

**Dynamic Risk Management:
Investment, Capital
Structure, and Hedging**

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Abstract

This paper develops a dynamic model to determine a firm's optimal risk management strategy when it faces uncertainty about its future profitability and investment opportunities. To address the complexity that results from a realistic description of the firm's environment, this article combines numerical methods and sensitivity analysis to study the optimal policies implied by the model. The interactions between financing, hedging, and investment decisions, as well as the effects of the firm's business environment on these interactions are analyzed.

Key Words: Dynamic programming; risk management; capital structure; hedging.

Résumé

Dans cet article, nous proposons un modèle dynamique afin de déterminer la stratégie de gestion optimale en présence d'incertitude concernant la profitabilité et les opportunités d'investissement futures. Nous utilisons une combinaison de méthodes numériques et d'analyses de sensibilité afin d'étudier les politiques de gestion optimales. Les interactions entre le financement, la couverture, les décisions d'investissement et l'environnement financier de la firme sont analysées.

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1 Introduction

The recent financial crisis has shown that developing an effective risk management strategy remains a challenge for many corporations. This may be surprising at first, for recent years have witnessed at least three significant advances in risk management. First, thanks to the tremendous increase in computing power, most industrial firms can now estimate their future profitability under millions of scenarios that describe the possible evolution of key business drivers such as input prices and exchange rates, supply chain performance, operational effectiveness, and competitors' strategies. Second, firms now can choose from a large variety of instruments to transfer specific risks (currency, interest rate, raw material cost, insurable events, etc.). Finally, the economic theory of risk management is now firmly established, providing a coherent conceptual framework for developing an optimal risk management strategy and reversing the Modigliani and Miller (1958, 1963) paradigm of risk management's irrelevance.

However, despite these advances, most industrial firms lack a robust process to determine an acceptable level of risk given their strategic plan and financial situation. Oftentimes, inconclusive discussions between senior managers and board members about a firm's risk management strategy revolve around the ill-defined concept of "risk appetite" and how to implement it. The 2008 twin financial and economic crises make the resolution of this issue central to the survival and long-term prospects of many firms.

This article proposes an integrated framework that allows firms to jointly determine their capital structure, hedging strategy, and investment decisions. It complements and differs from the rich academic literature on the subject (reviewed briefly below), as it recognizes that an increase in a firm's cost of capital raises the cost of its entire capital base, hence the rate at which future cash flows are discounted.

Modern risk management theory is grounded in the observation that raising external funds is costly, hence not all value-creating firms or projects are financed. Holmström and Tirole (2000) show how information asymmetry between outside investors and entrepreneurs/insiders produces this striking reversal of Modigliani and Miller (1958, 1963) irrelevance propositions. As a result, profitable – but cash constrained – firms may not be able to refinance themselves after a negative shock to their cash flow, hence may go bankrupt, and firms with insufficient internal funds may have to forego profitable investment opportunities (the underinvestment problem).

The interaction between costly external financing, underinvestment, and risk management was first described in the two-period models proposed by Froot et al. (1993) and Froot and Stein (1998). The former article considers a firm facing random cash flows and random investment opportunities, and derives the optimal hedging strategy that mitigates the expected cost of underinvestment. The latter article introduces capital structure as a risk management device. A marginal increase in equity raises the financial flexibility, hence the ability to pursue risky investments. On the other hand, it reduces the tax-shield from interest payments. The optimal equity level balances these two effects.

Other researchers have examined this trade-off in a multi-period environment. For example, Rochet and Villeneuve (2006) develop an infinite-horizon continuous-time model, where a constant-size firm faces exogenous cash flow shocks and a "hard" liquidity constraint: the firm is liquidated as soon as its cash reserve becomes negative. At each instant, the firm selects its dividend payment and decides its hedging ratio or insurance coverage for discrete risks. Rochet and Villeneuve (2006) observe that the risk management problem is an inventory management problem, where the cash reserve is the state variable, and dividend payment and risk transfer decisions are the control variables. They then proceed to show that the firm pays dividends if and only if the cash reserve exceeds a threshold, and it fully hedges if the cash reserve is below the threshold. In addition, they show that the firm insures small risks but not large ones.

Bolton et al. (2009) extend Rochet and Villeneuve (2006), most notably by including investment and a refinancing possibility, albeit costly. Exploiting the homogeneity of the profit and cost functions, they first characterize the optimal investment, refinancing, and dividend distribution policies. As Rochet and Villeneuve (2006), they find that a firm optimally distributes dividends if and only if its cash reserves (as a percentage of its size) exceed a given threshold. Below that threshold, growth is self-financed, and the optimal investment policy equalizes the marginal cost of adjusting physical capital to the ratio of marginal value of

capital (the marginal Tobin's q) to the marginal cost of financing. They then determine the optimal hedging policy, taking into account margin requirements.

Léautier et al. (2007) consider a firm in a multi-period environment (Rochet and Villeneuve (2006)) facing uncertainty about both future cash flows and future investment opportunities (Froot et al. (1993)). The firm's refinancing constraint is incorporated through the Weighted Average Cost of Capital (WACC), which is convex in the firm's leverage ratio. For low leverage ratios, raising debt increases the tax shield, hence reduces after-tax WACC. After a certain level, leverage is too high, investors require a higher return, and consequently the WACC increases. An increase (decrease) in the cost of capital is applied to the entire capital base, thereby increasing (decreasing) the rate at which future cash flows are discounted. The leverage ratio then becomes the state variable.

The previous approach is more realistic and flexible than the extreme, no-refinancing representation of Rochet and Villeneuve (2006): when its cash reserve runs low, a firm can still secure refinancing, albeit at an increasingly high cost. The main innovation of this approach is that this higher marginal cost of financing is then applied to the entire capital base, which is justified both theoretically and empirically. Economic theory suggests that price must be set by the marginal transaction: the cost of capital must therefore reflect the value of the marginal unit. This is confirmed in practice; for example, some loan covenants increase the cost of all loans when credit rating deteriorates.

This article builds upon the multi-period model of Léautier et al. (2007) to provide a theoretical and numerical guideline for the optimal investment and hedging policies of a firm. However, we relax two key assumptions which generate distinctive results. First, any surplus after debt repayment is paid out to stockholders in the form of dividends. Second, the possibility that the firm goes bankrupt is incorporated into the model when all of its investments are financed from external sources.

The analysis of the proposed model is by no means trivial, since a realistic description of the firm's environment precludes closed-form solutions for the optimal policies. To overcome this issue, this article combines numerical methods and sensitivity analysis to study the optimal policies implied by the model. Specifically, it employs the numerical implementation of the dynamic programming problem proposed in Ben-Ameur et al. (2009) to solve the Bellman equations arising from the model's formulation, while analytically characterizing local variations of the firm's decision variables - hedging and investment policies - in the model equations.

