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GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

Sustainability of cooperation in dynamic games played over event trees: A tutorial

Georges Zaccour

GERAD, Chair in Game Theory and Management and HEC Montréal, Montréal (Québec) Canada, H3T 2A7

georges.zaccour@gerad.ca

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Abstract: In this tutorial, we recall the main ingredients of the theory of dyamic games played over event trees and show step-by-step how to build a sustainable cooperative solution.

Key Words: Dynamic games, cooperation, sustainability.

Résumé : Dans cet exposé, on rappelle les éléments importants de la théorie des jeux dynamiques définis sur des arbres d'évènements. Ensuite, on montre comment bâtir une solution coopérative durable dans le temps.

Mots clés : Jeux dynamiques, coopération, durabilité.

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1 Introduction

Many problems in economics, engineering and management science have the following three features in common: (a) They involve only a few agents (players), which have interdependent payoffs, that is, the action of any player affects the payoffs of all. (b) The agents cooperate or compete repeatedly over time, and the problem involves an accumulation process, e.g., production capacity, pollution stock. (c) Some of the parameter values are uncertain. A natural framework to deal with such problems is the theory of dynamic games played over event trees (DGPET). As an illustration of such a setting, consider a region served by a few electricity producers (players) who compete in one or more market segments (peak-load, local market, export market, etc.). At each period, the price in each segment depends on the total available supply and on the realization of some random events (e.g., weather conditions or the state of the economy). Further, producers invest in different production capacities (nuclear, thermal, hydro, etc.) over time. In the terminology of dynamic games, the quantities committed to each market segment, which are constrained by available capacity, and the investments in different production technologies are the player's control variables and the installed production capacities are the state variables. The players must account for uncertainty in demand when they make their decisions.

Now, suppose that the players (firms, countries, individuals) involved in an example of DGPET agree to cooperate, that is, to coordinate their strategies in order to maximize their joint payoff over a given time interval $[0, T]$. A legitimate question is then how to ensure that each player will indeed fulfill her part of the agreement over time? This is the question we deal with in this paper.

It is useful from the outset to make some clarifying observations regarding the nature of the problem at hand. First, although it may be appealing to favor short-term agreements to keep all options open, long-term commitments cannot be avoided when the contracting cost is high. For instance, it is unthinkable that the government and the civil service union meet every Monday to negotiate that week's employment conditions. Common sense clearly suggests that both parties should avoid costly and time-consuming negotiations and agree on a collective labor agreement that will remain in place for a number of years.

Second, it is an empirical fact that some long-term agreements are abandoned before their maturity. A drastic illustration of this is the high level of divorce observed around the globe. Haurie (1976) cites two reasons why an agreement (contract or cooperative solution), which suits everyone at an initial instant of time may not reach its maturity date T: (i) If the players agree to renegotiate the original agreement at time $\tau \in (0, T]$, it is not certain that they will all want to continue with that agreement. In fact, they will not go on with the original agreement if it is not a solution of the cooperative game that starts out at time τ. (ii) If a player obtains a higher payoff by leaving the agreement at time $\tau \in (0, T]$ than by continuing to implement her cooperative strategies, then she will indeed deviate from cooperation. In the parlance of dynamic optimization and dynamic games, such a breakdown means that the agreement is time inconsistent. It is important to mention here that if the cooperative agreement is an equilibrium, then item (ii) above cannot occur because no player would, by definition, find it optimal to deviate from the solution. It is well-known that, except in games having very special structures (see, e.g., Chiarella et al. (1984) and Martín-Herrán and Rincón-Zapatero (2005)), a Pareto-optimal (or cooperative) solution is not an equilibrium.

The rest of the paper is organized as follows: In Section 2, we give a brief account of the literature dealing with the sustainability of cooperation in dynamic games. In Section 3, we recall the main ingredients of dynamic games played over event trees. We explain the approach to achieve a node-consistent outcome in DGPET in Section 4. In Section 5, we illustrate this approach using the Shapley value and the core; and in Section 6, we briefly conclude.

2 Brief literature review

The literature in dynamic games has followed two streams in its quest of sustain cooperation over time, namely, building cooperative equilibria or defining time-consistent solutions.

Through the implementation of some (punishing) strategies, the first stream seeks to make the cooperative solution an equilibrium of an associated noncooperative game. If this is achieved, then the result will be at once collectively optimal and stable, as no player will find it optimal to deviate unilaterally from the equilibrium. To build a cooperative equilibrium, players can for instance implement trigger strategies, which are strategies based on the history of the game. Loosely speaking, such strategies are defined as follows: At any decision node, if the history of the game has been till now cooperative, then each player will implement the cooperative action; otherwise, which means that a player has cheated, then all the other players implement their punishing strategies, which are set out in a pre-play arrangement. Intuitively, for such punishing strategies to work, they must be: (i) effective, that is, the deviator would lose from cheating on the agreement, and (ii) credible, that is, it is in the best interest of the other players to implement their punishing strategies if a deviation is observed, rather than sticking to cooperation.

Sustaining a Pareto outcome as an equilibrium has a long history in repeated games, and a well-known result in this area is the so-called folk theorem, which (informally) states that if the players are sufficiently patient, then any Pareto-optimal outcome can be achieved as a Nash equilibrium; see, e.g., Osborne and Rubinstein (1994). A similar theorem has been proved for stochastic games by Dutta (1995). Trigger strategies have also been considered in multistage games and in differential games; see the early contributions by Haurie and Tolwinski (1985), Tolwinski et al. (1986), Haurie and Pohjola (1987) and Haurie et al. (1994). The books by Dockner et al. (2000) and Haurie et al. (2012) provide a comprehensive introduction to cooperative equilibria in differential games.

