BnB
LCP 1	1000	3	49.317	yes	No
	1500	3	184.853	yes	No
	2000	3	494.625	yes	No
	2500	3	931.146	yes	No
	3000	3	1733.607	yes	No
LCP 2	1000	1	14.602	yes	No
	1500	1	48.092	yes	No
	2000	1	117.575	yes	No
	2500	1	222.218	yes	No
	3000	1	382.391	yes	No
LCP 3	1000	1	13.921	yes	No
	1500	1	47.254	yes	No
	2000	1	113.614	yes	No
	2500	1	223.516	yes	No
	3000	1	388.519	yes	No
LCP 4	1000	1	14.115	yes	No
	1500	1	46.62	yes	No
	2000	1	112.738	yes	No
	2500	1	223.116	yes	No
	3000	1	387.519	yes	No
LCP 5	1000	1	21.354	yes	No
	1500	1	72.419	yes	No
	2000	1	175.861	yes	No
	2500	1	329.954	yes	No
	3000	1	573.764	yes	No

5.2.2 An energy network equilibrium problem

Our results for this test problem are reported in Table 2. We provide the value of each variable as found by our algorithm. We can note that some of the values in our results are different from the ones reported in Gabriel et al. [17]. This shows that this problem has more than one solution. Note that the branch-and-bound was not needed here as Algorithm 1 alone was capable of finding this solution.

Table 2: Output of the energy network equilibrium problem

Prod. A		Prod. B		Prod. C		Prod. D		Network		Output Param.	
s_1^A	5	s_1^B	3	s_2^C	5	s_2^D	0	g_{12}	5	#iter.	11
q_1^A	10	q_1^B	3	q_2^C	5	q_2^D	0	ϵ_{12}	2	time(s)	0.023
f_{12}^A	5	f_{12}^B	0	λ_2^C	0	λ_2^D	0	π_1	12	Comp.	yes
λ_1^A	2	λ_1^B	0	δ_2^C	15	δ_2^D	15	π_2	15	BnB	No
δ_1^A	12	δ_1^B	12					τ_{12}	2.5		

5.2.3 A two-node energy system

These results are reported in Table 3. Our results are similar to the ones found by Gabriel et al. [18]. This does not necessarily suggest that there is a unique solution to this problem. Note that all the complementarity constraints are satisfied as well as the binary requirement on the variables s_1 and s_2 . For this instance,

Algorithm 1 was able to provide a solution that satisfies both requirements without the need of the branch-and-bound phase.

Table 3: Output of the two-node energy system

Param.	q_1	q_2	s_1	s_2	λ_1^{max}	λ_2^{max}	λ_1^{min}	λ_2^{min}
Values	1.625	1.5	1	1	0.0	0.0	0.0	1.625

5.2.4 Random BC-MLCP instances

For randomly generated instances we carried out three sets of experiments. First we wanted to know if the percentage (α) of the binary variables present in the problem could affect the performance or the computational time of our algorithm. For this experiment, we generated 10 instances with $n = 200, n_1 = n_2 = 100$ for each value of $\alpha \in \{10, 20, 30, \dots, 100\}$. We present the general overview of this experiment in Figure 1a. These box plots, for each value of α , represent the spread of the computational time needed to solve the instances. The general tendency of these graphs suggests that problems with a higher number of binary variables may be the easiest ones for our algorithm. Without the outlier in the results of $\alpha = 30$, the plot of the average computational times (Figure 1b) would follow the same tendency.

