

**Credit Spreads, Recovery Rates and
Bond Portfolio Risk Measures in a
Hybrid Credit Risk Model**

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Abstract

This paper presents a framework in which many structural credit risk models can be made hybrid by randomizing the default trigger, while keeping the capital structure intact. This produces random recovery rates negatively correlated with the default probability. The approach is implemented on a firm-by-firm basis using maximum likelihood and the unscented Kalman filter (UKF) on each of the 225 companies of the CDX NA IG and HY indices using weekly CDS data from December 2007 to January 2012. Adding the surprise element and the time-varying distribution of recovery rates has a large impact on credit spreads as it modifies both the level and shape of the curves. When a bond portfolio is considered, the presence of dependence among firm leverage ratios and between default probabilities and recovery rates produces clusters of defaults with low recovery rates. It has a major impact on standard risk measures such as Value-at-Risk and conditional tail expectation.

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1 Introduction and review of the literature

Credit risk, which is the potential loss arising from a default of payment by an obligor, has two primary sources: uncertainty regarding the timing of default and the amount lost by the creditor at the moment of default. The literature on the former is very rich and encompasses the structural, reduced-form (intensity), and more recently, hybrid models. Although many authors have investigated the empirical behaviour of recovery rates, very few of them have integrated observed features of recovery rates into credit risk models. This paper presents a hybrid credit risk framework that incorporates recovery rate risk and, more importantly, demonstrates how the presence of dependence between default probabilities and recovery rates significantly affects the occurrence of large losses and, therefore, changes the perception of risk associated with bond portfolios.

Hybrid credit risk models combine features of structural and reduced-form models. One major class of hybrid models was developed in response to the significant information gap between a firm's managers and investors. Only managers can observe the true market value of assets and liabilities whereas investors receive incomplete information. For example, in Duffie & Lando (2001), investors receive periodic financial statements that provide only a noisy estimate of market values. Jarrow & Protter (2004) argue that the gap between reduced-form and structural models is explained by the amount of information observed by investors. The latter model is appropriate when investors and managers possess the same information set, in which case default is fully predictable. In the opposite case, default takes investors by surprise. Other models also relax the complete information assumption, notably, Çetin, Jarrow, Protter & Yildirim (2004), Giesecke (2004) and Giesecke & Goldberg (2004).

Other contributions integrate both reduced-form and structural models. In Madan & Unal (2000), a model for the value of assets and liabilities is presented and default occurs when a single and random loss occurring at a random time is larger than the value of the equity. The resulting model is constructed using an intensity-based approach. Chen, Filipovic & Poor (2004) introduce a model in which the firm's credit state (interpreted as either the rating of the company or its distance to default) is a Cox-Ingersoll-Ross (CIR)-type of affine process with gamma-distributed jumps. A second process dependent on interest rates and the firm's credit state is required. Default occurs either as soon as the first process hits a barrier or when the second process jumps. Thus, default is the result of either a predictable or an unpredictable process. Bakshi, Madan & Zhang's (2006a) model is a reduced-form model based on Vasicek-type state variables. One of the latter is the firm's leverage.

In the literature, recovery rates have long been considered a constant fraction or a exogenous random variable. However, recovery rates are inversely related to default probabilities and this has been documented, mainly by Edward Altman. For example, in Altman & Kishore (1996), the recovery rate of an AAA company is 68%, whereas a creditor of a CCC company should expect a recovery rate of 38%. Moreover, Altman, Testi & Sironi (2004) and Altman (2006) survey the current literature of credit risk models and emphasize the importance of relating the recovery rate to the probability of default. The very recent literature is slowly integrating stochastic recovery rates that are inversely related to default probabilities. In Bakshi, Madan & Zhang (2006b) and Das & Hanouna (2009), the authors use arbitrary mathematical functions to transform the default intensity (which can have any positive value) into a recovery rate (that has a value of between 0 and 1). They both obtain decreasing term structures of recovery rates. Other significant contributions to stochastic recovery rates in a single-name model are Gaspar & Slinko (2008) and Hocht & Zagst (2009). These authors use a stochastic recovery rate framework, but do not further discuss this specific issue in empirical studies.

The effect of having a recovery rate that is inversely related to the default probability has been rarely investigated in a portfolio. Andersen & Sidenius (2004) and Hull, Predescu, White (2010) are notable exceptions where both papers have defined a loss variable that is related to a common factor (random variable in the former and a stochastic process in the latter) and an idiosyncratic noise. Their analysis is

focused toward pricing collateralized debt obligations whereas risk management and tail risk measures are not discussed.¹

This paper presents a framework in which many structural credit risk models can be made hybrid by changing the default trigger, while keeping the capital structure intact. This is accomplished by linking both the leverage ratio and default intensity using a simple transformation. Once dynamics for assets and liabilities are determined, the framework can be used to model the two sources of credit risk: the moment of default and the amount of loss at default. The consequences of the approach are that short-term credit spreads are significantly different from zero and more importantly, time-varying stochastic recovery rates can be directly derived from the value of assets and liabilities at the moment of default. These recovery rates are consistent with the empirical findings that the default probability and loss given default are proportional. Notably, the stochastic recovery rate model presented gives rise to a term structure of recovery rates. For highly rated firms, the major risk is a downgrade associated with a decreasing recovery rate. Consequently, the term structure of expected recovery rates is usually downward-sloping for these firms. For a risky firm, if it survives, its rating should improve, leading to a higher recovery rate. For such a firm, the term structure is usually upward-sloping. At the portfolio level, the dependence between leverage ratios, in addition to its relationship with recovery rates increase the occurrence of default clusters and affect all common risk measures directly and significantly.

This paper's key contributions are as follows. First, from a modelling perspective, default intensity is defined as an increasing and convex transformation of leverage.² Section 2 describes how the proposed framework can be interpreted as a shell added to various structural models to account for unmodelled factors that contribute to potential default. Then, the value of assets and liabilities at default can be used in a cascading structure to model the recovery rate of various types of debtholders or seniority. Thus, the proposed framework allows for endogenous random recovery rates that are inversely proportional to default probabilities.

Second, as described in Section 3, a version of the proposed approach is implemented on a firm-by-firm basis using maximum likelihood and the unscented Kalman filter (UKF). This technique allows estimation of the model's parameters using multiple information sources and accounting for possible trading noise. In our setup, time series of credit default swaps (CDS) with various maturities serve as observable variables to retrieve a firm-specific leverage ratio path and a set of parameters. This empirical study is performed using more than two hundred firms listed in the CDX indices and serves as a platform to better understand the forces at work. Indeed, the presence of the shell modifies the credit spreads as they are larger than in the structural version (Figure 2 and Table 3). Moreover, the recovery risk (Section 4) and its interplay with the default risk modify the shape of credit spreads, although it is a second-order effect (Figure 5). Finally, the presence of positive dependence between the firm's leverage ratios produces clusters of default combined with low recovery rates and has a significant impact on typical portfolio credit risk measures such as Value-at-Risk (VaR) and conditional tail expectation (CTE) (Table 7).

This paper is structured as follows. In the next section, the joint default and recovery rate model is presented. Section 3 describes the data set, the estimation procedure and parameter estimates; Section 4 analyzes the behaviour of recovery rates; Section 5 discusses the impact of the proposed model on bond portfolio loss distribution; and Section 6 concludes.

2 Joint default and loss model

It is widely agreed and backed empirically that the main determinants of default are leverage, its volatility and the interest rate (as an example, see Ericsson, Jacobs & Oviedo (2009) and references therein). However, a

¹There also exists other academic and professional multi-name credit risk models but the loss given default is rarely modeled in these papers.

²This differs from Bakshi, Madan & Zhang (2006a), as their default intensity is proportional to leverage, which is a linear transformation. Moreover, Bharath & Shumway (2008) and Duffie, Saita & Wang (2007) use a Cox proportional hazards (CPH) model to define a default intensity. Although this is a non-linear function of leverage, this approach is used to find possible determinants of default, bankruptcy, merger, etc.; it is not used for pricing and risk management purposes.

large proportion of the spreads remain unexplained even when accounting for these three variables combined. In addition, structural models in general have failed to appropriately represent the short-term level of credit spreads, mainly because default is a predictable process. To solve these issues, the literature has turned to structural models with incomplete information or reduced-form models. The common feature of these classes of models is the presence of a surprise element that adds randomness to the default trigger.

The approach adopted in this paper is to construct a shell to be placed over many structural models to capture this surprise element. This shell is essentially an intensity process that depends on the firm's leverage. This approach allows for the construction of an endogenous recovery rate that is both stochastic and dependent on the firm's probability of default. This dependence affects the shape of credit spreads and induces a time-varying recovery rate. Consequently, in addition to the default risk and the recovery rate risk, a recovery premium arises from the dependence structure between the default time and the recovery rate, which also affects the shape of the credit spread curve.

2.1 Hybrid default risk model

2.1.1 Structural framework

As a starting point for the model, assume that the total value of the firm's assets is represented by the continuous-time stochastic process $\{A_t : t \geq 0\}$. The company's obligations toward creditors are defined by the liabilities process $\{L_t : t \geq 0\}$, which may also be interpreted as a default threshold. The risk-free short rate is denoted by $\{r_t : t \geq 0\}$. Formally, the filtration $\{\mathcal{G}_t : t \geq 0\}$ is generated by r , A and L (with the usual regularity conditions). Different dynamics for the firm's assets and liabilities can be considered.

Structural models usually define the moment of default as the first moment that the assets cross the value of the liabilities (or some other threshold). In the proposed model, default is instead defined using a reduced-form default trigger that is highly correlated with the firm's leverage ratio. This is discussed next.

