Les Cahiers du GERAD ISSN: 0711–2440

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G–2014–78

November 2014

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Abstract: We consider an n-player game in coalitional form. We use the so-called δ characteristic function to determine the strength of all possible coalitions. The value of a coalition is obtained under the behavioral assumption that left-out players do not react strategically to the formation of that coalition, but stick to their Nash equilibrium actions in the n-player noncooperative game. This assumption has huge computational merit, especially in games where each player is described by a large-scale mathematical program. For the class of games with multilateral externalities discussed in Chander and Tulkens, we show that the δ characteristic function is superadditive and has a nonempty core, and that the δ-core is a subset of the γ-core.

Key Words: Coalitional games, characteristic function, multilateral externalities.

Résumé : On considère un jeu coopératif à n joueurs et on utilise la fonction caractéristique δ pour évaluer la force stratégique de chacune des coalitions possibles. La valeur pour chaque coalition est obtenue en maximisant la somme des gains de ses membres sous l'hypothèse que les joueurs non membres utilisent leurs actions d'équilibre dans le jeu non coopératif à n joueurs. Cette hypothèse sur le comportement des non membres simplifie énormément les calculs à effectuer pour déterminer la valeur de chacune des coalitions possibles. On montre que pour la classe de jeux avec externalités multilatérales à la Chander et Tulkens que la fonction caractéristique δ est superadditive, que le noyau est non vide et qu'il est inclut dans le noyau γ.

Mots clés : Jeux coopératifs, fonction caractéristique, externalités multilatérales.

Acknowledgments: Research supported by NSERC, Canada.

1 Introduction

The implementation of any classical solution of an n-player cooperative game with transferable utilities, e.g., Shapley value, core or nucleolus, requires the computation of the characteristic function (CF) values of the $2^n - 1$ possible nonempty coalitions. This task is very challenging, if at all feasible, when (i) n is large; (ii) each player is described by a large-scale mathematical programming model; and (iii) each of these values is the result of an equilibrium problem, as is the case for the γ characteristic function (see Chander and Tulkens (1997)). To illustrate the difficulty, suppose that we have to allocate among all countries the total emissions reductions needed to avoid a serious climate change. To start, we have to adopt a model that describes the economics and climate interactions for each of the countries (players). A well-accepted model in this area is the Integrated Assessment Model (TIAM-WORLD), which is a multi-regional model of the energy/emissions of 16 world regions (see, for instance, Loulou and Labriet (2008)). If we adopt this model and the γ characteristic function to define the imputations in the core or the Shapley value of the 16-player cooperative game, then we would need to compute the characteristic function values for $65,535$ coalitions. Knowing that TIAM-WORLD has more than one million variables, the task of computing $65,534$ equilibrium values is near impossible, even if we have access to the most sophisticated computers and algorithms available.

Petrosjan and Zaccour (2003) dealt with this "curse of dimensionality" in cooperative games by defining a characteristic function that requires considerably less computational effort than the γ characteristic function (γ-CF). Indeed, whereas in the γ-CF approach we must solve one optimization problem (the grand coalition's problem) and $2^n - 2$ equilibrium problems, in the Petrosjan-Zaccour CF it is the other way around, that is, we need to solve one equilibrium problem (the n-player noncooperative game) and $2^n - 2$ optimization problems. As solving an optimization problem is significantly simpler than solving an equilibrium one, the computational savings are huge. We shall refer from now on to the Petrosjan-Zaccour CF as the δ-CF.[1] This computational benefit comes, however, at the cost of making a strong assumption on players' behavior, which will be specified in the sequel.

Petrosjan and Zaccour (2003) did not discuss the properties that the δ-CF may exhibit, but simply used it to design a time-consistent Shapley value for the differential game they considered. Zaccour (2003) showed that the γ and δ characteristic function values coincide for the particular class of linear-state differential games.[2] In this paper, we establish some properties of the δ-CF and discuss its relationships with the γ-CF for the class of games studied in Chander and Tulkens (1997), namely, games with multilateral externalities. Our main results are: (i) the δ-core is nonempty; and (ii) the δ-core is a subset of the γ-core. Thus, we can interpret the δ-core as a "selector" of some imputations from the γ-core.

The rest of the paper is organized as follows: In Section 2, we recall some preliminaries. In Section 3, we introduce the class of games with negative externalities, and in Section 4, our main results. Section 5 presents an illustrative example, and Section 6 briefly concludes.

2 Preliminaries

Denote by $N = \{1, \ldots, n\}$ the set of players. Let $v(\cdot)$ be the characteristic function defined by

$$v : \mathcal{P}(N) \to \mathbb{R}, \qquad v(\varnothing) = 0,$$

where $\mathcal{P}(N)$ is the power set of N, and denote by I the set of imputations given by

$$I = \left\{ (y_1, \ldots, y_n) \mid y_i \geq v(\{i\}), \forall i \text{ and } \sum_{i=1}^{n} y_i = v(N) \right\}.$$

A coalition K is a subset of players, i.e., $K \subseteq N$. Von Neumann and Morgenstern (1944) interpreted the value $v(K)$ as the sum of gains that a coalition can *guarantee* its members. In cooperative games without externalities, i.e., when the payoff of a coalition K is independent of the actions of the left-out players (LOP),

[1] This choice is to continue the series of characteristic functions designated by the greek letters α, β and γ.
[2] In this class of games, the payoff functions can be non-linear in the control (decision) variables.

i.e., players in $N\backslash K$, the value $v(K)$ is simply obtained by optimizing the (possibly weighted) sum of the coalition members' payoffs. In this context, we need to solve as many optimization problems as there are nonempty coalitions, that is, $2^n - 1$ problems. In games with externalities, the outcome of K depends on the behavior of players in $N\backslash K$, which leads to the definition of the α, β and γ characteristic functions. Briefly, $v^{\alpha}(K)$ is the maximum value of the minimum taken over LOP strategies. The α-CF is, for most purposes, seen as too conservative as it amounts to assuming an antagonistic zero-sum game between K and $N\backslash K$ (see Ordeshook (1986)). The β characteristic function assumes that the payoff $v^{\beta}(K)$ that coalition K can strategically guarantee its members corresponds to the minimum value of the maximum taken over LOP strategies. For games with transferable utilities, which are our focus here, the α- and β-CFs are equivalent. The common assumption is that the left-out players in $N\backslash K$ form a coalition whose purpose is to hurt coalition K as far as feasible, without any concern for their own payoffs. The construction of α- and β-CFs from a normal-form game ignores possible strategic interactions that could take place between the coalition and the left-out players. To account for this, Chander and Tulkens (1997) proposed the γ-CF for a class of games with multilateral externalities (see also Chander (2007) for a discussion of coalition formation in this context, and Germain et al. (2003), for an application in environmental economics). In the γ-CF, $v^{\gamma}(K)$ is defined as the partial equilibrium outcome of the noncooperative game between coalition K and LOP acting individually. This means that each left-out player is taking note of the formation of coalition K and best-replying to the other players' strategies. If we denote by k the number of players in coalition K, then the determination of $v^{\gamma}(K)$ requires finding an equilibrium for a noncooperative game with $n-k+1$ players. Here, we have to solve one optimization problem (the grand coalition's problem) and $2^n - 2$ equilibrium problems. The pending difficulties are the selection of one equilibrium outcome when there are multiple equilibria, and what to do when there is no equilibrium in pure strategies.[3] Further, it was shown under some convexity assumptions that the γ-core is nonempty for a quite general class of games with multilateral externalities (see Chander and Tulkens (1997) and Helm (2001)).