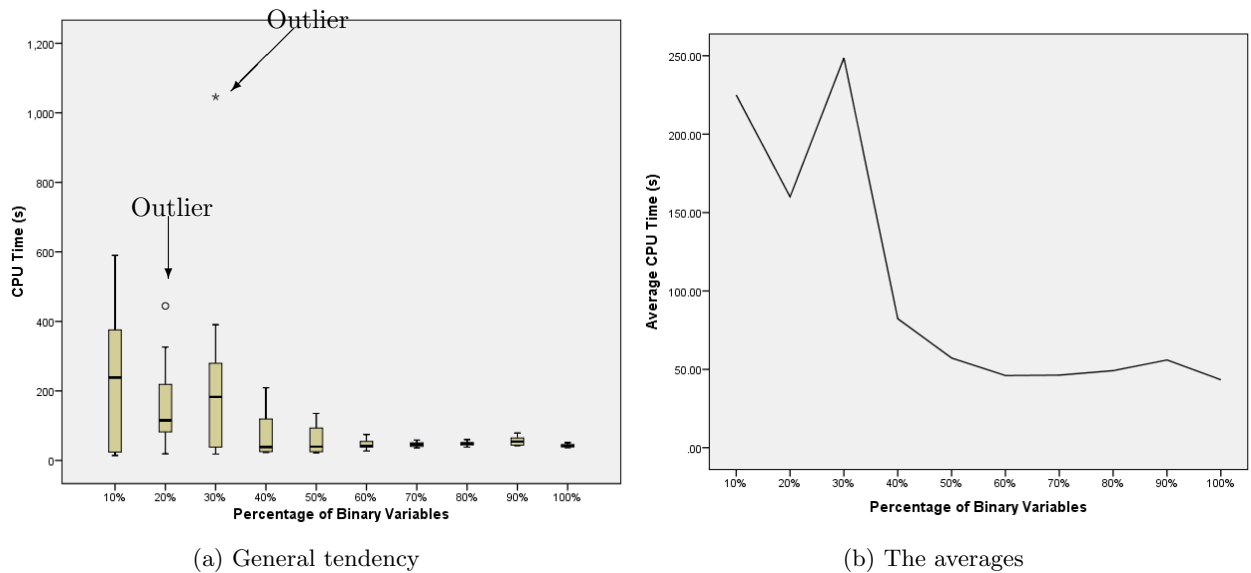


Figure 1: CPU time Vs percentage of binary variables

Our second round of experiments was purely to test the performance of our algorithm on BC-MLCP instances. For this experiment, we generated 10 instances with $n \in \{20, 40, 60, \dots, 200\}, n_1 = n_2 = n/2$ for each value of $\alpha \in \{20, 40, 60, 80\}$. The results of this test are shown in Table 4. In this table, the time reported in Column ‘Time’ is the average computational time over 10 instances. The size of each instance is in Column ‘ n ’, the average number of complementarity constraints fixed by Algorithm 1 is in Columns ‘Fixed’. Column ‘#BnB/10’ gives the number of instances, out of 10, for which the branch-and-bound phase was needed to solve the problem. Finally, Column ‘Comp.’ tells whether the all the complementarity constraints have been satisfied for all the 10 instances or not. Our computational results show that our algorithm is very efficient in solving BC-MLCP instances. We can note that out of 400 instances tested, the branch-and-bound phase was only required once. One can also note that with these parameters, our algorithm does not require more than 10 minutes before finding a solution for the BC-MLCP instances.

For these first two sets of experiments, one can notice that the running time of our algorithm gets relatively smaller when the percentage of binary variables present in the problem increases. This may be

Table 4: Results of randomly generated BC-MLCP instances

With 20% Binary					With 40% Binary				
n	Fixed	#BnB/10	Time (s)	Comp.	n	Fixed	#BnB/10	Time (s)	Comp.
20	2	0	0.03	yes	20	1	0	0.02	yes
40	2	0	0.24	yes	40	1	0	0.17	yes
60	6	0	3.20	yes	60	2	0	1.42	yes
80	7	0	9.65	yes	80	3	0	4.54	yes
100	7	0	25.54	yes	100	4	0	14.99	yes
120	9	0	74.41	yes	120	1	0	11.97	yes
140	4	0	57.99	yes	140	0	0	18.92	yes
160	10	0	239.11	yes	160	4	0	122.56	yes
180	10	0	502.20	yes	180	5	0	312.67	yes
200	28	1	765.26	yes	200	2	0	170.57	yes
With 60% Binary					With 80% Binary				
n	Fixed	#BnB/10	Time (s)	Comp.	n	Fixed	#BnB/10	Time (s)	Comp.
20	0	0	0.01	yes	20	1	0	0.01	yes
40	1	0	0.16	yes	40	0	0	0.07	yes
60	0	0	0.56	yes	60	0	0	0.31	yes
80	1	0	1.90	yes	80	2	0	2.02	yes
100	0	0	5.22	yes	100	0	0	5.17	yes
120	0	0	10.12	yes	120	0	0	5.21	yes
140	0	0	17.41	yes	140	1	0	26.00	yes
160	0	0	37.18	yes	160	1	0	27.10	yes
180	1	0	66.38	yes	180	1	0	45.88	yes
200	1	0	100.90	yes	200	2	0	103.05	yes