The numerical solution and the analysis of the model shed light on a number of management issues that are summarized below. In particular, the analysis documents the interplay between hedging and investment policies as functions of the firm leverage ratio and its business environment.

The first important result that emerges from this numerical study – and that contrasts with the standard result found in the literature of risk management (for example Froot et al. (1993), Holmström and Tirole (2000)) – is that independence between investment opportunities and profitability does not imply full hedging as the only optimal strategy: under certain circumstances, no or partial hedging may be the optimal decision. For firms with moderate leverage ratios, full hedging is the optimal strategy, as the underinvestment problem is more likely to arise in this case. However, a firm with a low leverage ratio has enough borrowing capacity to not be concerned about the possibility of negative profits, and hence can benefit from ceding control over its profitability's volatility. In this case, the Modigliani-Miller classical result of risk management's irrelevance holds. Moreover, this irrelevance proposition also holds for firms that are highly levered and have to "gamble for resurrection" by keeping a high profitability volatility. These results reconcile mixed empirical findings concerning the hedging-leverage relationship. For instance, Allayannis and Ofek (2001) find no support for a positive relationship between leverage and hedging, while Gay and Nam (1998), Graham and Roger (2002), and Dolde (1995), using different data sets, find evidence to support this relationship.

The dependence between profitability and investment opportunity has a direct impact on hedging and demand for borrowing. Indeed, it is found that positive dependence between opportunities and profitability acts as a natural hedge for the firm (also documented by Froot et al. (1993) and Léautier (2007)), while negative dependence exacerbates the underinvestment problem.

The analysis also documents how a firm in a riskier environment is able to create value. To this end, the firm manages the higher volatility of profitability by hedging more and protects its financial flexibility by using less debt. A high volatility in profits is especially beneficial to firms with low or large leverage levels that can increase their upside potential while limiting the downside risk.

The next set of findings are related to the firm's optimal investment decision. First, it is found that the investment decision depends on the firm's leverage level in a non-linear way. The firm fully invests for small leverage values, then as the leverage increases, the investment level decreases until it is no longer optimal for the firm to invest. Second, capturing the investment opportunity uses up financial flexibility, and hence requires the firm to better control for the volatility of its profitability. As a consequence, the presence of investment opportunities provides incentives for the firm to be more conservative as it hedges more and borrows less. The positive relationship between investment opportunities and hedging has previously been documented in empirical works of Nance et al. (1993), Gay and Nam (1998), and Graham and Roger (2002), in which R&D expenses, a proxy for growth opportunities, are positively related to hedging.

This paper further investigates the relationship between leverage and investment opportunities, where it is established that firms with large investment opportunities preserve their financial flexibility by borrowing less. This negative relationship is also documented in Myers (1977), where the author argues that agency costs between shareholders and debtholders are more important for firms with higher growth opportunities, so firms with attractive investment opportunities are more likely to use less debt. Empirically, Smith and Watts (1992) find that firms with greater access to positive net present value projects (more investment opportunities) employ a lower leverage.

Finally, the firm's optimal decisions are reexamined in a context where the firm is allowed to increase its profitability's volatility by gaining more exposure to the underlying source of uncertainty. It is found that, for a firm with small or large leverage, it is optimal to increase its volatility by selling short its risk. Moreover, when the firm adopts such behavior, it increases its financial flexibility by using less debt.

The rest of the paper proceeds as follows: Section 2 introduces the multi-period model for the value of the firm. Section 3 formulates the problem of jointly determining the initial optimal capital structure and the optimal risk management strategy as a dynamic programming problem. Section 4 conducts a numerical analysis of the proposed framework and examines the impact of changes in the business environment on the optimal strategies. Section 5 concludes.

2 Model Specification

The business environment of a firm is governed by various sources of uncertainty that fall into two categories: (1) returns on invested capital and (2) investment opportunities. The former recognizes that senior managers have limited knowledge about the future outcome of an investment. The latter captures the fact they do not know in advance when or whether new investment opportunities will arise or materialize, since they have imperfect foresight of factors that affect the realization of these projects (i.e. regulatory approval, operational limitations, etc.).

As opportunities arise, firms determine their optimal level of investment by taking into account their current financial flexibility and the profitability of the prospective investment. Obviously, financial flexibility increases the likelihood of a firm capturing profitable investment opportunities. However, financial flexibility is also costly and, if not correctly managed, can undermine the future performance of a firm. This trade-off between financial flexibility and level of investment is precisely the feature that this model is designed to capture.

2.1 Timing and Decisions

For $t \geq 1$, period t runs from dates $(t - 1)$ to t . At the beginning of period t , the firm's invested capital is I_{t-1} . Throughout period t , the firm deploys capital. At date t , the invested capital is I_t . The firm's capital structure at date t is debt D_t and equity E_t . As usual, equity has two sources: share issues and retained

earnings. This article takes a highly simplified view of equity and dividends: dividends d_t are supposed to be paid to shareholders only if the firm has a surplus after repaying the total value of its debt; and firms do not issue shares. This last assumption is consistent with empirical evidence as well as theoretical models. For instance, Rajan and Zingales (2003) report that the fraction of gross fixed-capital formation raised via equity (including initial and seasoned equity offerings) in 1999 was only 12% in the United States, 9% in the United Kingdom and France, 8% in Japan, and 6% in Germany. Most corporate Treasurers are reluctant to issue shares, as seasoned equity offerings usually yield a permanent fall in the stock price of about 3% (see Tirole (2006), page 101, and the references included). On the theoretical front, this is consistent with the "pecking order" theory of financing. Further work can relax this assumption and identify situations where share issuance would be optimal for firms.

The level of investment and debt determine the firm's leverage ratio $\lambda_t = \frac{D_t}{I_t}$ for a given date t . This ratio plays an important role in the proposed framework, since it reflects the current financial state of the firm, which in turn determines the firm's financing costs. The relationship between leverage and financial costs will later be discussed in greater detail.

Throughout period t , the invested capital I_{t-1} generates randomly distributed returns x_t , expressed in percent, that represent the Return On Invested Capital (ROIC)¹ plus the depreciation rate. The Post-Tax Cash Operating Profit realized during period t is then:

$$\pi_t = x_t I_{t-1}. \quad (1)$$

At date t , an investment opportunity arises. The magnitude of the opportunity i_t is expressed as a fraction of the invested capital I_t . Therefore, the nominal opportunity is $i_t I_t$. Investment opportunities $\{i_t : t \in \mathbb{N}\}$ are assumed to be independent and identically distributed. Note, however, that the returns x_t and the investment opportunities i_t can be correlated.