2.1.2 Shell

The default trigger for reduced-form models is based mostly on Lando (2004); it represents the first jump of a Cox process. In that case, the default time satisfies

$$\tau \equiv \inf \left\{ t > 0 : \int_0^t H_u du > E_1 \right\} \quad (1)$$

where E_1 is an exponential random variable with mean 1 that is independent of $\{\mathcal{G}_t : t \geq 0\}$ and $\{H_t : t \geq 0\}$ is the default intensity process.

The intensity process is constructed so that it depends on the firm's financial health. More precisely, define the leverage ratio $X_t \equiv L_t/A_t$ so that $\{X_t, t \geq 0\}$ results in a \mathcal{G} -adapted continuous-time stochastic process. Because the asset value is unobservable, so is the leverage ratio. Assume that the dynamics of the leverage ratio are

$$dX_t = \mu_t^{(X)} X_t dt + \sigma_t^{(X)} X_t dB_t \quad (2)$$

where $\{B_t : t \geq 0\}$ is a Brownian motion, and $\sigma^{(X)}$ and $\mu^{(X)}$ are predictable processes satisfying some regularity conditions such that Equation (2) admits a solution. As L is the liabilities value and not the risky debt value, the liabilities value may be larger than the asset value, resulting in an X process that is not constrained to lie within the unit interval. This leverage ratio is modelled under the objective measure \mathbb{P} for risk management purposes as well as under the risk-neutral measure \mathbb{Q} or the T -forward measure \mathbb{Q}_T if the objective is pricing credit-sensitive derivatives. As far as modelling is concerned, a jump term may easily be added within this framework. However, jumps have an impact on the numerical procedures involved at the estimation stage and, therefore, is beyond the scope of this paper.

Model (2) encompasses several existing structural models. Indeed, Black & Cox (1976) assume that the firm value behaves as a geometric Brownian motion (GBM) under the risk-neutral measure and that default

arises as soon as the firm value reaches an exponential barrier. The liabilities are modelled as a single zero-coupon bond in a deterministic interest rate framework. In that case, the process X corresponds to the ratio of the firm value over the riskless bond value and behaves as a GBM, that is, $\mu_t^{(X)} = \mu^{(X)}$ and $\sigma_t^{(X)} = \sigma^{(X)}$ are constants. Leland (1994) can also be considered as a special case of Equation (2). Indeed, he models the firm value using Equation (2) and a constant volatility parameter. The liabilities are a perpetuity paying a constant coupon of C in a constant interest rate r framework. Since the liabilities value is C/r , the process X has constant drift and volatility parameters under the risk-neutral measure. Moreover, default arises as soon as the firm's value reaches a constant barrier. Although these papers priced the risky debt using different hypotheses, the underlying leverage ratio process fits this framework. Leland & Toft (1996) present a different story about liability value, but the resulting leverage ratio process is again a GBM under the risk-neutral measure. Under stochastic interest rates, the capital structure proposed by Longstaff and Schwartz (1995) and Bryis and de Varenne (1997) for example, are compatible with the previous setting, since the structure leads to a stochastic drift term $\mu_t^{(X)}$ which is a function of the risk-free rate.

It is assumed that the intensity process at time u , H_u , is a function of the leverage ratio X_u , i.e. $H_u = h(X_u)$ where $h : \mathbb{R} \rightarrow [0, \infty[$ is a firm-specific deterministic transformation known as the sensitivity function. The firm-specific sensitivity function h plays a major role in the model: it gauges how the leverage ratio affects the default probability. The interpretation of h is that it represents the sensitivity of the firm's credit risk to its leverage ratio. This function h must obey a few restrictions³ so that the model respects some basic observed facts.

First, h has to be positive; otherwise it cannot act as an intensity. Second, h must be an increasing function since default probabilities should increase with the leverage ratio. As a consequence, the firm will not necessarily default as soon as its leverage ratio approaches some critical threshold; rather, it simply means that its default likelihood increases. The firm may very well survive and improve its financial situation. Third, the function h should be parsimonious in terms of the number of parameters so that the estimation procedure for the unobserved intensity process remains feasible.

For all these reasons, throughout this paper, the intensity process is built on the transformation h given by

$$h(x) = \beta + \left(\frac{x}{\theta}\right)^\alpha, \alpha > 0, \beta > 0, \theta > 0, x > 0. \quad (3)$$

A value of $\alpha > 0$ is required for the transformation to increase with x . When $\alpha > 1$, the function h is convex, meaning that a small increase in the leverage ratio has a greater impact when the firm already has a large leverage ratio. The parameters α and θ are responsible for the sensitivity of the survival of the firm toward its leverage ratio. θ is known as the critical level of leverage or the default threshold because the default intensity is large when leverage is greater than θ and small otherwise. The rate at which default intensity will converge to either 0 or infinity is guided by α . Moreover, setting θ as the default threshold and letting $\beta = 0$ and $\alpha \rightarrow \infty$, one recovers the basic structural models underlying the leverage ratio process X .

Other authors have proposed transformations of leverage for their intensity functions. For example, Bakshi et al. (2006a) consider an intensity process that has a linear relationship with leverage. Although the default probability increases with leverage, the relationship is not convex and thus a structural model cannot be obtained from their model.

Applying Itô's lemma to the particular shape of the intensity function implies that the dynamics of the default intensity $\{H_t : t \geq 0\}$ is

$$dH_t = \mu_t^{(H)} H_t dt + \sigma_t^{(H)} H_t dB_t \quad (4)$$

where $H_0 = h(X_0)$,

$$\mu_t^{(H)} = \alpha \mu_t^{(X)} + \frac{1}{2} \alpha (\alpha - 1) \left(\sigma_t^{(X)}\right)^2 \quad \text{and} \quad \sigma_t^{(H)} = \alpha \sigma_t^{(X)}.$$

Although β and θ do not appear in Equation (4), they affect the level of the initial default intensity H_0 .

³During the implementation in real data, these restrictions are not imposed. They are simply found from the estimates.

2.2 Recovery rate model

The second component of the joint default and loss model is the recovery rate component. In the vast majority of credit risk models, an exogenous specification of the recovery rate R is required, which is independent of the capital structure of the firm upon default. Most of the time, it is set as a constant often estimated from the seniority of the bond or chosen exogenously based on empirical studies, such as those by Carty & Lieberman (1996) and Altman & Kishore (1996). In CreditMetrics (1997), a beta-distributed random recovery rate, independent of the default process, is used. However, Altman et al. (2004) and Altman (2006) both argue the importance of having a recovery rate structure that is inversely related to the default probability.

A significant contribution of the proposed joint model is that the leverage ratio process may also be used to construct a random recovery rate. Since default comprises an element of surprise, the assets at default will very likely be lower than the value of the liabilities and taking the ratio of the two at default is indeed one way to construct recovery rates that depend on the firm's capital structure. Moreover, legal fees, liquidation costs and different seniority can be accounted for in obtaining a recovery rate distribution.

The recovery rate model presented is described as being endogenous as it comes from the value of the assets and liabilities at the moment of default. Assume that the liquidation and legal fees represent a fraction κ of the market value of assets at default. The approximated value of the assets available to debtholders at default time is $A_\tau^* = \min((1 - \kappa) A_\tau; L_\tau)$. Suppose there are two classes of bondholders: junior and senior. The senior bondholders represent $100\omega\%$ of the liabilities and the junior debtholders, $100(1 - \omega)\%$. Thus, in case of default and assuming the absolute priority rule holds, senior investors have a right on the first $100\omega\%$ of the assets and the juniors take what remains. One can represent the recovery rate (with respect to the market value) for both senior and junior debtholders as

$$\begin{aligned} R_\tau^{(S)} &= \frac{\min(A_\tau^*; \omega L_\tau)}{\omega L_\tau} = \min\left(\frac{1-\kappa}{\omega} \frac{1}{X_\tau}; 1\right) \text{ and} \\ R_\tau^{(J)} &= \frac{A_\tau^* - \min(A_\tau^*; \omega L_\tau)}{(1-\omega)L_\tau} = \max\left(0; \min\left(\frac{(1-\kappa)X_\tau^{-1}-\omega}{(1-\omega)}; 1\right)\right). \end{aligned} \quad (5)$$

Note that these recovery rates do not represent a fraction of the face value, but rather, a fraction of the market value of the liabilities at default and thus, act as an approximation of the true recovery rates. With one class of investors, R_τ becomes

$$R_\tau = \frac{A_\tau^*}{L_\tau} = \min\left((1 - \kappa) \frac{1}{X_\tau}; 1\right). \quad (6)$$

It results in a stochastic and time-varying recovery rate that will vary according to the firm's solvency. The leverage ratio process exerts an influence on both the moment of default (through the default intensity) and the recovery rate at the moment of default (through the inverse of the leverage ratio), inducing a negative correlation between the two. This is consistent with the findings of Altman et al. (2004) and Altman (2006). Finally, the definitions of Equation (5) and Equation (6) can be used for both risk management and pricing purposes. Indeed, for pricing purposes, one uses the risk-neutral dynamics of the leverage ratio and for comparisons with the empirical literature, one uses the objective probability measure \mathbb{P} .

