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Abstract

We introduce a signless Laplacian for the distance matrix of a connected graph, called the *distance signless Laplacian*. We study the *distance signless Laplacian spectrum* of a connected graph. We show the equivalence between the distance signless Laplacian, distance Laplacian and the distance spectra for the class of *transmission regular* graphs. We also establish a relationship between the smallest eigenvalue of the distance signless Laplacian of a connected graph G and the existence of a bipartite component in the complement $\overline{G}$.

Key Words: Distance matrix, eigenvalues, Laplacian, signless Laplacian, spectral radius.

Résumé

On introduit un laplacien sans signe pour la matrice des distances d'un graphe connexe, appelé *laplacien sans signe des distances* et on étudie son spectre. On montre l'équivalence entre le spectre du laplacien sans signe des distances et le spectre de la matrice des distances pour la classe des graphes transmission-réguliers. On établit également une relation entre la plus petite valeur propre du laplacien sans signe des distances et l'existence d'une composante bipartie dans le complémentaire $\overline{G}$.

Mots clés : Matrice des distances, valeurs propres, laplacien, laplacien sans signe, rayon spectral.

1 Introduction

We begin by recalling some definitions. In this paper, we consider only simple and finite graphs, *i.e*, graphs on a finite number of vertices without multiple edges or loops. A graph is (usually) denoted by $G = G(V, E)$, where V is its vertex set and E its edge set. The *order* of G is the number $n = |V|$ of its vertices and its *size* is the number $m = |E|$ of its edges. The *adjacency matrix* A of G is a 0–1 $n \times n$–matrix indexed by the vertices of G and defined by $a_{ij} = 1$ if and only if $ij \in E$. Denote by $(\lambda_1, \lambda_2, \dots, \lambda_n)$ the A–spectrum of G, *i.e.*, the spectrum of the adjacency matrix of G, and assume that the eigenvalues are labeled such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The matrix $L = Diag(Deg) - A$, where $Diag(Deg)$ is the diagonal matrix whose diagonal entries are the degrees in G, is called *the Laplacian* of G. Denote by $(\mu_1, \mu_2, \dots, \mu_n)$ the L–spectrum of G, *i.e.*, the spectrum of the Laplacian of G, and assume that the eigenvalues are labeled such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The matrix $Q = Diag(Deg) + A$ is called *the signless Laplacian* of G. Denote by $(q_1, q_2, \dots, q_n)$ the Q–spectrum of G, *i.e.*, the spectrum of signless Laplacian of G, and assume that the eigenvalues are labeled such that $q_1 \geq q_2 \geq \cdots \geq q_n$.

Given two vertices u and v in a connected graph G, $d(u, v) = d_G(u, v)$ denotes the *distance* (the length of a shortest path) between u and v. The *Wiener index* $W(G)$ of a connected graph G is defined to be the sum of all distances in G, *i.e.*,

$$W(G) = \frac{1}{2} \sum_{u,v \in V} d(u, v).$$

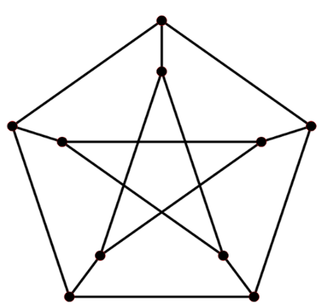
The *transmission* $Tr(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, *i.e.*,

$$Tr(v) = \sum_{u \in V} d(u, v).$$

A connected graph $G = (V, E)$ is said to be *k–transmission regular* if $Tr(v) = k$ for every vertex $v \in V$.

As usual, we denote by P_n the path, by C_n the cycle, by S_n the star, by $K_{a,n-a}$ the complete bipartite graph and by K_n the complete graph, each on n vertices. A *kite* $Ki_{n,\omega}$ is the graph obtained from a clique K_ω and a path $P_{n-\omega}$ by adding an edge between and endpoint of the path and a vertex from the clique.

The *distance matrix* $\mathcal{D}$ of a connected graph G is the matrix indexed by the vertices of G where $\mathcal{D}_{i,j} = d(v_i, v_j)$ and $d(v_i, v_j)$ denotes the distance between the vertices v_i and v_j. Let $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ denote the spectrum of $\mathcal{D}$. It is called the distance spectrum of the graph G.
Similarly to the (adjacency) Laplacian, we defined in [3] the *distance Laplacian* of a connected graph G as the matrix $\mathcal{D}^L = Diag(Tr) - \mathcal{D}$, where $Diag(Tr)$ denotes the diagonal matrix of the vertex transmissions in G. Let $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L$ denote the spectrum of $\mathcal{D}^L$. We call it the *distance Laplacian spectrum* of the graph G. Along this line, we define the *distance signless Laplacian* of a connected graph G to be $\mathcal{D}^Q = Diag(Tr) + \mathcal{D}$. Let $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_n^Q$ denote the spectrum of $\mathcal{D}^Q$. We call it the *distance signless Laplacian spectrum* of the graph G. In Figure 1, we give a graph with its different spectra.



