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Abstract

We introduce a Laplacian for the distance matrix of a connected graph, called the *distance Laplacian* and we study its spectrum. We show the equivalence between the distance Laplacian spectrum and the distance spectrum for the class of *transmission regular* graphs. There is also an equivalence between the Laplacian spectrum and the distance Laplacian spectrum of any connected graph of diameter 2. Similarities between n, as a distance Laplacian eigenvalue, and the algebraic connectivity are established. Finally, we investigate some particular distance Laplacian eigenvalues.

Key Words: Distance matrix, eigenvalues, Laplacian, spectral radius.

Résumé

On introduit un laplacien pour la matrice des distances d'un graphe connexe, appelé *laplacien des distances* et on étudie son spectre. On montre l'équivalence entre le spectre du laplacien des distances et le spectre de la matrice des distances pour la classe des graphes transmission-réguliers. On montre également qu'il y a une équivalence entre le spectre du laplacien et le spectre du laplacien des distances pour tout graphe connexe de diamètre 2. Des similitudes entre n, considéré comme une valeur propre du laplacien des distances, et la connectivité algébrique sont établies. Enfin, on étudie certaines valeurs propres du laplacien des distances.

Mots clés : Matrice des distances, valeurs propres, laplacien, rayon spectral.

1 Introduction

We begin by recalling some definitions. In this paper, we consider only simple, undirected and finite graphs, *i.e*, undirected graphs on a finite number of vertices without multiple edges or loops. A graph is (usually) denoted by $G = G(V, E)$, where V is its vertex set and E its edge set. The *order* of G is the number $n = |V|$ of its vertices and its *size* is the number $m = |E|$ of its edges. The *adjacency matrix* of G is a 0–1 $n \times n$–matrix indexed by the vertices of G and defined by $a_{ij} = 1$ if and only if $ij \in E$. Denote by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ the A–spectrum of G, *i.e.*, the spectrum of the adjacency matrix A of G, and assume that the eigenvalues are labeled such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The matrix $L = Diag(Deg) - A$, where $Diag(Deg)$ is the diagonal matrix whose diagonal entries are the degrees in G, is called *the Laplacian* of G. Denote by $(\mu_1, \mu_2, \ldots, \mu_n)$ the L–spectrum of G, *i.e.*, the spectrum of the Laplacian of G, and assume that the eigenvalues are labeled such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The matrix $Q = Diag(Deg) + A$ is called *the signless Laplacian* of G. Denote by $(q_1, q_2, \ldots, q_n)$ the Q–spectrum of G, *i.e.*, the spectrum of the signless Laplacian of G, and assume that the eigenvalues are labeled such that $q_1 \geq q_2 \geq \cdots \geq q_n$.

Given two vertices u and v in a connected graph G, $d(u, v) = d_G(u, v)$ denotes the *distance* (the length of a shortest path) between u and v. The *Wiener index* $W(G)$ of a connected graph G is defined to be the sum of all distances in G, *i.e.*,

$$W(G) = \frac{1}{2} \sum_{u,v \in V} d(u, v).$$

The *transmission* $Tr(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, *i.e.*,

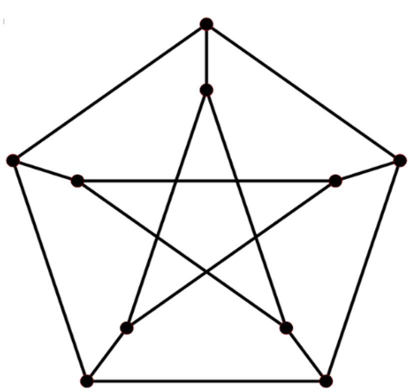
$$Tr(v) = \sum_{u \in V} d(u, v).$$

A connected graph $G = (V, E)$ is said to be k*–transmission regular* if $Tr(v) = k$ for every vertex $v \in V$.

As usual, we denote by P_n the path, by C_n the cycle, by S_n the star, by $K_{a,n-a}$ the complete bipartite graph and by K_n the complete graph, each on n vertices. A *kite* $Ki_{n,\omega}$ is the graph obtained from a clique K_ω and a path $P_{n-\omega}$ by adding an edge between an endpoint of the path and a vertex from the clique.

The *distance matrix* $\mathcal{D}$ of a connected graph G is the matrix indexed by the vertices of G where $\mathcal{D}_{i,j} = d(v_i, v_j)$ and $d(v_i, v_j)$ denotes the distance between the vertices v_i and v_j. Let $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ denote the spectrum of $\mathcal{D}$. It is called the distance spectrum of the graph G.