counter intuitive for readers that are familiar with integer programming. Recalling that the McCormick inequalities are equivalent to $y_{ij} = x_i x_j$ when $x_i, x_j \in \{0, 1\}$, our explanation for this behaviour is that, when there are more binary variables in the problem, many of the McCormick inequalities will be equivalent to $y_{ij} = x_i x_j$ and the RLT relaxation will be closer to an equivalent reformulation. As opposed to integer programming where one is looking for an optimal solution, in this problem, we are only searching for a feasible solution.

We carried out a third set of experiments with randomly generated BC-MLCP instances to investigate how the density (number of non zero elements) of the matrix A and the proportion of the complementarity variable could affect the performance of our approach. For this experiment, we generated 10 instances for each combination of $n \in \{20, 40, 60, \dots, 200\}$, $\Delta \in \{25\%, 50\%, 75\%\}$ and respectively 50%, 70% and 90% of variables being complementarity variables, where Δ is the density of the matrix A . This made a total of 900 additional test instances for our approach. The results of this experiment are reported in Table 5. We notice that when we consider 50% of the variables to be complementarity variables, Algorithm 1 often solves the BC-MLCP problem, regardless of the density of the matrix. When the number of complementarity variables increases (70% and 90%), the branch-and-bound step becomes necessary, but the number of complementarity constraints fixed increases, which helps the branch-and-bound algorithm to converge faster. We note that the computational time also becomes very large for the instances with higher percentage of complementarity. However, this does not suggest that our algorithm is not capable of solving pure LCP instances. Our results on “difficult” pure LCP instances (see Table 1) prove the strength of our algorithm for LCPs.

5.2.5 A market-clearing problem as BC-MLCP

In this subsection, we present the results obtained by our algorithm when solving the energy market-clearing problem from Gabriel et al. [17]. In Table 6, we report the values of the variables related to each producer. The values of variables related to each demand block and the values of nodes specific variables are given in Table 7 and in Table 8, respectively. In addition to the values reported in these tables we also had $\nu_{tnm}^{min} = 0, \nu_{tnm}^{max} = 0$ for all t, n, m , and $\xi_{t(n=1)}^1 = 0$ for all t . Some of these values match with those reported in [17]. For this problem, the branch-and-bound phase of our method was required in order to find a

Table 5: Results of randomly generated BC-MLCP instances for a combination of 50%, 70% and 90% of complementarity with various values of the density of the matrix A , $\Delta \in \{25\%, 50\%, 75\%\}$

$\Delta = 25\%$				$\Delta = 50\%$				$\Delta = 75\%$						
n	Fixed	BnB	Time	Comp	n	Fixed	BnB	Time	Comp	n	Fixed	BnB	Time	Comp
20	1	0	0.01	1	20	1	0	0.01	1	20	1	0	0.02	1
40	0	0	0.12	1	40	0	0	0.11	1	40	0	0	0.09	1
60	1	0	1.95	1	60	1	0	2.83	1	60	0	0	1.09	1
80	0	0	1.63	1	80	0	0	2.07	1	80	0	0	1.41	1
100	0	0	4.36	1	100	0	0	2.88	1	100	0	0	2.83	1
120	1	0	9.07	1	120	0	0	6.14	1	120	1	0	20.68	1
140	0	0	7.79	1	140	0	0	7.51	1	140	0	0	7.18	1
160	0	0	16.08	1	160	0	0	12.22	1	160	0	0	11.37	1
180	0	0	20.13	1	180	0	0	26.01	1	180	0	0	20.11	1
200	0	0	30.59	1	200	0	0	30.75	1	200	0	0	39.37	1