Having the same objective of embedding the cooperative solution with an equilibrium property, Ehtamo and Hämäläinen (1986, 1989, 1995) proposed the concept of incentive strategies and a corresponding equilibrium in two-player differential games. A player's incentive strategy is a function of the other player's action. In an incentive equilibrium, each player implements her part of the agreement if the other player also does. In terms of computation, the determination of an incentive equilibrium requires solving a pair of optimal-control problems, which is in general relatively easy to do. A main concern with incentive strategies is their credibility, since it may happen that the best response to a deviation from cooperation is to stick to cooperation rather than to also deviating. In such a situation, the threat of punishment for a deviation is an empty one. In applications, one can derive the conditions that the parameter values must satisfy to have credible incentive strategies. For a discussion of the credibility of incentive strategies in differential games with special structures, see Martín-Herrán and Zaccour (2005, 2009). A further drawback of incentive equilibrium is that the concept is defined for only two players. Incentive strategies and equilibria have been applied in a number of areas, including environmental economics (see, e.g., Breton et al. (2008), de Frutos and Martín-Herrán (2015)), marketing (see, e.g., Martín-Herrán and Taboubi (2005), Buratto and Zaccour (2009)) and in closed-loop supply chains (De Giovanni et al. (2016)).

In the second stream, to which this contribution belongs, the idea is to define a *time-consistent* decomposition over time of the total cooperative payoff (allocation) of player $j, j \in M$, over the planning horizon $[0, T]$. An allocation is time consistent if at any intermediate instant of time the cooperative payoff-to-go dominates (at least weakly) the noncooperative payoff-to-go for all players. It is important to mention that the inequality is verified along the cooperative state trajectory, which means that cooperation has prevailed up to the time of comparison. A stronger condition is used in the concept of *agreeability*, where the above payoff dominance must hold along any feasible state trajectory (see Kaitala and Pohjola (1990, 1995) and Jørgensen et al. (2003, 2005)). The literature on time consistency in cooperative dynamic games has essentially been in continuous time. The concept was initially proposed in Petrosjan (1977) and Petrosjan and Danilov (1979, 1982, 1986). In these publications in Russian, as well as in subsequent books in English (Petrosjan (1993), Petrosjan and Zenkevich (1996)), and in Petrosjan (1997), time consistency was termed dynamic stability. In Yeung and Petrosjan (2001) a *proportional* time-consistent solution was investigated, whereas Petrosjan and Zaccour (2003) proposed a time-consistent *Shapley value*. Jørgensen and Zaccour (2001) and Yeung and Petrosjan (2006) derived time-consistent solutions in environmental and joint-venture games, respectively. Yeung and Petrosjan (2004, 2005a) and Yeung et al. (2007) studied time consistency in stochastic differential games. For a general discussion of time consistency in differential games, see the book by Yeung and Petrosjan (2005b) and the survey by Zaccour (2008).

Other papers discussed time-consistent solutions (or very close concepts) for deterministic or stochastic discrete-dynamic games; see, e.g., Chandler and Tulkens (1997), Filar and Petrosjan (2000), Germain et al.

(2003), Petrosjan et al. (2004), Predtetchinski (2007), Lehrer and Scarsini (2013) and Xu and Veinott (2013). Finally, Avrachenkov et al. (2013) established conditions for time consistency for cooperative Markov decision processes.

3 Games played over event trees

In this section, we recall the main elements of DGPET. This class of games was introduced by Zaccour (1987) and Haurie et al. (1990), and further developed in Haurie and Zaccour (2005). The initial motivation was an analysis of the European natural gas market, and more specifically, the forecasting of long-term deliveries of gas from four producers (Algeria, Netherlands, Norway and the former USSR) to nine consuming European regions. The deliveries and investments are the control variables, and production capacities and reserves of gas are the state variables. Each consuming region is described by a time-varying demand function whose parameter values are uncertain, with the stochasticity represented by an event tree. This is a situation where the three features mentioned in the introduction, that is, strategic interaction, dynamic, and uncertainty, are clearly present. More recently, the class of DGPET has been applied to electricity markets in, e.g., Pineau and Murto (2003), Genc et al. (2007), Genc and Sen (2008) and Pineau et al. (2011b). Here, the main objective is to predict equilibrium investments in different generation technologies in deregulated electricity markets. Parilina and Zaccour (2015b) constructed an ε-cooperative equilibrium for this class of games and illustrated their results using a linear-quadratic game in environmental economics. For a comprehensive introduction to the class of DGPET, see Haurie et al. (2012).

Let $\mathcal{T} = \{0,\ 1,\ \ldots,\ T\}$ be the set periods, and denote by $(\xi\,(t) : t \in \mathcal{T})$ the exogenous stochastic process represented by an event tree, with a root node n^0 in period 0 and a set of nodes $\mathcal{N}^t$ in period $t = 0,\ 1,\ \ldots,\ T$. Let $a(n^t) \in \mathcal{N}^{t-1}$ be the unique predecessor of node $n^t \in \mathcal{N}^t$ for $t = 0,\ 1,\ \ldots,\ T$, and denote by $S(n^t) \in \mathcal{N}^{t+1}$ the set of all possible direct successors of node $n^t \in \mathcal{N}^t$ for $t = 0,\ 1,\ \ldots,\ T-1$. We call *scenario* any path from node n^0 to a terminal node n^T. Each scenario has a probability, and the probabilities of all scenarios sum up to 1. We denote by π^{n^t} the probability of passing through node n^t, which corresponds to the sum of the probabilities of all scenarios that contain this node. In particular, $\pi^{n^0} = 1$, and π^{n^T} is equal to the probability of the single scenario that terminates in (leaf) node $n^T \in \mathcal{N}^T$. Also, $\sum_{n^t \in \mathcal{N}^t} \pi^{n^t} = 1, \forall t$.