The firm makes two decisions at every date t :

1. The level of (net) investment in the opportunity, denoted g_t , expressed as a fraction of the initial invested capital. Given the investment opportunity i_t , the firm's investment cannot exceed this value. This condition gives rise to the following constraint:

$$0 \leq g_t \leq i_t. \quad (2)$$

The amount invested throughout period $(t+1)$ is then equal to $g_t I_t$. Consequently, the capital invested at date $(t+1)$ is $I_{t+1} = I_t + g_t I_t = (1 + g_t) I_t$. The time-lag between the investment decision (date t) and the moment where the returns are recognized (date $t+2$) is caused by two factors: the investment decided at date t is spread throughout the period $(t+1)$; and the investment generates returns only when completed from date $(t+1)$ onwards.

2. The hedging ratio, denoted η_t , which will be discussed in Section 2.4.

Note that the leverage ratio of the firm for $t \geq 1$ depends not only on the decisions made at $t-1$, but also on the Post-Tax Cash Operating Profit and the investment opportunities realized. At date $t=0$, the firm selects its initial capital structure, which corresponds to the selection of the leverage ratio λ_0 .

2.2 Cost of Capital

Denote by $w(\lambda)$ the Weighted Average Cost of Capital (WACC). Although theory suggests that market values of debt and equity should be used for the computation of the firm's WACC, this article uses book values, since they are commonly employed in practice. Define $k_d(\lambda)$ as the cost of debt and $k_e(\lambda)$ as the cost of equity for a given leverage ratio λ . Denote by τ the cash tax rate or debt subsidy percentage. Then:

$$w(\lambda) = \lambda k_d(\lambda)(1 - \tau) + (1 - \lambda)k_e(\lambda). \quad (3)$$

¹See for example Copeland et al. (1995).

Empirical and theoretical evidence indicates that the cost of capital is a convex function of the leverage ratio with a unique minimum $\bar{\lambda}$ on $[0, 1]$. On the theory front, Holmström and Tirole (2000) show that a borrowing constraint arises from information asymmetries between entrepreneurs and lenders: if a firm's leverage exceeds a given threshold, it simply cannot obtain external financing. The approach used here "smooths" this discontinuity by using a progressively increasing cost of funding. To the left of $\bar{\lambda}$, the cost of capital decreases with λ , since debt is tax-advantaged and adding debt contributes to a reduction in the cost of capital. To the right of $\bar{\lambda}$, the cost of capital increases as financial ratios deteriorate to a point where the probability of financial distress is no longer negligible, and both debt holders and equity holders start to demand additional premium for holding such risk. Eventually, the firm's leverage deteriorates to the point where the firm is no longer able to access financial markets and has to declare bankruptcy. Therefore, $w(\lambda)$ has an asymptote at $\lambda = 1$, which represents the point at which the firm is bankrupt.

2.3 Free Cash Flow During Period $(t + 1)$

The Free Cash Flow (FCF) is the Post-Tax Cash Operating Profit (1) minus the net increase in invested capital $I_{t+1} - I_t = g_t I_t$:

$$FCF_{t+1} = (x_{t+1} - g_t)I_t. \quad (4)$$

Denoting the after-tax initial cost of debt by $r(\lambda_t)$, the Financial Flow (FF) for this period is the after-tax interest payments² $r(\lambda_t)D_t$, minus the changes in financial structure $D_{t+1} - D_t$, plus the dividends paid during the period d_{t+1} :

$$FF_{t+1} = r(\lambda_t)D_t - (D_{t+1} - D_t) + d_{t+1}.$$

Free cash flow equals financing flow for the period, hence:

$$(x_{t+1} - g_t)I_t = r(\lambda_t)D_t - (D_{t+1} - D_t) + d_{t+1}. \quad (5)$$

Dividing equation (5) by I_t and observing that $I_{t+1} = (1 + g_t)I_t$ yields:

$$x_{t+1} - g_t = \lambda_t(1 + r(\lambda_t)) - \lambda_{t+1}(1 + g_t) + \tilde{d}_{t+1}, \quad (6)$$

where $\tilde{d}_{t+1} = d_{t+1}/I_t$. Dividends are only distributed when the total amount of debt is repaid by the firm. Equation (6) then yields:

$$\tilde{d}_{t+1} = (x_{t+1} - g_t - \lambda_t(1 + r(\lambda_t)))^+, \quad (7)$$

where $(\cdot)^+$ takes the value zero if the quantity in parenthesis is negative. Equation (7) indicates that dividends represent the excess cash flow once interests and debt have been paid. Solving (7) for λ_{t+1} , and taking into account that λ_{t+1} can only take values less than or equal to one, the leverage ratio for $t + 1$ is

$$\lambda_{t+1} = \min(\Lambda_{t+1}^+, 1), \quad (8)$$

where

$$\Lambda_{t+1} = \frac{g_t - x_{t+1} + \lambda_t(1 + r(\lambda_t))}{1 + g_t}. \quad (9)$$

As expected, the leverage at date $(t + 1)$ increases with the previous-period leverage (λ_t) and decreases with the profitability realized during period (x_{t+1}) . A higher investment level g_t does not always lead to higher leverage: increments arrive when x_{t+1} is bigger than $\lambda_t(1 + r(\lambda_t)) - 1$.

2.4 Hedging Technology

The primitive source of uncertainty that affects the ROIC during period t is denoted by the random variable z_t . For example, for an oil company, z_t is the price of crude oil on international markets multiplied by its per-period production. Returns on invested capital $\{z_t : t \in \mathbb{N}\}$ are assumed to be serially independent

²In fact, since the investment is spread throughout the period, the interest payments are slightly higher than $r(\lambda_t) \cdot D_t$. For example, firms sometimes use the average debt: $r(\lambda_t) \cdot \frac{D_{t+1} + D_t}{2}$. This extension is left for future work.

and identically distributed through time. This assumption implies that the firm does not diversify on its investments.

The firm only uses linear hedging strategies (e.g. forward sales or purchases) whose main purpose is to reduce the volatility of the ROIC. Supposing an expected forward premium of zero, the profitability x_{t+1} during period $(t + 1)$ is:

$$x_{t+1} = \eta_t \mathbf{E}[z] + (1 - \eta_t) z_{t+1}, \quad (10)$$

where $\mathbf{E}[z]$ denotes the expected value of z_{t+1} and η_t stands for the hedging ratio selected at date t .

From equation (10), it can be seen that hedging does not affect expected profitability, since $\mathbf{E}[x_{t+1}] = \mathbf{E}[z]$ for all values of η_t . This means that hedging per se does not create value, which is a familiar result in risk management. However, as expected, hedging modifies the profitability's volatility:

$$\mathbf{Var}(x_{t+1}) = (1 - \eta_t)^2 \mathbf{Var}(z). \quad (11)$$

Equation (11) shows that the firm can gain exposure to the source of uncertainty by using hedging ratios that increase its volatility: $\eta_t < 0$ or $\eta_t > 1$. The firm is assumed to be prevented from selling short its intrinsic profitability z_{t+1} , i.e., $\eta_t \leq 1$. In most of this article, the firm is also assumed to be prevented from increasing its exposure beyond its "natural exposure" by purchasing z_{t+1} forward, i.e., $\eta_t \geq 0$. In practice, these restrictions arise from concerns of the Board of Directors that managers should not use derivatives to speculate. Later in the paper, the constraint $\eta_t \geq 0$ is relaxed, which corresponds to the case in which the firm is allowed to gamble by increasing the volatility of its profitability if needed.