2.3 Bonds and credit default swap (CDS)

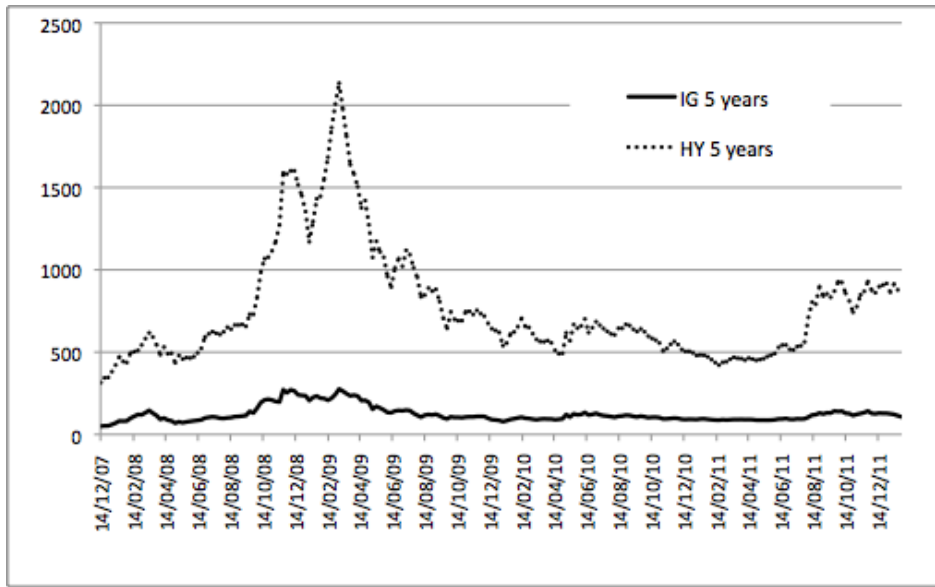
This setting is used to price derivative securities that account for default and recovery risks. However, the presence of stochastic recovery rates implies that there is no closed-form solution for corporate bonds and CDS. Standard numerical methods such as a Monte Carlo simulation and a Schonbucher (2002) tree may be used. Details about pricing bonds and CDS are provided in Appendix A.

3 Estimation

In this section we explain how we have estimated the proposed model and present some empirical results.

3.1 Data and assumptions

The companies on which investigations were performed are the 225 firms of the CDX North American Investment Grade (IG) and High Yield (HY) indices (CDX.NA.IG.17 and CDX.NA.HY.17) provided by Markit on January 27, 2012. The firms' sample spans multiple credit ratings and industrial sectors as well. The weekly⁴ term structure of CDS prices from December 14, 2007, to January 27, 2012, is provided by DATASTREAM for a maximum of 216 weeks. Prices for maturities of 1, 2, 3, 4, 5, 7, and 10 years are available for most firms: for 15 firms, there were not enough observations for the estimation procedure, leading to a sample of 210 firms (118 IG and 92 HY) for a total of 312,151 observations. In the following sections, some of the results are broken down by risk rating. These ratings are attributed by Standard and Poor's, and available from Bloomberg on January 27, 2012. For 13 firms, no rating was available, leaving a sample of 197 firms whenever results are presented according to credit rating. To illustrate how CDS premiums move over time, Figure 1 shows the evolution of the mean (taken across firms) CDS premium for maturities of five years, for both IG and HY portfolios.



Evolution of the mean five-year CDS premium (across firms) for both CDX.NA.IG and CDX.NA.HY portfolios, between December 2007 and January 2012.

Figure 1: Evolution of the mean five-year CDS premium

In all experiments and unless stated otherwise, a constant risk-free rate of 3% has been assumed. This rate is consistent with the rounded average daily rates of one- and three-month constant-maturity Treasury rates over the sampling period. Interest rate data is provided by the Federal Reserve of St. Louis Web site via FRED (Federal Reserve Economic Data).

To illustrate uses of the proposed hybrid model, a capital structure described by

$$dX_t = \mu_X^{\mathbb{P}} X_t dt + \sigma_X X_t dW_t^{\mathbb{P}} \quad (7)$$

is assumed where $\mu_X^{\mathbb{P}}$ and σ_X are constant. As argued in the review of the literature, the market leverage ratio is not observed and is inferred on a firm-by-firm basis using quasi-maximum likelihood estimation

⁴Every Wednesday.

and the unscented Kalman filter (UKF) along with the CDS data. It is further assumed that, in the risk neutralization procedure, $\mu_X^{\mathbb{P}} - \mu_X^{\mathbb{Q}} = \xi_X$ is a constant. More precisely, ξ_X is not a risk premium itself, but could be interpreted as the difference of two risk premiums: the liability risk premium minus the asset risk premium. The filter is described in Appendix B, and details about risk premiums are provided in Appendix C.

3.2 Results

This section shows basic estimation results in order to describe the distribution of parameters in the set of 210 firms of the CDX indices. This is meant to provide an overview of the range of parameters across firms.

Table 1 shows descriptive statistics on estimated parameters. All firms have a positive β parameter, which is a sufficient condition to ensure that the intensity process is positive. The α parameter is always greater than 1, the smallest estimated value being 1.48, thus confirming the convex relationship between the intensity process and the leverage ratio. For 90% of firms, this relationship is strongly convex, as the estimated value is greater than 4. The critical leverage threshold θ is generally between 1.0 and 1.8 which is reasonable considering that the leverage ratio is not the debt ratio. As ξ_X is the difference of two risk premiums (liabilities minus asset risk premiums: $\xi_X = \mu_X^{\mathbb{P}} - \mu_X^{\mathbb{Q}} = \sigma_t^L \gamma_t^L - \sigma_t^A \gamma_t^A$), negative values are expected for some firms.

Table 1: Descriptive statistics on the distribution of parameters across the portfolio of firms of the CDX indices

	$\mu_X^{\mathbb{Q}}$	ξ_X	$\mu_X^{\mathbb{P}}$	σ_X	α	θ	β	κ	$\hat{x}_{0 0}$	Obs.
Mean	0.77%	0.19%	0.96%	12.80%	15.99	1.414	1.39%	46.56%	79.59%	210
Stdev	1.90%	2.04%	2.69%	7.12%	21.49	0.375	3.20%	19.39%	26.78%	210
10%	-1.14%	-1.94%	-1.92%	3.02%	3.97	1.032	0.00%	21.73%	53.72%	
25%	-0.47%	-0.10%	-0.64%	8.10%	5.41	1.203	0.01%	36.83%	59.26%	
50%	0.25%	0.50%	0.85%	12.22%	8.11	1.384	0.21%	46.69%	74.77%	
75%	1.68%	0.95%	2.43%	17.00%	14.11	1.548	1.36%	55.16%	92.21%	
90%	2.98%	1.56%	3.41%	21.56%	36.42	1.777	3.93%	64.60%	119.99%	
IG	-0.27%	0.45%	0.18%	12.27%	12.72	1.395	1.33%	43.29%	72.24%	118
HY	2.10%	-0.14%	1.96%	13.49%	20.17	1.437	1.48%	50.74%	89.00%	92
b	2.37%	-0.59%	1.78%	1.21%	7.453	0.042	0.15%	7.45%	16.76%	
p-value	0.00%	3.87%	0.00%	22.08%	1.23%	42.24%	72.94%	0.54%	0.00%	
		$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$	$\delta^{(5)}$	$\delta^{(7)}$	$\delta^{(10)}$		Obs.
Mean		32.33%	16.01%	7.25%	1.90%	6.18%	11.02%	15.11%		210
Stdev		10.34%	5.87%	3.01%	2.45%	2.64%	4.91%	6.55%		210
10%		19.68%	9.14%	3.71%	0.00%	3.72%	4.65%	6.02%		
25%		24.48%	11.23%	5.32%	0.20%	4.47%	6.97%	9.69%		
50%		32.12%	15.72%	7.26%	0.70%	5.84%	11.07%	15.38%		
75%		39.07%	20.13%	8.94%	2.35%	7.14%	14.25%	20.00%		
90%		45.88%	23.72%	10.73%	6.11%	9.34%	17.34%	23.39%		
IG		31.71%	16.13%	7.19%	2.37%	7.06%	13.75%	18.81%		118
HY		33.13%	15.84%	7.32%	1.29%	5.07%	7.51%	10.36%		92

For each of the 210 firms, the parameters of the model have been estimated using weekly CDS prices of maturities of one to ten years, using the UKF filtering technique. The data is available from December 2007 to January 2012. The mean, standard deviation and quantiles are computed across firms. The last two rows compute the mean across firms of the CDX.NA.IG or CDX.NA.HY portfolios. As described in Appendix B, the δ 's represent the standard deviation of the noise terms.

For each of the 210 firms and each of the 9 parameters, the regression $\Theta_i = a + b \mathbb{I}_{HY}(i) + \varepsilon_i$ has been estimated where Θ_i is the parameter value of the i th firm and $\mathbb{I}_{HY}(i)$ is an indicator function that worth 1 if the i th firm belongs to the investment grade category. The estimated a parameter correspond to the mean parameter value of IG firms. The statistical test $H_0 : b = 0$ against $H_1 : b \neq 0$ has been performed. The p-values are reported. Estimates in bold are significant at a confidence level of 95%.

Statistical tests are performed to determine whether the parameters differ significantly between IG and HY firms. It appears that there are mainly six parameters that distinguish IG and HY firms: the drift of the leverage ratio under both measures (μ_X^P and μ_X^Q), its initial value X_0 , the difference between risk premiums ξ_X , the intensity parameter α and the bankruptcy and legal fees κ . The drift of the leverage ratio is higher for HY firms meaning that these firms have a tendency to contract more debt or their debt is more expensive (in terms of interest rate). Moreover, their initial leverage ratio is, on average, approximately 17% greater. The latter results have to be expected given the nature and the separation of these firms in the two portfolios: the HY firms are more indebted. The average ξ_X parameter is negative for HY firms while it is positive for the IG category. A negative ξ_X means that the asset's risk premium is larger than the liabilities' risk premium. The β parameter is, on average, larger for the HY firms. If all other variables are the same, it produces a larger intensity, implying a larger default probability. The IG firms have, on average, a smaller α parameter. Because the intensity's drift and volatility parameters (see Equation (4)) are increasing functions of α and $\mu_t^{(X)}$, IG firms tends to have a smaller intensity process, which was expected as they are less risky.