A–spectrum	$3^{(1)}$	$1^{(5)}$	$-2^{(4)}$
L–spectrum	$5^{(4)}$	$2^{(5)}$	$0^{(1)}$
Q–spectrum	$2^{(1)}$	$4^{(5)}$	$1^{(4)}$
$\mathcal{D}$–spectrum	$15^{(1)}$	$0^{(4)}$	$-3^{(5)}$
$\mathcal{D}^L$–spectrum	$18^{(5)}$	$15^{(4)}$	$0^{(1)}$
$\mathcal{D}^Q$–spectrum	$30^{(1)}$	$15^{(4)}$	$12^{(5)}$

Figure 1: The Petersen graph and its different spectra.

For a connected graph G, let $P^G_{\mathcal{D}}(t)$, $P^G_{\mathcal{L}}(t)$ and $P^G_{\mathcal{Q}}(t)$ denote the distance, the distance Laplacian and the distance signless Laplacian characteristic polynomials respectively. With the help of the software Maple and using different properties of determinants and eigenvalues, we established the characteristic polynomials $P^G_{\mathcal{D}}(t)$, $P^G_{\mathcal{L}}(t)$ and $P^G_{\mathcal{Q}}(t)$ for some particular graphs. We next list them.

The complete graph

The distance, the distance Laplacian and the distance signless Laplacian spectra of the complete graph K_n are respectively its adjacency, Laplacian and signless Laplacian spectra, *i.e.*,

$$\begin{aligned}
P^{K_n}_{\mathcal{D}}(t) &= (t-n+1)(t+1)^{n-1};\\
P^{K_n}_{\mathcal{L}}(t) &= t(t-n)^{n-1};\\
P^{K_n}_{\mathcal{Q}}(t) &= (t-2n+2)(t-n+2)^{n-1}.
\end{aligned}$$

The complement of an edge

The distance, the distance Laplacian and the distance signless Laplacian spectra of the complement of an edge $K_n - e$ are respectively

$$\begin{aligned}
P^{K_n-e}_{\mathcal{D}}(t) &= \left(t-\frac{n-1+\sqrt{(n-1)^2+8}}{2}\right)\left(t-\frac{n-1-\sqrt{(n-1)^2+8}}{2}\right)(t+2)(t+1)^{n-3};\\
P^{K_n-e}_{\mathcal{L}}(t) &= t(t-n-2)(t-n)^{n-2};\\
P^{K_n-e}_{\mathcal{Q}}(t) &= \left(t-\frac{3n-2+\sqrt{n^2-4n+20}}{2}\right)\left(t-\frac{3n-2-\sqrt{n^2-4n+20}}{2}\right)(t-n+2)^{n-2}.
\end{aligned}$$

The star

The distance, distance Laplacian and the distance signless Laplacian characteristic polynomials of the star S_n are respectively

$$\begin{aligned}
P^{S_n}_{\mathcal{D}}(t) &= \left(t-n+2-\sqrt{n^2-3n+3}\right)\left(t-n+2+\sqrt{n^2-3n+3}\right)(t+2)^{n-2};\\
P^{S_n}_{\mathcal{L}}(t) &= t(t-n)(t-2n+1)^{n-2};\\
P^{S_n}_{\mathcal{Q}}(t) &= \left(t-\frac{5n-8+\sqrt{9n^2-32n+32}}{2}\right)\left(t-\frac{5n-8-\sqrt{9n^2-32n+32}}{2}\right)(t-2n+5)^{n-2}.
\end{aligned}$$

The complete bipartite graph

The distance, distance Laplacian and the distance signless Laplacian characteristic polynomials of the complete bipartite graph $K_{a,b}$ are respectively

$$\begin{aligned}
P^{K_{a,b}}_{\mathcal{D}}(t) &= \left(t-n+2-\sqrt{a^2-ab+b^2}\right)\left(t-n+2+\sqrt{a^2-ab+b^2}\right)(t+2)^{n-2};\\
P^{K_{a,b}}_{\mathcal{L}}(t) &= t(t-n)(t-(2n-a))^{b-1}(t-(2n-b))^{a-1};\\
P^{K_{a,b}}_{\mathcal{Q}}(t) &= \left(t-\frac{5n-8+\sqrt{9(a-b)^2+4ab}}{2}\right)\left(t-\frac{5n-8-\sqrt{9(a-b)^2+4ab}}{2}\right)\\
&\qquad (t-2n+b+4)^{a-1}\,(t-2n+a+4)^{b-1}.
\end{aligned}$$