2.5 Value of the Firm and Relative Firm Value

Following the standard valuation approach, V_t – the value of the firm at date t – is the expectation of the discounted sum of the free cash flows generated at each period:

$$V_t = \mathbf{E}_t \left\{ \sum_{s=t+1}^{\infty} \frac{FCF_s}{\prod_{k=t+1}^s (1 + w(\lambda_{k-1}))} \right\}, \quad (12)$$

where the conditional expectation \mathbf{E}_t is taken with respect to all random variables z_u, i_u , for $u > t$. As the firm's cost of capital increases, the value of the cash flows, discounted back to date t , is reduced, thereby encouraging firms to manage their capital structure. This will be seen in Section 4 of this paper.

From (12), and using the identity

$$I_{t+s} = I_t \prod_{k=0}^{s-1} (1 + g_{t+k}), \quad s \geq 1$$

the firm value V_t is:

$$V_t = I_t \mathbf{E}_t \left\{ \sum_{s=t+1}^{\infty} \frac{x_s - g_{s-1}}{\prod_{k=t+1}^s (1 + w(\lambda_{k-1}))} \prod_{u=t}^{s-2} (1 + g_u) \right\}, \quad (13)$$

with the convention that $\prod_{u=t}^{t-1} (1 + g_u) = 1$.

To measure the firm's ability to create value from the invested capital, the firm's relative value at time t is set to be:

$$v_t = \frac{V_t}{I_t}, \quad (14)$$

that is, the ratio between the value of the firm V_t and the cost of its invested capital I_t . The ratio v_t corresponds to the average Tobin's Q. Equations (13) and (14) then yield:

$$v_t = \mathbf{E}_t \left\{ \sum_{s=t+1}^{\infty} \frac{x_s - g_{s-1}}{\prod_{k=t+1}^s (1 + w(\lambda_{k-1}))} \prod_{u=t}^{s-2} (1 + g_u) \right\}. \quad (15)$$

3 Dynamic Programming Formulation

Choosing the optimal hedging and level of investment can be formulated as a dynamic programming problem, where the relative value of the firm v_t is maximized.

Following the logic of previous sections, the state of the system is summarized at date t by (a) the leverage λ_t and (b) the investment opportunities available i_t . The decision variables or controls in this context are the hedging ratio η_t and the investment g_t . The disturbances are represented by the variables z_s and i_s , for $s = t + 1, t + 2$, and so on.

Conditional on the state of the firm at date t , the problem of maximizing the relative value of the firm at date t is:

$$J_t^*(\lambda_t, i_t) = \max_{\substack{\{g_s, \eta_s\}_{s \geq t} \\ 0 \leq g_s \leq i_s, 0 \leq \eta_s \leq 1}} \mathbf{E}_t \left\{ \sum_{s=t+1}^{\infty} \frac{x_s - g_{s-1}}{\prod_{k=t+1}^s (1 + w(\lambda_{k-1}))} \prod_{u=t}^{s-2} (1 + g_u) \right\}. \quad (16)$$

In practice, firms use a finite horizon model for valuation, and cash flows are explicitly computed until date T . From time T onwards, a continuing value is estimated for cash flows.

3.1 Continuing Value at Date T

To estimate the continuing value, assume that from date $(T + 1)$ onwards, the following values remain constant:

- Long-term growth rate in invested capital, denoted by g ,³
- Cost of capital given by $w(\lambda_{T+1})$,
- ROIC of $\mathbf{E}[z]$, with $g \leq \mathbf{E}[z]$.

Given that the continuing value is a discounted infinite stream of cash flows, g is assumed to be less than $w(\lambda_{T+1})$ so that the infinite series converges. Given that for all $s \geq T + 2$, $x_s = \mathbf{E}[z]$, $g_{s-1} = g$, $w(\lambda_{s-1}) = w(\lambda_{T+1})$, and $I_{s-1} = (1 + g)^{s-T-2} I_{T+1}$, equation (15) becomes:

$$J_{T+1}(\lambda_{T+1}, i_{T+1}) = \mathbf{E}_{T+1} \left\{ (\mathbf{E}[z] - g) \sum_{s=T+2}^{\infty} \frac{(1 + g)^{s-2-T}}{(1 + w(\lambda_{T+1}))^{(s-1)-T}} \right\} = \frac{\mathbf{E}[z] - g}{w(\lambda_{T+1}) - g}. \quad (17)$$

The continuing value is equivalent to the value of a growing perpetuity of the expected free cash flow at date $(T + 1)$, discounted by the cost of capital of the firm at date $T + 1$.

3.2 Bellman Equations

Using the continuing value at time $T + 1$ and taking $t = T$ as the last decision point, Appendix A shows that the optimal solution to equation (16) verifies:

³Note that g is not necessarily the growth rate g_T , since the latter may not be representative of the firm's long-term growth potential.

$$J_T^*(\lambda_T, i_T) = \max_{\substack{g_T, \eta_T \\ 0 \leq g_T \leq i_T, 0 \leq \eta_T \leq 1}} \frac{1}{1 + w(\lambda_T)} \left(\mathbf{E}[z] - g_T + (1 + g_T) \mathbf{E}_T \left\{ \frac{E[z] - g}{w(\lambda_{T+1}) - g} \right\} \right), \quad (18)$$

and for all $t = 1, \dots, T - 1$:

$$J_t^*(\lambda_t, i_t) = \max_{\substack{g_t, \eta_t \\ 0 \leq g_t \leq i_t, 0 \leq \eta_t \leq 1}} \frac{1}{1 + w(\lambda_t)} \left\{ \mathbf{E}[z] - g_t + (1 + g_t) \mathbf{E}_t \left\{ J_{t+1}^*(\lambda_{t+1}, i_{t+1}) \right\} \right\}, \quad (19)$$

with a transition function for λ_{t+1} given by equation (8). Equations (18) and (19) show the role of the investment ratio g_t and the hedging ratio η_t . At any decision period, investment g_t reduces the FCF at the next period and the relative value increases proportionally to the magnitude of the investment. Additionally, investment affects leverage, which in turn affects the discount rate and the continuing value. The hedging ratio influences the firm's future leverage, thus allowing the firm to alter the distribution of its ROIC for the next period. As a result, the firm is able to control the capital cost of its future cash flows.