Denote by $M = \{1, \ldots, m\}$ the set of players. Denote by $u_j(n_l^t) \in \mathbb{R}^{m_j}$ the decision variables of player j at node n_l^t, and let $\underline{u}(n_l^t) = (u_1(n_l^t), \ldots, u_m(n_l^t))$. Let $X \subset \mathbb{R}^p$, with p a given positive integer, be a state set. For each node $n_l^t \in \mathcal{N}^t$, $t = 0, 1, \ldots, T$, let $U_j^{n_l^t} \subset \mathbb{R}^{\mu_j^{n_l^t}}$, with $\mu_j^{n_l^t}$ a given positive integer, be the control set of player j. Denote by $\underline{U}^{n_l^t} = U_1^{n_l^t} \times \cdots \times U_j^{n_l^t} \times \cdots \times U_m^{n_l^t}$ the product control sets. A transition function $f^{n_l^t}(\cdot,\cdot) : X \times \underline{U}^{n_l^t} \mapsto X$ is associated with each node n_l^t. The state equations are given as

$$x(n_l^t) \quad = \quad f^{a(n_l^t)}\left(x\left(a\left(n_l^t\right)\right), \underline{u}\left(a\left(n_l^t\right)\right)\right), \tag{1}$$

$$\underline{u}\left(a\left(n_l^t\right)\right) \quad \in \quad \underline{U}^{a(n_l^t)}, \quad n_l^t \in\ \mathcal{N}^t, t = 1, \ldots, T. \tag{2}$$

At each node n_l^t, $t = 0, \ldots, T-1$, the reward to player j is a function of the state and of the controls of all players, given by $\phi_j^{n_l^t}(x(n_l^t), \underline{u}(n_l^t))$. At a terminal node n_l^T, the reward to player j is given by the function $\Phi_j^{n_l^T}(x(n_l^T))$.

We assume that player $j \in M$ maximizes her expected stream of payoffs. The state equations and the reward functions define the following multistage game, where we let

$$\begin{aligned} \tilde{x} &= \{x(n_l^t) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T\}, \\ \underline{\tilde{u}} &= \{\underline{u}(n_l^t) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T-1\}, \end{aligned}$$

and $J_j(\tilde{x}, \underline{\tilde{u}})$ be the payoff to player j, that is,

$$
\begin{aligned}
J_j(\tilde{x}, \underline{\tilde{u}}) &= \sum_{t=0}^{T-1} \sum_{n_l^t \in \mathcal{N}^t} \pi(n_l^t) \phi_j^{n_l^t}(x(n_l^t), \underline{u}(n_l^t)) + \sum_{n_l^t \in \mathcal{N}_T} \pi(n_l^T) \Phi_j^{n_l^T}(x(n_l^T)), \quad j \in M, &(3)\\
\text{s.t.} & \\
x(n_l^t) &= f^{a(n_l^t)}(x(a(n_l^t)), \underline{u}(a(n_l^t))), &(4)\\
\underline{u}(a(n_l^t)) &\in \underline{U}^{a(n_l^t)}, \quad n_l^t \in \mathcal{N}^t, t = 1, \ldots, T, &\\
x(n_0) &= x^0 \text{ given.} &(5)
\end{aligned}
$$

Remark 1 *As we are dealing with a finite horizon, we do not discount future payoffs. Adding discounting would not cause any conceptual difficulty.*

Remark 2 *The DGPET framework can take into account more complicated constraints on the control variables than the ones considered here, e.g., constraints with lags and coupled constraints (see Kanani Kuchesfehani and Zaccour (2015)).*

As alluded to before, dealing with long-term cooperation involves at intermediate instants of time, a comparison of noncooperative and cooperative payoffs-to-go.

3.1 Noncooperative and cooperative outcomes

In DGPET, the control and state variables are node dependent, and each node $n^t \in \mathcal{N}^t$ represents a possible sample value of the history h^t of the $\xi(.)$ process up to time t. Because of this, a strategy in DGPET is referred to as S-adapted strategy, where the S stands for sample.

Definition 1 *An admissible* S*-adapted strategy for player j is a vector $\tilde{u}_j = \{u_j(n_l^t) : n_l^t \in \mathcal{N}^t, t = 0, \ldots, T-1\}$, that is, a plan of actions adapted to the history of the random process represented by the event tree.*

Denote by $\underline{\tilde{u}} = (\tilde{u}_j : j \in M)$ the S-adapted strategy vector of the m players. We can thus define a game in normal form,[1] with payoffs $W_j(\underline{\tilde{u}}, x^0) = J_j(\tilde{x}, \underline{\tilde{u}})$, $j \in M$, where $\tilde{x}$ is obtained from $\underline{\tilde{u}}$ as the unique solution of the state equations that emanate from the initial state x^0.

If the game is played noncooperatively, then the players will seek a Nash equilibrium in S-adapted strategies defined as follows:

Definition 2 *An* S*-adapted Nash equilibrium is an admissible* S*-adapted strategy $\underline{\tilde{u}}^N$ such that for every player j the following holds:*

$$W_j(\underline{\tilde{u}}^N, x^0) \geq W_j([\tilde{u}_j, \tilde{\mathbf{u}}_{-j}^N], x^0),$$

where $\tilde{\mathbf{u}}_{-j}^N$ is the Nash equilibrium policy vector of all players $i \neq j$.