The δ-CF involves a two-step procedure. First, we compute an n-player noncooperative equilibrium. Second, for each coalition K, we solve an optimization problem consisting of maximizing the joint payoff, assuming that LOP stick to their Nash equilibrium actions in the n-player noncooperative game. As stated before, the main advantage of the δ-CF with respect to the γ-CF is computational. Its main drawback is the assumption that the left-out players do not best-reply to the formation of the coalition, but stick to their Nash equilibrium actions in the n-player noncooperative game. In the next section, we consider the class of games with multilateral externalities in Chander and Tulkens (1997), and show that the δ-core is nonempty. Further, we prove that the δ-core is included in the γ-core, and hence, is a selector of γ-core imputations.

3 Games with negative externalities

Chander and Tulkens (1997) consider a model with multilateral externalities and two kinds of commodities: a standard private good, whose quantities are denoted by $x_1, x_2, \ldots, x_n$, and an environmental good (pollution emissions), whose quantities are denoted by $e_1, e_2, \ldots, e_n$. Technology is described by a production function $f_i(e_i)$, which is assumed to be increasing, differentiable and concave in e_i. Each agent's preferences are represented by a quasi-linear utility function $u_i(x_i, e) = x_i - d_i(e)$, where $d_i(e)$ is agent i's disutility, which is assumed to be a positive, increasing, differentiable and convex function of the level of the externality $e = \sum_{i \in N} e_i$. The agents outside of a coalition $K \subset N$ choose their individual best reply to the coalition members' actions, which results in the γ-characteristic function defined as follows:

$$
\begin{aligned}
& v^{\gamma}(K) = \max_{\{(x_i,e_i)_{i\in K}\}} \sum_{i \in K} u_i(x_i, e), \\
& \text{subject to } \sum_{i \in K} x_i \leq \sum_{i \in K} f_i(e_i) \text{ and } e = \sum_{i \in K} e_i + \sum_{j \in N \backslash K} e_j, \\
& \text{where for all } i \in N\backslash K,\ (x_i, e_i) \text{ maximizes } u_i(x_i, e_i), \\
& \text{subject to } x_i \leq f_i(e_i) \text{ and } e = e_i + \sum_{j \in N \backslash i} e_j.
\end{aligned}
$$

[3] These difficulties vanish for the class of games considered by Chander and Tulkens (1997).

Denote by $\tilde{e}_i$ player i's Nash-equilibrium action in the n-player noncooperative game, and denote by $\tilde{e} = \sum_{i\in N} \tilde{e}_i$. In the δ characteristic function, players outside of a coalition $K \subset N$ freeze their actions at their Nash-equilibrium levels $\tilde{e}_j, j \in N\backslash K$. The δ-CF value of coalition K is then given by

$$v^{\delta}(K) = \max_{\{(x_i,e_i)_{i\in K}\}} \sum_{i\in K} u_i(x_i, e),$$
$$\text{subject to } \sum_{i\in K} x_i \leq \sum_{i\in K} f_i(e_i) \text{ and } e = \sum_{i\in K} e_i + \sum_{j\in N\backslash K} \tilde{e}_j.$$

In the rest of the paper, we shall use the following notations:

$e_i^{\delta,K}$: Player i's optimal action under δ-CF when coalition K forms and $i \in K$, and

$$e^{\delta,K} = \sum_{i\in K} e_i^{\delta,K} + \sum_{j\in N\backslash K} \tilde{e}_j.$$

$e_i^{\gamma,K}$: Player i's Nash-equilibrium action under γ-CF when coalition K forms, and

$$e^{\gamma,K} = \sum_{i\in N} e_i^{\gamma,K}.$$

e_i^* : Player i's Pareto-optimal emissions. We know that when a grand coalition forms, both γ and δ concepts coincide.

The necessary and sufficient conditions for the above optimization problem are

$$f_i'(e_i^{\delta,K}) = \sum_{i\in K} d_i'(e^{\delta,K}), \forall i \in K, \tag{1}$$
$$e^{\delta,K} = \sum_{i\in K} e_i^{\delta,K} + \sum_{j\in N\backslash K} \tilde{e}_j. \tag{2}$$

For our purposes, the main results in Chander and Tulkens (1997) are: (i) given any coalition K, there exists a unique Nash equilibrium in pure strategies of the $(n-k+1)$-player noncooperative game; (ii) the core is nonempty; (iii) the total emissions of the players when a coalition K forms satisfy $e^{\gamma,K} \leq \tilde{e}$; and finally (iv) the left-out players free-ride on the coalition players, i.e., $e_j^{\gamma,K} \geq \tilde{e}_j$ for $j \in N\backslash K$.

4 Results

We start by the following lemma about the total players' emissions when coalition K forms.

Lemma 1 *For any coalition $K \subseteq N$, the total emissions under the δ-CF are less or equal to the total emissions in the fully noncooperative game, that is, $e^{\delta,K} \leq \tilde{e}$.*

Proof. Suppose that $e^{\delta,K} > \tilde{e}$. The convexity of $d_i(\cdot)$, which implies $d_i'(e^{\delta,K}) \geq d_i'(\tilde{e})$, and the first-order conditions (1)–(2) yield

$$f_i'(e_i^{\delta,K}) = \sum_{i\in K} d_i'(e^{\delta,K}) \geq d_i'(e^{\delta,K}) \geq d_i'(\tilde{e}) = f_i'(\tilde{e}_i),\ \forall i \in K,$$
$$e_j^{\delta,K} = \tilde{e}_j,\ \forall j \in N\backslash K.$$

From the concavity of the production functions we have that $e_i^{\delta,K} \leq \tilde{e}_i,\ \forall i \in K$. This implies

$$e^{\delta,K} = \sum_{i\in K} e_i^{\delta,K} + \sum_{j\in N\backslash K} \tilde{e}_j \leq \sum_{i\in K} \tilde{e}_i + \sum_{j\in N\backslash K} \tilde{e}_j = \tilde{e},$$

which contradicts our assumption. □

The formation of any coalition leads to lower emissions than under a fully noncooperative Nash equilibrium. In the next lemma, we show that the combined emissions of the players in coalition K are greater when the left-out players freeze their strategies at their individual Nash-equilibrium strategies (δ concept) than when they use their best-reply strategies (γ concept).