The standard error of the trading noise is the highest for the one-year CDS, for two possible reasons: elements not necessarily related to the company's true default risk and fitting error. When these standard deviations are compared with Duan & Fulop (2009), one sees that those presented in this paper are higher. Two aspects of the analysis need to be pointed out. First, this paper's analysis is based on CDS prices during a very volatile period, while Duan & Fulop (2009) is based on equity prices during a stable period. Moreover, simultaneously fitting seven credit derivative prices can have a significant impact on the quality of the fit with each derivative.

Table 2 shows average parameter value by credit rating. The leverage ratio's drift tends to increase with riskiness, since the firms are more indebted. The liquidation and legal fees parameter κ and the initial leverage ratio filtered out from the CDS data, tend to increase with riskiness as well.

Table 2: Average parameter value across credit ratings

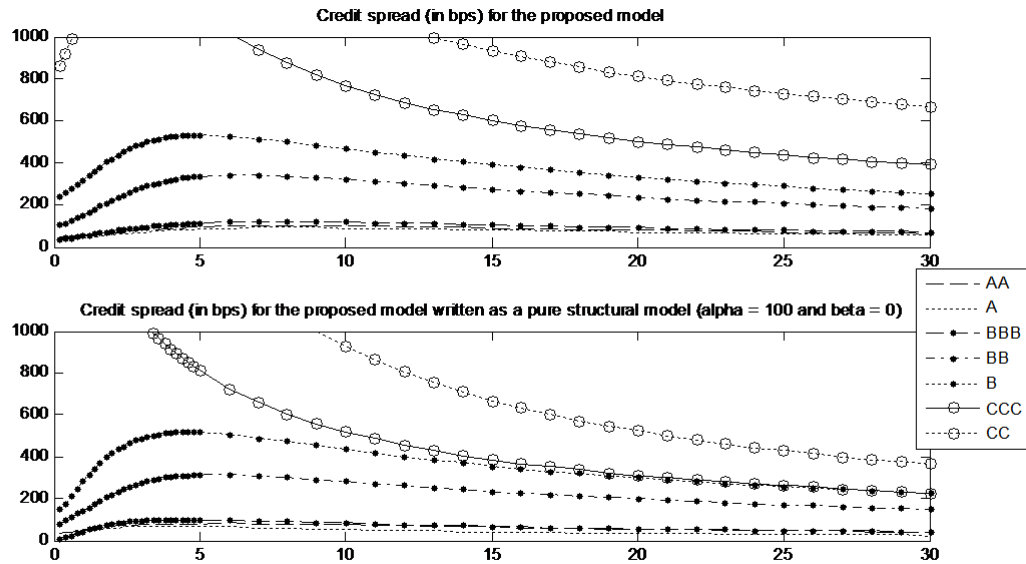
Rating	μ_X^Q	ξ_X	μ_X^P	σ_X	α	θ	β	κ	$\hat{x}_{0 0}$	Obs.
AAA	—	—	—	—	—	—	—	—	—	0
AA	-1.12%	0.50%	-0.62%	13.00%	13.22	1.659	1.11%	57.40%	77.58%	5
A	-0.50%	0.29%	-0.20%	9.44%	17.70	1.342	1.34%	39.08%	78.15%	42
BBB	0.06%	0.48%	0.55%	14.08%	9.44	1.430	1.53%	44.45%	69.05%	73
BB	1.60%	-0.13%	1.47%	12.04%	21.74	1.349	1.04%	48.98%	85.82%	37
B	2.52%	-0.08%	2.45%	14.35%	20.83	1.459	1.30%	52.38%	89.62%	32
CCC	2.78%	2.20%	4.98%	15.73%	12.69	1.542	1.13%	51.89%	97.72%	7
CC	3.07%	0.48%	3.56%	14.12%	6.91	1.447	2.03%	54.42%	80.15%	1

Rating	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$	$\delta^{(5)}$	$\delta^{(7)}$	$\delta^{(10)}$	Obs.
AAA	—	—	—	—	—	—	—	0
AA	35.69%	19.49%	9.34%	4.69%	5.59%	10.14%	16.87%	5
A	29.86%	16.36%	7.32%	1.06%	6.09%	13.02%	18.38%	42
BBB	32.56%	15.66%	6.82%	3.00%	7.70%	14.19%	18.63%	73
BB	35.24%	17.15%	8.05%	1.65%	5.52%	8.54%	12.05%	37
B	32.65%	15.32%	6.89%	0.84%	4.75%	6.68%	9.45%	32
CCC	30.39%	14.95%	6.41%	0.82%	4.20%	7.46%	10.92%	7
CC	26.38%	15.49%	8.65%	0.04%	5.72%	8.22%	10.69%	1

The mean of each parameter is reported for the 197 firms for which both the credit rating was observed in January 2012 and enough observations were available for the estimation procedure.

3.3 Credit spreads

Figure 2 and Table 3 show the average credit spread curves for firms having the same rating, in the proposed model and in a pure structural version of the model (where $\alpha = 100$ and $\beta = 0$). When compared to the structural version, the proposed model produces a similar short-term credit spread for IG companies and larger ones for HY firms. However, the differences increase with maturity: credit spreads derived from the hybrid model are one and a half times larger than those of the structural version for a maturity of 20 years.



For each of the 197 firms, the parameters of the model have been estimated using weekly CDS prices with maturities of one to ten years, using the UKF filtering technique. The data is available for December 2007 to January 2012. The top graph shows credit spread curves for firms with the same credit rating (averaged across firms) in the proposed model, while the bottom graph shows similar curves within a structural framework, that is, when all parameters are estimated again except for α which is fixed at 100 and β at 0.

Figure 2: Comparison of the credit spread curves generated by two versions of the proposed framework

Table 3: Average credit spreads (in bps) for two versions of the proposed framework

		Credit spreads (basis points)													
		AA	A	BBB	BB	B	CCC	CC	AA	A	BBB	BB	B	CCC	CC
		Proposed model							Structural model ($\alpha = 100$ and $\beta = 0$)						
Maturity		54	40	53	148	316	1640	1134	52	35	43	144	281	1456	2198
1		77	68	90	281	490	1375	1490	77	68	95	275	488	1053	1909
3		93	84	112	335	532	1119	1418	83	68	99	315	516	810	1489
5		99	90	119	340	516	942	1289	82	62	94	310	490	659	1202
7		98	89	117	319	467	768	1124	77	53	83	282	435	518	926
10		78	70	89	235	331	501	813	55	34	57	198	299	312	523
20															
No. of firms		5	42	73	37	32	7	1	5	42	73	37	32	7	1

For each of the 197 firms, the parameters of the model have been estimated using weekly CDS prices with maturities of one to ten years, using the UKF filtering technique. The data is for December 2007 to January 2012. Each panel shows the averaged (across firms) credit spreads for firms with the same credit rating. For the structural version, all parameters have been re-estimated under the constraint $\alpha = 100$ and $\beta = 0$.

Intuitively, the credit spread is influenced by two main factors: the default probability and the recovery rate. In the short term, adding an element of surprise increases the likelihood of default. However, for many firms, the initial leverage ratio is below the default threshold so that over the short term, if there is a default, it is caused by surprises and, consequently, the recovery rate can be very high. This simultaneous rise in the short-term default probability and the recovery rate cancel each other out, at least when it comes to expectations, which is the case for credit spreads. Conclusions differ when dispersion is under study, as is the case for many risk-management measures commonly used. In the long run, default is more likely when the leverage ratio is large. Hence, recovery rates tend to be lower, which in turn widens the credit spreads.

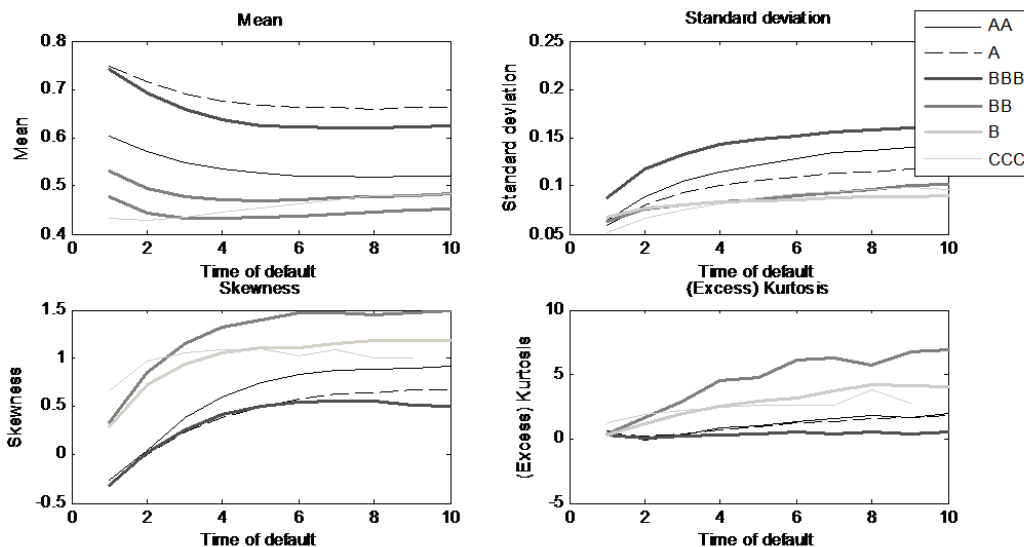
4 Analysis of recovery rates

In this section, the empirical behaviour of recovery rates generated by the model is investigated. More precisely, the parameters obtained for each firm have been used to simulate one million leverage ratio paths under the real-world probability measure \mathbb{P} . When a default was generated, the result of Equation (6) was computed. Defaults have then been sorted according to their moment of occurrence to measure the time-varying characteristics of the recovery rate. Thus, values of R_τ ($k \leq \tau < k + 1$), $k = 0, 1, 2, \dots, 9$ have been generated for each company. The purpose of this entire section is to emphasize the importance of modelling recovery rates and their potential impact in pricing and risk management applications.