We make the following remarks.

Remark 3 *Although the* S*-adapted and open-loop equilibria look similar, they differ in the definitions of the state equations and control variables. In an open-loop information structure, the control variables and the state equations are defined over time. Here, as mentioned above, they are defined (indexed) over the set of nodes of the event tree.*

Remark 4 *As a DGPET has a normal-form representation, the conditions for existence and uniqueness of a Nash equilibrium are the same as in classical games with continuous payoffs with constraints as established in Rosen (1965).*[2]

[1] To define a game in normal form, we need three elements: (a) a finite set of players $M = \{1, \ldots, m\}$, (b) a strategy set S_i of player $i \in M$, and (c) a payoff function $\pi_i : \prod_{i \in M} S_i \to \mathbb{R}$.

[2] For a detailed treatment in the context of this class of games, see Haurie et al. (2012).

If the players agree to cooperate, then they will optimize the sum of their payoffs throughout the entire horizon,[3] that is,

$$\max_{\tilde{u}_j, j\in M} W = \sum_{j\in M} W_j\left(\underline{\tilde{u}}, x^0\right).$$

Denote by $\underline{\tilde{u}}^*\left(x^0\right)$ the resulting vector of cooperative controls, i.e.,

$$\underline{\tilde{u}}^*\left(x^0\right) = \arg\max \sum_{j\in M} W_j\left(\underline{\tilde{u}}, x^0\right).$$

Remark 5 *The vector $\underline{\tilde{u}}^*\left(x^0\right)$ corresponds to the agreement signed by all players at initial date. This is the vector that we would like to see it implemented throughout the duration of the game.*

Denote by $\tilde{x}^* = \{x^*(n_l^t) : n_l^t \in \mathcal{N}^t,\ t = 0, 1, \ldots, T\}$ the cooperative state trajectory generated by $\underline{\tilde{u}}^*\left(x^0\right)$.

4 Node consistency

Informally speaking, a cooperative solution in DGPET is node consistent, if the cooperative payoff-to-go of player $j, j \in M$, in the subgame starting at any node is at least equal to the noncooperative payoff-to-go in this subgame. We reiterate that this comparison takes place along the cooperative state trajectory, meaning that at node of comparison n_l^t,$n_l^t \in \mathcal{N}^t, t = 1, \ldots, T$, the state value is $\tilde{x}^*\left(n_l^t\right)$. If all players implement the prescribed actions by joint maximization, then they will collectively obtain the following outcome:

$$W^* = \sum_{j\in M} W_j\left(\underline{\tilde{u}}^*\left(x^0\right)\right).$$

Two questions remain unresolved:

1. How can W^* be divided among the players? Note that $W_j\left(\underline{\tilde{u}}^*\left(x^0\right)\right)$ is the before side-payment payoff of player j and not what she will actually obtain after side payments have been made.[4]
2. How do we design a node-consistent agreement? That is, how is it possible to allocate each player's after side-payment payoff over nodes such that all players stick to the agreement as time goes by?

In order to address these issues, we need to implement the following steps:

1. Define a cooperative game and compute all characteristic function values.
2. Choose a solution concept. This amounts at selecting an imputation, that is, a vector whose entries correspond to after-side-payment outcomes of the players.
3. Compute for each node of the event tree the cooperative and noncooperative payoffs-to-go.
4. Define an imputation distribution procedure (IDP) that is node consistent.

4.1 Defining the cooperative game

A cooperative game is a triplet (M, v, Y), where M is the set of players; v is the characteristic function that assigns to each coalition $G, G \subseteq M$, a numerical value,

$$v\left(G\right) : P(M) \to \mathbb{R}, \qquad v\left(\varnothing\right) = 0,$$

where $P(M)$ is the power set of M; and Y is the set of imputations, that is,

$$Y = \left\{ (y_1, \ldots, y_m) \text{ such that } y_j \geq v\left(\{j\}\right) \text{ and } \sum_{j\in M} y_j = v\left(M\right) \right\}.$$

[3] We can easily extend our framework to the case where the players maximize a weighted sum of payoffs.

[4] The implicit assumption here is that players' utilities (gains) are comparable and transferable; otherwise side payments do not make sense.

The characteristic function measures the power or the strength of a coalition. Its precise definition depends on the assumption made about what the left-out players— that is, the complement subset of players $M\backslash G$— will do (see, e.g., Ordeshook (1986) and Osborne and Rubinstein (1994)). In their seminal book, von Neumann and Morgenstern (1944) interpreted $v(G)$ as the largest joint payoff that a coalition G can guarantee its members. In the absence of externalities, i.e., if the payoffs to the members of a coalition G is independent of the actions of the non-members $(M\backslash G)$, then $v(G)$ would be the result of an optimization problem. However, in the presence of externalities, a prediction of the actions of the non-members of G plays a central role in the computation of the worth of a coalition. This aspect has led to different definitions of a characteristic function (see Aumann (1961) and Chander and Tulkens (1997)). Note that the developments to come are valid for any choice of $v(\cdot)$.

The definition of the set of imputations involves two conditions, namely, individual rationality ($y_j \geq v(\{j\})$) and collective rationality $\left(\sum_{j\in M} y_j = v(M)\right)$. Individual rationality means that no player will accept an allocation or imputation that gives her less than what she can secure by acting alone. Collective rationality means that the total collective gain should be allocated, that is, no deficit or subsidies are considered. To make the connection with what was said earlier, observe that $v(M) = W^* = \sum_{j\in M} W_j\left(\underline{\tilde{u}}^*\left(x^0\right)\right)$, and that player j will get some y_j, which is still to be decided (in the next step) and which will not necessarily be equal to $W_j\left(\underline{\tilde{u}}^*\left(x^0\right)\right)$.

