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On r-Equitable Colorings of Trees and Forests

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Abstract

An r-equitable k-coloring c of a graph $G = (V, E)$ is a partition of V into k stable sets $V_1(c), \cdots, V_k(c)$ such that $|\,|V_i(c)| - |V_j(c)|\,| \leq r$ for any $i, j \in \{1, \cdots, k\}$. In [B.-L. Chen and K.-W. Lih, Equitable Coloring of Trees, *Journal of Combinatorial Theory*, Series B 61, 83–87 (1994)], the authors gave a complete characterization of trees which are 1-equitably k-colorable. In this paper, we generalize this result and give a complete characterization of trees which are r-equitably k-colorable for any given $r \geq 1$. Furthermore we explain how to extend our result to forests.

Key Words: trees, forests, equitable coloring, maximum degree, independent sets.

Résumé

Une k-coloration r-équitable c d'un graphe $G = (V, E)$ est une partition de V en k ensembles stables $V_1(c), \cdots, V_k(c)$ tel que $|\,|V_i(c)| - |V_j(c)|\,| \leq r$ pour tout $i, j \in \{1, \cdots, k\}$. Dans [B.-L. Chen and K.-W. Lih, Equitable Coloring of Trees, *Journal of Combinatorial Theory*, Series B 61, 83–87 (1994)], les auteurs ont donné une caractérisation complète des arbres qui sont 1-équitablement k-colorables. Dans cet article, nous généralisons ce résultat et donnons une caractérisation complète des arbres qui sont r-équitablement k-colorables quel que soit $r \geq 1$. De plus, nous montrons comment ce résultat peut être étendu aux forêts.

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1 Introduction

All graphs in this paper are finite, simple and loopless. Let $G = (V, E)$ be a graph. We denote by $|G|$ the number of vertices in G. For a vertex $v \in V$, let $N(v)$ denote the set of vertices in G that are adjacent to v, i.e., the *neighbors* of v. $N(v)$ is called the *neighborhood* of vertex v. We also define for every $v \in V$, the *closed neighborhood* of v as $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $deg(v)$, is the number of neighbors of v, i.e., $deg(v) = |N(v)|$. $\Delta(G)$ denotes the *maximum degree* of G, i.e., $\Delta(G) = max\{deg(v)| \ v \in V\}$. For a set $V' \subseteq V$, we denote by $G - V'$ the graph obtained from G by deleting all vertices in V' as well as all edges incident to at least one vertex of V'.

An *independent set* in a graph $G = (V, E)$ is a set $S \subseteq V$ of pairwise nonadjacent vertices. The maximum size of an independent set in a graph $G = (V, E)$ is called the *independence number* of G and denoted by $\alpha(G)$. We define $\alpha^*(G) = \min\{\alpha(G - N[v])| \ deg(v) = \Delta(G)\}$. In other words, $\alpha^*(G)$ is the minimum size of a maximum independent set in a graph G' obtained from G by deleting the closed neighborhood of a vertex of maximum degree in G. A bipartite graph $G = (V, E)$ is a graph whose vertex set can be partitioned into two independent sets X and Y. Such a graph will be referred to as $G = (X, Y, E)$.

A *k-coloring* c of a graph $G = (V, E)$ is a partition of V into k independent sets which we will denote by $V_1(c), V_2(c), \cdots, V_k(c)$ and refer to as *color classes*. The cardinality of a largest color class with respect to a coloring c will be denoted by Max_c. A graph G is *r-equitably k-colorable*, with $r \geq 1$ and $k \geq 2$, if there exists a k-coloring c of G such that $|\ |V_i(c)| - |V_j(c)|\ | \leq r$ for any $i, j \in \{1, 2, \cdots, k\}$. A graph which is 1-equitably k-colorable is simply said to be *equitably k-colorable*.

The notion of equitable colorability was introduced in [7]. Since then, it has been studied by many authors (see for instance [2, 3, 5, 6, 8]). To the best of our knowledge, no results are known about r-equitable colorability for $r \geq 2$, although this seems to be a natural extension. Indeed, a k-colorable graph G does not always admit an equitable k-coloring, but clearly there always exists an integer $r \geq 1$ such that G admits an r-equitable k-coloring.

In [3], the authors studied the case when $r = 1$ and G is a tree. They gave a complete characterization of trees which are equitably k-colorable. Their result is split into two parts.

Theorem 1.1 *([3]) Let $T = (X, Y, E)$ be a tree containing at least one edge and such that $|\ |X| - |Y|\ | \leq 1$. Then T is equitably k-colorable if and only if $k \geq 2$.*

Theorem 1.2 *([3]) Let $T = (X, Y, E)$ be a tree such that $|\ |X| - |Y|\ | > 1$. Then T is equitably k-colorable if and only if $k \geq \max\{3, \lceil \frac{|T|+1}{\alpha^*(T)+2} \rceil\}$.*

This result was then generalized to forests for $k \geq 3$ in [2].

Theorem 1.3 *([2]) Suppose F is a forest and $k \geq 3$ is an integer. Then F is equitably k-colorable if and only if $k \geq \lceil \frac{|F|+1}{\alpha^*(F)+2} \rceil$.*

In this paper, we consider trees and we give a complete characterization of those that are r-equitably k-colorable for $r \geq 1$ and $k \geq 2$, thus generalizing the result of [3]. Furthermore we will explain how to extend this result to forests, thus generalizing Theorem 1.3.

Our paper is organized as follows. In Section 2 we present some interesting properties of r-equitable k-colorings in trees as well as some preliminary results that we will use to prove our main result which will be given in Section 3. In Section 4, we explain how to extend it to forests.

2 Preliminary results

We will start by presenting some properties concerning r-equitable k-colorings of trees with $r \geq 1$ and $k \geq 2$.

