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1 Introduction

A tropical torsion module M is an idempotent commutative semimodule over the idempotent commutative extended semiring $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. Endowed with the max operator (written $\vee$) as first composition law, and classical addition (written$\cdot$, and which will usually be omitted when no confusion arises), with the (torsion) property that, for any two generators, x, y, there exist $\lambda_{xy} = \inf\{\xi \in \underline{\mathbb{R}} | x \leq \xi y\}$ and $\lambda_{yx} = \inf\{\xi \in \underline{\mathbb{R}} | y \leq \xi x\}$. Moreover, the product $\tau(x,y) = \lambda_{xy} \cdot \lambda_{yx}$ in $\underline{\mathbb{R}}$ is an invariant of the isomorphy class of M, called the *torsion*[1] of M.

We write $\mathbf{0}$ and $\mathbb{1}$ for the neutral elements of $\vee$ and $\cdot$ respectively.

In [5], we show that any m- dimensional tropical torsion module can be embedded in $\underline{\mathbb{R}}^d$, with $d \leq m(m-1)$, and that m-dimensional tropical torsion modules are classified by a p-parameter family, with $p \leq (m-1)[m(m-1)-1]$.

The aim of the paper is to revisit and extend some of these results by showing that – at least in the 3-dimensional case – the two upper bounds are tight. More precisely, we show that for $m = 3$, we can find tropical torsion modules which cannot be embedded in $\underline{\mathbb{R}}^d$ for $d < 6$, and that all the $p = 2 \cdot (2 \cdot 3 - 1) = 10$ parameters required for the unambiguous specification of the 3 generators of M are necessary for the characterization of M.

Also, the concept of injectivity set (or injectivity tropical module) briefly dealt with in [5] is further investigated. In particular, we show the conterintuitive result that, for a given tropical map $\varphi\colon M \to N$, the quotient $M_{|\varphi}$ defined by the equivalence $\sim$ given by $x \sim y \iff \varphi(x) = \varphi(y)$ is not isomorphic to $\mathrm{Im}\varphi$.

The paper is organised as follows. In Section 2, we briefly recall some of the results of [5] which will be used in the paper. in Section 3, we state the main result of the paper, related to the injectivity modules of a tropical map. then these results are illustrated in Section 4, by way of two examples, where $m < n$ and $n < m$, respectively. The first one with a tropical map in $\mathrm{Hom}(\underline{\mathbb{R}}^3, \underline{\mathbb{R}}^6)$, the second with a map in $\mathrm{Hom}(\underline{\mathbb{R}}^4, \underline{\mathbb{R}}^3)$. In both cases, (some of) the injectivity modules are exhibited.

2 The main results of [5]

In this section we briefly recall the main results of [5] which will be used in this paper.

1. The canonical form of the torsion matrix:

$$A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & a_{13} & \cdots & a_{1m} \\ \mathbb{1} & a_{22} & a_{23} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots & \\ \mathbb{1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix} \tag{1}$$

   with $\mathbb{1} = a_{12} \leq a_{22} \leq \cdots \leq a_{n2}$ $a_{ij} \leq a_{ij+1}$, $i = 1, \ldots, n$, $j = 2, \ldots, m$, and $\tau(x_{j-1}, x_j) \leq \tau(x_j, x_{j+1})$ $j = 2, \ldots, m-1$, where x_j stands for column j of A.
   This canonical form also defines the canonical basis of M_A.
2. $\forall j$ $(1 \leq j \leq m-1)$, $\exists i$ $(1 \leq i \leq n)$ such that $a_{ij+1} = a_{ij}$ (hence $\lambda_{jj+1} = \mathbb{1}$).

[1] Torsion in tropical modules has been introduced in [3]. $\tau(x,y)$ is equal to $exp(\delta(x,y))$, where $\delta(x,y)$ is the Hilbert pseudo-metric, invented by Hilbert in [1].

3. The λ_{ij} (from which we readily get the τ_j) are given by the matrix

$$\Lambda_A = A^t \cdot A^- = \begin{bmatrix} 1\!\!1 & 1\!\!1 & \lambda_{13} & \cdots & \lambda_{1m-1} & \lambda_{1m} \\ \tau_{12} & 1\!\!1 & 1\!\!1 & \cdots & \lambda_{2m-1} & \lambda_{2m} \\ \lambda_{31} & \tau_{23} & 1\!\!1 & 1\!\!1 & \cdots & \lambda_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1\!\!1 & 1\!\!1 \\ \lambda_{m1} & \lambda_{m2} & \cdots & \lambda_{mm-2} & \tau_{m-1m} & 1\!\!1 \end{bmatrix} \quad (2)$$

where A^t, and A^- stand for the transpose of A and for the matrix with entries the inverses of those of A.