4.2 Selecting imputations

Game theorists have proposed many solutions for sharing the total cooperative gain among the players. These solutions are typically based on a series of axioms or requirements that the allocation(s) must satisfy, e.g., fairness, stability. We distinguish between solution concepts that select a unique imputation in Y, e.g., Shapley value and the nucleolus, and those that select a subset of imputations, e.g., the core and stable set. The two most used solution concepts in applications of cooperative games are the Shapley value and the core. We will use them to illustrate the process of building a node-consistent cooperative solution.

Definition 3 *The Shapley value is an imputation $\sigma = (\sigma_1, \ldots \sigma_m)$ defined by*

$$\sigma_j = \sum_{\substack{G\subset M \\ j\in G}} \frac{(m-g)!(g-1)!}{m!}[v(G) - v(G\backslash\{j\})]. \tag{6}$$

Being an imputation, the Shapley value satisfies individual rationality, i.e., $\sigma_j \geq v(\{j\})$ for all $j \in M$. The term $[v(G) - v(G\backslash\{j\})]$ corresponds to the marginal contribution of player j to coalition G. Thus, the Shapley value allocates to each player the weighted sum of her marginal contributions to all coalitions that she may join. The Shapley value is the unique imputation satisfying three axioms: fairness (identical players are treated in the same way), efficiency $\left(\sum\limits_{j\in M} \sigma_j = v(M)\right)$ and linearity (if v and w are two characteristic functions defined for the same set of players, then $\sigma_j(v+w) = \sigma_j(v) + \sigma_j(w)$ for all $j \in M$).

To define the core, we need to introduce the concept of dominated imputations. Let $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ be two imputations of the cooperative game $< M, v, Y >$.

Definition 4 *The imputation $y = (y_1, \ldots, y_m)$ dominates the imputation $z = (z_1, \ldots, z_m)$ through a coalition G if the following two conditions are satisfied:*

$$\begin{aligned} \textit{feasibility condition} &: \sum_{j\in G} y_j \leq v(G), \\ \textit{preferability condition} &: y_j > z_j, \quad \forall j \in G. \end{aligned}$$

Definition 5 *The core is the set of all undominated imputations*

The following theorem, due to Gillies (1953), characterizes the set of imputations belonging to the core of a cooperative game.

Theorem 1 *An imputation $y = (y_1, \dots, y_m)$ is in the core if*

$$\sum_{j \in G} y_j \geq v(G), \forall G \subseteq M.$$

In other words, the above condition states that an imputation is in the core if it allocates to each possible coalition an outcome that is at least equal to what this coalition can secure by acting alone. Consequently, the core is defined by

$$C = \left\{ (y_1, \dots, y_m), \text{ such that } \sum_{j \in G} y_j \geq v(G), \forall G \subset M, \text{ and } \sum_{j \in M} y_j = v(M) \right\}.$$

Note that the core may be empty, may be a singleton or may contain many imputations.[5]

4.3 Cooperative and noncooperative payoffs-to-go

Introduce the following notation:

$\tilde{u}_j\left(x^*\left(n_l^t\right)\right)$: An admissible strategy for player j in the subgame starting in node n_l^t, with initial state $x^*\left(n_l^t\right)$, $n_l^t \in \mathcal{N}^t, t = 1, \dots, T$, and $\underline{\tilde{u}}\left(x^*\left(n_l^t\right)\right) = \left(\tilde{u}_j\left(x^*\left(n_l^t\right)\right) : j \in M\right)$.

$\tilde{u}_j^N\left(x^*\left(n_l^t\right)\right)$: S-adapted equilibrium strategy for player j in the subgame starting in node n_l^t, with initial state $x^*\left(n_l^t\right)$, $n_l^t \in \mathcal{N}^t, t = 1, \dots, T$, and $\underline{\tilde{u}}^N\left(x^*\left(n_l^t\right)\right) = \left(\tilde{u}_j^N\left(x^*\left(n_l^t\right)\right) : j \in M\right)$.

$\tilde{u}_j^N\left(x^*\left(n_l^t\right), \left[n_v^\tau, n_w^T\right]\right)$: The trajectory of $\tilde{u}_j^N\left(x^*\left(n_l^t\right)\right)$ on the path emanating from node $n_v^\tau, n_v^\tau \in \mathcal{N}^\tau, \tau > t$, and terminating at node $n_w^T \in \mathcal{N}^T$.

$\tilde{u}_j^*\left(x^*\left(n_l^t\right)\right)$: Cooperative strategy (control) for player j in the subgame starting in node n_l^t, with initial state $x^*\left(n_l^t\right)$, $n_l^t \in \mathcal{N}^t, t = 1, \dots, T$, and $\underline{\tilde{u}}^*\left(x^*\left(n_l^t\right)\right) = \left(\tilde{u}_j^*\left(x^*\left(n_l^t\right)\right) : j \in M\right)$.

$\tilde{u}_j^*\left(x^*\left(n_l^t\right), \left[n_v^\tau, n_w^T\right]\right)$: The trajectory of $\tilde{u}_j^*\left(x^*\left(n_l^t\right)\right)$ on the path emanating from node $n_v^\tau, n_v^\tau \in \mathcal{N}^\tau, \tau > t$, and terminating at node $n_w^T \in \mathcal{N}^T$.