4 Numerical Results

This section presents a numerical analysis of the problem. It begins by presenting the base-case parameters for the analysis.⁴ Results are then illustrated for the one-period model described in equation (18), and the extended analysis for the firm's relative value and optimal decision policies is presented for the multiple-period model given in equation (19). This section also defines and analyzes the optimal risk management strategy and how this strategy responds to changes in the firm's financial environment. Finally, the case of unrestricted positions on future contracts is studied, as well as the impact of these positions on the firm's optimal decisions.

4.1 Parameter Values

4.1.1 Sources of Uncertainty

The underlying source of volatility z is normally distributed with mean $\mu_z = 10\%$ and standard deviation $\sigma_z = 4\%$. The investment opportunity i_t equals 20% with probability $p_i = 40\%$, and 0% with probability $1 - p_i = 60\%$. At every period, the firm has a 40% chance of facing a growth opportunity of 20% its size.

4.1.2 Cost of Capital

As discussed in the first section, the cost of capital decreases for leverage values to the left of $\bar{\lambda}$ and increases afterwards.

Denote $w^-(\lambda)$ as the cost of capital to the left of the minimum. Since the value of the debt varies with the value of the firm, the Miles and Ezzel (1980) model is used for this segment, that is:

$$w^-(\lambda) = \mathbf{E}[r^*] - \tau \lambda r_f, \quad (20)$$

where

$$\mathbf{E}[r^*] = r_f + \beta(\mathbf{E}[r_M] - r_f)$$

is the expected return of the firm's assets according to the CAPM, β the sensitivity of the firm's return to market returns, r_f the risk-free rate, and $\mathbf{E}[r_M]$ the expected return of the market. The parameter τ represents the cash tax rate.

Denote $w^+(\lambda)$ as the cost of capital to the right of the minimum. Debt holders and stock holders require an additional premium to compensate for the risk arising from higher leverage. Although the exact shape of the premium function has not been assessed empirically, several qualitative properties of this function can be

⁴The base case parameters are similar to those presented in Chapter 6 of ?.

used to construct an approximative model for the cost of capital in this segment. As previously discussed, the cost-of-capital function is convex in λ with an asymptote at $\lambda = 1$. Since multiple functions may satisfy these conditions, one specific example is proposed. For $\lambda \geq \bar{\lambda}$:

$$w^+(\lambda) = \frac{c}{(1-\lambda)^a}, \tag{21}$$

where $a, c > 0$. The constant a controls for the degree of convexity of the cost of capital to the right of the minimum $\bar{\lambda}$. The degree of convexity $M(\lambda)$ at a given leverage value can be measured by:

$$M(\lambda) = \frac{\frac{\partial^2}{\partial \lambda^2} w^+(\lambda)}{\frac{\partial}{\partial \lambda} w^+(\lambda)} = \frac{a+1}{(1-\lambda)}, \tag{22}$$

which captures the degree of increased upward curvature relative to the slope at a given leverage value. Larger values of a imply a higher degree of convexity. In the base case, a is set to 1.

The firm's β is assumed to be 1. Then, the risk-free rate $r_f = 5\%$, the cash tax rate $\tau = 40\%$, and the market risk premium $\mathbf{E}[r_M] - r_f = 5\%$ yield:

$$w^-(\lambda) = 10\% - 2\%\lambda.$$

The constant c is selected so that $w^-(\bar{\lambda}) = w^+(\bar{\lambda})$.

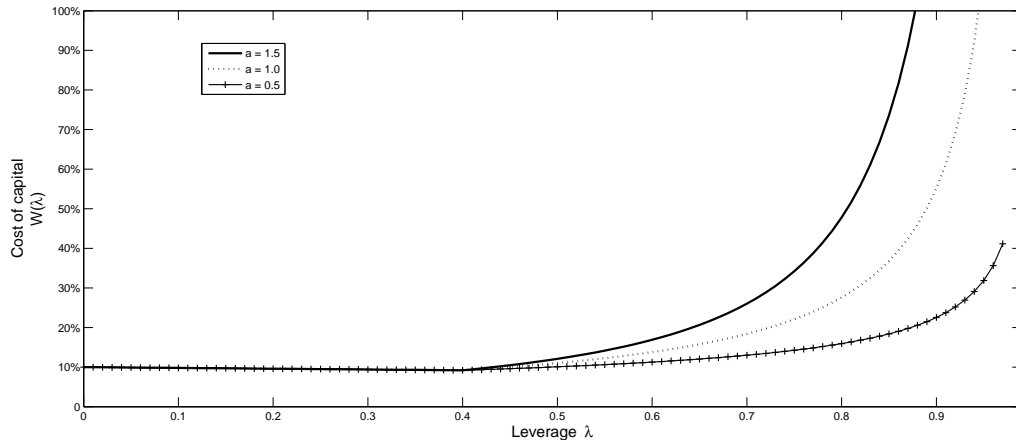
In conclusion, the WACC w is assumed to have the following functional form:

$$w(\lambda_t) = \begin{cases} 10\% - 2\%\lambda_t & \lambda_t \leq \bar{\lambda} \\ \frac{c}{(1-\lambda)} & \lambda_t > \bar{\lambda} \end{cases},$$

and the leverage at which the minimum is reached is $\bar{\lambda} = 40\%$.

Figure 1 shows three specifications for the cost of capital. As the value of a increases, the cost-of-capital curve becomes steeper. The steepest curve ($a = 1.5$) corresponds to a situation where the firm's credit environment is tighter and lenders demand a higher risk premium. The steepest curve also implies that increments on the leverage of the firm will require a higher spread from lenders, which is typical in situations of economic turmoil. The flattest curve ($a = 0.5$) corresponds to a case where either capital is abundant or a firm has a good reputation among lenders.

Figure 1: Cost of capital vs. Leverage



NOTES TO FIGURE: This figure presents the cost of capital for different convexity degrees at the right of $\lambda = 40\%$. The continuous line represents a cost-of-capital function with high convexity degree ($a = 1.5$). The dotted line depicts the cost of capital for the base case ($a = 1$). The line with crosses shows the cost of capital when the convexity is low ($a = 0.5$).

The cost of debt $r(\lambda)$ is assumed to have the same functional form of $w^+(\lambda)$, that is:

$$r(\lambda) = \frac{d}{(1-\lambda)^a}.$$

To the left of $\bar{\lambda}$, the cost of debt is assumed to be constant and equal to 3%. The parameter d is chosen so that $r(\bar{\lambda}) = 3\%$.

4.1.3 Long-term Growth Rate

The constant long-term rate at which the firm can grow without external financing is chosen as $g = 7\%$. This value guarantees that $\mathbf{E}[z] - g \geq 0$ and $w(\lambda_{T+1}) - g \geq 0$, which means that the continuation value of the firm at time $T + 1$ is a growing positive perpetuity.

4.2 One-Period Model

Consider first a firm that selects its hedging ratio η_T and level of investment g_T for only one period and, subsequently, has a guaranteed growing perpetuity that is discounted according to its WACC at period $T + 1$ (equation (17)). This corresponds to the situation described by equation (18) above.