The graph S_n^+

The distance, distance Laplacian and the distance signless Laplacian characteristic polynomials of the graph S_n^+ are respectively

$$\begin{aligned}
P_{\mathcal{D}}^{S_n^+}(t) &= \left(t^3-(2n-7)t^2-(7n-17)t-(3n-5)\right)(t+1)(t+2)^{n-4};\\
P_{\mathcal{L}}^{S_n^+}(t) &= t(t-n)(t-2n+3)(t-2n+1)^{n-3};\\
P_{\mathcal{Q}}^{S_n^+}(t) &= \left(t^3-(7n-15)t^2+\left(14n^2-36n+72\right)t-\left(8n^3-52n^2+108n-68\right)\right)(t-2n+5)^{n-3}.
\end{aligned}$$

The rest of the paper is organized as follows. In Section 2, we discuss general properties of the distance signless Laplacian spectrum. We first prove equivalence between the distance Laplacian spectrum and the distance spectrum among the class of transmission regular graphs. Thereafter, we show that the interlacing theorem does not apply for the distance Laplacian spectrum. We prove that the distance signless Laplacian eigenvalues do not decrease by the deletion of an edge. In Section 3, we prove a series of bounds on the eigenvalues of $\mathcal{D}^{\mathcal{Q}}$, specially the largest and the smallest of them. We also establish a relationship between the smallest eigenvalue of $\mathcal{D}^{\mathcal{Q}}$ of a connected graph G and the existence of a bipartite component in the complement $\overline{G}$. Finally, we list some open conjectures in Section 4.

2 General properties

In [7, 8, 9], Cvetković and Simić studied the spectral graph theory based on the signless Laplacian matrix. Among other results, they showed equivalence between the spectrum of the signless Laplacian and

- the adjacency spectrum for the class of (degree) regular graphs;
- the Laplacian spectrum for the class of (degree) regular graphs;
- the Laplacian spectrum for the class of bipartite graphs.

In [3], we showed equivalence between the distance Laplacian spectrum and

- the distance spectrum among the class of transmission regular graphs;
- the Laplacian spectrum among the class of graphs with diameter two.

Along these lines, we studied similarities between the distance signless Laplacian spectrum on the one hand and the spectra of different matrices associated to connected graphs on the other hand. The first result is that there is equivalence between the spectrum of the distance matrix $\mathcal{D}$ and that of the distance signless Laplacian $\mathcal{D}^{\mathcal{Q}}$ over the set of transmission regular graphs.

Theorem 2.1 *If G is a k–transmission regular graph on n vertices with distance spectrum $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ and distance signless Laplacian spectrum $\partial_1^{\mathcal{Q}} \geq \partial_2^{\mathcal{Q}} \geq \cdots \geq \partial_n^{\mathcal{Q}}$, then $\partial_i^{\mathcal{Q}} = k + \partial_i$ for all $i = 1, \ldots, n$.*

Proof. The relationship between the characteristic polynomials is as follows.

$$P_{\mathcal{Q}}(t) = det(\mathcal{D}^{\mathcal{Q}} - tI) = det(Diag(Tr) + \mathcal{D} - tI) = det(kI + \mathcal{D} - tI) = det(\mathcal{D} - (t-k)I) = P_{\mathcal{D}}(t-k).$$

Thus ∂ is an eigenvalue of $\mathcal{D}$ if and only if $\partial^{\mathcal{Q}} = k + \partial$ is an eigenvalue of $\mathcal{D}^{\mathcal{Q}}$. □

Using the above theorem, one can calculate the distance signless Laplacian characteristic polynomial of a transmission regular graph from its distance characteristic polynomial. For instance, we can do so for the cycle on n vertices.

Corollary 2.2 *The distance signless Laplacian characteristic polynomial of the cycle C_n is as follows.*

If $n = 2k$ (i.e., even)

$$P_{\mathcal{Q}}^{C_n}(t) = \left(t - \frac{n^2}{4}\right)^{k-1} \cdot \left(t - \frac{n^2}{2}\right) \cdot \prod_{j=1}^{k} \left(t - \frac{n^2}{4} + \csc^2\left(\frac{\pi(2j-1)}{n}\right)\right).$$

If $n = 2k+1$ (i.e., odd)

$$P_{\mathcal{Q}}^{C_n}(t) = \left(t - \frac{n^2-1}{2}\right) \cdot \prod_{j=1}^{k} \left(t - \frac{n^2-1}{4} + \frac{1}{4}\sec^2\left(\frac{\pi j}{n}\right)\right) \cdot \prod_{j=1}^{k} \left(t - \frac{n^2-1}{4} + \frac{1}{4}\csc^2\left(\frac{\pi(2j-1)}{2n}\right)\right).$$

Proof. The distance characteristic polynomial of C_n was given in [11], according to the parity of n, as follows.