Similarly to the (adjacency) Laplacian, we define the *distance Laplacian* of a connected graph G to be the matrix $\mathcal{L} = Diag(Tr) - \mathcal{D}$, where $Diag(Tr)$ denotes the diagonal matrix of the vertex transmissions in G. Let $\partial_1^{\mathcal{L}} \geq \partial_2^{\mathcal{L}} \geq \cdots \geq \partial_n^{\mathcal{L}}$ denote the spectrum of $\mathcal{L}$. We call it the *distance Laplacian spectrum* of the graph G. To illustrate, we present in Figure 1 the Petersen graph [9] with its different spectra.



A–spectrum	$3^{(1)}$	$1^{(5)}$	$-2^{(4)}$
L–spectrum	$5^{(4)}$	$2^{(5)}$	$0^{(1)}$
$\mathcal{D}$–spectrum	$15^{(1)}$	$0^{(4)}$	$-3^{(5)}$
$\mathcal{L}$–spectrum	$18^{(5)}$	$15^{(4)}$	$0^{(1)}$

Figure 1: The Petersen graph and its different spectra.

For a connected graph G, let $P_{\mathcal{D}}^{G}(t)$ and $P_{\mathcal{L}}^{G}(t)$ denote the distance and the distance Laplacian characteristic polynomials respectively. For instance, the distance and the distance Laplacian spectra of the complete graph K_n are respectively its adjacency and Laplacian spectra, *i.e.*,

$$\begin{aligned} P_{\mathcal{D}}^{K_n}(t) &= (t-n+1)(t+1)^{n-1}; \\ P_{\mathcal{L}}^{K_n}(t) &= t(t-n)^{n-1}. \end{aligned}$$

The rest of the paper is organized as follows. In Section 2, we discuss similarities between the distance Laplacian spectrum of a graph and its other spectra. We first prove equivalence between the distance Laplacian spectrum and the distance spectrum among the class of transmission regular graphs. Then, we prove a similar result between the distance Laplacian spectrum and the Laplacian spectrum among the family of graphs of diameter 2. Thereafter, we show that the interlacing theorem does not apply for the distance Laplacian spectrum. In Section 3, we show that the second smallest distance Laplacian eigenvalue $\partial_{n-1}^{\mathcal{L}}$ of G is for $\overline{G}$ exactly what the second smallest Laplacian eigenvalue of G is for G, *i.e.*, $\partial_{n-1}^{\mathcal{L}}$ of G can be seen as the algebraic connectivity of $\overline{G}$. Section 4 is devoted to the study of some particular eigenvalues. Among other results, we show that 0 is the smallest distance Laplacian eigenvalue, with multiplicity 1. Finally, we list some open conjectures in Section 5.

2 Similarities with other spectra

In [5, 6, 7], Cvetković and Simić studied the spectral graph theory based on the signless Laplacian matrix. Among other results, they showed equivalence between the spectrum of the signless Laplacian and

- the adjacency spectrum for the class of (degree) regular graphs;
- the Laplacian spectrum for the class of (degree) regular graphs;
- the Laplacian spectrum for the class of bipartite graphs.

Along these lines, we studied similarities between the spectra of different distance matrices associated to connected graphs. A first result is that there is equivalence between the spectrum of the distance matrix $\mathcal{D}$ and the spectrum of the distance Laplacian $\mathcal{L}$ on the set of transmission regular graphs.

Theorem 2.1 *If G is a k–transmission regular graph on n vertices with distance spectrum $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ and distance Laplacian spectrum $\partial_1^{\mathcal{L}} \geq \partial_2^{\mathcal{L}} \geq \cdots \geq \partial_n^{\mathcal{L}}$, then $\partial_i^{\mathcal{L}} = k - \partial_{n-i+1}$ for all $i = 1, \ldots, n$.*

Proof. The relationship between the characteristic polynomials is as follows.

$$P_{\mathcal{L}}(t) = det(\mathcal{L} - tI) = det(Diag(Tr) - \mathcal{D} - tI) = (-1)^n det(\mathcal{D} - (k-t)I) = (-1)^n P_{\mathcal{D}}(k-t).$$

Thus ∂ is an eigenvalue of $\mathcal{D}$ if and only if $\partial^{\mathcal{L}} = k - \partial$ is an eigenvalue of $\mathcal{L}$. Ranking the eigenvalues in a non increasing order completes the proof. □

Using the above theorem, we can calculate the distance Laplacian characteristic polynomial of the cycle C_n from its distance polynomial. First, the distance characteristic polynomial of C_n was given in [8], according to the parity of n, as follows.