4. The Whitney embedding theorme and the classification of tropical modules have been recalled in Section 1 above.

3 The injectivity modules of a tropical map

In this section, we investigate some properties of INJ_A for a tropical torsion matrix (TTM) A. Let M, N be two tropical modules of dimension m, n respectively, $\varphi \in \mathrm{Hom}(M, N)$, and π the canonical projection $M \ \to M|_\sim$, defined by the equivalence relation $x \sim y \iff \varphi(x) = \varphi(y)$. Clearly φ is injective on the set $\{\xi \in M | \forall \lambda \in M\,,\ \lambda \neq \xi \ \Rightarrow\ \varphi(\lambda) \neq \varphi(\xi)\}$ is the **injectivity set** of φ.

Let A be a square tropical torsion matrix. In [5], we defined $\mathrm{INJ}_A = \{\xi \in \mathbb{R}^n | \exists \sigma \in \mathcal{S}_n$ such that $\forall\, k\,,\ \bigvee_{j=1, j\neq k}^{n} a_{\sigma(k)j}\xi_j \leq a_{\sigma(k)k}\xi_k\}$, and proved the following statement.

Proposition 1 *For any square tropical torsion matrix $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of maximal column rank, there is a unique permutation $\sigma \in \mathcal{S}_n$ such that*

$$\{\xi \in \mathbb{R}^n \ s.\ t.\ \text{for}\, k = 1, \ldots, n\,,\ \bigvee_{j=1, j\neq k}^{n} a_{\sigma(k)j}\xi_j \leq a_{\sigma(k)k}\xi_k\}. \quad (3)$$

□

It is easy to see that the injectivity set of A satisfying 3 is a tropical module.

Clearly, for any $n \times n$ permutation matrix P $\mathrm{INJ}_{PA} = \mathrm{INJ}_A$, and, by Proposition 1, there exists a unique permutation matrix P such that, for $B = PA$, (3) is equivalent to

$$\mathrm{INJ}_A = \{\xi \in \mathbb{R}^n \ s.\ t.\ \text{for}\, k = 1, \ldots, n\,,\ \bigvee_{j=1, j\neq k}^{n} b_{kj}\xi_j \leq b_{kk}\xi_k\}. \quad (4)$$

Let $\tilde{A} = (\mathrm{diag}(b_{ii}^{-1}))B$.

As a straightforward application of a weel-known result (cf [2] for instance), we have the following statement.

Proposition 2 INJ_A *is generated by the columns of* $\tilde{A}^*$. □

Theorem 1 *Let A be a TTM $m \times n$, then there are at most $\begin{pmatrix} \max\{m, n\} \\ \min\{m, n\} \end{pmatrix}$ tropical modules where A is injective. Each of these injectivity modules is generated by the Kleene star of some square matrix derived from A.* □

Proposition 3 *The tropical modules* $\mathrm{Im}A$ *and* INJ_A *are not isomorphic in general.* □

Proposition 4 *If A is a rectangular $n \times m$ matrix with $m \neq n$, then INJ_A is a union of tropical modules, which is not a tropical module in general.* □

Definition. We say the the union of modules $\mathrm{INJ}_A = \bigcup_{i=1}^{k} M_i$ is isomorphic to the union of modules $\mathrm{INJ}_B = \bigcup_{i=1}^{k} N_i$. if, for every tropical module $M_i \in \mathrm{INJ}_A$, there is a tropical module $N_i \in \mathrm{INJ}_B$, which is isomorphic to $M_i\,,\ i = 1, \dots, k$.

Remark. The statement in Proposition 3 differ from that in Propostion 4, since INJ_A is a TTM in Proposition 3.

4 Examples

The first two examples illustrate the statement in Theorem 1. In addition, our first example shows that the bound given in [5] for the Whitney embedding is tight, i.e. there exists a 3-dimensional tropical module which cannot be embedded in $\mathbb{R}^d$ for $d < 6 = m(m-1) = 6$. Alsp, as a complement to the classification theorem of the same reference, this example will be used to show that all the $p = (m-1)[m(m-1)-1]$ parameters are needed for the classification of M_A.

Our third example shows that we can find n-dimensional tropical modules with $m \leq n$ generators with equal torsion coefficients.