If $n = 2p$ (*i.e., even*)

$$P_{\mathcal{D}}^{C_n}(t) = t^{p-1} \cdot \left(t - \frac{n^2}{4}\right) \cdot \prod_{j=1}^{p} \left(t + \csc^2\left(\frac{\pi(2j-1)}{n}\right)\right).$$

If $n = 2p+1$ (*i.e., odd*)

$$P_{\mathcal{D}}^{C_n}(t) = \left(t - \frac{n^2-1}{4}\right) \cdot \prod_{j=1}^{p} \left(t + \frac{1}{4}\sec^2\left(\frac{\pi j}{n}\right)\right) \cdot \prod_{j=1}^{p} \left(t + \frac{1}{4}\csc^2\left(\frac{\pi(2j-1)}{2n}\right)\right).$$

The cycle C_n is a k–transmission regular graph with $k = n^2/4$ if n is even and $k = (n^2-1)/4$ if n is odd. Applying Theorem 2.1, we get the result. □

The famous interlacing theorem (see e.g. [5, p. 9]) does not apply in the case of the distance signless Laplacian spectrum of a graph. Indeed, consider the path P_n obtained from the cycle C_n by the deletion of an edge. The distance signless Laplacian spectra of P_n and C_n do not interlace for $n \geq 5$. For instance the distance signless Laplacian spectrum of P_6 is approximately $(25.0838, 12.1755, 11.1743, 8.6727, 7.7418, 5.5118)$ while the distance signless Laplacian spectrum of C_6 is $(18, 9, 9, 8, 5, 5)$. The corresponding property for the distance signless Laplacian spectrum is that each eigenvalue $\partial_i^{\mathcal{Q}}$ does not decrease if an edge is deleted from the graph. To prove this fact, we need the following lemma.

Lemma 2.3 (Courant–Weyl inequalities, [5]) *For a real symmetric matrix M of order n, let $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$ denote its eigenvalues. If N_1 and N_2 are two real symmetric matrices of order n and if $N = N_1 + N_2$, then for every $i = 1, \ldots, n$, we have*

$$\lambda_i(N_1) + \lambda_1(N_2) \geq \lambda_i(N) \geq \lambda_i(N_1) + \lambda_n(N_2).$$

Theorem 2.4 *Let G be a connected graph on n vertices and $m \geq n$ edges. Consider G' the connected graph obtained from G by the deletion of an edge. Denote $\left(\partial_1^{\mathcal{Q}}, \partial_2^{\mathcal{Q}}, \ldots \partial_n^{\mathcal{Q}}\right)$ and $\left(\tilde{\partial}_1^{\mathcal{Q}}, \tilde{\partial}_2^{\mathcal{Q}}, \ldots \tilde{\partial}_n^{\mathcal{Q}}\right)$ the distance signless Laplacian spectra of G and G' respectively. Then $\tilde{\partial}_i^{\mathcal{Q}} \geq \partial_i^{\mathcal{Q}}$ for all $i = 1, \ldots n$.*

Proof. We write the distance signless Laplacian matrix of G' as $\mathcal{D}'^{\mathcal{Q}} = \mathcal{D}^{\mathcal{Q}} + M$, where M expresses the changes in $\mathcal{D}^{\mathcal{Q}}$ due to the deletion of an edge from G. It is easy to see that M is diagonally dominant with positive (diagonal) entries. Thus M is a positive semi-definite matrix. □

Some regularities in graphs are useful in calculating certain eigenvalues of the matrices related to these graphs. It is the case, for instance, for the largest eigenvalue of the adjacency matrix or the signless Laplacian whenever the graph is degree regular. The same is true for the largest eigenvalue of the distance Laplacian, and of the distance signless Laplacian, whenever the graph is transmission regular. Sometimes, a local regularity in a graph suffices to determine some eigenvalue. We prove below that it is possible to know a distance signless Laplacian eigenvalue of a graph if it contains a clique or an independent set whose vertices share the same transmission.

Theorem 2.5 *Let G be a connected graph on n vertices. If $S = \{v_1, v_2, \ldots v_p\}$ is an independent set of G such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\tau = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\tau - 2$ is an eigenvalue of $\mathcal{D}^{\mathcal{Q}}$ with multiplicity at least $p - 1$.*

Proof. Since the vertices in S share the same neighborhood, any vertex in $V - S$ is at the same distance from all vertices in S. Each vertex of independent set S is at distance 2 from any other vertex in S. Thus all vertices in S have the same transmission, say τ.
To show that $\tau - 2$ is a distance Laplacian eigenvalue with multiplicity $p - 1$, it suffices to observe that the matrix $(\tau - 2)I_n - \mathcal{D}^{\mathcal{Q}}$ contains p identical rows (columns). □

Theorem 2.6 *Let G be a connected graph on n vertices. If $K = \{v_1, v_2, \ldots v_p\}$ is a clique of G such that $N(v_i) - K = N(v_j) - K$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\tau = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\tau - 1$ is an eigenvalue of $\mathcal{D}^{\mathcal{Q}}$ with multiplicity at least $p - 1$.*

The proof of this theorem is similar to that of the previous one.