Lemma 2 *For any coalition $K \subseteq N$, we have*

$$\sum_{i\in K} e_i^{\delta,K} \geq \sum_{i\in K} e_i^{\gamma,K}.$$

Proof. From Chander and Tulkens (1997) and Helm (2001), we know that

$$e^{\gamma,K} \leq \tilde{e} \quad \text{and} \quad e_j^{\gamma,K} \geq \tilde{e}_j, \;\; \text{for all } j \in N\backslash K.$$

Suppose that $\sum_{i\in K} e_i^{\delta,K} < \sum_{i\in K} e_i^{\gamma,K}$. Consequently, we have

$$\begin{aligned} e^{\delta,K} &= \sum_{i\in K} e_i^{\delta,K} + \sum_{j\in N\backslash K} \tilde{e}_j < \sum_{i\in K} e_i^{\gamma,K} + \sum_{j\in N\backslash K} \tilde{e}_j \\ &\leq \sum_{i\in K} e_i^{\gamma,K} + \sum_{j\in N\backslash K} e_j^{\gamma,K} = e^{\gamma,K}. \end{aligned}$$

The convexity of $d(\cdot)$, implies $d_i'(e^{\delta,K}) \leq d_i'(e^{\gamma,K})$, which, along with the first-order optimality conditions of Lemma (1) leads to

$$f_i'(e_i^{\delta,K}) = \sum_{i\in K} d_i'(e^{\delta,K}) \leq \sum_{i\in K} d_i'(e^{\gamma,K}) = f_i'(e_i^{\gamma,K}), \; \forall i \in K.$$

By the concavity of the production functions, we have $e_i^{\delta,K} \geq e_i^{\gamma,K}, \; \forall i \in K$. Consequently, $\sum_{i\in K} e_i^{\delta,K} \geq \sum_{i\in K} e_i^{\gamma,K}$, which contradicts our assumption. □

Chander and Tulkens (1997) showed that $e_j^{\gamma,K} \geq \tilde{e}_j, \; j \in N\backslash K$ and $e^{\gamma,K} \leq \tilde{e}$ for any $K \subset N$, i.e., the left-out players free-ride on the coalition by emitting more than under their Nash-equilibrium strategies. In the δ approach however, the left-out players freeze their strategies at the Nash-equilibrium levels instead of best-replying to coalition K. Hence, the combined emissions of coalition players in K are greater. Next, we show that the δ-CF gives any coalition K a higher value than the γ-CF.

Theorem 1 *For any $K \subseteq N$, $v^\delta(K) \geq v^\gamma(K)$.*

Proof. By definition, we have $v^\delta(K) = v^\gamma(K)$, for $|K| = 1$ or $K = N$. For $1 < |K| < |N|$, we have

$$\begin{aligned} v^\delta(K) - v^\gamma(K) &= \Big(\sum_{i\in K} f_i(e_i^{\delta,K}) - d_i\big(\sum_{j\in K} e_j^{\delta,K} + \sum_{j\in N\backslash K} e_j^{\delta,K}\big)\Big) \\ &\quad - \Big(\sum_{i\in K} f_i(e_i^{\gamma,K}) - d_i\big(\sum_{j\in K} e_j^{\gamma,K} + \sum_{j\in N\backslash K} e_j^{\gamma,K}\big)\Big) \\ &= \sum_{i\in K}\Big(f_i(e_i^{\delta,K}) - f_i(e_i^{\gamma,K})\Big) - \sum_{i\in K}\Big(d_i(e^{\delta,K}) - d_i(e^{\gamma,K})\Big) \\ &\geq \sum_{i\in K} f_i'(e_i^{\delta,K})(e_i^{\delta,K} - e_i^{\gamma,K}) + \sum_{i\in K} d_i'(e^{\delta,K})(e^{\gamma,K} - e^{\delta,K}) \\ &= \sum_{i\in K} f_i'(e_i^{\delta,K})(e_i^{\delta,K} - e_i^{\gamma,K}) + (e^{\gamma,K} - e^{\delta,K}) \sum_{i\in K} d_i'(e^{\delta,K}). \end{aligned}$$

The inequality follows from the concavity of f_i and convexity of d_i. Next, using the first-order conditions (1)–(2) we have

$$\begin{aligned} v^\delta(K) - v^\gamma(K) &\geq \left((e^{\gamma,K} - e^{\delta,K}) + \sum_{i\in K} (e_i^{\delta,K} - e_i^{\gamma,K}) \right) \sum_{i\in K} d_i'(e^{\delta,K}) \\ &= \sum_{i\in N\setminus K} (e_i^{\gamma,K} - e_i^{\delta,K}) \sum_{i\in K} d_i'(e^{\delta,K}) \\ &= \sum_{i\in N\setminus K} (e_i^{\gamma,K} - \tilde{e}_i) \sum_{i\in K} d_i'(e^{\delta,K}) \geq 0. \end{aligned}$$

The last inequality follows from the monotonicity of the damage functions and the $e_j^{\gamma,K} \geq \tilde{e}_j = e_j^{\delta,K},\ j \in N\setminus K$. □

Using Lemma 1, the following theorem shows that $v^\delta(.)$ is superadditive.