Example 1

Let $A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & 2 & 14 \\ \mathbb{1} & a & a \\ \mathbb{1} & 8 & 15 \\ \mathbb{1} & 9 & 11 \end{bmatrix}$, with $5 < a < 8$. We have

$$\Gamma_A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & 1 & 2 & a & 8 & 9 \\ 5 & 4 & 14 & a & 15 & 11 \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5^{-1} \\ \mathbb{1} & 1^{-1} & 4^{-1} \\ \mathbb{1} & 2^{-1} & 14^{-1} \\ \mathbb{1} & a^{-1} & a^{-1} \\ \mathbb{1} & 8^{-1} & 15^{-1} \\ \mathbb{1} & 9^{-1} & 11^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 4^{-1} \\ 9 & \mathbb{1} & \mathbb{1} \\ 15 & 12 & \mathbb{1} \end{bmatrix}, \text{ with}$$

$\lambda_{12} = \mathbb{1}\,,\ \lambda_{21} = 9\,,\ \lambda_{13} = 4^{-1}\,,\ \lambda_{31} = 15\,,\ \ \lambda_{23} = \mathbb{1}$, and $\lambda_{32} = 12$, given by rows $1, 6, 2, 5, 4$, and 3, respectively.

We have $\tau_{12} = \lambda_{12} \cdot \lambda_{21} = 9 < \tau_{13} = \lambda_{13} \cdot \lambda_{31} = 11 < \tau_{23} = \lambda_{23} \cdot \lambda_{32} = 12$.

It follows that all six rows of A are required for the torsion of M_A. Hence, it A cannot be embedded into $\underline{\mathbb{R}}^d$ for $d < 6$. Note that the τ_{ij} are independent of a.

The tropical modules INJ_A

We compute the tropical modules $M_{ijk} = \mathrm{INJ}_{A_{ijk}}$ for $i = 1, j = 2, k = 3$, and for $i = 1, j = 2, k = 4$, where A_{ijk} is he map given by the square submatrix of A determined by rows i, j, k.

We have : $A_{123} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & 2 & 14 \end{bmatrix}$, then, since $\sigma = I$, i.e. $P = I$, we have $\tilde{A}_{123} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ 1^{-1} & \mathbb{1} & 3 \\ 14^{-1} & 12^{-1} & \mathbb{1} \end{bmatrix}$, and $\tilde{A}^*_{123} = \tilde{A}^2_{123} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ 1^{-1} & \mathbb{1} & 4 \\ 13^{-1} & 12^{-1} & \mathbb{1} \end{bmatrix}$.

Hence M_{123} is generated by $\begin{bmatrix} \mathbb{1} \\ 1^{-1} \\ 13^{-1} \end{bmatrix}$, $\begin{bmatrix} \mathbb{1} \\ \mathbb{1} \\ 12^{-1} \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 4 \\ \mathbb{1} \end{bmatrix}$.

$A_{124} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & a & a \end{bmatrix}$, with $P = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbb{1} \\ \mathbb{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{1} & \mathbf{0} \end{bmatrix}$ thus

$\tilde{A}_{124} = \mathrm{diag}(\mathbb{1}\ a^{-1}\ 5^{-1})\, P\, A_{124} = \begin{bmatrix} \mathbb{1} & 1 & 4 \\ a^{-1} & \mathbb{1} & \mathbb{1} \\ 5^{-1} & 5^{-1} & \mathbb{1} \end{bmatrix}$, and we get

$$\tilde{A}^*_{124} = \tilde{A}^2_{124} = \begin{bmatrix} \mathbb{1} & 1 & 4 \\ 5^{-1} & \mathbb{1} & \mathbb{1} \\ 5^{-1} & 4^{-1} & \mathbb{1} \end{bmatrix}$$

$M_{124} = \{\xi|\ 1\xi_2 \leq \xi_1 \leq a\xi_2\,,\ 4\xi_3 \leq \xi_1 \leq 5\xi_3\,,\ \xi_2 \leq 5\xi_3 \leq 5\xi_2\}$, its generators are given by the columns of $\tilde{A}^*_{124} = \begin{bmatrix} \mathbb{1} & 1 & 4 \\ 5^{-1} & \mathbb{1} & \mathbb{1} \\ 5^{-1} & 4^{-1} & \mathbb{1} \end{bmatrix}$.