If $n = 2p$ (*i.e., even*)

$$P_{\mathcal{D}}^{C_n}(t) = t^{p-1} \cdot \left(t - \frac{n^2}{4}\right) \cdot \prod_{j=1}^{p} \left(t + \csc^2\left(\frac{\pi(2j-1)}{n}\right)\right).$$

If $n = 2p+1$ (*i.e., odd*)

$$P_{\mathcal{D}}^{C_n}(t) = \left(t - \frac{n^2-1}{4}\right) \cdot \prod_{j=1}^{p} \left(t + \frac{1}{4}\sec^2\left(\frac{\pi j}{n}\right)\right) \cdot \prod_{j=1}^{p} \left(t + \frac{1}{4}\csc^2\left(\frac{\pi(2j-1)}{2n}\right)\right).$$

Since the cycle C_n is a k–transmission regular graph with $k = n^2/4$ if n is even and $k = (n^2-1)/4$ if n is odd, we have

if $n = 2p$ (*i.e., even*)

$$P_{\mathcal{L}}^{C_n}(t) = t \cdot \left(t - \frac{n^2}{4}\right)^{p-1} \cdot \prod_{j=1}^{p} \left(t - \frac{n^2}{4} - \csc^2\left(\frac{\pi(2j-1)}{n}\right)\right);$$

if $n = 2p+1$ (*i.e., odd*)

$$P_{\mathcal{L}}^{C_n}(t) = -t \cdot \prod_{j=1}^{p} \left(t - \frac{n^2-1}{4} - \frac{1}{4}\sec^2\left(\frac{\pi j}{n}\right)\right) \cdot \prod_{j=1}^{p} \left(t - \frac{n^2-1}{4} - \frac{1}{4}\csc^2\left(\frac{\pi(2j-1)}{2n}\right)\right).$$

The second equivalence established is between L–theory and $\mathcal{L}$–theory on the set of graphs of diameter 2.

Theorem 2.2 *Let G be a connected graph on n vertices with diameter $D = 2$. Let $\lambda_1^L \geq \lambda_2^L \geq \cdots \lambda_{n-1}^L > \lambda_n^L = 0$ be the Laplacian spectrum of G. Then the distance Laplacian spectrum of G is $2n - \lambda_{n-1}^L \geq 2n - \lambda_{n-2}^L \geq \cdots 2n - \lambda_1^L > \partial_n^{\mathcal{L}} = 0$. Moreover, for every $i \in \{1, 2, \ldots, n-1\}$ the eigenspaces corresponding to λ_i^L and to $2n - \lambda_{n-i}^L$ are the same.*

Proof. Concerning the zero eigenvalue, the result is trivial.

For a connected graph of diameter 2, the transmission of each vertex $v \in V$ is

$$Tr(v) = d(v) + 2(n-1-d(v)) = 2n - d(v) - 2,$$

where $d(v)$ denotes the degree of v in G.

Also, the distance matrix is

$$\mathcal{D} = 2J - 2I - A,$$

where J is the all ones matrix. Thus the distance Laplacian matrix can be written as

$$\mathcal{L} = Diag(Tr) - \mathcal{D} = (2n-2)I - Diag(Deg) - 2J + 2I + A = 2nI - 2J - L.$$

Consider any eigenvalue λ_i^L of the Laplacian L with $1 \leq i \leq n-1$, *i.e.,* a non zero Laplacian eigenvalue of G, and let U_i denote a Laplacian eigenvector for λ_i^L. Since L is symmetric and the all ones column vector e is an eigenvector for $\lambda_n^L = 0$, we have $e^T \cdot U_i = 0$ and therefore, $J \cdot U_i = 0$. Thus

$$\mathcal{L} \cdot U_i = 2nU_i - L \cdot U_i = (2n - \lambda_i^L)U_i,$$

which means that $2n - \lambda_i^L$ is an eigenvalue of $\mathcal{L}$ and U_i is a corresponding eigenvector. This completes the proof. □

The famous interlacing theorem (see e.g. [4, p. 9]) does not apply in the case of the distance Laplacian spectrum of a graph. Indeed, consider the path P_n obtained from the cycle C_n by the deletion of an edge. The distance Laplacian spectra of P_n and C_n do not interlace for $n \geq 5$. For instance the distance Laplacian spectrum of P_6 is approximately $(21.3929, 15, 12.8532, 11, 9.7539, 0)$ while the distance Laplacian spectrum of C_6 is $(13, 13, 10, 9, 9, 0)$. The corresponding property for the distance Laplacian spectrum is that each eigenvalue ∂_i does not decrease if an edge is deleted from the graph. To prove this fact, we need the following lemma.