4.1 Term structure of recovery rates

In this section, recovery rates are analyzed on a firm-by-firm basis and also with respect to their (S&P) credit rating as at January 27, 2012.

Figure 3 computes the average (across firms of a given credit rating) expectation, standard deviation, skewness and kurtosis of recovery rates given default. The top left figure deserves attention since it clearly shows a term structure of expected recovery rates. Thus, on average, recovery rates should be higher over the short term and lower over the long term except for risky firms. This can be explained as follows. For highly rated firms, the initial leverage ratio is usually small. If default arises in the short term, it is mainly attributed to the element of surprise. In that case, the recovery rate, which depends on the inverse of the leverage ratio, is large. In the long run, conditional upon survival, the default event is more closely linked to the financial health of the firm. Thus, the recovery rate tends to be smaller. For risky firms, long-term survival is possible if the accumulation of intensity is relatively small. As the intensity is strongly related to the leverage ratio, survival events are linked to small leverage ratios, implying larger recovery rates. A slightly increasing term structure of recovery rates is observed for the CCC companies. One sees that generally, the



The hybrid model of Equation (3) combined with the leverage ratio of Equation (7) have been estimated for all firms of the CDX indices using weekly CDS data from December 2007 to January 2012 and the UKF filtering technique. Leverage ratio paths and defaults have been simulated over the next 10 years (under the physical probability measure), starting on January 27, 2012 and recovery rates have been computed using Equation (6). Moments of the recovery rates given default have been computed on a firm-by-firm basis. Each graph shows the average (across firms with the same S&P rating as of January 27, 2012) expectation, standard deviation, skewness and kurtosis of recovery rates given default.

Figure 3: Term structure of the moments of the recovery rate for all credit ratings

lower the credit rating, the lower the recovery rate (this relation holds on average except for AA-rated firms and this may be attributed to the sample size as there were only five of them).

Other components of the recovery rate distribution, that is, standard deviation, skewness and kurtosis, show a term structure. Note that the standard deviations measure the variability of the recovery rate for a given firm. This can never be empirically observed since there are not enough default observations for a single firm. However, these measures quantify the degree of risk linked with the loss given default when an investor assumes a firm's credit risk. This is similar for the skewness and kurtosis, which illustrate the asymmetry and fat-tailness of the recovery rates. Figure 3 shows that recovery rate risk increases over time, that is, the standard deviation and kurtosis both increase with the time of default. When analyzed with respect to credit ratings, one sees that standard deviation tends to decrease with riskiness. This should have been expected for the standard deviation since smaller values tend to have smaller variability (scaling effect). Moreover, skewness and kurtosis are larger for HY firms than for IG ones.

In its 2007 report, Moody's stated that the average recovery rate (across firms) is 52% and the standard deviation (across firms) is 26%. With our sample of firms, the average (across firms) expected recovery rate is between 56% to 63% (depending on the time of default) while the standard deviation (across firms) of the expected recovery rate is about 23% to 26%. Our numbers are therefore similar to Moody's values.

4.2 Comparison with other deterministic recovery rates

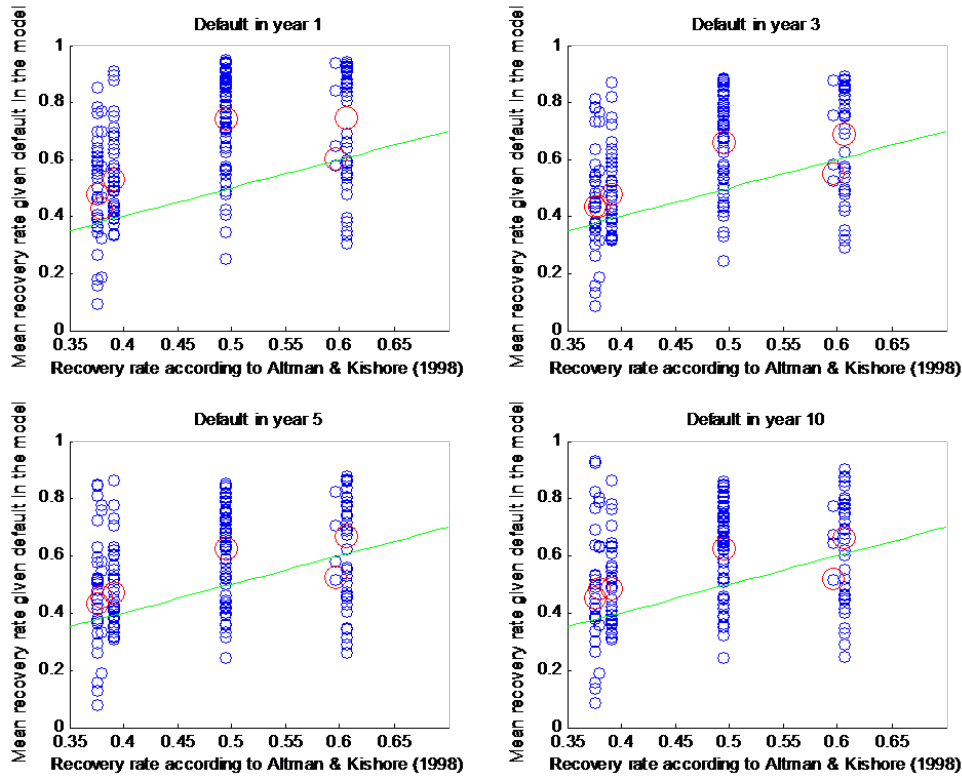
As a final example regarding the link between default probability and recovery rates, we have compared our results to Altman & Kishore (1998), who provide the mean recovery rate given default as a function of the rating prior to default. Their study clearly shows that the higher the credit rating, the higher the recovery rate. For example, the mean recovery rate for AAA companies that have defaulted from 1973 to 1998 is 68.34% whereas it is 49.42% for BBB companies and 38.25% for CCC firms. Figure 4 shows the relationship between the mean recovery rates generated by the model and the recovery rates given by Altman & Kishore (1998), using the latest rating for each company.

In Figure 4, the circles correspond to the average (across firms of a given credit rating) of the mean recovery rate given default, for various periods. We see that these circles cluster around the 45-degree line, which means there is a somewhat close relationship between Altman & Kishore (1998)'s recovery rates and those generated by our model and data. The small circles illustrate the cross-section variability, which can be very high, but as mentioned previously, it is consistent with Moody's (2007). Finally, Altman & Kishore (1998) cover the 1973 to 1998 period, which is very different from the 2008 to 2012 period considered in this paper.

4.3 Timing risk premium and impacts on credit spreads

In our model, we have shown empirically that when recovery rates depend on the solvency of the firm, they will vary over time, thus leading to a term structure of recovery rates. For a highly solvent (insolvent) firm, recovery rates will tend to decrease (increase) because of downgrades (survival). Thus, default timing has an effect on the amount of losses and so should be accounted for in the pricing of bonds and CDS. In other words, defaults over the short (long) term should cost less (more). When compared with a recovery rate that is constant over time, this effect provides what we call a "timing risk premium," which is the additional spread expected to be paid when default timing is considered in computing losses. This spread results from the interdependence of the moment of default and the amount of loss.

To illustrate this point, credit spreads have been expressed as the sum of three premiums. The first measures the effect of recovery endogeneity when compared with a constant recovery rate estimated under the risk-neutral measure. It includes dependence and timing effects (details are provided in Appendix A, Equation (13)). The second illustrates the recovery premium, as expected recovery under the risk-neutral measure \mathbb{Q} is opposed to the common practice of using averaged recoveries observed under \mathbb{P} (Appendix A, Equation (14)). The third premium, known as the default premium, represents the difference between risky and riskless bond yields (Appendix A, Equation (15)). Finally, for comparison, the credit spreads have also