$d = 50\%$

$\Delta = 25\%$				$\Delta = 50\%$				$\Delta = 75\%$						
n	Fixed	BnB	Time	Comp	n	Fixed	BnB	Time	Comp	n	Fixed	BnB	Time	Comp
20	1	0	0.02	1	20	2	1	0.06	1	20	2	0	0.03	1
40	4	1	0.63	1	40	3	1	1.49	1	40	4	1	1.84	1
60	7	1	17.04	1	60	8	0	26.12	1	60	10	1	53.49	1
80	11	4	282.24	1	80	5	2	277.74	1	80	7	1	173.34	1
100	13	2	225.04	1	100	9	0	64.57	1	100	8	0	61.33	1
120	17	1	333.02	1	120	13	0	136.84	1	120	11	1	885.77	1
140	13	0	321.53	1	140	10	0	352.00	1	140	16	1	1842.19	1
160	14	2	1724.12	1	160	17	1	2533.81	1	160	15	1	2387.33	1
180	14	2	2402.58	1	180	17	0	2000.21	1	180	11	0	1358.29	1
200	21	2	4937.28	1	200	15	0	2389.03	1	200	17	0	2998.35	1

$d = 70\%$

$\Delta = 25\%$				$\Delta = 50\%$				$\Delta = 75\%$						
n	Fixed	BnB	Time	Comp	n	Fixed	BnB	Time	Comp	n	Fixed	BnB	Time	Comp
20	5	6	0.0938	1	20	3	0	0.1374	1	20	5	6	0.3172	1
40	7	6	37.892	1	40	5	8	95.6526	1	40	3	10	78.6666	1
60	6	10	896.827	1	60	5	10	2044.1254	1	60	5	10	1382.2456	1
80	9	10	3479.2606	1	80	4	10	3413.4902	1	80	5	10	3320.7934	1
100	7	10	3769.555	1	100	4	10	7271.6318	1	100	5	10	3770.6522	1
120	6	10	4008.3996	1	120	5	10	4168.84	1	120	1	10	4061.5448	1
140	5	10	4977.394	1	140	6	10	5773.8006	1	140	5	10	5343.462	1
160	4	10	6260.342	1	160	4	10	7255.6278	1	160	2	10	7124.5336	1
180	2	10	8279.1924	1	180	1	10	9130.6418	1	180	8	10	9736.3408	1
200	8	10	10674.255	1	200	6	10	12720.2366	1	200	3	10	12587.482	1

$d = 90\%$

solution. However, Algorithm 1 was able to replace up to 99 (out of 248) complementarity constraints with linear equations.

Table 6: Producers variables

Producers	P_{tib}^G		μ_{tib}^G		μ_{tii}^{min}		η_{ti}^U		C_{ti}^U	
	t=1	t=2	t=1	t=2	t=1	t=2	t=1	t=2	t=1	t=2
$i = 1$	0	0	2	0	0	10	0	0	0	0
$i = 2$	0	0	4	0	0	8	1	0	0	0
$i = 3$	50	25	6	0	0	6	0.8	0	0	0
$i = 4$	50	25	8	0	0	4	0	0	0	0
$i = 5$	50	25	10	0	0	2	0	0	0	0
$i = 6$	50	25	12	0	0	0	0	0	0	0
$i = 7$	50	50	14	2	0	0	1	0	350	0
$i = 8$	50	50	16	4	0	0	1	0.4	500	0

Producers	η_{ti}^D		C_{ti}^D		β_{ti}^{max}		v_{ti}	
	t=1	t=2	t=1	t=2	t=1	t=2	t=1	t=2
$i = 1$	0	0.2	0	0	0	0	0	0
$i = 2$	0	0.57	0	0	0	0	0	0
$i = 3$	0	0.5	0	0	0	0	1	1
$i = 4$	0	0.4	0	0	300	0	1	1
$i = 5$	0	0.22	0	0	450	0	1	1
$i = 6$	0	0	0	0	600	0	1	1
$i = 7$	0	0	0	0	350	100	1	1
$i = 8$	0	0	0	0	500	0	1	1

Table 7: Demands variables

Demands	P_{tjk}^D		μ_{tjk}^D	
	t=1	t=2	t=1	t=2
$j = 1$	0	50	0	6
$j = 2$	100	50	0	6
$j = 3$	100	50	0	7
$j = 4$	100	50	1	7