Lemma 2.3 (Courant–Weyl inequalities, [4]) *For a real symmetric matrix M of order n, let $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$ denote its eigenvalues. If A and B are real symmetric matrices of order n and if $C = A + B$, then for every $i = 1, \ldots, n$, we have*

$$\lambda_i(A) + \lambda_1(B) \geq \lambda_i(C) \geq \lambda_i(A) + \lambda_n(B).$$

Theorem 2.4 *Let G be a connected graph on n vertices and $m \geq n$ edges. Consider the connected graph G' obtained from G by the deletion of an edge. Denote by $\left(\partial_1^{\mathcal{L}}, \partial_2^{\mathcal{L}}, \ldots \partial_n^{\mathcal{L}}\right)$ and $\left(\tilde{\partial}_1^{\mathcal{L}}, \tilde{\partial}_2^{\mathcal{L}}, \ldots \tilde{\partial}_n^{\mathcal{L}}\right)$ the distance Laplacian spectra of G and G' respectively. Then $\tilde{\partial}_i^{\mathcal{L}} \geq \partial_i^{\mathcal{L}}$ for all $i = 1, \ldots n$.*

Proof. We write the distance Laplacian matrix of G' as $\mathcal{L}' = \mathcal{L} + M$, where M expresses the changes in $\mathcal{L}$ due to the deletion of an edge from G. It is easy to see that M is diagonally dominant with positive diagonal entries. Thus M is a positive semi-definite matrix. Also, it is easy to see that 0 is an eigenvalue of M. Now, the result follows immediately from Lemma 2.3. □

An immediate consequence of the spectra domination resulting from the deletion of an edge is that the distance Laplacian of any connected graph dominates that of the complete graph of the same order.

Corollary 2.5 *If G is a connected graph on $n \geq 2$ vertices, then $\partial_i(G) \geq \partial_i(K_n) = n$, for all $1 \leq i \leq n-1$, and $\partial_n(G) = \partial_n(K_n) = 0$.*

3 Similarities with the algebraic connectivity

The study of the second smallest distance Laplacian eigenvalue $\partial_{n-1}^{\mathcal{L}}$ of G led to the observation that $\partial_{n-1}^{\mathcal{L}}(G) = n$ if and only if $\overline{G}$ is disconnected. In fact, the second smallest distance Laplacian eigenvalue $\partial_{n-1}^{\mathcal{L}}$ of G is for $\overline{G}$ exactly what the second smallest Laplacian eigenvalue of G is for G, *i.e.*, $\partial_{n-1}^{\mathcal{L}}$ of G can be seen as the algebraic connectivity of $\overline{G}$.

Theorem 3.1 *Let G be a connected graph on n vertices. Then $\partial_{n-1}^{\mathcal{L}} = n$ if and only if $\overline{G}$ is disconnected. Furthermore, the multiplicity of n as an eigenvalue of $\mathcal{L}$ is one less than the number of components of $\overline{G}$.*

Proof. First, from Corollary 2.5, for any connected graph G,

$$\partial_{n-1}^{\mathcal{L}} \geq \partial_{n-1}^{\mathcal{L}}(K_n) = n.$$

If $\overline{G}$ is disconnected, then the diameter of G is 2 and thus by Theorem 2.2, $\partial_{n-1}^{\mathcal{L}} = n$.

If $\overline{G}$ is connected, since adding edges does not increase the eigenvalues of $\mathcal{L}$ (according to Theorem 2.4), it suffices to prove that $\partial_{n-1}^{\mathcal{L}} \neq n$ when $\overline{G}$ is a tree. Assume that $\overline{G}$ is a tree of diameter $\overline{D}$. Since G is connected $\overline{D} \geq 3$. If $\overline{D} \geq 4$, then the diameter of G is $D = 2$ and by Theorem 2.2 n is not a distance Laplacian eigenvalue of G as the algebraic connectivity of $\overline{G}$ is not 0. Now, assume that $\overline{D} = 3$. All the vertices, but two denoted u and v, are pending in $\overline{G}$. Under these conditions, $d_G(u, v) = 3$ and $\{u, v\}$ is the only pair of vertices at distance 3. Let $d(u) = k$ and $d(v) = l$ (note that $k + l = n - 2$) and label the vertices of G $v_1, v_2, \ldots, v_n$ such that $v_1, \ldots v_k$ are the neighbors of u, $v_{k+1}, \ldots v_{k+l}$ are the neighbors of v, $u = v_{n-1}$ and $v = v_n$. Using that labeling we can write the value of the characteristic polynomial of $\mathcal{L}$ at n as follows.