Consider a tree T and an integer $r \geq 1$. Let c be an arbitrary r-equitable k-coloring of the vertex set of T such that $|V_1(c)| \geq |V_2(c)| \geq \cdots \geq |V_k(c)|$ with $k \geq 3$. Then there may be vertices in T which are forced to be colored with color k. Indeed, if for instance T is a star on $(k-1)r+k$ vertices, then the vertex v of degree > 1 necessarily belongs to $V_k(c)$ and actually $V_k(c) = \{v\}$. Furthermore, we have $|V_i(c)| = r+1$ for $i \in \{1, \cdots, k-1\}$. It turns out that this is no longer true for colors $1, 2 \cdots, k-1$. In fact, we obtain the following.

Lemma 2.1 *Let T be a tree containing at least two vertices and let u be any vertex in T. Assume T is r-equitably k-colorable for some integers $k \geq 3$ and $r \geq 1$, and let ℓ be any integer in $\{1, \cdots, k-1\}$. Then there exists an r-equitable k-coloring c of T such that $|V_i(c)| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$ and $u \notin V_\ell(c)$.*

Proof. Suppose the Lemma is false. Let c be an r-equitable k-coloring of T with $|V_i(c)| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$. Among all such colorings we choose one such that for every $j = 1, \cdots, k$, there exists no r-equitable k-coloring c' of T with $|V_1(c)| = |V_i(c')|$ $i = 1, .., j-1$ and $\max_{i=j+1}^{k}\{|V_i(c')|\} < |V_j(c)|$. In other words, $Max_c = |V_1(c)|$ is minimum among all r-equitable k-colorings of T, $|V_2(c)|$ is mininum among all r-equitable k-colorings c' of T with $Max_{c'} = Max_c$, and so on.

Let $\ell \in \{1, \cdots, k-1\}$ be an integer for which the Lemma does not hold. We define $x = 1,\ y = 2,\ z = 3$ if $\ell = 1$ and $x = \ell - 1,\ y = \ell,\ z = \ell + 1$ if $\ell > 1$. Since we assume that the lemma is false, it follows that $u \in V_\ell(c)$, which means that $u \in V_x(c)$ if $\ell = 1$ and $u \in V_y(c)$ if $\ell > 1$. Then $|V_x(c)| > |V_y(c)|$, otherwise we could assign color x to all vertices in $V_y(c)$ and color y to all vertices in $V_x(c)$ to obtain an r-equitable k-coloring c' with $u \notin V_\ell(c')$, a contradiction. Similarly, we must have $|V_y(c)| > |V_z(c)|$ when $l > 1$ since otherwise we could assign color y to all vertices in $V_z(c)$ and color z to all vertices in $V_y(c)$ and thus the lemma would hold.

We define F as the subgraph of T induced by $V_x(c) \cup V_y(c) \cup V_z(c)$. If F is disconnected, we add some edges to make F become a tree T' such that no two adjacent vertices have the same color with respect to c; otherwise we set $T' = F$. Let $V(T')$ denote the vertex set of T'. Moreover, for $q = y$ or z, we denote $\overline{q} = y + z - q$. This implies that $\overline{q} = z$ if $q = y$ and $\overline{q} = y$ if $q = z$. We start by proving the following two claims.

Claim 1: There exists no r-equitable 3-coloring c' of T' (using colors x, y, z) with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_q(c')| = |V_q(c)| + 1$ and $|V_{\overline{q}}(c')| = |V_{\overline{q}}(c)|$ for $q = y$ or z.

Indeed, if such a coloring c' exists, then the assumption on c implies $|V_q(c')| = |V_x(c)| > |V_x(c')|$. Now we assign color x to all vertices in $V_q(c')$, color q to all vertices in $V_x(c')$ and color $c''(v) = c(v)$ to all vertices in $T - (V_x(c') \cup V_q(c'))$ to obtain an r-equitable k-coloring c'' of T. We distinguish two cases:

- If $l = 1$, we have $|V_1(c'')| > \max_{i=2}^{k}\{|V_i(c'')|\}$ and $u \notin V_1(c'')$.
- If $l > 1$, we have $q = y$ since otherwise $|V_z(c')| = |V_z(c)| + 1 = |V_x(c)|$ which contradicts $|V_x(c)| > |V_y(c)| > |V_z(c)|$. Then $|V_1(c'')| \geq \cdots \geq |V_{\ell-1}(c'')| > |V_\ell(c'')| \geq |V_{\ell+1}(c'')| \geq \cdots \geq |V_k(c'')|$ and $u \in V_{\ell-1}(c'')$.

Thus in both cases, c'' is an r-equitable k-coloring of T such that $u \notin V_l(c'')$, a contradiction. This proves Claim 1.

Claim 2: No leaf of T', except possibly u, is in $V_x(c)$.

Indeed, assume T' has a leaf $v \neq u$ in $V_x(c)$ and let w be its unique neighbor in T'. We can change the color of v from x to $\overline{c(w)}$ to obtain an r-equitable 3-coloring c' of T' with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(w)}}(c')| = |V_{\overline{c(w)}}(c)| + 1$ and $|V_{c(w)}(c')| = |V_{c(w)}(c)|$, contradicting Claim 1. This proves Claim 2.

Let $\vec{T}$ be the oriented rooted tree obtained from T' by orienting the edges from root u to the leaves. Let us partition the vertices in $V_x(c)$ into subsets $U_1, \cdots, U_p$ such that U_q $(q = 1, \cdots, p)$ contains all vertices in $V_x(c)$ having no successor in $V_x(c) - \bigcup_{j=1}^{q-1} U_j$. For a vertex $v \in U_1$, let $L(v)$ denote the set of leaves in $\vec{T}$ having v as predecessor.