Note that results similar to Theorem 2.5 and Theorem 2.6 are proved in [3] for the distance Laplacian spectrum.

3 Bounds on the eigenvalues

In this section, we prove some bounds on the eigenvalues of the distance signless Laplacian of a connected graph. We first give bounds proved using the property of the spectra domination resulting from the deletion of an edge stated in Theorem 2.4.

Proposition 3.1 *If G is a connected graph on $n \geq 3$ vertices, then $\partial_i^{\mathcal{Q}}(G) \geq \partial_i^{\mathcal{Q}}(K_n) = n - 2$, for all $2 \leq i \leq n$. Moreover, $\partial_2^{\mathcal{Q}}(G) = \partial_2^{\mathcal{Q}}(K_n) = n - 2$ if and only if G is the complete graph K_n.*

Proof. The inequalities $\partial_i^{\mathcal{Q}}(G) \geq \partial_i^{\mathcal{Q}}(K_n)$, for all $2 \leq i \leq n$ follow from Theorem 2.4. To see that $\partial_2^{\mathcal{Q}}(G) = \partial_2^{\mathcal{Q}}(K_n) = n - 2$ if and only if G is the complete graph K_n, it suffices to observe that if $G \ncong K_n$, then $\partial_2^{\mathcal{Q}}(G) \geq \partial_2^{\mathcal{Q}}(K_n - e) > n - 2$. □

The next proposition gives a sharp upper bound on the index of $\mathcal{D}^{\mathcal{Q}}$ in terms of the Wiener index and the order of the graph.

Proposition 3.2 *Let G be a connected graph on $n \geq 2$ vertices with Wiener index W, then $\partial_1^{\mathcal{Q}}(G) \leq 2W - (n - 1)(n - 2)$ with equality if and only if G is the complete graph K_n.*

Proof. From spectral theory, we have

$$\partial_1^{\mathcal{Q}}(G) + \partial_2^{\mathcal{Q}}(G) + \cdots + \partial_n^{\mathcal{Q}}(G) = Tr_1 + Tr_2 + \cdots + Tr_n = 2W.$$

Then

$$\partial_1^{\mathcal{Q}}(G) = 2W - \partial_2^{\mathcal{Q}}(G) - \cdots - \partial_n^{\mathcal{Q}}(G).$$

We conclude using Proposition 3.1. □

Note that the gape between $\partial_1^{\mathcal{Q}}(G)$ and $2W - (n - 1)(n - 2)$ may be arbitrarily large when the graph is note dense. To illustrate, the gape for an even cycle on n vertices is exactly $n^2(n - 2)/4$.

To prove the next theorem, we need the following well-known result from matrix theory.

Lemma 3.3 (Gershgorin Theorem, [13]) *Let $M = (m_{ij})$ be a complex $n \times n$–matrix and denote by $\lambda_1, \lambda_2, \ldots \lambda_p$ its distinct eigenvalues. Then*

$$\{\lambda_1, \lambda_2, \ldots \lambda_p\} \subset \bigcup_{i=1}^{n} \left\{ z : |z - m_{ii}| \leq \sum_{j \neq i} |m_{ij}| \right\}.$$

We now give sharp bounds on ∂_1^Q in terms of minimum, average and maximum transmissions.

Theorem 3.4 *Let G be a connected graph with minimum, average and maximum transmissions Tr_{min}, $\overline{Tr}$ and Tr_{max} respectively. Then*

$$2Tr_{min} \leq 2\overline{Tr} \leq \partial_1^Q(G) \leq 2Tr_{max}$$

with equalities if and only if G is a transmission regular graph.

Proof. Using the Rayleigh's quotient, we have

$$\partial_1^Q(G) = \max_{X \neq 0} R(X) = \max_{X \neq 0} \frac{X^t \mathcal{D}^Q X}{X^t X}.$$

If we take $X = \mathbb{1}$, the all 1's vector, we get $R(\mathbb{1}) = 2\overline{Tr}$ and then $\partial_1^Q(G) \geq 2\overline{Tr} \geq 2Tr_{min}$.
The upper bound follows immediately from Lemma 3.3.

It is easy to see that equalities hold if and only if $Tr_{min} = \overline{Tr} = Tr_{max}$ and $\mathbb{1}$ is an eigenvector belonging to the largest eigenvalue $\partial_1^Q(G)$. □

Combining the above theorem and Proposition 3.1, we easily get the following corollary.