$$P_{\mathcal{L}}(n) = \begin{bmatrix} M & N \\ N^T & R \end{bmatrix},$$

where M is the $(n-2) \times (n-2)$–matrix all diagonal entries of which are equal to 0 and non diagonal entries are all equal to 1, N is the $(n-2) \times 2$–matrix, such that the k first entries of its first column are equal to 1 and the l following entries are equal to 2, the k first entries of its second column are equal to 2 and the l following entries are equal to 1, and finally

$$R = \begin{bmatrix} -l-1 & 3 \\ 3 & -k-1 \end{bmatrix}.$$

The determinant of M is $det(M) = (-1)^{(n-1)} \cdot (n-3)$, and the inverse of M is $(n-3)^{-1} \cdot M'$, where M' is the $(n-2) \times (n-2)$–matrix, the diagonal entries of which are all equal to $4-n$ and all non diagonal entries are equal to 1. Now, using the properties of the determinants (see for example [4, Lemma 2.2.])

$$\begin{aligned} P_{\mathcal{L}}(n) &= det(M) \cdot det(R - N^T M^{-1} N) \\ &= (-1)^{(n-1)} \cdot (n-3) \cdot det\left(\begin{bmatrix} -l-1 & 3 \\ 3 & -k-1 \end{bmatrix} - \frac{1}{n-3} \cdot \begin{bmatrix} 4l - k(l-1) & kl + 2(k+l) \\ kl + 2(k+l) & 4k - l(k-1) \end{bmatrix} \right) \\ &= (-1)^{(n-1)} \cdot 2(n-3) \quad \neq \quad 0. \end{aligned}$$

Thus n is not an eigenvalue of $\mathcal{L}$. Note that we used the *MAPLE* software to evaluate the determinant.

From the above lines, if n is an eigenvalue of $\mathcal{L}$, the diameter of G is necessarily $D = 2$. Thus the relation between the multiplicity of n as an eigenvalue of $\mathcal{L}$ and the number of components of $\overline{G}$ follows from Theorem 2.2. □

As immediate consequences of the above theorem, we have the following corollaries.

Corollary 3.2 *Let G be a connected graph on n vertices. Then $\partial_1^{\mathcal{L}}(G) \geq n$ with equality if and only if G is the complete graph K_n.*

Proof. If $\partial_1^{\mathcal{L}}(K_n) = n$, then n is an eigenvalue of $\mathcal{L}$ of multiplicity $n-1$. Thus $\overline{G}$ has n components that are necessarily isolated vertices and therefore G is the complete graph.

If G is not complete,

$$\sum_{i=1}^{n-1} \partial_i^{\mathcal{L}} = 2W > n(n-1),$$

where W denotes the Wiener index (the sum of all distances) in G. Thus $\partial_1^{\mathcal{L}} > n$, and this completes the proof. □

Corollary 3.3 *If G is bipartite and n is among its distance Laplacian eigenvalues, then G is complete bipartite. Therefore, the star S_n is the only tree for which n is a distance Laplacian eigenvalue.*

Proof. The only bipartite graphs with disconnected complement are the complete bipartite graphs. □

Corollary 3.4 *If $\Delta = n-1$, then n is an eigenvalue of $\mathcal{L}$ with multiplicity $n-1$ if G is complete and at least n_Δ if G is not complete, where n_Δ denotes the number of vertices of maximum degree in G.*

Proof. It suffices to note that each dominating vertex (a vertex of degree $n-1$) of G corresponds to an isolated vertex, and thus to a component, of its complement $\overline{G}$. □

4 Some particular eigenvalues

In this section, we study some particular distance Laplacian eigenvalues. First, as for the Laplacian, 0 is also an eigenvalue of the distance Laplacian. Before proving this fact, recall the following well-known result from matrix theory.