Theorem 2 *The δ-characteristic function is superadditive.*

Proof. Let S and T be subsets of N with $S \cap T = \emptyset$. To show superadditivity, compute

$$\begin{aligned} & v^\delta(S\cup T) - v^\delta(S) - v^\delta(T) \\ &= \sum_{i\in S\cup T} f_i(e_i^{\delta,S\cup T}) - d_i(e^{\delta,S\cup T}) - \left(\sum_{i\in S} f_i(e_i^{\delta,S}) - d_i(e^{\delta,S}) \right) - \left(\sum_{i\in T} f_i(e_i^{\delta,T}) - d_i(e^{\delta,T}) \right) \\ &\geq \sum_{i\in S} f_i'(e_i^{\delta,S\cup T}) \left(e_i^{\delta,S\cup T} - e_i^{\delta,S} \right) + (e^{\delta,S} - e^{\delta,S\cup T}) d_i'(e^{\delta,S\cup T}) \\ &\quad + \sum_{i\in T} f_i'(e_i^{\delta,S\cup T}) \left(e_i^{\delta,S\cup T} - e_i^{\delta,T} \right) + (e^{\delta,T} - e^{\delta,S\cup T}) d_i'(e^{\delta,S\cup T}) \\ &= \sum_{i\in S\cup T} d_i'(e^{\delta,S\cup T}) \left(\sum_{i\in S} (e_i^{\delta,S\cup T} - e_i^{\delta,S}) + \sum_{i\in T} (e_i^{\delta,S\cup T} - e_i^{\delta,T}) \right) \\ &\quad + (e^{\delta,S} - e^{\delta,S\cup T}) \sum_{i\in S} d_i'(e^{\delta,S\cup T}) + (e^{\delta,T} - e^{\delta,S\cup T}) \sum_{i\in T} d_i'(e^{\delta,S\cup T}) \end{aligned}$$

The inequality follows from the concavity of $f_i(.)$, the convexity of $d_i(.)$ and the assumption that $S \cap T = \emptyset$. Then, the last equality follows from the first-order conditions (1)–(2). Rearranging the terms in the last expression, we obtain

$$\begin{aligned} & v^\delta(S\cup T) - v^\delta(S) - v^\delta(T) \\ &\geq \left(\sum_{i\in S} d_i'(e^{\delta,S\cup T}) + \sum_{i\in T} d_i'(e^{\delta,S\cup T}) \right) \left(e^{\delta,S\cup T} - \sum_{i\in N\setminus\{S\cup T\}} \tilde{e}_i + \sum_{i\in N\setminus S} \tilde{e}_i + \sum_{i\in N\setminus T} \tilde{e}_i - e^{\delta,S} - e^{\delta,T} \right) \\ &\quad + (e^{\delta,S} - e^{\delta,S\cup T}) \sum_{i\in S} d_i'(e^{\delta,S\cup T}) + (e^{\delta,T} - e^{\delta,S\cup T}) \sum_{i\in T} d_i'(e^{\delta,S\cup T}) \\ &= \left(\sum_{i\in S} d_i'(e^{\delta,S\cup T}) + \sum_{i\in T} d_i'(e^{\delta,S\cup T}) \right) \left(e^{\delta,S\cup T} - e^{\delta,S} - e^{\delta,T} + \sum_{i\in N} \tilde{e}_i \right) \\ &\quad + (e^{\delta,S} - e^{\delta,S\cup T}) \sum_{i\in S} d_i'(e^{\delta,S\cup T}) + (e^{\delta,T} - e^{\delta,S\cup T}) \sum_{i\in T} d_i'(e^{\delta,S\cup T}) \\ &= \left(\tilde{e} - e^{\delta,T} \right) \sum_{i\in S} d_i'(e^{\delta,S\cup T}) + \left(\tilde{e} - e^{\delta,S} \right) \sum_{i\in T} d_i'(e^{\delta,S\cup T}) \geq 0, \end{aligned}$$

where the last inequality follows from Lemma 1. □

It is well known that the superadditivity of $v^\delta(.)$ cannot guarantee that the cooperative game has a nonempty core. Chander and Tulkens (1997) show the nonemptiness of the γ-core under a few assumptions,[4]

[4] Under the assumption that for all $S \subset N$, $|S| \geq 2$: $\sum_{j\in S} d_j'(e^*) \geq d_i'(\tilde{e}), i \in S$.

by proving that a specific imputation, that is, a particular version of the ratio equilibrium (see Kaneko (1977)), belongs to the γ-core. Using the convexity assumptions of the production and damage functions and the Bondereva-Shapley theorem, Helm (2001) shows the nonemptiness of the γ-core. We follow a similar approach towards showing the nonemptiness of the δ-core.

Let $\mathcal{C}$ be the set of possible coalitions. Let $\mathcal{C}_i = \{K \in \mathcal{C} : i \in K\}$ be the subset of those coalitions that have player i as a member. A vector $(\mu_K)_{K\in\mathcal{C}} : \mu_K \in [0,1]$ for all $K \in \mathcal{C}$ is called a balanced collection of weights if for all $i \in N$ we have $\sum_{K\in\mathcal{C}_i} \mu_K = 1$.

Proposition 1 (Bondareva-Shapley Theorem) *A coalitional game with transferable payoff (N, v) has a nonempty core if, and only if, for every balanced collection of weights $\sum_{K\in\mathcal{C}} \mu_K v(K) \le v(N)$.*

Theorem 3 *The coalitional game $(N, v^\delta(.))$ is balanced, and therefore, has a nonempty core.*

Proof. From Lemma 1, we have $e^{\delta,K} \le \tilde{e}$, which implies $\sum_{i\in K} e_i^{\delta,K} \le \sum_{i\in K} \tilde{e}_i$, since $\sum_{i\in N\setminus K} e_i^{\delta,K} = \sum_{i\in N\setminus K} \tilde{e}_i$. Now,

$$\begin{aligned}\sum_{K\in\mathcal{C}\setminus\mathcal{C}_i} \mu_K \sum_{j\in K} e_j^{\delta,K} &\le \sum_{K\in\mathcal{C}\setminus\mathcal{C}_i} \mu_K \sum_{j\in K} \tilde{e}_j \\ &= \sum_{K\in\mathcal{C}} \mu_K \sum_{j\in K} \tilde{e}_j - \sum_{K\in\mathcal{C}_i} \mu_K \sum_{j\in K} \tilde{e}_j \\ &= \sum_{j\in N} \tilde{e}_j - \sum_{K\in\mathcal{C}_i} \mu_K \sum_{j\in K} \tilde{e}_j \\ &= \sum_{K\in\mathcal{C}_i} \mu_K \sum_{j\in N} \tilde{e}_j - \sum_{K\in\mathcal{C}_i} \mu_K \sum_{j\in K} \tilde{e}_j \\ &= \sum_{K\in\mathcal{C}_i} \mu_K \sum_{j\in N\setminus K} \tilde{e}_j = \sum_{K\in\mathcal{C}_i} \mu_K \sum_{j\in N\setminus K} e_j^{\delta,K}. \end{aligned} \tag{3}$$

For each $i \in N$, let $e_i^{\mathcal{C}} = \sum_{K\in\mathcal{C}_i} \mu_K e_i^{\delta,K}$, that is, the sum of weighted optimal emissions of player i in all coalitions to which he may belong. By concavity of the production function, we have

$$f_i(e_i^{\mathcal{C}}) = f_i\left(\sum_{K\in\mathcal{C}_i} \mu_K e_i^{\delta,K}\right) \ge \sum_{K\in\mathcal{C}_i} \mu_K f_i(e_i^{\delta,K}).$$