Corollary 3.5 *If G is a connected graph on $n \geq 2$ vertices, then $\partial_1^Q(G) \geq \partial_1^Q(K_n) = 2n - 2$ with equality if and only if G is the complete graph K_n.*

Proposition 3.6 *Let $G = (V, E)$ be a connected graph on $n \geq 2$ vertices and k an integer such that $1 \leq k \leq n$. Denote by $\mathcal{P}_k(V)$ the family of subsets of V with cardinality k. Then*

$$\partial_1(G) \geq \max_{S \in \mathcal{P}_k(V)} \left\{ \frac{1}{k} \sum_{u \in S} Tr(u) + \frac{1}{k} \sum_{u,v \in S} d(u,v) \right\}$$

$$\text{and} \quad \partial_n(G) \leq \min_{S \in \mathcal{P}_k(V)} \left\{ \frac{1}{k} \sum_{u \in S} Tr(u) + \frac{1}{k} \sum_{u,v \in S} d(u,v) \right\}.$$

Proof. Using Rayleigh's quotient, we have

$$\partial_1^Q(G) = \max_{X \neq 0} R(X) = \max_{X \neq 0} \frac{X^t \mathcal{D}^Q X}{X^t X} \qquad \text{and} \qquad \partial_n^Q(G) = \min_{X \neq 0} R(X) = \max_{X \neq 0} \frac{X^t \mathcal{D}^Q X}{X^t X}.$$

Thus, to be done, it suffices to take $X = [x_1, x_2, \ldots x_n]^t$ with $x_i = 1$ if $u_i \in S$ and 0 otherwise. □

We next establish some interconnections, as inequalities, between the distance signless Laplacian spectrum of a connected graph G and the signless Laplacian spectrum of its complement $\overline{G}$.

Theorem 3.7 *Let G be a connected graph on $n \geq 3$ vertices with diameter D. Let $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_n^Q$ and $\overline{q}_1 \geq \overline{q}_2 \geq \cdots \geq \overline{q}_n$ be the distance signless Laplacian of G and the signless Laplacian of the complement $\overline{G}$ of G.*

(1) If $D = 2$, then

$$n-2+\overline{q}_i \le \partial_i^{\mathcal{Q}} \le 2n-2+\overline{q}_i \qquad \textit{for every } 1 \le i \le n; \tag{1}$$
$$n-2+\overline{q}_n \le \partial_i^{\mathcal{Q}} \le n-2+\overline{q}_1 \qquad \textit{for every } 1 \le i \le n-1; \tag{2}$$
$$2n-2+\overline{q}_n \le \partial_1^{\mathcal{Q}} \le 2n-2+\overline{q}_1. \tag{3}$$

(2) If $D \ge 3$, then

$$\partial_i^{\mathcal{Q}} \ge n-2+\overline{q}_i \qquad \textit{for every } 1 \le i \le n. \tag{4}$$

Proof.

(1) In a connected graph with diameter 2, we have $Tr(v) = d(v) + 2(n - d(v) - 1) = 2n - 2 - d(v)$, and therefore $Diag(Tr) = (2n-2)I - Diag(Deg)$. On another side, the distance between two vertices is 1 if they are neighbors and 2 otherwise. Thus the distance matrix can be written as $\mathcal{D} = A + 2\overline{A}$, where A and $\overline{A}$ denote the adjacency matrices of G and its complement $\overline{G}$ respectively. Now, if we denote by $\overline{Q}$ and $\overline{Deg}$ the signless Laplacian matrix and the degree vector of $\overline{G}$, the distance signless Laplacian of G can be written as

$$\begin{aligned}
\mathcal{D}^{\mathcal{Q}} &= \mathcal{D} + Diag(Tr)\\
&= A + 2\overline{A} + (2n-2)I - Diag(Deg)\\
&= A + \overline{A} + (n-1)I + \big((n-1)I - Diag(Deg) + \overline{A}\big)\\
&= J + (n-2)I + Diag\left(\overline{Deg}\right) + \overline{A}\\
&= J + (n-2)I + \overline{Q},
\end{aligned}$$

where J is the all ones $n \times n$ matrix, whose eigenvalues are 0 with multiplicity $n-1$ and n with multiplicity 1.
Applying Lemma 2.3 with $N_1 = (n-2)I + \overline{Q}$ and $N_2 = J$, we get (1), and with $N_1 = J$ and $N_2 = (n-2)I + \overline{Q}$, we get (2) and (3).