If $|L(v)| = 1$ for some $v \in U_1$, then let $P = v \to s_1 \to \cdots \to s_a$ denote the path from v to the leaf s_a in $L(v)$. If $v = u$ (and hence $\ell = 1$ since $u \in V_x(c)$) then T' is a chain with only one vertex in $V_x(c)$, which means that $V_y(c) = V_z(c) = \emptyset$ since $|V_x(c)| > |V_y(c)| \geq |V_z(c)|$. Thus T' has only one vertex, namely u, and since $u \in V_1(c)$ this implies that T has only one vertex, a contradiction. Hence $v \neq u$. Let w be the predecessor of v in $\vec{T}$:

- if $c(w) = c(s_1)$, we change the color of v to $\overline{c(w)}$ to obtain an r-equitable 3-coloring c' of T' with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(w)}}(c')| = |V_{\overline{c(w)}}(c)| + 1$ and $|V_{c(w)}(c')| = |V_{c(w)}(c)|$, contradicting Claim 1;
- if $c(w) \neq c(s_1)$, we assign color $c(s_1)$ to v, color $c(s_{j+1})$ to s_j $(j = 1, ..., a-1)$, and color x to s_a; we obtain an r-equitable 3-coloring c' of T' with $|V_i(c')| = |V_i(c)|$ $(i = x, y, z)$, $c'(u) = c(u)$ and a leaf $s_a \in V_x(c')$. But this contradicts Claim 2.

We therefore conclude that $|L(v)| \geq 2$ for all $v \in U_1$. By denoting $W_1 = \bigcup_{v \in U_1} L(v)$, we get $|W_1| \geq 2|U_1|$. For each set U_q, with $q > 1$, we will now construct a set W_q containing vertices in $V_y(c) \cup V_z(c)$ that are successors of vertices in U_q but not successors of vertices in U_{q-1}. So let v be any vertex in U_q $(q > 1)$. If v has at least 2 immediate successors in $\vec{T}$, we add two of them to W_q. If v has a unique immediate successor in $\vec{T}$, then let $P = v \to s_1 \to \cdots \to s_a \to v'$ denote a path from v to a vertex $v' \in U_{q-1}$. If $a > 1$, we add s_1 and s_2 to W_q. If $a = 1$ and s_1 has an immediate successor $w \notin V_x(c)$, then we add s_1 and w to W_q. Assume now that $a = 1$ and all the immediate successors of s_1 are in $V_x(c)$. We will prove that such a case is not possible.

- If $v \neq u$, then v has a predecessor w in $\vec{T}$. We must have $c(w) = \overline{c(s_1)}$, otherwise we could assign color $\overline{c(s_1)}$ to v to obtain an r-equitable 3-coloring c' of T' with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$, contradicting Claim 1. But now we can assign color $c(s_1)$ to v and assign color $\overline{c(s_1)}$ to s_1 to obtain an r-equitable 3-coloring c' of T' with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$, contradicting Claim 1.
- If $v = u$, then $\ell = 1$ since $u \in V_x(c)$. By assigning color $c(s_1)$ to u and color $\overline{c(s_1)}$ to s_1, we obtain an r-equitable 3-coloring c' of T' with $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$. It follows from the assumptions on c that $|V_{\overline{c(s_1)}}(c')| = |V_x(c)| > |V_{c(s_1)}(c)| = |V_{c(s_1)}(c')|$. Thus the lemma would hold, a contradiction.

In summary, we have $|W_q| \geq 2|U_q|$. Since all sets W_q are disjoint, we have

$$|V_y(c)| + |V_z(c)| \geq \sum_{q=1}^{p} |W_q| \geq \sum_{q=1}^{p} 2|U_q| = 2|V_x(c)|.$$

Hence $|V_y(c)|$ or $|V_z(c)|$ is larger than or equal to $|V_x(c)|$, a contradiction. □

Lemma 2.1 allows us to show the following.

Lemma 2.2 *Let T_1 and T_2 be two trees, each one containing at least two vertices. Assume that T_i is r-equitably k-colorable for $i = 1, 2$ and $k \geq 2, r \geq 1$. Then the tree T obtained by adding an arbitrary edge between T_1 and T_2 is r-equitably k-colorable.*

Proof. Consider an r-equitable k-coloring c of T_1 and an r-equitable k-coloring c' of T_2 such that $|V_i(c)| \geq |V_j(c)|$ and $|V_i(c')| \geq |V_j(c')|$ for all $1 \leq i < j \leq k$. Let u be a vertex in T_1 and v a vertex in T_2, and let T be the tree obtained by adding an edge between u and v. According to Lemma 2.1, we may assume that $v \notin V_1(c')$. Hence $v \in V_{k-\ell+1}(c')$ for some $\ell \in \{1, \cdots, k-1\}$ and it follows from Lemma 2.1 that we may assume that $u \notin V_\ell(c)$. We can therefore construct a k-coloring c'' of T such that $V_i(c'') = V_i(c) \cup V_{k-i+1}(c')$, $i = 1, \cdots, k$. For $i > j$, we have

- $|V_i(c'')| - |V_j(c'')| = |V_i(c)| + |V_{k-i+1}(c')| - (|V_j(c)| + |V_{k-j+1}(c')|) \geq |V_i(c)| - |V_j(c)| \geq -r$;
- $|V_i(c'')| - |V_j(c'')| = |V_i(c)| + |V_{k-i+1}(c')| - (|V_j(c)| + |V_{k-j+1}(c')|) \leq |V_{k-i+1}(c')| - |V_{k-j+1}(c')| \leq r$;

This proves that the considered k-coloring c'' of T is r-equitable. □

Lemma 2.3 *If T is an r-equitably k-colorable tree for some $k \geq 2$ and $r \geq 1$, then the tree obtained by adding a pending edge to T is $(r+1)$-equitably k-colorable.*

Proof. Consider an r-equitable k-coloring c of T and let T' be the tree obtained by adding a new vertex u and making it adjacent to some vertex v of T. Without loss of generality, we may assume that $|V_1(c)| \geq |V_2(c)| \geq \cdots \geq |V_k(c)|$. We extend c to a coloring c' of T' by assigning any color $j \neq c(v)$ to u with $j \in \{1, \cdots, k\}$. If $|V_j(c)| = |V_1(c)|$ in T, then c' is $(r+1)$-equitable, otherwise c' is r-equitable. □

Let us now present some results which we will need to prove our main result. We start with a special case in which we can get an r-equitable k-coloring from an r-equitable $(k-1)$-coloring.