We have : $\Gamma_{A_{124}} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & 1 & a \\ 5 & 4 & a \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5^{-1} \\ \mathbb{1} & 1^{-1} & 4^{-1} \\ \mathbb{1} & a^{-1} & a^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 4^{-1} \\ a & \mathbb{1} & \mathbb{1} \\ a & 5 & \mathbb{1} \end{bmatrix}$, with the torsion coefficients given by $4^{-1}a, 5, a$, respectively, and

$$\Gamma_{A^*_{124}} = \begin{bmatrix} \mathbb{1} & 5^{-1} & 5^{-1} \\ 1 & \mathbb{1} & 4^{-1} \\ 4 & \mathbb{1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 1^{-1} & 4^{-1} \\ 5 & \mathbb{1} & \mathbb{1} \\ 5 & 4 & \mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & 1^{-1} & 4^{-1} \\ 5 & \mathbb{1} & \mathbb{1} \\ 5 & 4 & \mathbb{1} \end{bmatrix},$$

with the $\tau(i,j) = 1, 4, 4$, respectively.

Thus $\mathrm{Im}A_{124}$ is **not isomorphic** to $\mathrm{INJ}_{A_{124}}$.

On the other hand it is easy to see that:

$$A_{124}\,\tilde{A}^*_{124} = \mathrm{diag}(5\ \mathbb{1}\ a)P\tilde{A}^*_{124}, \text{i.e.}$$

M_{124} is equal to its image under A_{124}.

This example also illustrates the fact that the domain of a tropical map $\varphi\colon M \ \to\ N$ splits into two parts:

- INJ_φ, every point of which is an equivalence class of "$\sim$".
- $M \setminus \mathrm{INJ}_\varphi$ where the equivalence classes contain more than one point of M.

Moreover:, as easily seen from the torsion coefficients between generators, the $Mijk$ are neither isomorphic to $\mathrm{Im}A$, nor isomorphic to oneanother in general.

Our next example, which first appeared in [4] has been shortly examined in [5] . it is revisited here for an illustration of the case $m > n$ in Theorem 1.

Example 2

Let $x_i = \begin{bmatrix} \mathbb{1} \\ i \\ i^2 \end{bmatrix}$, $i = 1, 2, \ldots, m$, with $i = i^2 = \mathbb{1}$ for $i = 0$, and $A = [x_1|x_2|\cdots|x_m|]$. The tropical submodule M_A of $\mathbb{R}^3$ can be made infinite dimen sional by letting $m \to \infty$.

It is not difficult to see that A is injective on $\bigcup\limits_{0\le i<j<k} M_{ijk}$, where

$$M_{ijk}=\{\xi| \bigvee_{\ell\ge 1,\ell\ne i}\xi_\ell\le\xi_i\,,\ \bigvee_{\ell\ge 1,\ell\ne j}\ell\xi_\ell\le j\xi_j\,,\ \bigvee_{\ell\ge 1,\ell\ne k}\ell^2\xi_\ell\le k^2\xi_k\}$$

For instance, with $m=4$, we have:

$$\begin{aligned}
M_{124}&=\{\xi\in\mathbb{R}^4|\,\xi_i\le\xi_1\,,\,i=2,3,4\,,\ \xi_1\vee 2\xi_3\vee 3\xi_4\le 1\xi_2\,,\ \xi_1\vee 2\xi_2\vee 4\xi_3\le 6\xi_4\}\\
M_{134}&=\{\xi\in\mathbb{R}^4|\ \xi_i\le\xi_1\,,\,i=2,3,4\,,\ \xi_1\vee 1\xi_2\vee 3\xi_4\le\ 2\xi_3\ \ \xi_1\vee 2\xi_2\vee 4\xi_3\le 6\xi_4\}\\
M_{234}&=\{\xi\in\mathbb{R}^4|\ \xi_1\vee\xi_3\vee\xi_4\le\xi_2\,,\ \xi_1\vee 1\xi_2\vee 3\xi_4\le\ 2\xi_3\,,\ \xi_1\vee 2\xi_2\vee 4\xi_3\le 6\xi_4\}.
\end{aligned}$$

The method described in Theorem 1.is illustrated as follows for the generators of the M_{ijk}, where the i (resp. j , k] stands for the rank of the column which dominates row 1 (resp. $2,3$) of $A\xi$.

For M_{123}, define $\underline{A}_{123}=\begin{bmatrix}\mathbb{1}&\mathbb{1}&\mathbb{1}&\mathbb{1}\\ \mathbb{1}&1&2&3\\ \mathbb{1}&1^2&2^2&3^2\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbb{1}\end{bmatrix}$.