Table 8: Nodes variables

Nodes	δ_{tn}		ξ_{tn}^{max}		λ_{tn}	
	t=1	t=2	t=1	t=2	t=1	t=2
$n = 1$	0	0	0	0	26	14
$n = 2$	0.002	-0.005	0	0	26	14
$n = 3$	-0.0017	0.005	0	0	26	14
$n = 4$	0.1067	0.03	0	0	26	14
$n = 5$	0.125	0.025	0	0	26	14
$n = 6$	0.0933	0.02	0	0	26	14

6 Conclusions

We have presented a novel exact solution method for the BC-MLCP. Our method starts with a sophisticated pre-processor which considers the RLT relaxation of the complementarity constraints and enables us to replace some of the complementarity constraints with linear equations by solving a series of LPs. This pre-processor is then followed by the solution of an equivalent MILP reformulation of the BC-MLCP with a reduced number of complementarity constraints, which ensures that our algorithm always finds a solution that satisfies all the complementarity constraints as well as all the integrality conditions. Our computational results on a variety of both practical and theoretical examples show that the pre-processor often suffices to find a solution. They also show that our method can be used for pure LCP and MLCP.

For future research, one could incorporate cutting planes in the second phase (Algorithm 1) of our approach to strengthen the RLT relaxation. Several cutting planes have been proposed in the literature for this purpose, see for [3, 8]. This will enable Algorithm 1 to quickly find a solution with only few iterations.

References

- [1] F. Al-Khayyal (1987), an implicit enumerative procedure for the general linear complementarity problem. *Math. Program. Stud.*, 31, 1–20.
- [2] M. Anitescu & F.A. Potra (1997), Formulating dynamic multi-rigid-body contact problems with friction as linear solvable complementarity problems. *Nonlinear Dynam.*, 14, 231–247.
- [3] C. Audet, P.Hansen, B. Jaumard & G. Savard (2000), A branch and cut algorithm for nonconvex quadratically constrained quadratic programming. *Math. Program. Ser. A*, 87, 131–152.
- [4] D. Bertsimas & J.N. Tsitsiklis (1997), *Introduction to Linear Programming.*, Athena Scientific, Belmont, USA.
- [5] S.C. Billups & K.G. Murty (2000), Complementarity problems. *J. Comput. Appl. Math.*, 124, 303–318.
- [6] R.W. Cottle, J. S. Pang & R.E. Stone (1992), *The linear complementarity problem.* Academic Press INC., San Diego.
- [7] R. W. Cottle & G.B. Dantzig (1968), Complementarity pivot theory of mathematical programming. *Linear Algebra Appl.*, 1, 103–125.
- [8] F. Djeumou Fomeni, K. Kaparis & A.N. Letchford (2015), Cutting planes for RLT relaxations of mixed 0-1 polynomial programs. Published online. *Math. Program.*
- [9] Y. Fathi (1979), Computational complexity of LCPs associated with positive definite symmetric matrices. *Math. Program.*, 17, 335–344.
- [10] M.C. Ferris & J.S. Pang (1997), Engineering and economic applications of complementarity problems. *SIAM Rev.*, 39(4), 669–713.
- [11] M.C. Ferris, A.J. Wathen & P. Armand (2007), Limited memory solution of bound constrained convex quadratic problems arising in video games. *RAIRO-Oper. Res.*, 41, 19–34
- [12] S.A. Gabriel (1998), A hybrid smoothing method for mixed nonlinear complementarity problems. *Comput. Optim. Appl.*, 9, 153–173.
- [13] S.A. Gabriel & F.U. Leuthold (2010), Solving discretely-constrained MPEC problems with applications in electric power markets. *Energ. Econ.*, 32, 3–14.
- [14] S.A. Gabriel, Y. Shim, A.J. Conejo, S. de la Torre & R. García-Bertrand (2010), A Benders decomposition method for discretely-constrained mathematical programs with equilibrium constraints. *J. Oper. Res. Soc.*, 61, 1404–1419.
- [15] S.A. Gabriel, K.E. Rosendahl, R. Egging, H. Avetisyan & S. Siddiqui (2012), Cartelization in gas markets: studying the potential for a ‘Gas OPEC’, *Energ. Econ.*, 34, 137–152.
- [16] S.A. Gabriel, A.J. Conejo, B.F. Hobbs, D. Fuller & C. Ruiz (2012), *Complementarity Modeling In Energy Markets.* Springer, New York.
- [17] S.A. Gabriel, A.J. Conejo, C. Ruiz & S. Siddiqui (2013), Solving discretely-constrained mixed linear complementarity problems with applications in energy. *Comput. Oper. Res.*, 40, 1339–1350.
- [18] S.A. Gabriel, S. Siddiqui, A.J. Conejo & C. Ruiz (2013), Discretely-constrained Nash-Cournot games with an application to power markets. *Netw. Spat. Econ.*, 13, 307–326.
- [19] C. Geiger & C. Kanzow (1996), On the resolution of monotone complementarity problems. *Comput. Optim. Appl.*, 5, 155–173.
- [20] T. Rothwell & J. Youngman (2007), *The GNU C Reference Manual*, www.gnu.org/software/gnu-c-manual/gnu-c-manual.pdf
- [21] IBM ILOG CPLEX Optimization Studio, www-01.ibm.com/software/commerce/optimization/cplex-optimizer/
- [22] J. Júdice & A. Faustino (1988), An experimental investigation of enumerative methods for the linear complementarity problem. *Comput. Oper. Res.*, 15, 417–426.
- [23] J. Júdice (2002), On the solution of NP-hard linear complementarity problems. *Sociedad de Estadística e Investigación Operativa*, 10, 125–145.
- [24] M. Kocvara & J. Zowe (1997), An iterative two-step algorithm for linear complementarity problems. *Numer. Math.*, 68, 95–106.
- [25] C.E. Lemke (1965), Bimatrix equilibrium points and mathematical programming. *Manage. Sci.*, 11, 681–689.

