The Injectivity Modules of a Tropical Map

E. Wagneur
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Edouard Wagneur

GERAD
HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada, H3T 2A7
edouard.wagneur@gerad.ca

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1 Introduction

A tropical torsion module $M$ is an idempotent commutative semimodule over the idempotent commutative extended semiring $\mathbb{R} = \mathbb{R} \cup \{-\infty\}$. Endowed with the max operator (written $\lor$) as first composition law, and classical addition (written $\cdot$, and which will usually be omitted when no confusion arises), with the (torsion) property that, for any two generators, $x, y$, there exist $\lambda_{xy} = \inf\{\xi \in \mathbb{R} | x \leq \xi y\}$ and $\lambda_{yx} = \inf\{\xi \in \mathbb{R} | y \leq \xi x\}$. Moreover, the product $\tau(x, y) = \lambda_{xy} \cdot \lambda_{yx}$ in $\mathbb{R}$ is an invariant of the isomorphism class of $M$, called the torsion of $M$.

We write $\mathbf{0}$ and $\mathbf{1}$ for the neutral elements of $\lor$ and $\cdot$ respectively.

In [5], we show that any $m$-dimensional tropical torsion module can be embedded in $\mathbb{R}^d$, with $d \leq m(m - 1)$, and that $m$-dimensional tropical torsion modules are classified by a $p$-parameter family, with $p \leq (m - 1)^2 + 1$.

The aim of the paper is to revisit and extend some of these results by showing that – at least in the 3-dimensional case – the two upper bounds are tight. More precisely, we show that for $m = 3$, we can find tropical torsion modules which cannot be embedded in $\mathbb{R}^d$ for $d < 6$, and that all the $p = 2 \cdot (2 \cdot 3 - 1) = 10$ parameters required for the unambiguous specification of the 3 generators of $M$ are necessary for the characterization of $M$.

Also, the concept of injectivity set (or injectivity tropical module) briefly dealt with in [5] is further investigated. In particular, we show the counterintuitive result that, for a given tropical map $\varphi: M \rightarrow N$, the quotient $M_{\varphi}$ defined by the equivalence $\sim$ given by $x \sim y \iff \varphi(x) = \varphi(y)$ is not isomorphic to $\text{Im}\varphi$.

The paper is organised as follows. In Section 2, we briefly recall some of the results of [5] which will be used in the paper. In Section 3, we state the main result of the paper, related to the injectivity modules of a tropical map. Then these results are illustrated in Section 4, by way of two examples, where $m < n$ and $n < m$, respectively. The first one with a tropical map in $\text{Hom}(\mathbb{R}^3, \mathbb{R}^6)$, the second with a map in $\text{Hom}(\mathbb{R}^4, \mathbb{R}^3)$. In both cases, (some of) the injectivity modules are exhibited.

2 The main results of [5]

In this section we briefly recall the main results of [5] which will be used in this paper.

1. The canonical form of the torsion matrix:

$$A = \begin{bmatrix}
\mathbf{1} & \mathbf{1} & a_{13} & \cdots & a_{1m} \\
\mathbf{1} & a_{22} & a_{23} & \cdots & a_{2m} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{1} & a_{n2} & a_{n3} & \cdots & a_{nm}
\end{bmatrix}$$

(1)

with $\mathbf{1} = a_{12} \leq a_{22} \leq \cdots \leq a_{n2}$, $a_{ij} \leq a_{ij+1}$, $i = 1, \ldots, n$, $j = 2, \ldots, m$, and $\tau(x_j, x_{j+1}) \leq \tau(x_j, x_{j+1})$, $j = 2, \ldots, m - 1$, where $x_j$ stands for column $j$ of $A$.

This canonical form also defines the canonical basis of $M_A$.

2. $\forall j \ (1 \leq j \leq m - 1), \exists i \ (1 \leq i \leq n)$ such that $a_{ij+1} = a_{ij}$ (hence $\lambda_{jj+1} = \mathbf{1}$).