Then from $\underline{A}_{123}\xi=\begin{bmatrix}\xi_1\\ 1\xi_2\\ 2^2\xi_3\\ \xi_4\end{bmatrix}$, we get $\tilde{\underline{A}}_{123}=\begin{bmatrix}\mathbb{1}&\mathbb{1}&\mathbb{1}&\mathbb{1}\\ 1^{-1}&\mathbb{1}&1&2\\ 2^{-2}&2^{-1}&\mathbb{1}&2\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbb{1}\end{bmatrix}$, and

$$\tilde{\underline{A}}^3_{123}=\begin{bmatrix}\mathbb{1}&\mathbb{1}&1&3\\ 1^{-1}&\mathbb{1}&1&3\\ 3^{-1}&2^{-1}&\mathbb{1}&2\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbb{1}\end{bmatrix}=\tilde{\underline{A}}^*_{123}$$

Clearly: $\begin{bmatrix}\mathbb{1}\\ 1^{-1}\\ 3^{-1}\\ \mathbf{0}\end{bmatrix}$, $\begin{bmatrix}\mathbb{1}\\ \mathbb{1}\\ 2^{-1}\\ \mathbf{0}\end{bmatrix}$, $\begin{bmatrix}1\\ 1\\ \mathbb{1}\\ \mathbf{0}\end{bmatrix}$, and $\begin{bmatrix}3\\ 3\\ 2\\ \mathbb{1}\end{bmatrix}$ generate $\mathrm{INJ}_{A_{123}}$.

For a straightforward verification, let

$$u=x_1\begin{bmatrix}\mathbb{1}\\ 1^{-1}\\ 3^{-1}\\ \mathbf{0}\end{bmatrix}\vee x_2\begin{bmatrix}\mathbb{1}\\ \mathbb{1}\\ 2^{-1}\\ \mathbf{0}\end{bmatrix}\vee x_3\begin{bmatrix}1\\ 1\\ \mathbb{1}\\ \mathbf{0}\end{bmatrix}\vee x_4\begin{bmatrix}3\\ 3\\ 2\\ \mathbb{1}\end{bmatrix}=\begin{bmatrix}x_1\vee x_2\vee 1x_3\vee 3x_4\\ 1^{-1}x_1\vee x_2\vee 1x_3\vee 3x_4\\ 3^{-1}x_1\vee 2^{-1}x_2\vee x_3\vee 2x_4\\ x_4\end{bmatrix}.$$

We leave it to the reader to check that

$$\begin{bmatrix}\mathbb{1}&\mathbb{1}&\mathbb{1}&\mathbb{1}\\ \mathbb{1}&1&2&3\\ \mathbb{1}&1^2&2^2&3^2\end{bmatrix}u=\begin{bmatrix}u_1\\ 1u_2\\ 2^2u_3\end{bmatrix},\ \text{i.e. } u\in M_{123}.$$

Example 3

This last example shows that we can ffind $n-1$ torsion elements in $\underline{\mathbb{R}}^n$ exhibiting two by two the same torsion.

Let $A=\begin{bmatrix}\mathbb{1}&\mathbb{1}&\mathbb{1}&\mathbb{1}&\mathbb{1}&\mathbb{1}\\ \mathbb{1}&\mathbb{1}&\mathbb{1}&\mathbb{1}&\mathbb{1}&\tau\\ \mathbb{1}&\mathbb{1}&\mathbb{1}&\mathbb{1}&\tau&\tau\\ \mathbb{1}&\mathbb{1}&\mathbb{1}&\tau&\tau&\tau\\ \mathbb{1}&\mathbb{1}&\tau&\tau&\tau&\tau\\ \mathbb{1}&\tau&\tau&\tau&\tau&\tau\end{bmatrix}$. Clearly the torsion coefficients of any two columns of A are equal to τ.

For $n = 2$, the injectivity module of A has been investigated in [5]. The general case is illustrated by the case $n = 6$. Let P be the permutation matrix $\begin{bmatrix} \mathbb{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Then $\tilde{A} = \begin{bmatrix} \mathbb{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tau^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tau^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tau^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tau^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tau^{-1} \end{bmatrix} PA = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \tau^{-1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \tau^{-1} & \tau^{-1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \tau^{-1} & \tau^{-1} & \tau^{-1} & \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \tau^{-1} & \tau^{-1} & \tau^{-1} & \tau^{-1} & \mathbb{1} & \mathbb{1} \\ \tau^{-1} & \tau^{-1} & \tau^{-1} & \tau^{-1} & \tau^{-1} & \mathbb{1} \end{bmatrix} \sim A.$

Since $\tilde{A} \geq I$, and $\tilde{A}^2 = \tilde{A}$, then $\tilde{A}^* = \tilde{A}$, and its columns generate INJ_A.

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