Lemma 4.1 (Gershgorin Theorem, [10]) *Let $M = (m_{ij})$ be a complex $n \times n$–matrix and denote by $\lambda_1, \lambda_2, \ldots \lambda_p$ its distinct eigenvalues. Then*

$$\{\lambda_1, \lambda_2, \ldots \lambda_p\} \subset \bigcup_{i=1}^{n} \left\{ z : |z - m_{ii}| \leq \sum_{j \neq i} |m_{ij}| \right\}.$$

Theorem 4.2 *For any connected graph G, we have $\partial_n^{\mathcal{L}} = 0$ with multiplicity 1.*

Proof. If $e = [1, 1, \dots, 1]^t$ is the all ones n–vector, then $\mathcal{L}e = 0$. Thus $\partial = 0$ is an eigenvalue of $\mathcal{L}$. Since $\mathcal{L}$ is positive semi-definite, then $\partial_n^{\mathcal{L}} = 0$.

To prove that the multiplicity of $\partial_n^{\mathcal{L}} = 0$ is 1, it suffices to prove that the rank of $\mathcal{L}$ is $n-1$. Consider the matrix M obtained from $\mathcal{L}$ by the deletion of, say, the last row and the last column. Then M is strictly diagonally dominant. Using Lemma 4.1, 0 is not an eigenvalue of M. Thus $det(M) \neq 0$ and therefore the rank of $\mathcal{L}$ is $n-1$. □

Some regularities in graphs are useful in calculating certain eigenvalues of the matrices related to these graphs. It is the case, for instance, for the largest eigenvalue of the adjacency matrix or the signless Laplacian whenever the graph is degree regular. The same is true for the largest eigenvalue of the distance Laplacian whenever the graph is transmission regular. Sometimes, a local regularity in a graph suffices to know some eigenvalue. We prove below that it is possible to know a distance Laplacian eigenvalue of a graph if it contains a clique or an independent set whose vertices share the same neighborhood.

Theorem 4.3 *Let G be a connected graph on n vertices. If $S = \{v_1, v_2, \dots v_p\}$ is an independent set of G such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$, then $\partial = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$ and $\partial + 2$ is an eigenvalue of $\mathcal{L}$ with multiplicity at least $p-1$.*

Proof. Since the vertices in S share the same neighborhood, any vertex in $V - S$ is at the same distance from all vertices in S. Any vertex of S is at distance 2 from any other vertex in S. Thus all vertices in S have the same transmission, say ∂.

To show that $\partial + 2$ is a distance Laplacian eigenvalue with multiplicity $p-1$, it suffices to observe that the matrix $(\partial + 2)I_n - \mathcal{L}$ contains p identical rows (columns). □

Corollary 4.4

(*a*) *The distance Laplacian characteristic polynomial of the star S_n is*

$$P_{\mathcal{L}}^{S_n}(t) = t \cdot (t-n) \cdot (t - 2n + 1)^{n-2}.$$

(*b*) *The distance Laplacian characteristic polynomial of the complete bipartite graph $K_{a,b}$ is*

$$P_{\mathcal{L}}^{K_{a,b}}(t) = t \cdot (t-n) \cdot (t - (2a+b))^{a-1} \cdot (t - (2b+a))^{b-1}.$$

(*c*) *Let $SK_{n,\alpha}$ denote the complete split graph, i.e., the complement of the disjoint union of a clique K_α and $n - \alpha$ isolated vertices. Then*

$$P_{\mathcal{L}}^{SK_{n,\alpha}}(t) = t \cdot (t-n)^{n-\alpha} \cdot (t - n - \alpha)^{\alpha - 1}.$$

Proof.

(*a*) The star S_n contains an independent set S of $n-1$ vertices with a common neighborhood. Each vertex of S has a transmission of $2n-1$. Thus by Theorem 4.3, $2n-1$ is a diatance Laplacian eigenvalue with multiplicity at least $n-2$. The complement of S_n contains exactly two components. Then, by Theorem 3.1, n is a simple eigenvalue of $\mathcal{L}^{S_n}$. Finally, using Theorem 4.2, we get the characteristic polynomial of $\mathcal{L}^{S_n}$.

(*b*) The complete bipartite graph $K_{a,b}$ contains two independent sets S_1 and S_2 with $|S_1| = a$ and $|S_2| = b$. The vertices of S_1 (resp. S_2) share the same neighborhood S_2 (resp. S_1). The transmission of each vertex of S_1 (resp. S_2) is $2a + b - 2$ (resp. $2b + a - 2$). Thus, by Theorem 4.3, $2a + b$ and $2b + a$ are eigenvalues of $\mathcal{L}^{K_{a,b}}$ with multiplicities at least $a-1$ and $b-1$ respectively. In addition, n and 0 are eigenvalues of $\mathcal{L}^{K_{a,b}}$, by Theorem 3.1 and Theorem 4.2, respectively.