1 Torsion in tropical modules has been introduced in [3], $\tau(x, y)$ is equal to $\exp(\delta(x, y))$, where $\delta(x, y)$ is the Hilbert pseudo-metric, invented by Hilbert in [1].
3. The $\lambda_{ij}$ (from which we readily get the $\tau_j$) are given by the matrix

$$\Lambda_A = A^t \cdot A^-= \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{m1} & \lambda_{m2} & \cdots & \lambda_{mm} \end{bmatrix}$$

(2)

where $A^t$, and $A^-$ stand for the transpose of $A$ and for the matrix with entries the inverses of those of $A$.

4. The Whitney embedding theorem and the classification of tropical modules have been recalled in Section 1 above.

3. The injectivity modules of a tropical map

In this section, we investigate some properties of $\text{INJ}_A$ for a tropical torsion matrix (TTM) $A$. Let $M, N$ be two tropical modules of dimension $m, n$ respectively, $\varphi \in \text{Hom}(M, N)$, and $\pi$ the canonical projection $M \rightarrow M/\sim$, defined by the equivalence relation $x \sim y$ if and only if $\varphi(x) = \varphi(y)$. Clearly $\varphi$ is injective on the set $\{\xi \in M | \forall \lambda \in M, \lambda \neq \xi \Rightarrow \varphi(\lambda) \neq \varphi(\xi)\}$ is the injectivity set of $\varphi$.

Let $A$ be a square tropical torsion matrix. In [5], we defined $\text{INJ}_A = \{\xi \in \mathbb{R}^n | \exists \sigma \in \mathcal{S}_n \text{ such that } \forall k, \bigvee_{j=1, j \neq k}^n a_{\sigma(k)j} \xi_j \leq a_{\sigma(k)k} \xi_k \}$, and proved the following statement.

**Proposition 1** For any square tropical torsion matrix $A \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of maximal column rank, there is a unique permutation $\sigma \in \mathcal{S}_n$ such that

$$\{\xi \in \mathbb{R}^n \text{ s. t. for } k = 1, \ldots, n, \bigvee_{j=1, j \neq k}^n a_{\sigma(k)j} \xi_j \leq a_{\sigma(k)k} \xi_k \}.$$  

(3)

It is easy to see that the injectivity set of $A$ satisfying 3 is a tropical module.

Clearly, for any $n \times n$ permutation matrix $P$ $\text{INJ}_{PA} = \text{INJ}_A$, and, by Proposition 1, there exists a unique permutation matrix $P$ such that, for $B = PA$, (3) is equivalent to

$$\text{INJ}_A = \{\xi \in \mathbb{R}^n \text{ s. t. for } k = 1, \ldots, n, \bigvee_{j=1, j \neq k}^n b_{kj} \xi_j \leq b_{kk} \xi_k \}.$$  

(4)

Let $\tilde{A} = (\text{diag}(b_{ii}^{-1}))B$.

As a straightforward application of a well-known result (cf [2] for instance), we have the following statement.

**Proposition 2** $\text{INJ}_A$ is generated by the columns of $\tilde{A}^*$.

**Theorem 1** Let $A$ be a TTM $m \times n$, then there are at most \left(\frac{\max\{m,n\}}{\min\{m,n\}}\right)$ tropical modules where $A$ is injective. Each of these injectivity modules is generated by the Kleene star of some square matrix derived from $A$.

**Proposition 3** The tropical modules $\text{Im}A$ and $\text{INJ}_A$ are not isomorphic in general.

**Proposition 4** If $A$ is a rectangular $n \times m$ matrix with $m \neq n$, then $\text{INJ}_A$ is a union of tropical modules, which is not a tropical module in general.
Definition. We say the union of modules $\text{INJ}_A = \bigcup_{i=1}^k M_i$ is isomorphic to the union of modules $\text{INJ}_B = \bigcup_{i=1}^k N_i$ if, for every tropical module $M_i \in \text{INJ}_A$, there is a tropical module $N_i \in \text{INJ}_B$, which is isomorphic to $M_i$, $i = 1, \ldots, k$.

Remark. The statement in Proposition 3 differ from that in Propostion 4, since $\text{INJ}_A$ is a TTM in Proposition 3.