(2) Consider the $n \times n$ matrix $M = (m_{,j})$ defined by $m_{i,j} = \max\{0, d_{i,j} - 2\}$ for $1 \le i, j \le n$, where $\mathcal{D} = (d_{i,j})$ denotes the distance matrix of G. For a vertex i in G, we write its transmission as $Tr_i = d_i + 2\overline{d}_i + Tr'_i$, where $\overline{d}_i$ denotes the degree of i in $\overline{G}$. Using this notation, we have

$$\begin{aligned}
\mathcal{D}^{\mathcal{Q}} &= Diag(Tr) + \mathcal{D}\\
&= Diag(Deg) + Diag(\overline{Deg}) + Diag(Tr') + A + 2\overline{A} + M\\
&= \left(A + \overline{A} + Diag(Deg) + Diag\left(\overline{Deg}\right)\right) + \left(\overline{A} + Diag\left(\overline{Deg}\right)\right) + (Diag(Tr') + M)\\
&= Q(K_n) + \overline{Q} + M',
\end{aligned}$$

where $M' = Diag(Tr') + M$. It is easy to see that M' a diagonally dominant matrix, and then, its least eigenvalue is not negative. Now, applying twice Lemma 2.3 (with $N_1 = Q(K_n)$ and $N_2 = \overline{Q} + M'$ and then with $N_1 = \overline{Q}$ and $N_2 = M'$), we get $\partial_i^{\mathcal{Q}} \ge n - 2 + \overline{q}_i$, for $1 \le i \le n$. □

As a corollary of the above theorem, we establish a relationship between the fact that $n-2$ is a distance signless Laplacian eigenvalue of a connected graph G and the existence of a bipartite component or an isolated vertex in the complement $\overline{G}$.

Corollary 3.8 *Let G be a connected graph on n vertices. If $\partial^{\mathcal{Q}} = n-2$ is a distance signless Laplacian eigenvalue with multiplicity μ, then the complement $\overline{G}$ of G contains at least μ components, each of which is bipartite or an isolated vertex.*

Proof. From (1) of Theorem 3.7, if $n-2$ is a distance signless Laplacian eigenvalue, then 0 is a signless Laplacian eigenvalue at least as many times as $n-2$ for $\mathcal{D}^{\mathcal{Q}}$. To complete the proof, we use the fact (see [6, 10]) that 0 is a Q–eigenvalue of a graph G if and only if G contains a bipartite component or an isolated vertex, and in this case, the multiplicity of 0 is at most equal to the number of bipartite components plus the number of isolated vertices. □

Note that there exist graphs with bipartite complements with $\partial_n > n-2$. For instance, if G is the complement of the path on 7 vertices, *i.e.* $G = \overline{P}_7$, we have $\partial_7^Q(G) \simeq 5.042816 > 5$ while $\overline{G} = P_7$ is bipartite. Another example is illustrated on Figure 2.

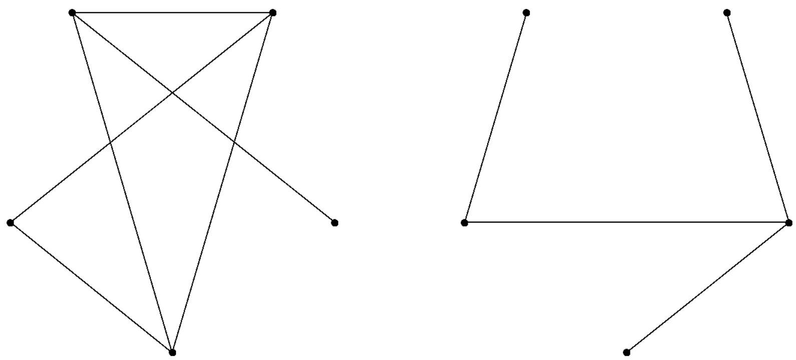


Figure 2: A graph G (left) on 5 vertices with $\partial_5^Q \simeq 3.050286 > 3$ and a bipartite complement (right).

Corollary 3.9 *Let G be a connected graph on n vertices with diameter D. If $D \geq 4$, then $\partial_n^Q > n-2$.*

Proof. Since $D \geq 4$, $\overline{G}$ is connected and contains at least a triangle (a cycle on 3 vertices). Thus $\overline{G}$ is not bipartite, and therefore $\overline{q}_i \geq \overline{q}_n > 0$. The results follows from (2) of Theorem 3.7. □

Another consequence of Theorem 3.7 is that, for a given order $n \geq 3$, the bipartite graphs with $\partial_n^Q = n-2$ are entirely characterized.

Corollary 3.10 *Let G be a bipartite graph on $n \geq 3$ vertices, then $\partial_n^Q(G) = n-2$ if and only if G is the path P_4 or the complete bipartite graph $K_{n-2,2}$.*

Proof. If G is the star S_n with $n \geq 3$, then $\partial_n^Q(G) > n-2$ except for $S_3 = K_{1,2}$.