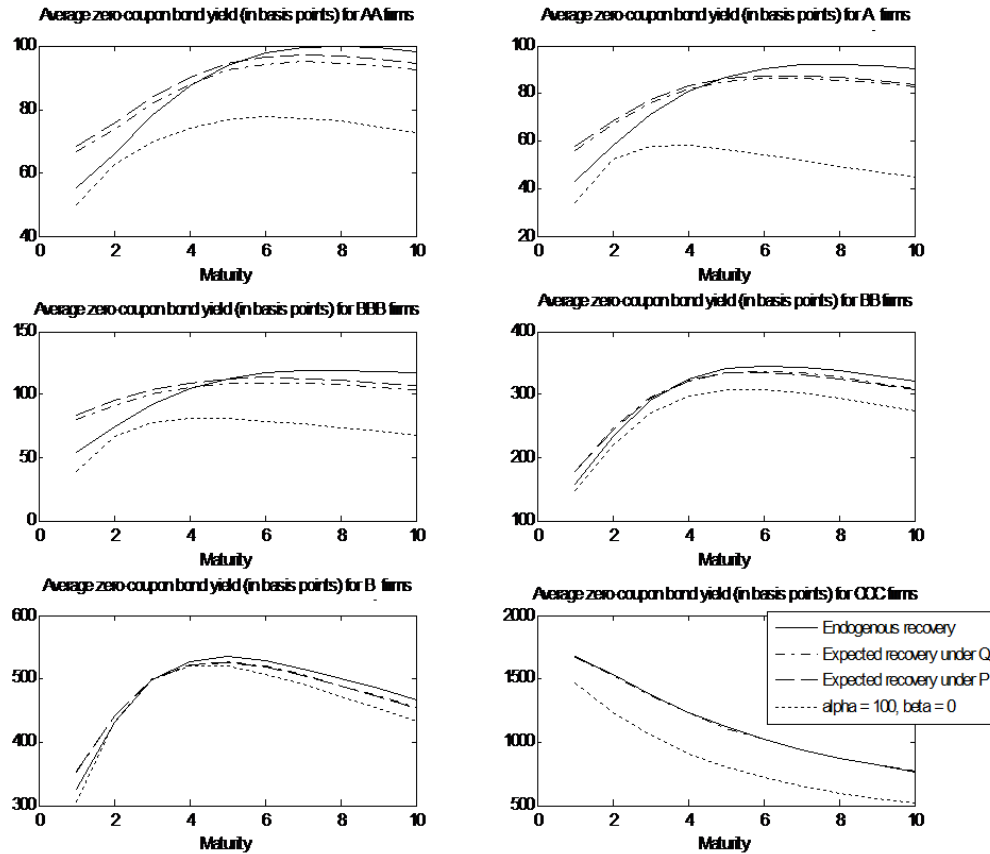
The hybrid model of Equation (3) combined with the leverage ratio of Equation (7) have been estimated for all firms of the CDX indices using weekly CDS data from December 2007 to January 2012 and the UKF filtering technique. Leverage ratio paths and defaults have been simulated over the next 10 years (under the physical probability measure), starting on January 27, 2012, and recovery rates have been computed using Equation (6). The four graphs show the scatter plot of the mean recovery rate given default for each of the 196 companies of the CDX indices, given that default occurred in the k -th year ($k = 1, 3, 5, 10$). The sole CC firm has not been considered as it was not available in Altman & Kishore's study. This is compared with the recovery rate given by their rating in the study by Altman & Kishore (1998). The flat line is a 45-degree line relating (0,0) and (1,1). Were there a perfect relationship between the recovery rates generated by our model and Altman & Kishore's recovery rates, points would cluster around this line. Finally, the large circles show the expected recovery rates given default, which have been averaged across firms of a given rating.

Figure 4: Relationship between the expected recovery rate in the model and values from Altman and Kishore (1998)

been computed for the corresponding structural model by setting $\alpha = 100$ and $\beta = 0$ (other parameters being estimated accordingly). This has been done for each of the 196 firms of the CDX index.⁵

Figure 5 shows two major features of this modelling approach. First, the impact of the recovery assumptions on the credit spreads mostly affects the shape of the curves, since the term structure and the interdependence of recovery and default time produce steeper credit spread curves. Second, the surprise element brought by the shell modifies the level of the credit spread curves. Consequently, as far as credit spreads are concerned, the random recovery assumptions act as a second-order effect when compared with the presence of surprises. However, the presence of randomness in the recovery rates does have an impact when risk measurement is considered. Indeed, in such applications, the focus is mostly on tail events rather than on expectations. This is the goal of the next section.

⁵The single CC company has been discarded.



The hybrid model of Equation (3) combined with the leverage ratio of Equation (7) have been estimated for 196 firms of the CDX indices using weekly CDS data from December 2007 to January 2012. Leverage ratio paths and defaults have been simulated (under the risk-neutral measure) over the next 10 years, starting on January 27, 2012, and recovery rates have been computed using Equation (6). The zero-coupon bond spreads have been constructed for each of the 196 firms. An average curve by risk category is shown in this figure. The continuous line represents the sum of the three risk premiums $\gamma_1 + \gamma_2 + \gamma_3$. The “Expected recovery under Q” corresponds to Assumption (11) about recovery and, therefore, is the sum of two risk premiums: $\gamma_2 + \gamma_3$. “Expected recovery under P” stands for Assumption (12) used in conjunction with constant recovery rates estimated using a Monte Carlo simulation and the proposed framework under the probability measure \mathbb{P} . It is equivalent to γ_3 . The dashed line refers to the structural version of the model in which $\alpha = 100, \beta = 0$ and the other parameters have been re-estimated for each firm to reach the best possible fit with the CDS data.

Figure 5: Zero-coupon bond spreads averaged across companies with the same rating

5 A portfolio approach

This section illustrates the importance of having a good measure of interdependence on usual portfolio risk measures. Indeed, the previous sections established a connection between default probabilities and recovery rates through leverage ratios. At the portfolio level, if one admits that firms’ leverage ratios are correlated, then clusters of defaults may appear along with correlated recovery rates. Using a bond portfolio, we illustrate how it impacts the usual credit risk measures

A first level of dependence among firms appears if the noise terms, modelled by the Brownian motions in Equation (2), are correlated. The simplest case is to assume that these correlations are constant over time:

$$\text{Corr}^{\mathbb{P}} \left[B_t^i, B_t^j \right] = \rho_{ij}$$

where the superscript i refers to the firm i . This interdependence between firm leverage ratios impacts future portfolio value distribution in several ways. First, defaults arise in clusters as the firms' intensities increase simultaneously (assuming positive correlations). Second, since recoveries are negatively correlated with default probabilities, recoveries tend to be smaller whenever clusters of defaults appear. Because the probability of observing large portfolio losses is higher in the presence of default clusters, it directly impacts the usual risk measures, such as the Value-at-Risk (VaR) and conditional tail expectation (CTE).

According to Schönbucher & Schubert (2001), a second level of dependence may be introduced by using a copula to create links among the exponential random variables appearing in Equation (1). Indeed, these random variables may be interpreted as external shocks that affect a firm's solvability, such as a large variation in the exchange rate, a financial crisis, a rise in energy costs, etc. The Clayton copula impacts the left tail of the distributions and produces clusters of small exponential realizations, meaning that many firms are likely to default in the short term. In opposition, the Gumbel copula affects the right tail of the distribution, implying that the default clusters are likely to appear over the long term. Because most risk-management measures deal with one-year horizons, the Clayton copula is used in this study.

5.1 Correlation estimation

If there are n firms, there are $n(n-1)/2$ correlation parameters to be estimated. Estimating the set comprising all these parameters by considering the n firms simultaneously is numerically unmanageable. One way to circumvent this problem is to estimate each firm's parameters individually. In a second step, while considering the bivariate problem with two firms, we estimate the correlation parameter by keeping all firm-specific parameters fixed at their estimated value. This approach keeps the estimation problem numerically manageable. A Monte Carlo study not reported here has shown that the resulting estimates are unbiased.

Table 4 reports descriptive statistics about correlation estimates. These correlations are predominantly positive and generally large. We can therefore expect to observe clusters of defaults that have a significant impact on the tails of the distribution of portfolio losses.

Table 4: Descriptive statistics on the correlation estimates

Level	Percentiles												
	0.1%	0.5%	1%	5%	10%	25%	50%	75%	90%	95%	99%	99.5%	99.9%
Corr. (%)	-31.8	3.5	10.9	24.9	31.0	43.9	56.8	66.9	73.8	77.1	83.1	85.1	89.3

There are $210 \times 209/2 = 21,945$ correlation estimates amongst the 210 firms. Level represents the level of the quantile, while Corr. stands for the correlation estimate (expressed as a %). The median correlation estimate is 56.8%, while the mean is 54.4%. When trading noise is considered, correlations between log-leverage ratios are generally positive and large.

5.2 Empirical results

The portfolio is made up of 210 coupon bonds with nominal values of \$1,000 each. These corporate bonds are linked to each firm for which the model has been previously estimated. The maturities and coupon rates are given by the reference bonds of each CDS⁶. Descriptive statistics are provided in Table 5. Since the portfolio comprises 98 bonds of high-yield firms, it may be considered risky and an important number of defaults has to be expected.

Before proceeding with the empirical study, the parameter that gauges the dependence level among the marginal exponential distributions needs to be determined. Expressed in terms of Kendall's tau coefficient (τ_{Kendall}),⁷ it controls the size of default clusters attributed to external factors that impact the entire econ-

⁶More precisely, for the 125 firms of the CDX.NA.IG, the reference bonds are those of the CDX.NA.IG.17 V1, available at www.markit.com under Credit Index Annex Archives. For the 100 firms of the CDX.NA.HY, the bonds are those associated with the CDX.NA.HY.17 V4.

⁷It is a measure of rank correlation that varies between -1 and 1. If there is a perfect positive dependence, then the Kendall tau coefficient is 1, whereas independence produces a measure close to 0.

Table 5: Bonds characteristics

		Mean	Standard deviation	Percentiles				
				1%	25%	50%	75%	99%
Coupon rate	IG	6.20%	1.23%	2.63%	5.60%	6.13%	6.98%	8.97%
	HY	7.42%	1.71%	3.11%	6.50%	7.25%	8.07%	12.51%
Maturity	IG	2021	7	2014	2016	2018	2027	2037
	HY	2018	4	2013	2015	2017	2018	2031

Table 6: Statistics on the number of defaults

Corr Matrix	Independence			Correlated			
	τ_{Kendall}	0%	10%	20%	0%	10%	20%
Mean		13.38	13.38	13.38	13.18	13.39	13.37
Std		3.16	13.17	19.00	4.49	13.68	19.44
Level		Value-at-Risk (VaR)					
90%		17	28	31	19	29	31
95%		19	39	49	21	40	50
99%		21	67	101	26	69	103

Results are obtained using Monte Carlo simulation with 5×10^6 paths, with each model sharing the same simulated noises. **Independence** means that the log-leverage ratios correlation matrix is the identity matrix, while **Correlated** applies to the case where the estimated correlation matrix is used. τ_{Kendall} refers to the copula parameter that links the exponential random variables together.

omy. Several levels ranging from near independence ($\tau_{\text{Kendall}} = 0$) to an extreme case of high positive rank correlation with a $\tau_{\text{Kendall}} = 50\%$ (not reported here) have been tested and results for moderate dependence are presented next.