Figure 2 shows the optimal relative value of the firm J_T^* (upper panel), the optimal hedging ratio η_T^* (lower left panel), and the optimal investment level g_T^* (lower right panel), as functions of the leverage ratio and the investment opportunity in the decision period.

Increasing the leverage affects the firm's relative value in different ways: for low leverage, an increase in this variable raises the firm's relative value; the inverse is true for high leverage. Intuitively, a firm with a low leverage ratio can benefit from the tax shield of more debt, while a firm with a high leverage ratio requires a significant risk premium that discourages it from increasing its debt. Mathematically, this behavior reflects the functional form of the cost of capital: increments to the left of $\bar{\lambda}_{T+1} = 40\%$ decrease the cost of capital, while increments to the right augment it. When the investment opportunity is present at the decision period T , the relative value reaches a maximum value of 1.40 at $\lambda_T = 37\%$; in the case where there is no investment opportunity, the maximum relative value is 1.32 and it is achieved at $\lambda_T = 48\%$.

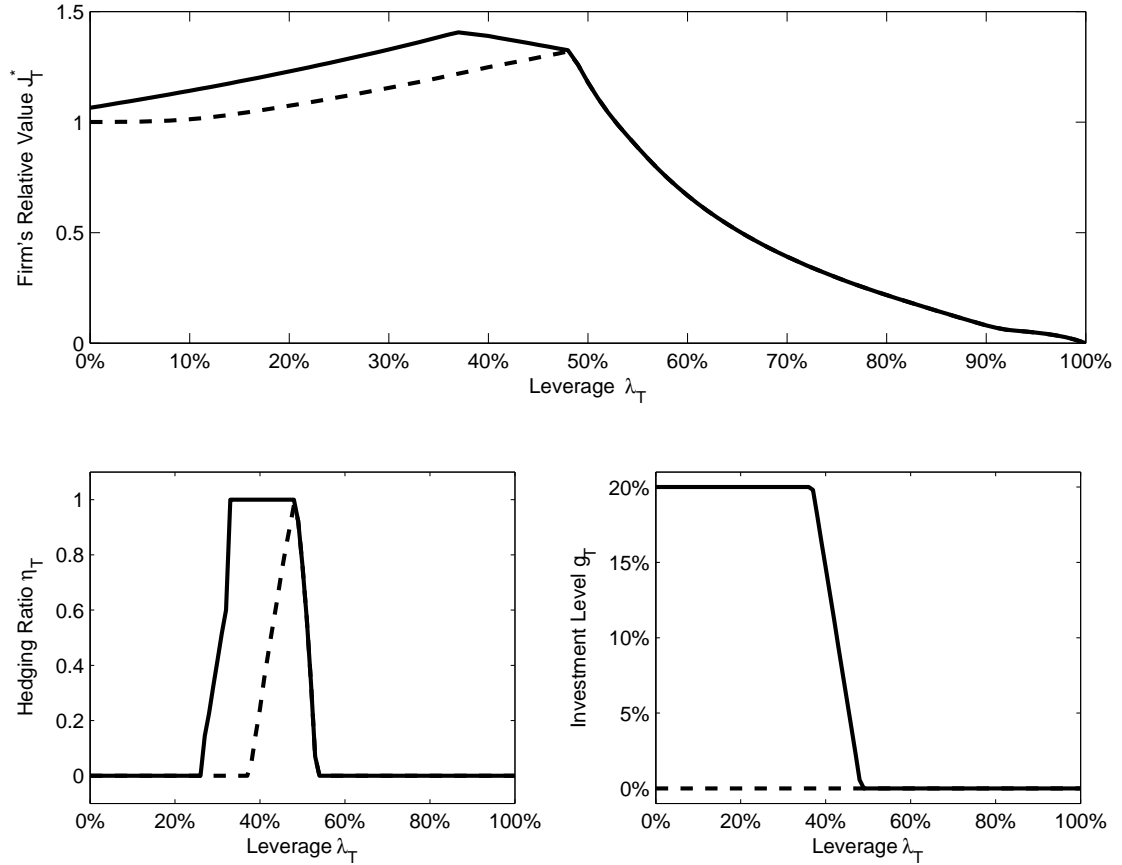
Consider now the optimal level of investment g_T . For a firm with investment opportunities, the optimal investment level varies with its leverage at time T . For small leverage values, the optimal decision is to fully invest in the opportunity ($g_T = 20\%$). For leverage values greater than $\lambda_T = 37\%$, the continuation value attained from fully investing no longer compensates the firm for the expected cost of capital associated with a higher future leverage value λ_{T+1} ; thus, the firm reduces the investment in the opportunity and consequently its value. After $\lambda_T = 48\%$, there is no additional benefit from further investing in the opportunity, and the value of the firm is the same whether an investment opportunity presents itself or not.

Regarding the optimal hedging ratio η_T , the optimal decision is not to hedge for small leverage values. For such values, the firm can allow itself to be fully exposed to the underlying source of uncertainty, since with a larger profitability's volatility, the firm can benefit from positive profits, while tolerating negative profits, as they are not likely to have a negative impact on the future cost of capital. When the firm has an investment opportunity in the decision period, it starts to hedge earlier ($\lambda_T = 27\%$) than when it has none ($\lambda_T = 38\%$). This difference arises from the fact that the expected leverage value of a firm that invests in the opportunity is higher than the expected leverage value of a firm that does not invest. Consequently, hedging becomes crucial for helping firms control their expected leverage values and undertake the investment opportunity.

To gain more insight into the interplay between the optimal decisions at time T and the expected leverage at time $T + 1$, Figure 3 shows the expected value of λ_{T+1} as a function of the leverage at time T . This value is computed using the optimal decisions associated with the state (λ_T, i_T) and equation (50) in Appendix F.