Next, compute

$$\begin{aligned} d_i\left(\sum_{i\in N} e_i^{\mathcal{C}}\right) &= d_i\left(\sum_{i\in N}\sum_{K\in\mathcal{C}_i} \mu_K e_i^{\delta,K}\right) = d_i\left(\sum_{K\in\mathcal{C}} \mu_K \sum_{j\in K} e_j^{\delta,K}\right) \\ &= d_i\left(\sum_{K\in\mathcal{C}_i} \mu_K \sum_{j\in K} e_j^{\delta,K} + \sum_{K\in\mathcal{C}\setminus\mathcal{C}_i} \mu_K \sum_{j\in K} e_j^{\delta,K}\right) \\ &\le d_i\left(\sum_{K\in\mathcal{C}_i} \mu_K \sum_{j\in K} e_j^{\delta,K} + \sum_{K\in\mathcal{C}_i} \mu_K \sum_{j\in N\setminus K} e_j^{\delta,K}\right) \\ &= d_i\left(\sum_{K\in\mathcal{C}_i} \mu_K \left(\sum_{j\in K} e_j^{\delta,K} + \sum_{j\in N\setminus K} e_j^{\delta,K}\right)\right) \\ &\le \sum_{K\in\mathcal{C}_i} \mu_K d_i\left(\sum_{j\in K} e_j^{\delta,K} + \sum_{j\in N\setminus K} e_j^{\delta,K}\right). \end{aligned}$$

In the above, the first inequality follows from (3) and the last inequality follows from the convexity of the damage functions. Then, from the cohesiveness of the coalitional game, that is, since the value of the grand coalition is at least as large as the sum of the values of the members of any partition of N, we have

$$\begin{aligned} v^{\delta}(N) &\geq \sum_{i\in N} f_i(e_i^{\mathcal{C}}) - \sum_{i\in N} d_i\left(\sum_{j\in N} e_j^{\mathcal{C}}\right) \\ &\geq \sum_{i\in N}\sum_{K\in\mathcal{C}_i} \mu_K f_i(e_i^{\delta,K}) - \sum_{i\in N}\sum_{K\in\mathcal{C}_i} \mu_K d_i\left(\sum_{j\in K} e_j^{\delta,K} + \sum_{j\in N\setminus K} e_j^{\delta,K}\right) \\ &= \sum_{K\in\mathcal{C}_i} \mu_K \sum_{i\in K}\left(f_i(e_i^{\delta,K}) - d_i\left(\sum_{j\in K} e_j^{\delta,K} + \sum_{j\in N\setminus K} e_j^{\delta,K}\right)\right) \\ &= \sum_{K\in\mathcal{C}} \mu_K v_{\delta}(K). \end{aligned}$$

Consequently, the δ-core is nonempty. □

Using Theorem 1 and Theorem 3 we have the following result.

Theorem 4 *For the game with multilateral externalities, δ-core $\subseteq \gamma$-core.*

Proof. Given that the δ- and the γ-core are nonempty, the result is a direct consequence of Theorem 1, i.e., $v^{\delta}(K) \geq v^{\gamma}(K)$ for every $K \subseteq N$, and of the definition of the core of a cooperative game. □

Chander and Tulkens (1997) show that by emitting more than under their Nash equilibrium strategies, that is, $e_j^{\gamma,K} \geq \tilde{e}_j,\ j \in N\setminus K$, the left-out players free-ride on the members of coalition K. In the δ case, such behavior cannot occur, as by definition, left-out players freeze their emissions at the fully noncooperative Nash-equilibrium levels. Lemma 1 only says that when coalition K forms, the total emissions are lower than the total Nash-equilibrium emissions, i.e., $e^{\delta,K} \leq \tilde{e}$. Now, superadditivity of $v^{\delta}(.)$ implies that it is in the best interest of a coalition to enlarge, but how does this translate in terms of emissions? The following proposition gives a hint.

Proposition 2 *Let $S \subset T \subseteq N$. If $e^{\delta,S} \leq e^{\delta,T}$, then $e_i^{\delta,T} \leq e_i^{\delta,S}$, $\forall i \in S$, $S \subset T$ and $\sum_{i\in T\setminus S} e_i^{\delta,T} \geq \sum_{i\in T\setminus S} \tilde{e}_i$.*

Proof. The first-order conditions (1)–(2) imply $f_i'(e_i^{\delta,S}) = \sum_{i\in S} d_i'(e^{\delta,S})$, for all $i \in S$. The monotonicity of $d_i'(\cdot)$ yields

$$f_i'(e_i^{\delta,S}) = \sum_{i\in S} d_i'(e^{\delta,S}) \leq \sum_{i\in S} d_i'(e^{\delta,T}) = f_i(e_i^{\delta,T}) - \sum_{i\in T\setminus S} d_i'(e^{\delta,T}),\ \text{for all } i \in S,$$

and the positivity of $d_i'(\cdot)$

$$f_i'(e_i^{\delta,T}) - f_i'(e_i^{\delta,S}) \geq \sum_{i\in T\setminus S} d_i'(e^{\delta,T}) \geq 0,\ \text{for all } i \in S.$$

From the concavity of f_i we have $e_i^{\delta,T} \leq e_i^{\delta,S}$ for all $i \in S$ and $S \subset T$. Then,

$$\begin{aligned} &\sum_{i\in S} e_i^{\delta,S} + \sum_{i\in T\setminus S} \tilde{e}_i + \sum_{i\in N\setminus T} \tilde{e}_i = e^{\delta,S} \leq e^{\delta,T} = \sum_{i\in S} e_i^{\delta,T} + \sum_{i\in T\setminus S} e_i^{\delta,T} + \sum_{i\in N\setminus T} \tilde{e}_i \\ &0 \leq \sum_{i\in S}\left(e_i^{\delta,S} - e_i^{\delta,T}\right) \leq \sum_{i\in T\setminus S}\left(e_i^{\delta,T} - \tilde{e}_i\right) \end{aligned}$$

The last inequality clearly implies $\sum_{i\in T\setminus S} e_i^{\delta,T} \geq \sum_{i\in T\setminus S} \tilde{e}_i$. □

The interpretation of the above proposition is as follows: when a coalition S enlarges to T, the new entrants, i.e., players in $T\backslash S$, jointly free-ride on the players in coalition S. Further, players in $T \cap S$ reduce their emissions individually in the newly formed bigger coalition, that is, $e_i^{\delta,T} \leq e_i^{\delta,S}$, for all $i \in S,\ S \subset T$. Note that this holds true only if the total emissions under T are larger than under S, i.e., $e^{\delta,S} \leq e^{\delta,T}$.