Lemma 2.4 *Let c be an r-equitable $(k-1)$-coloring of a tree T for $r \geq 1$ and $k \geq 3$. If $Max_c \leq 2r+2$ then T is r-equitably k-colorable.*

Proof. Let c be an r-equitable $(k-1)$-coloring of a tree T. We distinguish four cases.

- $Max_c \leq r$. Then c is an r-equitable k-coloring of T.
- $r+1 \leq Max_c \leq 2r$. We assign color k to r vertices of a color class containing Max_c vertices to get an r-equitable k-coloring of T.
- $Max_c = 2r+1$. If there is a unique color class C containing Max_c vertices, then we assign color k to r vertices of C to get an r-equitable k-coloring of T. Otherwise, let C_1 and C_2 be two color classes containing each Max_c vertices. If there exists a vertex u in C_1 that is adjacent to at most $r+1$ vertices in C_2, then we assign color k to u and to r vertices of C_2 that are nonadjacent to u to obtain an r-equitable k-coloring of T. Otherwise, if such a vertex does not exist, there are at least $(2r+1)(r+2) = 2r^2+5r+2 = (2r^2+r+1)+4r+1 > 4r+1$ edges linking C_1 to C_2. But $|C_1|+|C_2| = 4r+2$, thus T would not be a tree, a contradiction.
- $Max_c = 2r+2$. If there is a unique color class C containing Max_c vertices, then we assign color k to $r+1$ vertices of C to get an r-equitable k-coloring of T. Otherwise, let C_1 and C_2 be two color classes containing each Max_c vertices. If there exist two vertices u, w in C_1 such that $|(N(u) \cup N(w)) \cap C_2| \leq r+2$, then we assign color k to u, w and to r vertices of C_2 that are nonadjacent to u and w to obtain an r-equitable k-coloring of T. Otherwise, if such two vertices do not exist, there are at least $\frac{2r+2}{2}(r+3) = r^2+4r+3 > 4r+3$ edges linking C_1 to C_2. But $|C_1|+|C_2| = 4r+4$, thus T would not be a tree, a contradiction.

□

We will now give a sufficient condition, involving the maximum degree, for a tree to be r-equitably k-colorable for $k \geq 3$ and $r \geq 1$. First we consider the case $k = 3$. In [1], the authors gave the following sufficient condition for a tree to be equitably 3-colorable. We will use this result in our proof.

Theorem 2.1 *([1]) A tree T is equitably 3-colorable if $|T| \geq 3\Delta(T) - 8$ or if $|T| = 3\Delta(T) - 10$.*

Lemma 2.5 *Let T be a tree with $\Delta(T) \leq \lfloor \frac{|T|+3}{3} \rfloor + \lfloor \frac{r-1}{2} \rfloor$, where $r \geq 1$. Then T is r-equitably 3-colorable.*

Proof. We will prove the result by induction on r. For $r = 1$, the result immediately follows from Theorem 2.1. Suppose that $r > 1$ and that the result holds up to $r - 1$. Consider a tree T with maximum degree $\Delta(T) \leq \lfloor\frac{|T|+3}{3}\rfloor + \lfloor\frac{r-1}{2}\rfloor$. We will show that T is r-equitably 3-colorable.

First suppose that r is even or/and $\Delta(T) < \lfloor\frac{|T|+3}{3}\rfloor + \lfloor\frac{r-1}{2}\rfloor$. Then $\Delta(T) \leq \lfloor\frac{|T|+3}{3}\rfloor + \lfloor\frac{r-2}{2}\rfloor$, and by induction it follows that T is $(r-1)$-equitably 3-colorable. Hence T is also r-equitably 3-colorable.

We can therefore assume that r is odd and $\Delta(T) = \lfloor\frac{|T|+3}{3}\rfloor + \frac{r-1}{2}$. Notice that this necessarily implies $r \geq 3$ and $\Delta(T) \geq 2$. Let $s = 3 - (|T| \mod 3)$ and consider a tree T' obtained from T by adding a chain on s vertices and linking it to a leaf of T. More precisely, we add s vertices $x_1, \cdots, x_s$, we make x_1 adjacent to some leaf v of T and we add the edges $x_i x_{i+1}$ for $i = 1, \cdots, s-1$. Since $\lfloor\frac{|T'|+3}{3}\rfloor = \lfloor\frac{|T|+3}{3}\rfloor + 1$, we have $\Delta(T') = \Delta(T) = \lfloor\frac{|T'|+3}{3}\rfloor + \frac{r-3}{2}$. Hence, by induction hypothesis, there exists an $(r-2)$-equitable 3-coloring c of T'. Since $1 \leq s \leq 3$ and since at most 2 vertices among $x_1, \cdots, x_s$ have the same color, the restriction of c to T is an r-equitable 3-coloring. □

Using Lemma 2.5, we may know prove the general case $k \geq 3$.