Several observations can be drawn from Figure 3. First, consistent with previous arguments, the expected leverage is higher when the firm invests in the opportunity, which occurs for leverage values of λ_T between 0% and 48%. Second, for $\lambda_T \leq 37\%$, the firm is opportunity constrained as it has the financial flexibility to invest

Figure 2: Firm’s Relative Value and Optimal Policies – One-Period Model

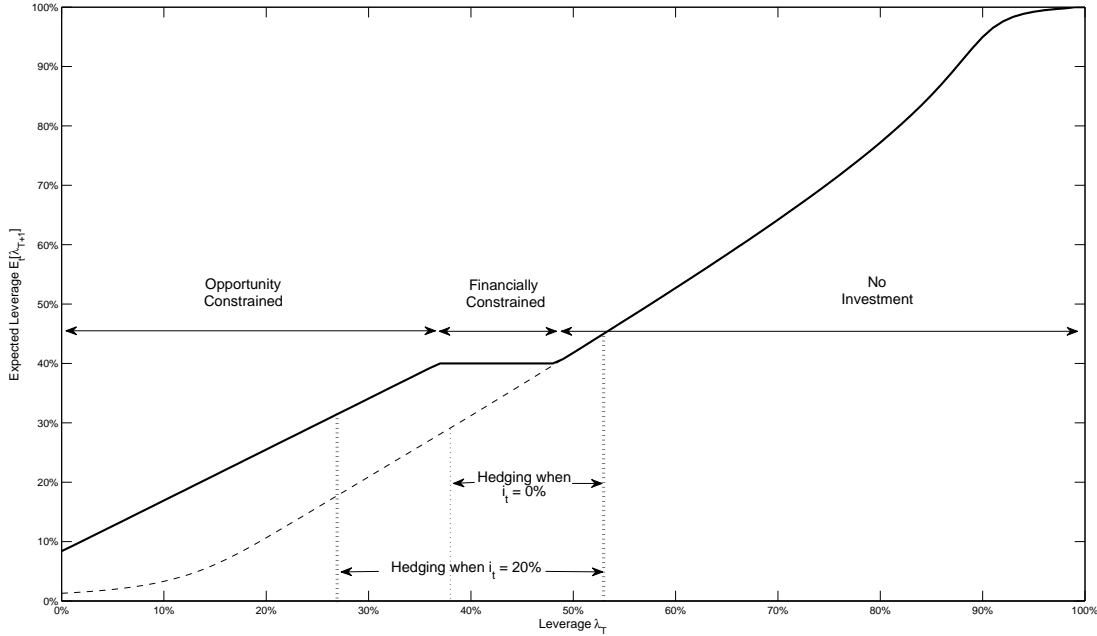


NOTES TO FIGURE: This figure shows the optimal relative value J_T^* (top figure), the optimal policy for the hedging ratio η_T (bottom left figure), the optimal policy for the investment level g_T (bottom right figure), as a function of leverage. The dotted line represents the case with no investment opportunity in the decision period ($i_T = 0$), while the continuous line represents the case when the investment opportunity is present ($i_T = 20\%$). The optimal policies and relative value are computed by solving (18). The cost of capital is minimized at $\lambda_t = 40\%$. The magnitude of the investment opportunity is equal to 20%, which arrives during a period with a probability of 40%. The expected value and standard deviation of the source of uncertainty z is 10% and 4% respectively. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$.

more. When λ_T lies between 37% and 48%, the firm is financially constrained and it cannot fully capture the opportunity. For these leverage values, the expected leverage for $T + 1$ is kept at 40%, which corresponds to the leverage that maximizes the continuing value at time $T + 1$. The firm is able to keep this expected leverage value by (1) adapting its level of investment in the opportunity and (2) fully hedging the underlying source of uncertainty. Once the firm becomes financially constrained to the point where investment is not optimal, which occurs at $\lambda_T = 48\%$, the firm progressively decreases its level of hedging and eventually leaves itself fully exposed to the underlying source of uncertainty.

4.3 Multi-period Model

This section seeks to determine the principal characteristics that govern the optimal policies and the firm’s relative value when it has to decide on these policies over multiple periods. To this end, using equation (18) as a starting solution, the approximation of the Bellman equation (19) is solved backwards until the

Figure 3: Expected Leverage Ratio for $T + 1$ – One-Period Model

NOTES TO FIGURE: This figure shows the expected leverage at time $T + 1$ as a function of the leverage ratio at time T . The dotted line represents the case with no investment opportunity in the decision period ($i_T = 0$), while the continuous line represents the case when the investment opportunity is present ($i_T = 20\%$). The expected leverage is computed with equation (50) and the associated optimal values for η_T and g_T . The arrows represent leverage regions in which the firm is fully investing in the opportunity (Opportunity constrained), investing a lower amount than the total magnitude (Financially constrained), or no investing at all (No investment). The cost of capital is minimized at $\lambda_t = 40\%$. The magnitude of the investment opportunity is equal to 20%, which arrives during a period with a probability of 40%. The expected value and standard deviation of the source of uncertainty z is 10% and 4% respectively. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$

convergence of the approximated value function \tilde{J}_t (and hence of the policy function). Convergence is achieved when $\max \|\tilde{J}_{t+1} - \tilde{J}_t\| < 10^{-5}$.⁵ A detailed description of the proposed approximation, as well as of the numerical implementation, is presented in Appendix E. Throughout the multi-period model analysis, several analytical results are used, which, for the sake of brevity, are presented in Appendix B.

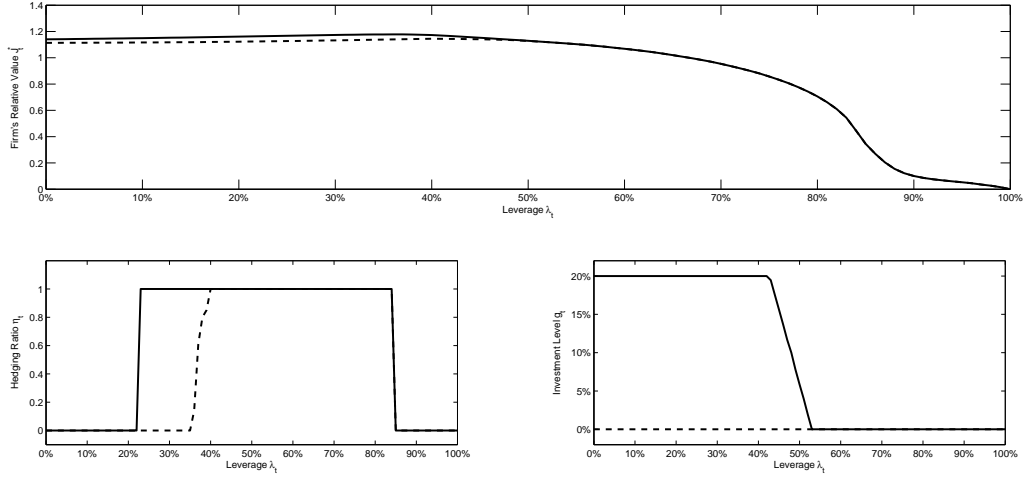
4.3.1 Firm's Relative Value

The solution to the dynamic problem defined in equation (19) provides information about the firm's relative value as a function of the leverage λ_t and the investment opportunity i_t in the decision period.

The upper part of Figure 4 shows the optimal relative value of the firm J^* as a function of the leverage and the investment opportunity at the decision period t . From this figure, the firm's relative value in the multi-period model follows a similar pattern as in the one-period model: for small leverage values, an increase in leverage marginally increases the relative value of the firm as the cost of capital decreases, whereas for large leverage values, increasing the leverage decreases the relative value. Observe that, depending on the presence of an investment opportunity, there exists a leverage value at which the relative value reaches a maximum. To the left of this maximum – contrary to the one-period model – the relative value is almost flat, suggesting that changes to the leverage ratio λ_t in this region do not have a significant impact on the

⁵A more stringent tolerance level of 10^{-7} for some cases, but equivalent results were found with a higher computational cost.