(*c*) The independent set of $SK_{n,\alpha}$ contains α vertices sharing the same neighborhood and the same transmission $n + \alpha - 2$. Then, $n + \alpha$ is an $\mathcal{L}$–eigenvalue with multiplicity at least $\alpha - 1$. In addition, the complement of $SK_{n,\alpha}$ contains $n - \alpha + 1$ components. Thus n is an $\mathcal{L}$–eigenvalue with multiplicity $n - \alpha$. □

Theorem 4.5 *Let G be a connected graph on n vertices. If $K = \{v_1, v_2, \dots v_p\}$ is a clique of G such that $N(v_i) - K = N(v_j) - K$ for all $i, j \in \{1, 2, \dots, p\}$, then $\partial = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$ and $\partial + 1$ is an eigenvalue of $\mathcal{L}$ with multiplicity at least $p - 1$.*

The proof of this theorem is similar to that of the previous one.

Corollary 4.6

(*a*) *The distance Laplacian characteristic polynomial of the graph S_n^+, obtained from the star S_n by adding an edge, is*

$$P_{\mathcal{L}}^{S_n^+}(t) = t \cdot (t - n) \cdot (t - 2n + 3) \cdot (t - 2n + 1)^{n-3}.$$

(*b*) *The distance Laplacian characteristic polynomial of the pineapple $PA_{n,p}$, obtained from a clique K_{n-p} by attaching $p > 0$ pending edges to a vertex from the clique, is*

$$P_{\mathcal{L}}^{PA_{n,p}}(t) = t \cdot (t - n) \cdot (t - n - p)^{n-p-2} \cdot (t - 2n + 1)^p.$$

Proof. (a) is a particular case of (b), with $p = n - 3$. Thus, it sufices to prove (b).

It is trivial that 0 is an eigenvalue of $\mathcal{L}^{PA_{n,p}}$. Since the complement of $PA_{n,p}$ contains two components, n is a simple eigenvalue of $\mathcal{L}^{PA_{n,p}}$. $PA_{n,p}$ contains an independent set of p (pending) vertices sharing the same neighborhood and the same transmission $2n - 3$. Thus, by Theorem 4.3, $2n - 1$ is an $\mathcal{L}$–eigenvalue with multiplicity at least $p - 1$. $PA_{n,p}$ contains a clique on $n - p - 1$ vertices sharing the same neighborhood (composed of the dominating vertex) and the same transmission $n + p - 1$. By Theorem 4.5, $n + p$ is an $\mathcal{L}$–eigenvalue with multiplicity at least $n - p - 2$. Now, exaclty $n - 1$ $\mathcal{L}$–eigenvalues are known. The remaining eigenvalue is equal to the difference between the sum of all transmissions and the sum of the $n - 1$ known eigenvalues. It is easy to evaluate the remaining eigenvalue, which in fact equals $2n - 1$. □

Theorem 4.7 *If G is a connected graph on $n \geq 2$ vertices then $m(\partial_1^{\mathcal{L}}) \leq n - 1$ with equality if and only if G is the complete graph K_n.*

Proof. The inequality results immediately from Theorem 4.2. If the graph is complete, it is easy to see that equality holds. Now, let G be a connected graph such that $m(\partial_1^{\mathcal{L}}) = n - 1$. Assume, without loss of generality that the vertices of G are labeled such that $Tr_{max} = Tr(v_1) \geq Tr(v_2) \geq \cdots \geq Tr(v_n) = Tr_{min}$. Since $\mathcal{L}$ admits only two distinct eigenvalues, 0 and $\partial_1^{\mathcal{L}}$, and $e = [1, 1, \dots, 1]^t$ is an eigenvector that belongs to 0, any vector $X = [x_1, x_2, \dots x_n]^t$, with $x_1 = 1$, $x_i = -1$ and $x_j = 0$ for $j \neq 1$ and $j \neq i$, is an eigenvector that belongs to $\partial_1^{\mathcal{L}}$. Using the characteristic relation $\mathcal{L} \cdot X = \partial_1 X$, we get $Tr_{max} + d(v_1, v_i) = \partial_1$ for every vertex v_i including the neighbors of v_1, *i.e.*, all the vertices, but v_1, are neighbors of v_1. Therefore, $Tr_{max} = n - 1$ which is true if and only if G is the complete graph. □