4 Examples

The first two examples illustrate the statement in Theorem 1. In addition, our first example shows that the bound given in [5] for the Whitney embedding is tight, i.e. there exists a 3-dimensional tropical module which cannot be embedded in $\mathbb{R}^d$ for $d < 6 = m(m - 1) = 6$. Also, as a complement to the classification theorem of the same reference, this example will be used to show that all the $p = (m - 1)[m(m - 1) - 1]$ parameters are needed for the classification of $M_A$.

Our third example shows that we can find $n$-dimensional tropical modules with $m \leq n$ generators with equal torsion coefficients.

Example 1

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & a & a \\ 8 & 15 \\ 9 & 11 \end{bmatrix}$, with $5 < a < 8$. We have

$\Gamma_A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & a & a \\ 8 & 15 \\ 9 & 11 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 2 & 1 & 4 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 4 \\ 1 & 1 & 4 \\ 1 & 1 & 4 \end{bmatrix}$, with

$\lambda_{12} = 1$, $\lambda_{21} = 9$, $\lambda_{13} = 4^{-1}$, $\lambda_{31} = 15$, $\lambda_{23} = 1$, and $\lambda_{32} = 12$, given by rows 1, 6, 2, 5, 4, and 3, respectively.

We have $\tau_{12} = \lambda_{12} \cdot \lambda_{21} = 9 < \tau_{13} = \lambda_{13} \cdot \lambda_{31} = 11 < \tau_{23} = \lambda_{23} \cdot \lambda_{32} = 12$.

It follows that all six rows of $A$ are required for the torsion of $M_A$. Hence, it $A$ cannot be embedded into $\mathbb{R}^d$ for $d < 6$. Note that the $\tau_{ij}$ are independent of $a$.

The tropical modules $\text{INJ}_A$

We compute the tropical modules $M_{ijk} = \text{INJ}_{A_{ijk}}$ for $i = 1, j = 2, k = 3$, and for $i = 1, j = 2, k = 4$, where $A_{ijk}$ is he map given by the square submatrix of $A$ determined by rows $i, j, k$.

We have: $A_{123} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 2 & 14 \end{bmatrix}$, then, since $\sigma = I$, i.e. $P = I$, we have $\tilde{A}_{123} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 14^{-1} & 12^{-1} & 1 \end{bmatrix}$, and

$\tilde{A}_{123}^* = \tilde{A}_{123}^2 = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 13^{-1} & 12^{-1} & 1 \end{bmatrix}$.
Hence $M_{123}$ is generated by \[
\frac{1}{13^{-1}} \begin{bmatrix} 1 & 1 \\ 0 & 12^{-1} \end{bmatrix}, \frac{1}{124} \begin{bmatrix} 1 \\ 12^{-1} \end{bmatrix}, \text{ and } \frac{1}{124} \begin{bmatrix} 5 \\ 12^{-1} \end{bmatrix}.
\]

$A_{124} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 1 & a \\ 1 & a \end{bmatrix}$, with $P = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ thus

$\tilde{A}_{124} = \text{diag}(1 a^{-1} 5^{-1}) P A_{124} = \begin{bmatrix} 1 & 1 \\ a^{-1} & 4 \\ 5^{-1} & 5^{-1} \\ 5^{-1} & \end{bmatrix}$, and we get

$\tilde{A}_{124}^* = \tilde{A}_{124}^2 = \begin{bmatrix} 1 & 1 \\ 5^{-1} & 5^{-1} \\ 5^{-1} & 4^{-1} \\ 4^{-1} & \end{bmatrix}$

$M_{124} = \{x | 1 \xi_2 \leq \xi_1 \leq a \xi_2, 4 \xi_3 \leq \xi_1 \leq 5 \xi_3, \xi_2 \leq 5 \xi_3 \leq 5 \xi_2 \}$, its generators are given by the columns of $\tilde{A}_{124}^* = \begin{bmatrix} 1 & 1 \\ 5^{-1} & 1 \\ 5^{-1} & 4^{-1} \\ 4^{-1} & \end{bmatrix}$.