Table 6 illustrates the impact of the dependence hypothesis on number of defaults distribution. As mentioned above, credit risk is important for this portfolio and is reflected in the average of 13 defaults during the first year, regardless of the type of dependence. The latter impacts mostly the dispersion of default distributions. The interrelation between log-leverage ratios has a moderate effect on default distributions as the three first columns are similar three last ones. This has to be expected, since in the short term, due to the generally low initial leverage ratios, defaults result mainly from surprises. In Table 6, it appears that even with a moderate $\tau_{\text{Kendall}} = 10\%$, the copula has a tremendous effect on the tail of the default distribution.

Next, we examine loss distribution. Three different recovery assumptions are considered: the proposed endogenous recovery assumption, an independent⁸ beta-distributed random recovery and a constant recovery. To simplify comparisons of the cases, the beta-distributed random recovery shares the same two moments as its endogenous counterpart and the constant recovery is set to the expected endogenous recovery. In Table 7, it appears that the endogenous recovery assumption does impact the tail of the distribution since the VaR of level 99% increases by a factor of about 10% in the independent case and by about 30% whenever log-leverages are correlated. Despite that the beta-distributed random variable is firm-specific and, consequently, depends on the creditworthiness of the firm, it is not enough to produce a severe impact on the tail of the loss distribution and this is where the endogeneity comes into play.

Table 7 shows the effect of dependence between the leverage ratios as well as among the surprise elements on loss distribution. The presence of dependence between default probability and each firm's recovery rate is revealed by the recovery assumptions. It has a moderate impact on the average relative loss as it increases from 3% to 3.2% of the initial portfolio value. This explains why the credit spread premiums associated with endogenous recovery rates are relatively small.

⁸These beta-distributed random variables are independent from leverage ratios and across firms.

Table 7: Statistics on the portfolio's loss distributions expressed as a percentage of the portfolio's initial value

Corr Matrix	Independence				Correlated			
	Const.	Beta	End.	End.	Const.	Beta	End.	End.
Kendall Tau	0%	0%	0%	10%	0%	0%	0%	10%
Mean	3.01	3.01	3.21	3.21	3.01	3.01	3.21	3.22
Std	0.69	0.70	0.78	1.69	3.82	3.82	5.19	5.41
Level	Value-at-Risk (VaR)							
90%	3.91	3.92	4.23	5.03	8.13	8.14	10.16	10.48
95%	4.18	4.19	4.53	6.30	10.07	10.07	12.89	13.36
99%	4.70	4.72	5.11	9.89	14.07	14.05	18.64	19.40
Level	Conditional tail expectation (CTE)							
90%	4.26	4.28	4.62	7.05	10.76	10.76	13.91	14.42
95%	4.50	4.52	4.89	8.53	12.53	12.52	16.44	17.07
99%	4.97	4.99	5.41	12.34	16.21	16.21	21.85	22.67

Results are obtained using a Monte Carlo simulation with 5×10^6 paths, with each model sharing the same simulated noise. **End.** stands for the endogenous recovery assumption, while **Beta** represents the case where an independent beta-distributed random variable with the same mean and standard deviation as its corresponding endogenous recovery rate is used. **Const.** is the case where a constant recovery rate equivalent to the expected endogenous recovery rate is applied. **Independence** means that the log-leverage ratios' correlation matrix is the identity matrix, while **Correlated** refers to the case where the estimated correlation matrix is used. **Kendall Tau** refers to the copula parameter that links the exponential random variables together. All numbers are expressed as a percentage of the initial portfolio value.

The presence of correlations among firm leverage ratios affects loss variability. Indeed, the relative loss standard deviation increases by a factor of 5 or 6, from between about 0.7%–0.8% to 3.8%–5.2%. The presence of endogenous recovery rates amplifies the impact of leverage dependence as the loss standard deviation increases from 0.8% to 5.2%.

The dependence between firms' leverage ratios has an important impact on the tail of the loss distribution. Indeed, regardless of the type of recovery, Value-at-Risk (VaR) and conditional tail expectation (CTE) are 2 to 4 times larger in the presence of correlations and the effect is more pronounced for the far tail of the distribution. Correlations between log-leverage ratios favour the appearance of default clusters accompanied by low recovery rates responsible for extreme loss events. Assuming endogenous recovery rates, the 99% VaR increases from 5% of the initial portfolio value up to 19% and, therefore, the presence of leverage dependence has severe impacts on capital requirements.

Dependence among exponential random variables, through the Clayton copula, acts as a second-order effect on the portfolio's losses except for extreme values represented by the VaR and the CTE of 99%. Indeed, as shown in Table 6, the presence of dependence between surprise elements favours the occurrence of defaults not caused by large leverage ratios. In these cases, recovery rates are higher and, therefore, the presence of many defaults is offset by smaller individual losses.

6 Conclusion

The approach presented in this paper allows many structural models to become hybrid with a simple transformation of leverage to define the firm's default intensity. This transformation can be interpreted as the sensitivity of the firm's credit risk to changes in leverage. The proposed framework can be viewed as a shell that can be added to various structural models to randomize the default trigger. Therefore, the model produces endogenous random recovery rates that are negatively correlated with the default probabilities, which is consistent with the empirical findings. Thus, the framework can be used to simultaneously model the two

sources of credit risk: the moment of default and the amount of loss at default, for risk management and pricing purposes.

With data from the 225 firms of the CDX.NA.IG and CDX.NA.HY indices, the model's parameters have been estimated with weekly CDS premiums between December 2007 and January 2012. The parameters indicate that the intensity is a convex function of the leverage, but they are not large enough to be interpreted as a pure structural model.

Within the proposed framework, it is possible to construct recovery rates that are firm-specific as opposed to recovery rates determined by a sample of defaulted firms with the same credit rating. The distribution of recovery rates varies over time, leading to what has been called a timing risk.

An extensive analysis of the behaviour of recovery rates has been conducted. It was found that the mean recovery rate given default generally has a decreasing term structure over the period considered. The standard deviation and kurtosis of the recovery rate at default, which indicate the degree of risk and tail risk for the loss given default of a specific company, both increase over time. This phenomenon will impact risk-management decisions based on tail events.

The impacts of the shell and of the term structure of recovery rates on credit spreads have been analyzed. The addition of the shell has a large impact on the level of credit spreads as the curves tend to shift upward. The endogenous recovery rates have a second-order effect and modify the shape of the credit spreads, as the latter tend to be steeper.

The correlations between log-leverage ratios have been estimated through CDS data using a UKF filter and the maximum likelihood approach. It has a large impact on bond portfolio credit risk measures. Dependence between surprise elements is achieved using a Clayton copula. The impact on the portfolio's losses is mainly observed in the extreme tail of the distribution. Moreover, correlations among firms leverage ratios adds up to the interdependence between default probability and recovery rate to produce default clusters associated with low recovery rates. It has a major impact on the tail distribution of the portfolio value.

A Credit spread and premiums

In this section, the credit spread is broken down to show the relative importance of the endogeneity of the recovery rate and the surprise element introduced by the model's shell component.

A.1 Bond prices

To price credit derivatives or coupon bonds on a single firm, recovery specifications are crucial. As a first step, the amount of money that will be paid to the security holder upon default needs to be determined as well as the moment when it will be paid. Although the amount payable and its exact timing are defined in the clauses of the contract, the fraction R to be used in the valuation must still be appropriately estimated. Ideally, it should be consistent with the default generating process used to value the assets and liabilities. Thus, when pricing a credit-sensitive asset, an assumption must be made on R and on the type of payment upon default.

For derivative pricing purposes, we must define the notation representing the information structure over time. Throughout this paper, it is assumed that $\mathcal{H}_t = \sigma \{ \mathbb{I}_{\{\tau \leq s\}}, s \leq t \}$ is the σ -algebra that contains the information regarding the survival of the firm and $\mathcal{F}_t = \sigma (\mathcal{G}_t \cup \mathcal{H}_t)$ is the σ -algebra that contains all information. The indicator of default $\mathbb{I}_{\{\tau \leq s\}}$ takes the value 1 if default occurred before s and 0 otherwise (survival).

Following the discussion in Section 2.2, it is assumed throughout this paper that the recovery rate used is defined as per Equations (5) and (6). Assume that the investor recovers a fraction R_τ of an equivalent Treasury bond at default time τ . According to Duffie & Singleton (1999), the time t value of a risky zero-

coupon bond⁹ is

$$V(t, T; R_\tau) = \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) \mathbb{I}_{\{\tau > T\}} + \exp \left(- \int_t^\tau r_u du \right) R_\tau p_\tau(T) \mathbb{I}_{\{\tau \leq T\}} \middle| \mathcal{F}_t \right] \mathbb{I}_{\{\tau > t\}} \quad (8)$$

$$\begin{aligned} &= \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T (r_u + H_u) du \right) \middle| \mathcal{G}_t \right] \\ &+ \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) \int_t^T R_s H_s \exp \left(- \int_t^s H_u du \right) ds \middle| \mathcal{G}_t \right]. \end{aligned} \quad (9)$$

Since r_u , R_s and H_s are dependent random variables, it is difficult to obtain a neat closed-form expression. Numerical methods such as Schönbucher (2002) or Monte Carlo simulations are required to evaluate such an expression.