In the next subsection, we provide some sharper results for when the players are identical.

4.1 Symmetric game

To save on notation in this symmetric case, we denote by $\underline{e}^S = e_i^{\delta,S}$, $f(\cdot) = f_i(\cdot)$ and $d(\cdot) = d_i(\cdot)$, $\forall i \in N$, $S \subset N$, and by s the number of players in S. The first-order conditions (1)–(2) can be rewritten as

$$f'(\underline{e}^S) = sd'\left(s\underline{e}^S + (n-s)\,\tilde{\underline{e}}\right),\ S \subset N. \tag{4}$$

The Nash-equilibrium and Pareto-optimal emissions satisfy, respectively, the following two conditions:

$$f'(\tilde{\underline{e}}) = d'(n\tilde{\underline{e}}), \tag{5}$$
$$f'(\underline{e}^*) = nd'(n\underline{e}^*). \tag{6}$$

The following proposition shows that increasing the size of a coalition benefits each of its member, that is, the per-capita payoff of a coalition is non-decreasing with its size.

Proposition 3 *For $S \subset T \subseteq N$ we have $\frac{v^\delta(T)}{t} \geq \frac{v^\delta(S)}{s}$.*

Proof. Compute

$$\begin{aligned}
\frac{v^\delta(T)}{t} - \frac{v^\delta(S)}{s} &= f(\underline{e}^T) - f(\underline{e}^S) - \left(d(t\underline{e}^T + (n-t)\tilde{\underline{e}}) - d(s^S + (n-s)\tilde{\underline{e}})\right) \\
&\geq f'(\underline{e}^T)\left(\underline{e}^T - \underline{e}^S\right) + d'(t\underline{e}^T + (n-t)\tilde{\underline{e}})\big((s\underline{e}^S + (n-s)\tilde{\underline{e}}) - (t\underline{e}^T + (n-t)\tilde{\underline{e}})\big) \\
&= td'(t\underline{e}^T + (n-t)\tilde{\underline{e}})\left(1 - \frac{s}{t}\right)(\tilde{\underline{e}} - \underline{e}^S).
\end{aligned}$$

The inequality follows from the concavity of $f(.)$ and convexity of $d(.)$. The last equality follows directly from the first-order condition (4). From Lemma 1, we have $e^{\delta,S} = s\underline{e}^S + (n-s)\,\tilde{\underline{e}} \leq \tilde{e} = n\tilde{\underline{e}}$, which implies $\underline{e}^S \leq \tilde{\underline{e}}$. Consequently, we have $\frac{v^\delta(T)}{t} \geq \frac{v^\delta(S)}{s}$. □

The impact on the per capita emissions of enlarging a coalition is characterized in the following proposition. The result is a sharper version of the one obtained in Proposition 2 for the asymmetric case.

Proposition 4 *For $S \subset T \subseteq N$ we have*

$$t\underline{e}^T \leq s\underline{e}^S + (t-s)\,\tilde{\underline{e}},$$

and consequently

$$e^T \leq \underline{e}^S + (1 - \frac{s}{t})(\tilde{\underline{e}} - \underline{e}^S).$$

Proof. Suppose that $t\underline{e}^T > s\underline{e}^S + (t-s)\tilde{\underline{e}}$, then

$$te^T + (n-t)\,\tilde{\underline{e}} > s\underline{e}^S + (n-s)\,\tilde{\underline{e}}. \tag{7}$$

Then, from monotonicity of $d'(\cdot)$ we have

$$\begin{aligned} d'(t\underline{e}^T + (n-t)\,\underline{\tilde{e}}) &\geq d'(s\underline{e}^S + (n-s)\,\underline{\tilde{e}}) \\ td'(t\underline{e}^T + (n-t)\,\underline{\tilde{e}}) &\geq td'(s\underline{e}^S + (n-s)\,\underline{\tilde{e}}) \\ &= sd'(s^S + (n-s)\,\underline{\tilde{e}}) + (t-s)\,d'(s\underline{e}^S + (n-s)\,\underline{\tilde{e}}) \\ &\geq sd'(s\underline{e}^S + (n-s)\,\underline{\tilde{e}}). \end{aligned}$$

The first-order condition (4) yields

$$f'(\underline{e}^T) = td'(t\underline{e}^T + (n-t)\,\underline{\tilde{e}}) \geq sd'(s\underline{e}^S + (n-s)\,\underline{\tilde{e}}) = f'(\underline{e}^S).$$

Since $f'(\cdot)$ is a decreasing function, we have $\underline{e}^T \leq \underline{e}^S$, which leads to

$$\begin{aligned} te^T + (n-t)\,\underline{\tilde{e}} \leq t\underline{e}^S + (n-t)\,\underline{\tilde{e}} = se^S + (n-s)\,\tilde{e} + (t-s)\,(\underline{e}^S - \underline{\tilde{e}}) \\ t^T + (n-t)\,\underline{\tilde{e}} - (s\underline{e}^S + (n-s)\,\underline{\tilde{e}}) \leq (t-s)\,(\underline{e}^S - \underline{\tilde{e}}). \end{aligned}$$

Using (7), we obtain

$$\begin{aligned} &0 < t\underline{e}^T + (n-t)\,\underline{\tilde{e}} - (s\underline{e}^S + (n-s)\,\underline{\tilde{e}}) \leq (t-s)\,(\underline{e}^S - \underline{\tilde{e}}) \\ &0 < (t-s)\,(\underline{e}^S - \underline{\tilde{e}}). \end{aligned}$$

The last inequality results in $\underline{e}^S > \underline{\tilde{e}}$, which contradicts Lemma 1. Therefore, $e^T \leq \underline{e}^S + (1 - \frac{s}{t})(\underline{\tilde{e}} - \underline{e}^S)$ holds true. □

Finally, we make the observation that for any n-player symmetric game with transferable utility, the γ- and δ-CF yield the same Shapley value $\phi_i, \forall i \in N$, that is,

$$\phi_i(v^\gamma) = \frac{1}{n} v^\gamma\,(N) = \frac{1}{n} v^\delta\,(N) = \phi_i(v^\delta).$$

5 An example

To illustrate our results, and more specifically, the inclusion of the δ-core in the γ-core, we consider a simple 3-player game with the utility function of player i given by

$$u_i = \alpha_i e_i - \frac{1}{2}\beta_i e_i^2 - \frac{1}{2}\sigma_i e^2, \quad i = 1, 2, 3.$$

To ensure that the differences are due, and only due, to the choice of the characteristic function, we make the further assumption that the game is symmetric, that is,

$$\alpha_i = \alpha; \quad \beta_i = \beta; \quad \sigma_i = \sigma, \quad i = 1, 2, 3.$$

Table 1 reports the results for all feasible coalitions. As expected, the emissions and payoff values only differ for two-player coalitions.