Theorem 2.2 *Let T be a tree with $\Delta(T) \leq \lfloor\frac{|T|+3}{3}\rfloor + \lfloor\frac{r-1}{2}\rfloor$, where $r \geq 1$. Then T is r-equitably k-colorable for all $k \geq 3$.*

Proof. The proof is by induction on k. The basis of our induction is Lemma 2.5 which asserts that T is r-equitably 3-colorable if $\Delta(T) \leq \lfloor\frac{|T|+3}{3}\rfloor + \lfloor\frac{r-1}{2}\rfloor$. Now suppose that T is r-equitably $(k-1)$-colorable for any $k \geq 4$ whenever $\Delta(T) \leq \lfloor\frac{|T|+3}{3}\rfloor + \lfloor\frac{r-1}{2}\rfloor$. It remains to show that T is r-equitably k-colorable. Let $q = \lfloor\frac{|T|-(r-1)(k-1)}{k}\rfloor$ and let $p = |T| - (r-1)(k-1) - kq$, which implies that $0 \leq p \leq k-1$. Furthermore, let c be an r-equitable $(k-1)$-coloring of T.

By Lemma 2.4, we may assume that $Max_c \geq 2r+3$. This implies that $q \geq 3$. Indeed, if $q \leq 2$ then $|T| \leq 3k + (r-1)(k-1) - 1 = rk + 2k - r$. But then necessarily $Max_c \leq 2r+2$, otherwise $|T| \geq (2r+3) + (k-2)(r+3) = rk + 3k - 3 = rk + 2k - r + (k-3+r) > |T|$, a contradiction. Thus $q \geq 3$. Furthermore, if $q = 3$, then $|T| \leq 4k + (r-1)(k-1) - 1 = rk + 3k - r$. Then we have $Max_c = 2r+3$, otherwise $|T| \geq (2r+4) + (k-2)(r+4) = rk + 4k - 4 = rk + 3k - r + (k-4+r) > |T|$, a contradiction. Finally, we have $r \leq 3$ whenever $q = 3$, otherwise $|T| \geq (2r+3) + (k-2)(r+3) = rk + 3k - 3 = rk + 3k - r + (r-3) > |T|$, a contradiction. If $q = 3$ and $r = 3$, then the above computation shows that c has exactly one color class C with $2r+3$ vertices and $k-2$ color classes with $r+3$ vertices. Hence we can assign color k to $r+1$ vertices of C to obtain an r-equitable k-coloring of T.

In summary, we may assume now that either $q \geq 4$ or $q = 3$ and $r \leq 2$. Note that since T is $(k-1)$-colorable, $\alpha(T) \geq \lceil\frac{|T|}{k-1}\rceil = \lceil\frac{qk+p}{k-1}\rceil + r - 1 \geq q + r - 1 \geq q + \lfloor\frac{3(r-1)}{4}\rfloor$. Now let M be the set of vertices with degree strictly larger than $\lfloor\frac{|T|-q-\lfloor\frac{3(r-1)}{4}\rfloor+3}{3}\rfloor$. Since $k \geq 4$ we have

$$(r-1)(k-1) \geq 3(r-1) \geq 4\lfloor\frac{3(r-1)}{4}\rfloor.$$

We therefore have

$$\begin{aligned}4(q + \lfloor\frac{3(r-1)}{4}\rfloor) &= 4\lfloor\frac{|T|-(r-1)(k-1)}{k}\rfloor + 4\lfloor\frac{3(r-1)}{4}\rfloor \\ &\leq |T| - (r-1)(k-1) + 4\lfloor\frac{3(r-1)}{4}\rfloor \\ &\leq |T|.\end{aligned}$$

Hence

$$|T| - 4q \geq 4\lfloor\frac{3(r-1)}{4}\rfloor$$

and

$$\frac{|T| - (q + \lfloor\frac{3(r-1)}{4}\rfloor) + 3}{3} \geq \frac{4(q + \lfloor\frac{3(r-1)}{4}\rfloor) - (q + \lfloor\frac{3(r-1)}{4}\rfloor) + 3}{3}$$
$$= q + \lfloor\frac{3(r-1)}{4}\rfloor + 1.$$

This means that every vertex of M is adjacent to more than $q + \lfloor\frac{3(r-1)}{4}\rfloor$ vertices. We also note that M contains at most 3 vertices. Indeed, if M contains 4 vertices, then it follows from the above inequalities that the number of edges in T is at least

$$4\frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor + 3}{3} - 3 = |T| + 1 + \frac{|T| - 4q - 4\lfloor\frac{3(r-1)}{4}\rfloor}{3}$$
$$\geq |T| + 1.$$

which is a contradiction.

Let us now distinguish several cases.

(i) If M is empty, then let S be any independent set containing $q + \lfloor\frac{3(r-1)}{4}\rfloor$ vertices (S exists since $\alpha(T) \geq q + \lfloor\frac{3(r-1)}{4}\rfloor$).

(ii) If $M = \{v_1\}$, then let S be any independent set consisting of $q + \lfloor\frac{3(r-1)}{4}\rfloor$ vertices adjacent to v_1.

(iii) If $M = \{v_1\, v_2\}$, then let S be any independent set consisting of v_2 and $q + \lfloor\frac{3(r-1)}{4}\rfloor - 1$ vertices adjacent to v_1 but nonadjacent to v_2.

(iv) If $M = \{v_1, v_2, v_3\}$, then we may suppose that v_2 is not adjacent to v_3. Then let S be any independent set consisting of v_2, v_3 and $q + \lfloor\frac{3(r-1)}{4}\rfloor - 2$ vertices adjacent to v_1 but nonadjacent to v_2 and nonadjacent to v_3.