$W_j^N\left(\underline{\tilde{u}}\left(x^*\left(n_l^t\right)\right)\right)$: S-adapted equilibrium payoff of player j in the subgame starting in node n_l^t, with initial state $x^*\left(n_l^t\right)$, $n_l^t \in \mathcal{N}^t, t = 1, \dots, T$.

$W_j^*\left(\underline{\tilde{u}}\left(x^*\left(n_l^t\right)\right)\right)$: Payoff of player j in the cooperative game starting in node n_l^t, with initial state $x^*\left(n_l^t\right)$, $n_l^t \in \mathcal{N}^t, t = 1, \dots, T$.

Remark 6 *The trajectories $\tilde{u}_j^N\left(x^*\left(n_l^t\right), \left[n_v^\tau, n_w^T\right]\right)$ and $\tilde{u}_j^N\left(x^*\left(n_v^\tau\right), \left[n_v^\tau, n_w^T\right]\right)$ do not, in general, coincide. One reason is that the trajectory $\tilde{u}_j^N\left(x^*\left(n_l^t\right), \left[n_v^\tau, n_w^T\right]\right)$ has been computed assuming that the players have co-operated only during the time interval $[0, t]$, whereas $\tilde{u}_j^N\left(x^*\left(n_v^\tau\right), \left[n_v^\tau, n_w^T\right]\right)$ is computed under the assumption of a cooperative mode of play on $[0, \tau]$, with $\tau > t$.*

If the players adopt the Shapley value, then, in the whole game, player j gets the following outcome:

$$\sigma_j\left(x^0\left(n^0\right)\right) = \sum_{\substack{G \subset M \\ j \in G}} \frac{(m-g)!(g-1)!}{m!} \left[v\left(G; x^0\left(n^0\right)\right) - v\left(G \backslash \{j\}; x^0\left(n^0\right)\right)\right], \tag{7}$$

with

$$\sum_{j \in M} \sigma_j\left(x^0\left(n^0\right)\right) = v\left(M; x^0\left(n^0\right)\right).$$

[5]The following example illustrates this statement. Consider a three-player cooperative game with characteristic function values given by

$$\begin{aligned} v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0, \\ v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = a, \quad v(\{1,2,3\}) = 1 \end{aligned}$$

where $0 < a \leq 1$. It is easy to verify that three cases can occur: (i) If $0 < a < 2/3$, then the core contains all imputations satisfying $y_j \geq 0$, $\sum_{j \in G} y_j \geq a$ and $\sum_{j \in M} y_j = 1$. (ii) If $a = 2/3$, then the core is a singleton, that is, the only imputation belonging to the core is $(1/3, 1/3, 1/3)$. (iii) If $a > 2/3$, then the core is empty.

Similarly, the Shapley value in the subgame starting in node n_l^t and in state $\tilde{x}^*\left(n_l^t\right)$ is given by

$$\begin{aligned} \sigma_j\left(x^*\left(n_l^t\right)\right) &= \sum_{\substack{G\subset M\\ j\in G}} \frac{(m-g)!(g-1)!}{m!}\left[v\left(G;x^*\left(n_l^t\right)\right)-v\left(G\backslash\{j\};x^*\left(n_l^t\right)\right)\right], \qquad (8)\\ \sum_{j\in M}\sigma_j\left(x^*\left(n_l^t\right)\right) &= v\left(M;x^*\left(n_l^t\right)\right). \end{aligned}$$

Now, suppose that the players wish to implement an imputation in the core. The set of imputations in the core of the whole game is given by

$$C\left(x^0\left(n^0\right)\right)=\left\{\left(y_1\left(x^0\left(n^0\right)\right),\ldots,y_m\left(x^0\left(n^0\right)\right)\right) \mid \sum_{j\in G} y_j\left(x^0\left(n^0\right)\right)\geq v(G;x^0), \quad \forall G\subset M, \right.$$
$$\left. \text{and } \sum_{j\in M} y_j\left(x^0\left(n^0\right)\right)=v(M;x^0)\right\}, \quad (9)$$

and in the subgame starting from node n_l^t, with state value $x^*\left(n_l^t\right)$, given by

$$C(x^*(n_l^t))=\left\{\left(y_1\left(x^*(n_l^t)\right),\ldots,y_m\left(x^*(n_l^t)\right)\right) \mid \sum_{j\in G} y_j \geq v(G;x^*(n_l^t)) \quad \forall G\subset M,\right.$$
$$\left. \text{and } \sum_{j\in M} y_j\left(x^*(n_l^t)\right)=v(M;x^*(n_l^t))\right\}. \quad (10)$$

A main difficulty in defining a node-consistent core is that $C(x^0\left(n^0\right))$ and $C(x^*(n_l^t))$ are not singletons. This implies that the players must agree, at each node, on the imputation that they wish to implement in the subgame starting at that node. Further, we assume that the core of any subgame is nonempty.

4.4 Defining a node-consistent allocation

A cooperative solution in DGPET is node consistent at $x^0\left(n^0\right)$, if the cooperative payoff-to-go of player $j, j\in M$, in the subgame starting at node $n_l^t\in\mathcal{N}^t, t=1,\ldots,T$, is at least equal to the noncooperative payoff-to-go in this subgame. This will be achieved by introducing an imputation distribution procedure (IDP), that is, payment functions $\beta_j\left(x^*\left(n_l^t\right)\right), j\in M, n_l^t\in\mathcal{N}^t, t=1,\ldots,T$. The specific values of an IDP will of course depend on the chosen imputation.