If $n = 4$, the only bipartite graphs are S_4, P_4 and $K_{2,2}$, for which $\partial_n^Q(S_n) > 2$ and $\partial_n^Q(P_n) = \partial_n^Q(K_{2,2}) = 2$. If $n \geq 5$ and $G \not\cong S_n$, then the bipartition of the vertex set of G defines two independent sets V_1 and V_2, each of which induces a clique in $\overline{G}$. By Theorem 3.7, G contains at least a bipartite component. To be done, it suffices to note that $\overline{G}$ contains a bipartite component if and only if G is a complete bipartite graph and $\min\{|V_1|, |V_2|\} = 2$. □

4 Some conjectures

In this section, we list a series of conjectures about some particular distance Laplacian eigenvalues of a connected graph. These conjectures, as well as some of the results proved in this paper, were obtained using the AutoGraphiX system [1, 2, 4] devoted to conjecture–making in graph theory.

First, we conjecture about an upper bound on the largest distance Laplacian eigenvalue over the class of all connected graphs with a given order n.

Conjecture 4.1 *Let G be connected graph on n vertices. Then*

$$\partial_1^Q(G) \leq \partial_1^Q(P_n)$$

with equality if and only if G is the path P_n.

Since a path is a tree, the above conjecture can be stated also for the set of trees. A general lower bound on ∂_1^Q is given in Corollary 3.5, if we assume that the graph is a tree, the bound is no more valid since K_n is not a tree for $n \geq 3$. We next conjecture a lower bound on ∂_1^Q over the set of trees.

Conjecture 4.2 *Let T be a tree on n vertices. Then*

$$\partial_1^{\mathcal{T}}(G) \geq \partial_1^{\mathcal{Q}}(S_n) = \frac{5n - 8 + \sqrt{9n^2 - 32n + 32}}{2}$$

with equality if and only if T is the star S_n.

For the class of unicyclic graphs, we conjecture a lower and an upper bound as well as a characterization of the extremal graphs for each bound.

Conjecture 4.3 *Let G be a connected unicyclic graph on $n \geq 6$ vertices. Then*

$$\partial_1^{\mathcal{Q}}(S_n^+) \leq \partial_1^{\mathcal{Q}}(G) \leq \partial_1^{\mathcal{Q}}(Ki_{n,3})$$

with equality for the lower (resp. upper) bound if and only if G is the graph S_n^+ (resp. the long kite $Ki_{n,3}$).

Before stating the next conjecture, we need to define the *Soltés graph* [14]. Let u be an isolated vertex or one endpoint of a path. Let us join u with at least one vertex of a clique. The graph so obtained is the *Soltés graph* $PK_{n,m}$, also called the *path-complete* graph, where n is its order and m its size. There is exactly one $PK_{n,m}$ for given n and m such that $1 \leq n-1 \leq m \leq n(n-1)/2$. The kite $Ki_{n,\omega}$, defined in the introduction, is a particular path-complete graph with $m = \omega(\omega-1)/2 + n - \omega$.

For given n and m such that $1 \leq n-1 \leq m \leq n(n-1)/2$, $PK_{n,m}$ maximizes (non uniquely) the diameter D [12] and (uniquely) the average distance $\bar{l}$ [14].

Conjecture 4.4 *Let n and m be integers such that $2 \leq n-1 \leq m$. The path-complete (Soltés) graph $PK_{n,m}$ maximizes $\partial_1^{\mathcal{Q}}(G)$ over all connected graphs with order n and size m.*

The next three conjectures are about the second largest distance signless Laplacian eigenvalue. First, we conjecture an upper bound on $\partial_2^{\mathcal{Q}}$, as well as a characterization of the corresponding extremal graphs, over all the connected graphs on n vertices.

Conjecture 4.5 *Let G be connected graph on n vertices. Then*

$$\partial_2^{\mathcal{Q}}(G) \leq \partial_2^{\mathcal{Q}}(P_n)$$

with equality if and only if G is the path P_n.

We proved in Proposition 3.1 that, among the class of connected graphs on n vertices, $\partial_2^{\mathcal{Q}}$ is minimum for the complete graph K_n. If we consider only the class of trees, the minimum of $\partial_2^{\mathcal{Q}}$ seems to be reached for the star S_n.

Conjecture 4.6 *Let T be a tree on $n \geq 4$ vertices. Then*

$$\partial_2^{\mathcal{Q}}(T) \geq \partial_2^{\mathcal{Q}}(S_n) = 2n - 5.$$

with equality if and only if T is the star S_n.

For the class of unicyclic graphs, we conjecture a lower and an upper bound as well as a characterization of the extremal graphs for each bound.

Conjecture 4.7 *Let G be a connected unicyclic graph on $n \geq 5$ vertices. Then*

$$2n - 5 = \partial_2^{\mathcal{Q}}(S_n^+) \leq \partial_1^{\mathcal{Q}}(G) \leq \partial_1^{\mathcal{Q}}(Ki_{n,3})$$

with equality for the lower (resp. upper) bound if and only if G is the graph S_n^+ (resp. the long kite $Ki_{n,3}$).

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