We notice that in each of the above cases, all the vertices in $T - S$, except possibly v_1, are adjacent to at most $\lfloor\frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor + 3}{3}\rfloor$ vertices. In cases (ii), (iii) and (iv), the degree of v_1 in $T - S$ is at most

$$\Delta(T) - (q + \lfloor\frac{3(r-1)}{4}\rfloor - 2) \leq \frac{|T| + 3}{3} + \lfloor\frac{r-1}{2}\rfloor - q - \lfloor\frac{3(r-1)}{4}\rfloor + 2$$
$$= \frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor + 3}{3} + \frac{6 - 2q}{3} + \lfloor\frac{r-1}{2}\rfloor - \frac{2}{3}\lfloor\frac{3(r-1)}{4}\rfloor.$$

- If $q = 3$, then we have already seen that we may assume that $r \leq 2$, which means that $\lfloor\frac{r-1}{2}\rfloor - \frac{2}{3}\lfloor\frac{3(r-1)}{4}\rfloor = 0$. Hence the degree of v_1 in $T - S$ is then at most $\frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor + 3}{3}$.
- If $q \geq 4$, then since $\lfloor\frac{r-1}{2}\rfloor - \frac{2}{3}\lfloor\frac{3(r-1)}{4}\rfloor \leq \frac{1}{3}$, we conclude that the degree of v_1 in $T - S$ is at most $\frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor + 3}{3} + \frac{7-2q}{3} < \frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor + 3}{3}$.

Thus all vertices in $T - S$ have degree at most $\frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor + 3}{3}$.

Observe that $T - S$ may be a forest. Let $T_1, \cdots, T_d$ be the connected components of $T - S$. For every T_i that is not a single vertex, let x_i and y_i denote two distinct leaves in T_i. For every T_i consisting of a single vertex u, let $x_i = y_i = u$. If $\Delta(T - S) \leq 1$ then $T - S$ can easily be equitably $(k-1)$-colored. Otherwise, we link x_i with y_{i+1} $(i = 1, \cdots, d-1)$ to get a tree T^* such that $\Delta(T^*) = \Delta(T - S) \leq \lfloor\frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor + 3}{3}\rfloor = \lfloor\frac{|T^*| + 3}{3}\rfloor$. Since $\Delta(T^*) \leq \lfloor\frac{|T^*| + 3}{3}\rfloor + \lfloor\frac{1-1}{2}\rfloor$, it follows from our induction hypothesis that T^*, and hence also $T - S$, is equitably $(k-1)$-colorable. The color classes of an equitable $(k-1)$-coloring of $T - S$ contain $\lfloor\frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor}{k-1}\rfloor$ or $\lceil\frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor}{k-1}\rceil$ vertices. Observe that

$$\frac{|T| - q - \lfloor\frac{3(r-1)}{4}\rfloor}{k-1} = q + r - 1 - \frac{\lfloor\frac{3(r-1)}{4}\rfloor}{k-1} + \frac{p}{k-1} = q + \lfloor\frac{3(r-1)}{4}\rfloor + r - 1 - \frac{k}{k-1}\lfloor\frac{3(r-1)}{4}\rfloor + \frac{p}{k-1}.$$

Hence, $\frac{|T|-q-\lfloor\frac{3(r-1)}{4}\rfloor}{k-1} \leq |S|+r$. Furthermore, since $k \geq 4$ we have $r-1 \geq \frac{k}{k-1}\frac{3(r-1)}{4}$, thus $\frac{|T|-q-\lfloor\frac{3(r-1)}{4}\rfloor}{k-1} \geq |S|$. This means that the independent set S together with the equitable $(k-1)$-coloring of $T-S$ induce an r-equitable k-coloring of T. □

We give now the following two Lemmas, the first of which is just an easy observation.

Lemma 2.6 *Let $G = (X, Y, E)$ be a connected bipartite graph and let $r \geq 1$ be an integer. Then G is r-equitably 2-colorable if and only if $|\,|X| - |Y|\,| \leq r$.*

Lemma 2.7 *If a graph $G = (V, E)$ is r-equitably k-colorable for $r \geq 1$ and $k \geq 2$, then $k \geq \lceil\frac{|G|+r}{\alpha^*(G)+r+1}\rceil$.*

Proof. Consider an r-equitable k-coloring of G and let $v = argmin_{v\in V}\{\alpha(G - N[v])|\ deg(v) = \Delta(G)\}$. Without loss of generality, we may assume that vertex v has color 1. Clearly, the total number of vertices in G having color 1 is at most $\alpha^*(G) + 1$. Since we have an r-equitable k-coloring, it follows that any color other than color 1 occurs at most $\alpha^*(G) + r + 1$ times. Thus $|G| \leq \alpha^*(G) + 1 + (k-1)(\alpha^*(G) + r + 1)$. It follows that $|G| \leq k(\alpha^*(G) + r + 1) - r$ and hence $k \geq \lceil\frac{|G|+r}{\alpha^*(G)+r+1}\rceil$. □

Finally the following result was shown in [3].

Theorem 2.3 *([3]) Let $G = (X, Y, E)$ be a bipartite graph. If G has p connected components and $p \geq \frac{|G|}{k}$ for some positive integer k, then G is equitably k-colorable.*

3 r-equitably k-colorable trees

In this section, we will give a complete characterization of trees which are r-equitably k-colorable for $r \geq 1$ and $k \geq 2$. Let $T = (X, Y, E)$ be a tree and let $r \geq 1$ be an integer. Similar to [3], our main result will consist of two parts: (a) $|\,|X| - |Y|\,| \leq r$; (b) $|\,|X| - |Y|\,| > r$. We will first deal with the case $|\,|X| - |Y|\,| \leq r$.

In the proof of Theorem 1.1 in [3], the authors show that if $k \geq 2$, then there exists an equitable k-coloring c with color classes $V_1(c), \cdots, V_k(c)$ such that at most one of these color classes contains vertices from both X and Y, and all other color classes are contained either in X or in Y. Using this fact, we obtain the following.

Theorem 3.1 *Let $T = (X, Y, E)$ be a tree containing at least one edge and such that $|\,|X| - |Y|\,| \leq r$, where $r \geq 1$. Then T is r-equitably k-colorable if and only if $k \geq 2$.*

Proof. Suppose that $n_1 = |X| \leq |Y| = n_2$. Notice that if $n_2 - 1 \leq n_1 \leq n_2$, the result immediately follows from Theorem 1.1. Thus we may assume now that $n_1 < n_2 - 1$.