4.4.1 Node-consistent Shapley value

Let us suppose that the players choose the Shapley value as solution of the cooperative game.

Definition 6 *An imputation distribution procedure of the Shapley value at $x_0\left(n_0\right)$ is given by $\left\{\beta_j\left(x^*\left(n_l^t\right)\right)\right\}_{n_l^t\in\mathcal{N}^t,t=1,\ldots,T}, j\in M$, satisfying*

$$\sigma_j\left(x^0\left(n^0\right)\right)=\sum_{\theta=0}^{t-1}\sum_{n_k^\theta\in\mathcal{N}^\theta}\pi(n_k^\theta)\beta_j(x^*(n_k^\theta)), \quad \textit{for all } j\in M. \qquad (11)$$

Clearly, an IDP always exists as it simply requires the satisfaction of an accounting condition stating that any stream of payments to a player is feasible as long as its total expected value is equal to what that player is entitled to in the whole game. Note that the payments $\beta_j(x^*(n_k^\theta))$ are not (necessarily) equal to the realized payoffs, that is, $\phi_j^{n_l^t}\left(x^*(n_l^t),\underline{\tilde{u}}^*(n_l^t)\right)$. Now, we add the node-consistency condition.

Definition 7 *The Shapley value $\sigma_j\left(x^0\left(n^0\right)\right)$ and the corresponding imputation distribution procedure $\{\beta_j\left(x^*\left(n_l^t\right)\right)\}_{n_l^t\in\mathcal{N}^t,t=1,\ldots,T}, j\in M$, are node consistent at $x_0\left(n_0\right)$, if for any $(x^*\left(n_l^t\right))$, $n_l^t\in\mathcal{N}^t, t=0,\ldots,T$, it holds that*

$$\sigma_j\left(x^0\left(n^0\right)\right)=\sum_{\theta=0}^{t-1}\sum_{n_k^\theta\in\mathcal{N}^\theta}\pi(n_k^\theta)\beta_j(x^*(n_k^\theta))+\sum_{n_k^\theta\in\mathcal{N}^t}\pi(n_k^\theta)\sigma_j\left(x^*(n_l^t)\right), \quad \forall\ j\in M. \qquad (12)$$

The definition states that what we allocate till any intermediate node using the IDP, plus the Shapley value payments in the subgame starting in that node must be equal to what player j is entitled to in the whole game, that is, her Shapley value $\sigma_j\left(x^0\left(n^0\right)\right)$. What remains to be done is to show that there exists an IDP satisfying the above definition. The following theorem, due to Reddy et al. (2013), gives the result.

Theorem 2 *The IDP $\left(\beta_1\left(x^*\left(n_l^t\right)\right),\ldots,\beta_m\left(x^*\left(n_l^t\right)\right)\right)$ defined by*

$$\beta_j\left(x^*\left(n_l^t\right)\right)=\sigma_j\left(x^*\left(n_l^t\right)\right)-\sum_{n_k^{t+1}\in\mathcal{S}(n_l^t)}\pi(n_k^{t+1}|n_l^t)\sigma_j\left(x^*\left(n_k^{t+1}\right)\right),\quad t=0,\ldots,T-1, \tag{13}$$

$$\beta_j\left(x^*\left(n_l^T\right)\right)=\sigma_j\left(x^*\left(n_l^T\right)\right), \tag{14}$$

satisfies (12).

Proof. See Reddy et al. (2013). □

The interpretation of this theorem is straightforward. At any terminal node n_l^T, the IDP payment is exactly the Shapley value in the static game at that node. At all other nodes, the IDP allocates to player j her Shapley value in the subgame starting at that node, minus the expected Shapley value in the subgames that are reached in the sequel. Note that $\beta_j\left(x^*\left(n_l^t\right)\right)$ can assume any sign.

4.4.2 Node-consistent core

Defining a node-consistent core is more demanding than defining a node-consistent Shapley value for two main reasons. First, the Shapley value in any subgame, including the whole game, always exists and is unique. The core may be empty in some of the subgames, if not in all of them. As we said before, we suppose here that the cores in all subgames are nonempty; otherwise the construction to follow will not be feasible. Second, at each intermediate node $n_l^t\in\mathcal{N}^t, t>0$, the players need to agree on which imputation to select in $C(x^*(n_l^t))$, whereas there is no selection process in the case of Shapley value because $\sigma_j\left(x^*(n_l^t)\right)$ is uniquely defined. Note that both these issues pertain to cooperative game theory in general and are not specific to what is done here. Dealing with sets of imputations at each node that are not singletons leads to the following definition of an IDP, which is clearly more restrictive than the one stated above.

Definition 8 *The node payments $\{\beta_j\left(x^*\left(n_l^t\right)\right)\}_{n_l^t\in\mathcal{N}^t,t=1,\ldots,T}, j\in M$, constitute an IDP of $y\left(x^0\left(n_0\right)\right)\in C(x^0\left(n^0\right))$, if they satisfy the following conditions:*

$$y_j(x^0\left(n_0\right))=\sum_{\theta=0}^{T}\sum_{n_l^\theta\in\mathcal{N}^\theta}\pi(n_l^\theta)\beta_j(x^*(n_l^\theta)), \tag{15}$$

$$\sum_{j\in M}\beta_j(x^*(n_l^t))=\sum_{j\in M}\phi_j^{n_l^t}\left(x^*(n_l^t),\underline{\tilde{u}}^*(n_l^t)\right), \tag{16}$$

$$\sum_{j\in M}\beta_j(x^*(n_l^T))=\sum_{j\in M}\Phi_j^{n_l^T}\left(x^*(n_l^T)\right), \tag{17}$$

where the two last conditions are satisfied for any $n_l^t\in\mathcal{N}^t$, $t=0,\ldots,T-1$ (for 16), and any $n_l^T\in\mathcal{N}^T$.