-
- [26] C.E. Lemke (1968), On complementary pivot theory. In: *Mathematics of the Decision Sciences*. G. B Dantzig, A. F. Veinott Jr, (eds) AMS.
- [27] G.P. McCormick (1979), Computability of global solutions to factorable nonconvex programs: Part I-Convex underestimating problems. *Math. Program.*, 10, 146-175.
- [28] J.L. Morales, J. Nocedal & M. Smelyanskiy (2008), An Algorithm for the fast solution of the symmetric linear complementarity problems. *Numer. Math.*, 111, 251-266.
- [29] K.G. Murty (1988), *Linear complementarity, linear and nonlinear programming*. Sig. Ser. in Appl. Math. 3rd ed. Berlin: Heldermann Verlag.
- [30] J.S. Pang & S.A. Gabriel (1993), NE/SQP: a robust algorithm for the nonlinear complementarity problem. *Math. Program.*, 60, 295-337.
- [31] C. Ruiz , A.J. Conejo & S.A. Gabriel (2012), Pricing non-convexities in an electricity pool. *IEEE T. Power Syst.*, 27, 1334-1342.
- [32] C. Ruiz, A.J. Conejo, J.D. Fuller, S.A. Gabriel & B.F. Hobbs (2014), A tutorial review of complementarity models for decision-making in energy markets. *EURO J. Decis. Process.*, 2, 91-120.
- [33] H.D. Sherali & W.P. Adams (1990), A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM J. Discrete Math.*, 3, 411-430.
- [34] H.D. Sherali, R.S. Krishnamurthy & F.A. Al-Khayyal (1998), Enumeration approach for linear complementarity problems based on a Reformulation-Linearization Technique. *J. Optimiz. Theory App.*, 99, 481-507.
- [35] H.D. Sherali & C.H. Tuncbilek (1997), New reformulation linearization/convexification relaxations for univariate and multivariate polynomial programming problems. *Oper. Res. Lett.*, 21, 1-9.
- [36] D.E. Stewart & J.C. Trinkle (1996), An implicit time-stepping scheme for rigid body dynamics with inelastic collisions and coulomb friction. *Int. J. Numer. Meth. Eng.*, 39, 2673-2691.
- [37] D.E. Stewart & J.C. Trinkle (1997). Dynamics, friction, and complementarity problems. In M.C. Ferris and J.S. Pang, (eds), *Complementarity and Variational Problems*, SIAM, 425-439.
- [38] P. Wilmott (2006), *Quantitative Finance*. John Wiley & Sons.