Table 1: Individual and total emissions and individual payoffs

	δ-CF			γ-CF		
K	$e_i,\ i \in K$	e	$u_i,\ i \in K$	$e_i,\ i \in K$	e	$u_i,\ i \in K$
$\{1\}, \{2\}, \{3\}$	$\frac{\alpha}{3\sigma+\beta}$	$\frac{3\alpha}{3\sigma+\beta}$	$\frac{\alpha^2(\beta-3\sigma)}{2(\beta+3\sigma)^2}$	$\frac{\alpha}{3\sigma+\beta}$	$\frac{3\alpha}{3\sigma+\beta}$	$\frac{\alpha^2(\beta-3\sigma)}{2(\beta+3\sigma)^2}$
$\{i, j\}\,, \{k\}$	$\frac{\alpha(\sigma+\beta)}{(\beta+4\sigma)(3\sigma+\beta)}$	$\frac{3\alpha(2\sigma+\beta)}{(\beta+4\sigma)(3\sigma+\beta)}$	$\frac{(\beta^2+\beta\sigma-3\sigma^2)\alpha^2}{2(\beta+4\sigma)(\beta+3\sigma)^2}$	$\frac{\alpha(\beta-\sigma)}{5\sigma\beta+\beta^2}$	$\frac{3\alpha}{5\sigma+\beta}$	$\frac{(\beta^2+\beta\sigma-11\sigma^2)\alpha^2}{2\beta(\beta+5\sigma)^2}$
$\{i, j, k\}$	$\frac{\alpha}{9\sigma+\beta}$	$\frac{3\alpha}{9\sigma+\beta}$	$\frac{\alpha^2}{2(\beta+9\sigma)}$	$\frac{\alpha}{9\sigma+\beta}$	$\frac{3\alpha}{9\sigma+\beta}$	$\frac{\alpha^2}{2(\beta+9\sigma)}$

For an imputation $y = (y_1, y_2, y_3)$ to be in the core, the following conditions must be satisfied for $i = 1, 2, 3$:

$$\begin{aligned} \delta\text{-core} &: \quad A \le y_i^\delta \le A + B, \\ \gamma\text{-core} &: \quad A \le y_i^\gamma \le A + B + C, \end{aligned}$$

where

$$\begin{aligned} A &= \frac{\alpha^2(\beta - 3\sigma)}{2(\beta + 3\sigma)^2}, \quad B = \frac{45\alpha^2\sigma^2}{(\beta + 3\sigma)(\beta + 4\sigma)(\beta + 9\sigma)}, \\ C &= \frac{36\sigma^3\alpha^2(\beta^2 + 7\beta\sigma + 11\sigma^2)}{\beta(\beta + 4\sigma)(\beta + 3\sigma)^2(\beta + 5\sigma)^2}. \end{aligned}$$

This clearly shows that the γ-CF yields a larger interval on player i's imputation to be in the core than does the δ-CF, with the difference given by C.

To give a more visual picture of the result, let us transform the game at hand into a strategically equivalent game in (0-1), that is,

$$\begin{aligned} \hat{v}(\{i\}) &= 0, \quad i = 1, 2, 3, \\ \hat{v}(N) &= 1. \end{aligned}$$

It is easy to verify that we obtain the following δ- and γ-CF values for $i, j = 1, 2, 3,\ i \neq j$:

$$\begin{aligned} \hat{v}(\{i\}, \delta) &= 0; \quad \hat{v}(\{i, j\}, \delta) = \frac{1}{6}\frac{9\sigma + \beta}{4\sigma + \beta}; \quad \hat{v}(N, \delta) = 1, \\ \hat{v}(\{i\}, \gamma) &= 0; \quad \hat{v}(\{i, j\}, \gamma) = \frac{1}{6}\frac{(9\sigma + \beta)\left(\beta^2 + 2\sigma\beta - 11\sigma^2\right)}{\beta\,(5\sigma + \beta)^2}; \quad \hat{v}(N, \gamma) = 1 \end{aligned}$$

The δ- and γ-core are then defined by the following inequalities:

$$\begin{aligned} 0 &\le y_i^\delta \le \frac{5}{6}\frac{3\sigma + \beta}{4\sigma + \beta} \\ 0 &\le y_i^\gamma \le \frac{5}{6}\frac{3\sigma + \beta}{4\sigma + \beta} + \frac{2}{3}\frac{\sigma\,(9\sigma + \beta)\left(11\sigma^2 + \beta^2 + 7\sigma\beta\right)}{\beta\,(4\sigma + \beta)\,(5\sigma + \beta)^2}. \end{aligned}$$

Note that the above inequalities are independent of α. Letting $\xi = \frac{\sigma}{\beta}$, we can then express the characteristic function values for two-player coalitions as functions of only one parameter, that is,

$$\begin{aligned} \hat{v}(\{i, j\}, \delta) &= g_\delta(\xi) = \frac{(1 + 9\xi)}{6(1 + 4\xi)}, \\ \hat{v}(\{i, j\}, \gamma) &= g_\gamma(\xi) = \frac{(1 + 9\xi)}{6(1 + 4\xi)}\frac{(1 + 4\xi)(1 + 2\xi - 11\xi^2)}{(1 + 5\xi)^2}. \end{aligned}$$

To ensure that $g_\gamma(\xi) \in [0, 1]$, we impose the following restriction on ξ:

$$0 \le \xi \le \bar{\xi} = \frac{1 + 2\sqrt{3}}{11} \approx 0.4058.$$

Interestingly, increasing the externality as measured by ξ has a different effect on $g_\delta(\xi)$ and than on $g_\gamma(\xi)$. Whereas $g_\delta(\xi)$ is monotonically increasing in ξ, $g_\gamma(\xi)$ is concave, first increasing and next decreasing. Indeed, the derivatives are given by

$$\begin{aligned} g'_\delta(\xi) &= \frac{5}{6(1 + 4\xi)^2} > 0 \\ g'_\gamma(\xi) &= -\sqrt{3}\left(\frac{(9\xi + 1)^2\left(-11\xi^2 + 2\xi + 1\right)^2}{24\,(5\xi + 1)^4} - \frac{1}{2}\right) \text{ is } \begin{cases} \ge 0, & \text{for } \xi \in [0\ \underline{\xi}] \\ \le 0, & \text{for } \xi \in [\underline{\xi}\ \bar{\xi}] \end{cases}. \end{aligned}$$

where $\underline{\xi} = \frac{\sqrt{649} - 22}{165} \approx 0.0211$, and $g_\gamma(\bar{\xi}) = 0$.