We have: $\Gamma_{A_{124}} = \begin{bmatrix} 1 & 1 \\ 1 & a \\ 5 & 4 \\ 1 & a \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1^{-1} \\ 1 & a^{-1} \\ 1 & a \end{bmatrix} = \begin{bmatrix} 1 & 1^{-1} \\ 1 & a \\ a & 5 \\ a & \end{bmatrix}$, with the torsion coefficients given by $4^{-1}a$, $5$, $a$, respectively, and

$\Gamma_{A_{124}} = \begin{bmatrix} 1 & 5^{-1} \\ 1 & 4^{-1} \\ 4 & 4^{-1} \end{bmatrix}$ $\begin{bmatrix} 1 & 1^{-1} \\ 1 & 1 \\ 5 & 4 \\ 5 & \end{bmatrix} = \begin{bmatrix} 1 & 1^{-1} \\ 5 & 5 \\ 5 & 4 \\ 4 & \end{bmatrix}$,

with the $\tau(i, j) = 1, 4, 4$, respectively.

Thus $\text{Im}A_{124}$ is not isomorphic to $\text{INJ}_{A_{124}}$.

On the other hand it is easy to see that:

$A_{124} \tilde{A}_{124}^* = \text{diag}(5 \ |a \ P \tilde{A}_{124}^*, \text{i.e.}

$M_{124}$ is equal to its image under $A_{124}$.

This example also illustrates the fact that the domain of a tropical map $\varphi: M \rightarrow N$ splits into two parts:

- $\text{INJ}_\varphi$, every point of which is an equivalence class of “~”.
- $M \setminus \text{INJ}_\varphi$ where the equivalence classes contain more than one point of $M$.

Moreover, as easily seen from the torsion coefficients between generators, the $M_{ijk}$ are neither isomorphic to $\text{Im}A$, nor isomorphic to one another in general.

Our next example, which first appeared in [4] has been shortly examined in [5]. It is revisited here for an illustration of the case $m > n$ in Theorem 1.

**Example 2**

Let $x_i = \begin{bmatrix} i \\ i \\ i^2 \end{bmatrix}$, $i = 1, 2, \ldots, m$, with $i = i^2 = \|i\|$ for $i = 0$, and $A = [x_1 | x_2 | \cdots | x_m]$. The tropical submodule $M_A$ of $\mathbb{R}^3$ can be made infinite dimensional by letting $m \rightarrow \infty$. 
It is not difficult to see that \( A \) is injective on \( \bigcup_{0 \leq i < j < k} M_{ijk} \), where

\[
M_{ijk} = \{ \xi \mid \bigvee_{\ell \geq 1, \ell \neq i} \xi_{\ell} \leq \xi_{i}, \bigvee_{\ell \geq 1, \ell \neq j} \xi_{\ell} \leq j \xi_{j}, \bigvee_{\ell \geq 1, \ell \neq k} \ell^{2} \xi_{\ell} \leq k^{2} \xi_{k} \}\]

For instance, with \( M = \mathbb{R}^{4} \), we get

\[
\begin{align*}
M_{124} &= \{ \xi \in \mathbb{R}^{4} \mid \xi_{1} \leq \xi_{1}, \xi_{1} \lor 2 \xi_{3} \lor 3 \xi_{4} \leq 1 \xi_{2}, \xi_{1} \lor 2 \xi_{2} \lor 4 \xi_{3} \leq 6 \xi_{4} \} \\
M_{134} &= \{ \xi \in \mathbb{R}^{4} \mid \xi_{1} \leq \xi_{1}, \xi_{1} \lor 1 \xi_{2} \lor 3 \xi_{4} \leq 2 \xi_{3}, \xi_{1} \lor 2 \xi_{2} \lor 4 \xi_{3} \leq 6 \xi_{4} \} \\
M_{234} &= \{ \xi \in \mathbb{R}^{4} \mid \xi_{1} \lor \xi_{3} \lor \xi_{4} \leq \xi_{2}, \xi_{1} \lor 1 \xi_{2} \lor 3 \xi_{4} \leq 2 \xi_{3}, \xi_{1} \lor 2 \xi_{2} \lor 4 \xi_{3} \leq 6 \xi_{4} \}.
\end{align*}
\]

The method described in Theorem 1 is illustrated as follows for the generators of the \( M_{ijk} \), where the \( i \) (resp. \( j, k \)) stands for the rank of the column which dominates row 1 (resp. 2, 3) of \( A \xi \).