The accounting condition (15) that must be satisfied for the whole game is the same as (11). The next two conditions in the above definition state that the sum of payments, at any node, must be equal to the sum of realized cooperative payoffs at that node. In economic terms, banking payoffs for future use, or borrowing from future periods are not allowed.

Definition 9 *The imputation $y(x^0)\in C(x^0)$ and corresponding imputation distribution procedure*

$$\left(\left\{\beta_j\left(x^*\left(n_l^t\right)\right)\right\}_{n_l^t\in\mathcal{N}^t,t=1,\ldots,T}:j\in M\right),$$

are called node consistent in the whole game if for any state $x^*(n_l^t)$, $n_l^t \in \mathcal{N}^t, t = 0, \ldots, T$, *there exists* $y(x^*(n_l^t)) = (y_1(x^*(n_l^t)), \ldots, y_m(x^*(n_l^t))) \in C(x^*(n_l^t))$ *satisfying the following condition:*

$$y_j(x^0(n_0)) = \sum_{\theta=0}^{t-1} \sum_{n_k^\theta \in \mathcal{N}^\theta} \pi(n_k^\theta)\beta_j(x^*(n_k^\theta)) + \sum_{n_k^t \in \mathcal{N}^t} \pi(n_k^\theta) y_j\left(x^*(n_l^t)\right). \tag{18}$$

If the payoffs in the nodes are allocated according to the imputation distribution procedure, then node-consistency of imputation $y(x^0)$ from the core means that one can define a feasible distribution procedure under which the continuation values at every node are in the core of the continuation game.

Definition 10 *The core* $C(x^0)$ *in the whole game is a node-consistent allocation mechanism if any imputation* y *from the core* $C(x^0)$ *is node consistent.*

Theorem 3 *If the core* $C(x^0)$ *of the whole game and the core* $C(x^*(n_l^t))$ *of the subgame starting from any node* n_l^t *are nonempty, then the core* $C(x^0)$ *is node consistent when the corresponding imputation distribution procedure for each imputation* $y(x^0) \in C(x^0)$ *satisfies the following conditions*
for $t = 0, \ldots, T-1$*:*

$$\beta_j(x^*(n_l^t)) = y_j(x^*(n_l^t)) - \sum_{n_k^{t+1} \in \mathcal{S}(n_l^t)} \pi(n_k^{t+1}|n_l^t) y_j(x^*(n_k^{t+1})), \tag{19}$$

and for $t = T$*:*

$$\beta_j(x^*(n_l^T)) = y_j(x^*(n_l^T)), \tag{20}$$

where $y(x^*(n_l^t)) = (y_1(x^*(n_l^t)), \ldots, y_m(x^*(n_l^t))) \in C(x^*(n_l^t))$ *for any* $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T$ *and* $\pi(n_k^{t+1}|n_l^t)$ *is the conditional probability that node* n_k^{t+1} *is reached if node* n_l^t *has already been reached.*

Proof. See Parilina and Zaccour (2015a). □

If the core $C(x^0)$ of the whole game and the core $C(x^*(n_l^t))$ of a subgame starting from any node n_l^t are nonempty, we can always find at least one imputation $y(x^*(n_l^t)) \in C(x^*(n_l^t))$ and, using the given imputations $y(x^*(n_l^t))$ for all nodes $n_l^t \in \mathcal{N}^t$, $t = 0, \ldots, T$, construct the imputation distribution procedure $\left(\{\beta_j(x^*(n_l^t))\}_{n_l^t \in \mathcal{N}^t, t=0,\ldots,T} : j \in M\right)$, with formulas (19) and (20) for any imputation from the core $C(x^0)$.

The IDP and the realized outcomes at node $n_l^t \in \mathcal{N}^t, t = 0, \ldots, T-1$ are related by the following side payments:

$$\omega_j(n_l^t, x^*(n_l^t)) = \beta_j(x^*(n_l^t)) - \phi_j^{n_l^t}(x^*(n_l^t), \underline{\tilde{u}}^*(n_l^t)), \tag{21}$$

and for $\forall n_l^T \in \mathcal{N}^T$:

$$\omega_j(n_l^T, x^*(n_l^T)) = \beta_j(x^*(n_l^T)) - \Phi_j^{n_l^T}(x^*(n_l^T)), \tag{22}$$

where $\omega_j(n_l^t, x^*(n_l^t))$ is the transfer payment that player j makes in node n_l^t over the cooperative trajectory $x^*(n_l^t)$, such that

$$\sum_{j \in M} \omega_j(n_l^t, x^*(n_l^t)) = 0,$$

for any node n_l^t over cooperative trajectory $x^*(n_l^t)$. Clearly, $\omega_j(n_l^t, x^*(n_l^t))$ can assume any sign depending on the sign of the difference in the right-hand sides of (21)–(22). □

5 Concluding remarks

We showed in this paper how to decompose over time the Shapley value and an imputation in the core such that cooperation is sustained at any node of the event tree. Many extensions to our framework can be envisioned. First, it should not be complicated to define node consistency for other solution concepts, such as proportional payments and a Nash bargaining procedure. Second, we assumed that the core $C(x^*(n_l^t))$ in any subgame is nonempty. An interesting open question is whether cooperation can still be sustained if the cores in some of the subgames (not the whole game) are empty. Finally, it would be interesting to consider node consistency for DGPET when the end of the horizon is random.

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