Clearly, if T is r-equitably k-colorable, then $k \geq 2$. Let us show now the converse. The result trivially holds for $k = 2$ (we simply set $V_1(c) = X$ and $V_2(c) = Y$ and hence c is an r-equitable 2-coloring). Thus, we may assume that $k \geq 3$.

If $n_1 \leq r$, it follows that $n_2 \leq 2r$. Then we obtain an r-equitable k-coloring c by setting $V_1(c) = X$ and by assigning color 2 to $\min\{r, n_2\}$ vertices in Y and color 3 to the remaining vertices in Y. Hence, we may assume that $n_1 > r$.

Now delete $n_2 - n_1 - 1$ vertices from Y. Notice that $n_2 - n_1 - 1 \leq r - 1$ since $n_2 - n_1 \leq r$. Let $F = (X, Y', E)$ be the remaining graph. Clearly $|\,|X| - |Y'|\,| = 1$ since $|Y'| = n_1 + 1$. If necessary, we add some arbitrary edges between X and Y' in order to make F become a tree. It follows from Theorem 1.1, that F admits an equitable k-coloring. Moreover, it follows from the above that there exists such an equitable k-coloring c with the property that at most one of its color classes contains vertices from both X and Y'. Notice that in this case, there must be a color class $V_i(c)$ of c, for some $i \in \{1, \cdots, k\}$, which is contained in Y'. Indeed if no such color class exists, this implies that $Y' \subset V_j(c)$ for some $j \in \{1, \cdots, k\}$ and

$|V_j(c)| \geq n_1 + 2$. But then c would not be equitable, since any remaining color class $V_{j'}(c)$, $j' \neq j$, would contain at most $|F| - n_1 - 2 = n_1 - 1$ vertices. Now we obtain an r-equitable k-coloring c' of T by copying the coloring c and by adding the deleted vertices to $V_i(c)$. □

Let us now consider the case $|\,|X| - |Y|\,| > r \geq 1$.

Theorem 3.2 *Let $T = (X, Y, E)$ be a tree such that $|\,|X| - |Y|\,| > r \geq 1$. Then T is r-equitably k-colorable if and only if $k \geq \max\{3, \lceil \frac{|T|+r}{\alpha^*(T)+r+1} \rceil\}$.*

Proof. By Lemmas 2.6 and 2.7, T is r-equitably k-colorable only if $k \geq \max\{3, \lceil \frac{|T|+r}{\alpha^*(T)+r+1} \rceil\}$. Therefore the condition is necessary. We now prove the sufficiency. By Theorem 2.2, we may assume that $\Delta(T) > \lfloor \frac{|T|+3}{3} \rfloor + \lfloor \frac{r-1}{2} \rfloor$.

Let v be any vertex of degree $\Delta(T)$ and let k be any integer at least equal to $\max\{3, \lceil \frac{|T|+r}{\alpha^*(T)+r+1} \rceil\}$. Also, let $q = \lfloor \frac{|T|-(r-1)(k-1)}{k} \rfloor$ and $p = |T| - (r-1)(k-1) - kq$, which implies that $0 \leq p \leq k-1$.

Since $k \geq \frac{|T|+r}{\alpha^*(T)+r+1}$, it follows that $\alpha^*(T) + 1 \geq \frac{|T|-r(k-1)}{k}$. Since $\alpha^*(T)+1$ is an integer, it follows that $\alpha^*(T) + 1 \geq \lceil \frac{|T|-r(k-1)}{k} \rceil = \lfloor \frac{|T|-(r-1)(k-1)}{k} \rfloor = q$. Hence, there exists an independent set S in T containing q vertices and such that $v \in S$. Moreover, since $\lfloor \frac{|T|+3}{3} \rfloor \geq \frac{|T|}{3}$ and $\lfloor \frac{r-1}{2} \rfloor + 1 \geq \frac{r}{3}$, we necessarily have $\Delta(T) \geq \lfloor \frac{|T|+3}{3} \rfloor + \lfloor \frac{r-1}{2} \rfloor + 1 \geq \frac{|T|+r}{3}$. Since $p \leq k-1$, we have $|T| - kq = p + (r-1)(k-1) \leq r(k-1)$. Hence, $k(|T| - q) = |T| - kq + (k-1)|T| \leq (k-1)(|T| + r)$. Thus, the number of connected components in $T - S$ is larger or equal to $\frac{|T|+r}{3} \geq \frac{|T|+r}{k} \geq \frac{|T|-q}{k-1}$. It follows from Theorem 2.3 that $T - S$ is equitably $(k-1)$-colorable. Each color class of the equitable $(k-1)$-coloring of $T - S$ contains either $\lfloor \frac{|T|-q}{k-1} \rfloor$ or $\lceil \frac{|T|-q}{k-1} \rceil$ vertices. Since $\frac{|T|-q}{k-1} = \frac{kq-q+(r-1)(k-1)+p}{k-1} = q + r - 1 + \frac{p}{k-1}$ and $0 \leq p \leq k-1$, each color class has $q + r - 1$ or $q + r$ vertices. Hence, the independent set S (which has size q) together with the equitable $(k-1)$-coloring of $T - S$ induce an r-equitable k-coloring of T. □

Thus Theorems 3.1 and 3.2 give a complete characterization of trees which are r-equitably k-colorable for $r \geq 1$ and $k \geq 2$.

4 r-equitably k-colorable forests

We will explain now how to extend our result on trees to the case of forests. Again we will split the result into two parts.