For \( M_{123} \), define \( A_{123} = \begin{bmatrix} \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} \\ \mathbb{I}^1 & 1 & 2 & 3 \\ \mathbb{I}^2 & 2^2 & 3^2 \\ 0 & 0 & 0 & \mathbb{I} \end{bmatrix} \).

Then from \( A_{123} \xi = \begin{bmatrix} \xi_{1} \\ \xi_{2} \\ \xi_{3} \\ \xi_{4} \end{bmatrix} \), we get \( \tilde{A}_{123} = \begin{bmatrix} \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} \\ \mathbb{I}^{-1} & 1 & 2 \\ 2^{-1} & 2^{-1} & 2 \\ 0 & 0 & 0 & \mathbb{I} \end{bmatrix} \), and

\[
\tilde{A}_{123}^3 = \begin{bmatrix} \mathbb{I} & \mathbb{I} & 1 & 3 \\ \mathbb{I}^{-1} & 1 & 3 \\ 3^{-1} & 2^{-1} & 2 \\ 0 & 0 & 0 & \mathbb{I} \end{bmatrix} = \tilde{A}_{123}^*.
\]

Clearly: \( \begin{bmatrix} \mathbb{I} \\ 1^{-1} \\ 3^{-1} \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbb{I} \\ 2^{-1} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \) generate \( \text{INJ}_{A_{123}} \).

For a straightforward verification, let

\[
u = x_{1} \begin{bmatrix} \mathbb{I} \\ 1^{-1} \\ 3^{-1} \\ 0 \end{bmatrix} \lor x_{2} \begin{bmatrix} \mathbb{I} \\ 2^{-1} \\ 0 \end{bmatrix} \lor x_{3} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \lor x_{4} \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} x_{1} \lor x_{2} \lor x_{3} \lor x_{4} \\ 1^{-1} x_{1} \lor x_{2} \lor x_{3} \lor x_{4} \\ 3^{-1} x_{1} \lor 2^{-1} x_{2} \lor x_{3} \lor x_{4} \end{bmatrix}.
\]

We leave it to the reader to check that

\[
\begin{bmatrix} \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} \\ \mathbb{I} & 1 & 2 & 3 \\ \mathbb{I} & 1^2 & 2^2 & 3^2 \\ \mathbb{I} & \tau & \tau & \tau \end{bmatrix} u = \begin{bmatrix} u_{1} \\ u_{2} \\ 2^{2} u_{3} \end{bmatrix}, \text{ i.e. } u \in M_{123}.
\]

**Example 3**

This last example shows that we can find \( n - 1 \) torsion elements in \( \mathbb{R}^{n} \) exhibiting two by two the same torsion.

Let \( A = \begin{bmatrix} \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} \end{bmatrix} \). Clearly the torsion coefficients of any two columns of \( A \) are equal to \( \tau \).
For \( n = 2 \), the injectivity module of \( A \) has been investigated in [5]. The general case is illustrated by the case \( n = 6 \). Let \( P \) be the permutation matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Then \( \tilde{A} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \tau^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & \tau^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & \tau^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & \tau^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & \tau^{-1}
\end{bmatrix} \quad PA = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
\tau^{-1} & 1 & 1 & 1 & 1 & 1 \\
\tau^{-1} & \tau^{-1} & 1 & 1 & 1 & 1 \\
\tau^{-1} & \tau^{-1} & \tau^{-1} & 1 & 1 & 1 \\
\tau^{-1} & \tau^{-1} & \tau^{-1} & \tau^{-1} & 1 & 1 \\
\tau^{-1} & \tau^{-1} & \tau^{-1} & \tau^{-1} & \tau^{-1} & 1
\end{bmatrix} \sim A.
\]

Since \( \tilde{A} \geq I \), and \( \tilde{A}^2 = \tilde{A} \), then \( \tilde{A}^* = \tilde{A} \), and its columns generate \( \text{INJ}_A \).

References