Theorem 4.8 *If G is a tree on $n \geq 3$ vertices, then $\partial_1^{\mathcal{L}} \geq 2n - 1$ with equality if and only if G is the star S_n.*

Proof. It is easy to see that if G is the star S_n with $n \geq 3$ equality holds. If the tree G is not a star, then its diameter is at least 3. For $n = 3$, there is only one tree S_3. For $n = 4$, there are two trees, P_4 and S_4, and equality holds only for S_4. Assume that $n \geq 5$. Let the vertex set $\{v_1, v_2, \dots v_n\}$ of G be labeled such that $v_1v_2v_3v_4$ is a path. For $i \geq 5$, v_i is adjacent to v_1 or to v_2 and $d(v_i, v_4) \geq 3$, or v_i is adjacent v_3 or to v_4 and $d(v_i, v_1) \geq 3$, or v_i is not adjacent to any of the four vertices and $d(v_i, v_1) \geq 3$ and $d(v_i, v_4) \geq 3$. Thus there are at least $n - 3$ distances greater than or equal to 3. Then we have

$$\sum_{i=1}^{n-1} \partial_i^{\mathcal{L}} = 2W \geq 2\left((n-1) + 2\left(\frac{n(n-1)}{2} - (n-1) - (n-3)\right) + 3(n-3)\right) = 2n(n-1) - 4.$$

Using Theorem 4.7, we get $m(\partial_1^{\mathcal{L}}) < n-1$ and therefore

$$\partial_1^{\mathcal{L}} > \frac{2W}{n-1} \geq 2n - \frac{4}{n-1} \geq 2n-1$$

for all $n \geq 5$. This completes the proof. □

5 Some conjectures

In this section, we list a series of conjectures about some particular distance Laplacian eigenvalues of a connected graph. These conjectures, as well as some of the results proved in this paper, were obtained using the AutoGraphiX system [1, 2, 3] devoted to conjecture–making in graph theory.

First, we conjecture about bounding the largest distance Laplacian eigenvalue.

Conjecture 5.1 *For any connected graph G on $n \geq 4$ vertices,*

- $\partial_1^{\mathcal{L}}(G) \leq \partial_1^{\mathcal{L}}(P_n)$ *with equality if and only if G is the path P_n;*
- *if G is unicyclic, then $\partial_1^{\mathcal{L}}(G) \leq \partial_1^{\mathcal{L}}(Ki_{n,3})$ with equality if and only if G is the kite $Ki_{n,3}$;*
- *if G is unicyclic and $n \geq 6$, then $\partial_1^{\mathcal{L}}(G) \geq \partial_1^{\mathcal{L}}(S_n^+)$ with equality if and only if G is the graph S_n^+, obtained from the star S_n by adding an edge.*

The next conjecture is about the multiplicity of the largest distance Laplacian eigenvalue. If true, this conjecture implies that any connected graph has at least two different distance Laplacian eigenvalues, and the complete graph K_n is the only graph with exactly two.

Conjecture 5.2 *If G is a connected graph on $n \geq 2$ vertices and $G \ncong K_n$, then $m(\partial_1^{\mathcal{L}}(G)) \leq n-2$ with equality if and only if G is the star S_n and if $n = 2p$ for the complete bipartite graph $K_{p,p}$.*

Finally, we give conjectures about the second largest distance Laplacian eigenvalue of a connected graph: lower and upper bounds among all connected graphs; a lower bound among all trees; and lower and upper bounds among unicyclic graphs.

Conjecture 5.3 *For any connected graph G on $n \geq 4$ vertices,*

- $\partial_2^{\mathcal{L}}(G) \geq n$ *with equality if and only if G is the complete graph K_n or K_n minus an edge;*
- *if $n \neq 7$, then $\partial_2^{\mathcal{L}}(G) \leq \partial_2^{\mathcal{L}}(P_n)$ with equality if and only if G is the path P_n;*
- *if G is a tree and $n \geq 5$, then $\partial_2^{\mathcal{L}}(G) \geq 2n-1$ with equality if and only if G is the star S_n;*
- *if G is unicyclic and $n \geq 10$, then $\partial_2^{\mathcal{L}}(G) \leq \partial_2^{\mathcal{L}}(Ki_{n,3})$ with equality if and only if G is the kite $Ki_{n,3}$;*
- *if G is unicyclic and $n \geq 6$, then $\partial_2^{\mathcal{L}}(G) \geq \partial_2^{\mathcal{L}}(S_n^+)$ with equality if and only if G is the graph S_n^+ obtained from the star S_n by adding an edge.*

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