Theorem 4.1 *Let $F = (X, Y, E)$ be a forest containing at least one edge and such that $|\,|X| - |Y|\,| \leq r$, where $r \geq 1$. Then F is r-equitably k-colorable if and only if $k \geq 2$.*

Proof. Clearly, if F is r-equitably k-colorable, then $k \geq 2$. Let us show now the converse. Assume $k \geq 2$ and let $T = (X, Y, E')$ be a tree obtained from F by adding some arbitrary edges between X and Y. It follows from Theorem 3.1 that there exists an r-equitable k-coloring c of T. It is obvious to see that c is also an r-equitable k-coloring of F. □

Theorem 4.2 *Let $F = (X, Y, E)$ be a forest such that $|\,|X| - |Y|\,| > r \geq 1$ and let $k \geq 3$ be an integer. Then F is r-equitably k-colorable if and only if $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$.*

Proof. It follows from Lemma 2.7 that F is r-equitably k-colorable only if $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$. Therefore the condition is necessary. To prove the sufficiency, we distinguish three cases.

- If $\Delta(F) \leq 1$ then F is clearly r-equitably k-colorable.
- If $2 \leq \Delta(F) \leq \lfloor\frac{|F|+3}{3}\rfloor + \lfloor\frac{r-1}{2}\rfloor$, then let $F_1, \cdots, F_d$ be the connected components of F. For every F_i that is not a single vertex, let x_i and y_i denote two distinct leaves in F_i. For every F_i consisting of a single vertex u, let $x_i = y_i = u$. We add edges $x_i y_{i+1}$ for $i = 1, \cdots, d-1$ to get a tree T such that $|F|=|T|$ and $\Delta(F) = \Delta(T) \leq \lfloor\frac{|T|+3}{3}\rfloor + \lfloor\frac{r-1}{2}\rfloor$. It then follows from Theorem 2.2 that T is r-equitably k-colorable. Clearly, the same coloring is also an r-equitably k-coloring of F.
- If $\Delta(F) > \lfloor\frac{|F|+3}{3}\rfloor + \lfloor\frac{r-1}{2}\rfloor$, then by applying the same arguments as in the proof of Theorem 3.2, we can show that F contains an independent set S with $q = \lfloor\frac{|F|-(r-1)(k-1)}{k}\rfloor$ vertices and $F - S$ is equitably $(k-1)$-colorable, each color class having $q+r-1$ or $q+r$ vertices. Hence, F is r-equitably k-colorable.

□

Notice that Lemma 2.6 shows that a tree $T = (X, Y, E)$ is r-equitably 2-colorable if and only if $|\,|X| - |Y|\,| \leq r$. As already mentioned in [2], characterizing the equitable 2-colorability of forests is more complicated because it turns out that this is equivalent to a partitioning problem. Since no explicit proof of the complexity status has been given so far, we will show here that the problem of deciding whether a forest $F = (X, Y, E)$ is equitably 2-colorable is $\mathcal{NP}$-complete.

Theorem 4.3 *Let $F = (X, Y, E)$ be a forest. Then deciding whether F is equitably 2-colorable is $\mathcal{NP}$-complete.*

Proof. Consider the PARTITION problem which is defined as follows: we are given a finite set A and a size $s(a) \in \mathbb{Z}^+$ for each $a \in A$; the question is whether there exists a subset $A' \subseteq A$ such that $\sum_{a\in A'} s(a) = \sum_{a\in A\setminus A'} s(a)$. It was shown that PARTITION is $\mathcal{NP}$-complete even if the elements in A are ordered as $a_1, \cdots, a_{2n}$ and we require that A' contains exactly one of a_{2i-1}, a_{2i} for $1 \leq i \leq n$ (see [4]). We will refer to this problem as R-PARTITION and we will use a reduction from this problem to show the $\mathcal{NP}$-completeness. Notice that we may assume that $\sum_{a\in A} s(a)$ is even otherwise there is clearly no solution.

Consider an instance $\mathcal{I}$ of R-PARTITION. Construct a forest F as follows: for every $i \in \{1, \cdots, n\}$, consider an arbitrary tree T_i with bipartition X_i, Y_i where $|X_i| = s(a_{2i-1})$ and $|Y_i| = s(a_{2i})$. This clearly gives us a forest $F = (X, Y, E)$ with n connected components $T_1, \cdots, T_n$ and $X = \bigcup_{i=1}^n X_i$ and $Y = \bigcup_{i=1}^n Y_i$.

Now suppose that the answer to $\mathcal{I}$ is yes. Then we obtain an equitable 2-coloring of F as follows: for every $i \in \{1, \cdots, n\}$, if $a_{2i-1} \in A'$ we color the vertices of X_i with color 1, and if $a_{2i} \in A'$ we color the vertices of Y_i with color 1; after, all remaining yet uncolored vertices will get color 2. This clearly gives us an equitable 2-coloring of F. Conversely suppose now that F admits an equitably 2-coloring c. Since F has an even number of vertices we have $|V_1(c)| = |V_2(c)|$. Then we construct A' as follows: for every tree T_i, $i = 1 \cdots, n$, if the vertices of X_i have color 1 we add a_{2i-1} to A' and if the vertices of Y_i have color 1, then we add a_{2i} to A'. Thus $A' \subseteq A$ is such that $\sum_{a\in A'} s(a) = \sum_{a\in A\setminus A'} s(a)$ and hence $\mathcal{I}$ has answer yes. □

5 Conclusion

In this paper we considered r-equitable k-colorings of trees and forests for $r \geq 1$ and $k \geq 2$. While the problem of equitable colorability has been extensively studied, no results seem to be known about r-equitable colorability for $r > 1$. Here we generalized known result for $r = 1$ to the case $r \geq 1$. This generalisation is quite natural since many k-colorable graphs do not admit equitable k-colorings. Thus our paper is a first step towards a generalisation of equitable colorings but many interesting questions remain open, for instance the r-equitable colorability of chordal graphs or series-parallel graphs.

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