A.2 Credit spreads

Credit spreads are defined as the difference between risky and riskless zero-coupon yields:

$$CS(t, s) = - \frac{\ln V(t, T; R_\tau)}{T-t} + \frac{\ln P(t, T)}{T-t}.$$

where $P(t, T) = \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) \middle| \mathcal{G}_t \right] = V(t, T; 1)$ is the time t value of a risk-free bond paying \$1 at maturity T . The impact of recovery assumptions on credit spreads appears naturally in the second term (9) of the risky bond value. Indeed, assuming the independence between the recovery rate and the state variables r , L and A leads to Equation (10). If, in addition, there is no term structure of expected recovery rate, that is, if $\mathbb{E}^{\mathbb{Q}} [R_s] = \bar{R}^{\mathbb{Q}}$ for any s , then Equation (10) becomes Equation (11). In the latter, the recovery rate effect is the same as if it were set to some constant $\bar{R}^{\mathbb{Q}}$, in addition to the fact that the expectation $\mathbb{E}^{\mathbb{Q}} [R]$ is taken under the risk-neutral measure \mathbb{Q} . Finally, assuming that no risk premium is associated with recovery risk is equivalent to setting the recovery rate to a constant $\bar{R}^{\mathbb{P}}$, as estimated in empirical studies. This last hypothesis leads to Equation (12).

$$\begin{aligned} (9) &= \int_t^T \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) R_s H_s \exp \left(- \int_t^s H_u du \right) \middle| \mathcal{G}_t \right] ds \\ &= \int_t^T \mathbb{E}^{\mathbb{Q}} [R_s] \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) H_s \exp \left(- \int_t^s H_u du \right) \middle| \mathcal{G}_t \right] ds \end{aligned} \quad (10)$$

$$= \bar{R}^{\mathbb{Q}} \int_t^T \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) H_s \exp \left(- \int_t^s H_u du \right) \middle| \mathcal{G}_t \right] ds \quad (11)$$

$$= \bar{R}^{\mathbb{P}} \int_t^T \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T r_u du \right) H_s \exp \left(- \int_t^s H_u du \right) \middle| \mathcal{G}_t \right] ds. \quad (12)$$

As shown, all these assumptions have an impact on bond values and, consequently, on credit spreads. Therefore, there is a decomposition of the credit spread in terms of risk premiums, each of them coming from some assumption about the recovery rate:

$$CS(t, T) = \gamma_1(t, T) + \gamma_2(t, T) + \gamma_3(t, T)$$

where

$$\gamma_1(t, T) = - \frac{\ln V(t, T; R_\tau)}{T-t} + \frac{\ln V(t, T; \bar{R}^{\mathbb{Q}})}{T-t}, \quad (13)$$

⁹See Lando (2004), Equation (5.6), p.117.

$$\gamma_2(t, T) = -\frac{\ln V(t, T; \bar{R}^{\mathbb{Q}})}{T-t} + \frac{\ln V(t, T; \bar{R}^{\mathbb{P}})}{T-t}, \quad (14)$$

$$\gamma_3(t, T) = -\frac{\ln V(t, T; \bar{R}^{\mathbb{P}})}{T-t} + \frac{\ln V(t, T; 1)}{T-t}. \quad (15)$$

$\gamma_1(t, T)$ is the excess premium brought by the dependence between recovery and default probability, that is, it is the excess premium induced by the endogenous recovery rate. $\gamma_2(t, T)$ is the premium associated with uncertainty regarding the recovery rate while $\gamma_3(t, T)$ is the default premium as the recovery rate is known with certainty.

A.3 Credit default swap

The estimation procedure is based on time series of credit default swap (CDS) premiums. Therefore, a pricing formula for such credit derivative is required within this framework.

A CDS is a credit derivative intended to provide protection against a default within a predetermined period. In the most basic type of CDS (settled in cash), the protection seller provides for a payment of par-minus-recovery on default, which covers the loss in case of default. In exchange, the protection buyer pays a periodic premium, usually four times a year, that ceases if a default occurs. This spread is usually fixed such that the expected present value (PV) of losses equals the expected PV of premiums.

Given that the CDS matures at time T , the expected PV of losses is

$$\mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^{\tau} r_u du \right) R_{\tau} \mathbb{I}_{\{\tau \leq T\}} \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[\int_t^T (1 - R_s) H_s \exp \left(- \int_t^s (r_u + H_u) du \right) ds \middle| \mathcal{G}_t \right]. \quad (16)$$

To simplify the presentation, assume that a premium of 1 is paid at times $t_i < T$. In this case, the expected PV of premiums is given by

$$\mathbb{E}^{\mathbb{Q}} \left[\sum_{t_i} \exp \left(- \int_t^{t_i} r_u du \right) \mathbb{I}_{\{t \leq t_i < \tau\}} \middle| \mathcal{F}_t \right] \mathbb{I}_{\{\tau > t\}}. \quad (17)$$

Then, the periodic premium is the ratio of (16) over (17).

As with defaultable bonds, r_u , H_u and R_u premiums cannot be obtained. Again, the application of numerical methods such as Monte Carlo simulations or Schönbucher (2002) are required.

B UKF

This appendix describes how the UKF filter¹⁰ has been applied to estimate both parameter sets and time series of leverage ratios.

The state equation, as needed with filtering techniques, is

$$x_k = x_{k-1} + \left(\mu_X^{\mathbb{P}} - \frac{1}{2} \sigma_X^2 \right) \Delta t + \sigma \sqrt{\Delta t} \varepsilon_k \quad (18)$$

with $x_k \equiv \ln X_{t_k}$ and $\Delta t = 1/52$.

In a filtering framework, one needs to compute the price of each CDS over the period in question. Because of the stochastic dependence between the moment of default and the amount of loss, no closed-form solutions are available for the theoretical price of a CDS. Thus, the Schönbucher (2002) tree technique is used. Moreover, because the contractual terms of each CDS are not clearly defined, we assume that the default

¹⁰The parameters of the UKF technique have been assumed to be $\kappa_{\text{UKF}} = 2$, $\alpha_{\text{UKF}} = 1$ and $\beta_{\text{UKF}} = 0$ as in Van der Merwe, Doucet, de Freitas & Wan (2000).

payoff is $1 - R_T$. To reduce the number of parameters to be estimated, independence in the pricing error between each CDS has been assumed in addition to using the first seven CDS maturities. The observation equation thus becomes

$$y_k^{(i)} = f^{(i)}(x_k) + \nu_k^{(i)}, i = 1, 2, \dots, 5, 7, 10$$

with $\nu_k^{(i)}$ (having variance $(\delta^{(i)})^2$) independent across maturities and $f^{(i)}(x_k) \equiv \ln(g^{(i)}(\exp(x_k)))$ where $g^{(i)}$ is the price of a i -year CDS computed with Schönbucher (2002). It is important to note that the dynamics of the leverage ratio implied by the pricing function $g^{(i)}$ have to be under \mathbb{Q} i.e. the drift of the leverage ratio is $\mu_X^{\mathbb{Q}}$. (Quasi-)¹¹Maximum likelihood estimation has been performed on the available data on a firm-by-firm basis.

Finally, the initial state value ($\hat{x}_{0|0}$) has been estimated and its variance ($P_{0|0}$) has been assumed to be very low (i.e. 0.001). Overall, the parameters to be estimated for each firm are $\mu_X^{\mathbb{P}}, \xi_X, \sigma_X, \alpha, \theta, \kappa, \hat{x}_{0|0}, \delta$. Thus, a single set of parameters is used for each firm to explain the credit risk and loss given default over the 2008-2012 period. This contrasts with calibration techniques where the CDS curve is fitted at every available period.

C Asset and liability risk premiums

Assuming that

$$dA_t = \mu_t^{\mathbb{P},A} A_t dt + \sigma_t^A A_t dW_t^{\mathbb{P},A} \text{ and } dL_t = \mu_t^{\mathbb{P},L} L_t dt + \sigma_t^L L_t dW_t^{\mathbb{P},L}$$

where, in general, $\mu_t^{\mathbb{P},A}, \mu_t^{\mathbb{P},L}, \sigma^A$ and σ^L are predictable processes such that the SDEs admit a solution. Itô's formula implies that

$$dX_t = d\frac{L_t}{A_t} = \left(\mu_t^{\mathbb{P},L} - \mu_t^{\mathbb{P},A} + (\sigma_t^A)^2 - \sigma_t^A \sigma_t^L \rho_t^{A,L} \right) \frac{L_t}{A_t} dt + \sigma_t^L \frac{L_t}{A_t} dW_t^{\mathbb{P},L} - \sigma_t^A \frac{L_t}{A_t} dW_t^{\mathbb{P},A}.$$

Let $W_t^{\mathbb{P},A} = W_t^{\mathbb{Q},A} - \int_0^t \gamma_s^A ds$ and $W_t^{\mathbb{P},L} = W_t^{\mathbb{Q},L} - \int_0^t \gamma_s^L ds$ where γ^A and γ^L are the risk premia associated to the corresponding Wiener processes. It implies that the drift coefficients under the \mathbb{Q} measure are $\mu_t^{\mathbb{Q},A} = \mu_t^{\mathbb{P},A} - \sigma_t^A \gamma_t^A$ and $\mu_t^{\mathbb{Q},L} = \mu_t^{\mathbb{P},L} - \sigma_t^L \gamma_t^L$ respectively. Hence

$$\xi_X = \mu_X^{\mathbb{P}} - \mu_X^{\mathbb{Q}} = \sigma_t^L \gamma_t^L - \sigma_t^A \gamma_t^A.$$

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¹¹Quasi in the sense that under a UKF framework, the first two moments of the posterior distribution have a second-order precision; however, a posterior Gaussian distribution has to be assumed.

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