Figure 1 exhibits the set of imputations and the δ- and γ-core. As expected, the δ-core is a subset of the γ-core. Denote by Δ_δ and Δ_γ the area of the δ-core and the γ-core, respectively, with their ratio given by

$$\frac{\Delta_\gamma}{\Delta_\delta} = \frac{\frac{\sqrt{3}}{2}(1-3(g_\gamma)^2)}{\frac{\sqrt{3}}{2}(1-3(g_\delta)^2)} = 1 + \frac{8\xi(9\xi+1)^2(11\xi^2+7\xi+1)(-22\xi^3+11\xi^2+8\xi+1)}{(5\xi+1)^4(111\xi^2+78\xi+11)}.$$

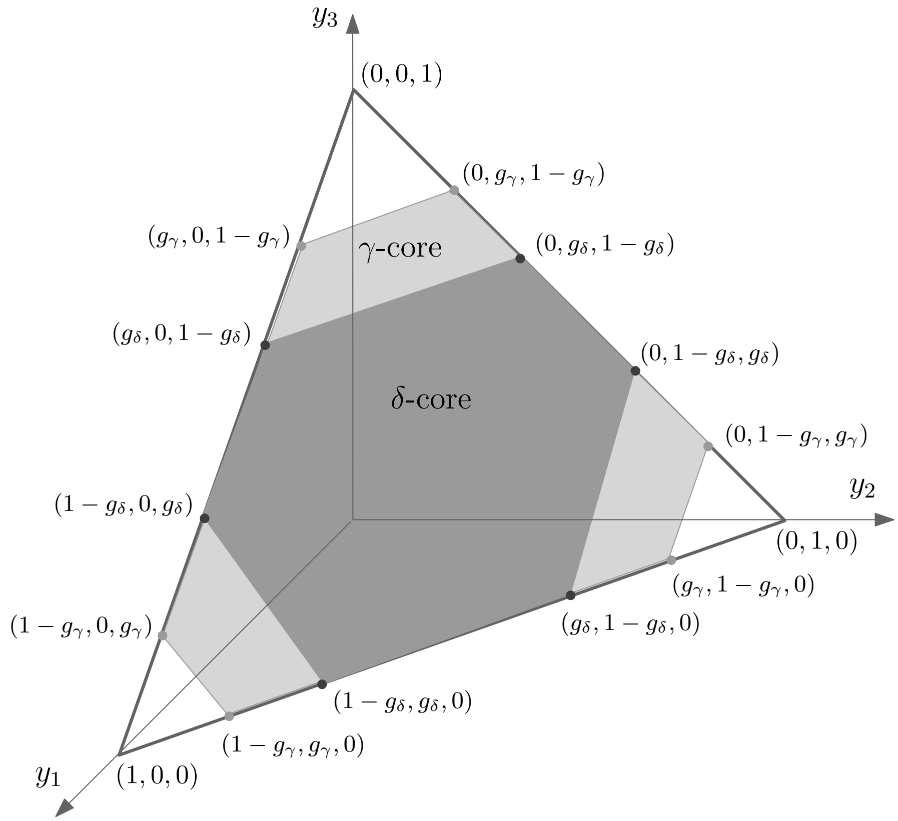

Figure 1: Outer triangle represents the imputation set. The gray hexagon represents the γ-core and the dark hexagon represents the δ-core

Figure 2 shows that the above ratio is an increasing function of ξ and reaches its maximum of 1.35 at $\bar{\xi}$. Notice, that this ratio is always greater than unity, implying that the δ-core is included in the γ-core, and increases with increasing externality. A relevant question is how far this ratio could go if we did not have the restriction $0 \le \xi \le \bar{\xi} \approx 0.4058$. To answer this question, we first observe that the γ-core can (maximally) be the whole imputation set of area $\Delta = \frac{\sqrt{3}}{2}$, while the smallest area of the δ-core is obtained when g_δ is maximal, that is,

$$\lim_{\xi\to\infty} g_\delta(\xi) = \lim_{\xi\to\infty} \frac{(1+9\xi)}{6(1+4\xi)} = \frac{3}{8}.$$

Therefore, the maximal ratio is given by

$$\frac{\Delta_\gamma}{\Delta_\delta} = \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}(1-3(\frac{3}{8})^2)} \approx 1.73$$

Finally, we observe that the δ-core is a hexagon, which excludes the case of having the δ-core be the whole imputation set. To see this, note that $0 < g_\delta(\xi) < \frac{1}{2}$, which implies that the δ-core cannot be a triangle, and is indeed necessarily a hexagon.

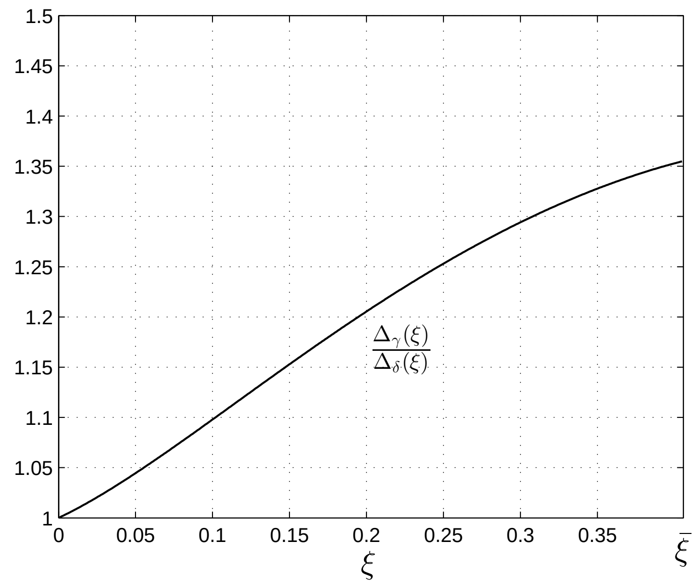

Figure 2: The ratio of areas of γ-core to δ-core with increasing externality

6 Conclusion

In this paper, we provided some properties of the δ-characteristic function and discussed its relationship with the γ-CF. More specifically, we proved that for the class of games with multilateral externalities, the δ-CF is superadditive and its core is nonempty and is included in the γ-core. The simple 3-player symmetric example gave an additional hint about the difference in size between the δ-core and the γ-core.

Two extensions of this paper are worth considering: first, to characterize the conditions under which the δ-core is nonempty for other classes of games than the one considered here; second, to extend the comparative analysis of the δ- and γ-CF to value solutions such as the Shapley value and the nucleolus. One can start by considering test games where the curse of dimensionality is not present and analytically assess the difference between the results produced by the two CFs, and next move on to numerical simulations for large-size games.

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