Figure 4: Firm's Relative Value and Optimal Policies – Multi-period Model



NOTES TO FIGURE: This figure shows the optimal relative value J_T^* (top figure), the optimal policy for the hedging ratio η_T (bottom left figure), the optimal policy for the investment level g_T (bottom right figure), as a function of leverage. The dotted line represents the case with no investment opportunity in the decision period ($i_T = 0$), while the continuous line represents the case when the investment opportunity is present ($i_T = 20\%$). The optimal policies and relative value are computed as described in Appendix E. The cost of capital is minimized at $\lambda_t = 40\%$. The magnitude of the investment opportunity is equal to 20%, which arrives during a period with a probability of 40%. The expected value and standard deviation of the source of uncertainty z is 10% and 4% respectively. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$

firm's relative value. This point will be further addressed when the discussion about the partial derivative of the relative value with respect to the leverage ratio is presented.

Notice that there is a change in the functional form of the firm's relative value for leverage values between 50% and 85%. In this part of the curve, as the leverage increases, decrements to the firm's debt capacity marginally increase the firm's relative value. This effect is due to the fact that, for these leverage values, the firm does not invest, so the only gain a firm obtains from decreasing its debt capacity is a reduction of the cost of capital.

4.3.2 Leverage, Hedging, and the Investment Decision

The interplay between the firm's leverage, investment decisions, and hedging policy is captured by the optimal policies for the hedging ratio η and the investment level g .

Result 1: *Full hedging is not always optimal when investment opportunities are independent of the firm's profitability.*

The first result documented in this section is the fact that the firm does not fully hedge, even though investment opportunities are independent of its profitability. A standard result in risk management literature states that if investment opportunities are independent of a firm's profitability, the firm fully hedges (Froot et al. (1993), Froot and Stein (1998), and Holmström and Tirole (2000)). Consider the first-order condition for optimal hedging. As shown in Appendix B, if the investment opportunity i_{t+1} and the source of uncertainty z_{t+1} are independent, the first-order condition for η_t to be an interior optimum can be simplified to

$$0 = (1 - \eta_t) \left(-\mathbf{E}_t \left[f_1 \left(\mathbf{E} [z], i_{t+1} \right) (z_{t+1} - \mathbf{E} [z])^2 \right] + \frac{1}{2} \mathbf{E}_t \left[f_2 \left(\xi_{t+1}, i_{t+1} \right) (\xi_{t+1} - \mathbf{E} [z])^2 (z_{t+1} - \mathbf{E} [z]) \right] \frac{1 - \eta_t}{1 + g_t} \right), \quad (23)$$

where f_1 and f_2 are defined as

$$f_1(z_{t+1}, i_{t+1}) = \left(\frac{\partial^2 J_{t+1}^*}{\partial \Lambda_{t+1}^2}(\Lambda_{t+1}(z_{t+1}), i_{t+1}) \right) I_{\{a(\eta_t) < z_{t+1} < b(\eta_t)\}}, \quad (24)$$

$$f_2(z_{t+1}, i_{t+1}) = \left(\frac{\partial^3 J_{t+1}^*}{\partial \Lambda_{t+1}^3}(\Lambda_{t+1}(z_{t+1}), i_{t+1}) \right) I_{\{a(\eta_t) < z_{t+1} < b(\eta_t)\}}, \quad (25)$$

and ξ_{t+1} satisfies

$$\min(\mathbf{E}[z], z_{t+1}) < \xi_{t+1} < \max(\mathbf{E}[z], z_{t+1}), \quad (26)$$

where $a(\eta_t)$ and $b(\eta_t)$ are defined in Appendix B.

Equation (23) shows that full hedging ($\eta_t^* = 1$) is not the only candidate for an interior optimum, as this equation can also be satisfied when the second term, which depends on η_t , equals zero. This term relies upon the conditional expectation of higher order derivatives of J_{t+1}^* , so that when full hedging is not the optimal decision, the hedging strategy depends on the firm's expected future performance.

The optimal hedging policy in Figure 4 shows that the optimal hedging value depends in a non-trivial way on the leverage at the decision period. Compared to the one-period model, the leverage region where full hedging is the optimal decision is wider. For these leverage values, the firm's financial health is not strong enough to support increments in future leverage values; hedging thus helps to reduce profitability's volatility and future capital costs.

There are also other regions where it is not optimal to hedge. Albeit for different reasons, small and large leverage levels deter a firm from controlling the volatility of its profitability. A firm with low leverage has a financial health strong enough to support potential losses, while allowing it to benefit from possible increases in its profitability. On the other hand, a firm with high leverage benefits from not hedging by increasing its chances to obtain potential gains that are associated with a higher volatility of its profitability. This behavior is consistent with Harris and Raviv (1991), where the authors argue that "the loss in value of the equity from the poor investment can be more than offset by the gain in equity value captured at the expense of debtholders."

This non-linear relationship between hedging and leverage may help explain why empirical studies have found mixed results regarding the significance of the relationship between hedging and leverage. On one hand, Mayers and Smith (1982) and Smith and Stulz (1985), document a positive relationship between leverage and the amount of hedging taken by firms, arguing that the possibility of costly bankruptcy incites firms to manage their risk. On the other hand, Nance et al. (1993) and Allayannis and Ofek (2001), with different data sets, find no empirical support for this relationship.

Result 2: *The optimal investment decision depends in a non-linear way on the firm's leverage.*

The lower right panel of Figure 4 shows the optimal investment level g_t in the multi-period model for different leverage values. As in the one-period model, the decision to invest depends on the firm's leverage during the decision period. From this figure, the interaction between these two variables can be classified into three regions.

The first region corresponds to small leverage values (below 41% in Figure 4) where the firm fully invests in the opportunity (if present). Full investment means that the firm is opportunity constrained, and as computed in Appendix B, a marginal increment in the investment level increases the firm's relative value by

$$\frac{1}{1 + w(\lambda_t)} \left(-1 + \mathbf{E}_t \{ J_{t+1}^*(\lambda_{t+1}, i_{t+1}) \} + \mathbf{E}_t \left\{ \frac{\partial J_{t+1}^*}{\partial \Lambda_{t+1}}(\Lambda_{t+1}, i_{t+1}) I_{\{a < z_{t+1} < b(g_t)\}} (1 - \Lambda_{t+1}) \right\} \right). \quad (27)$$

Equation (27) demonstrates that positive increments occur when the second and third terms, which are related to the expected relative value of the firm, are greater than one. Observe that in Figure 4, for small leverage values, the firm's relative value is higher than one and has a positive slope. When a firm is

opportunity-constrained, the long-term benefits from fully investing in the opportunity more than offset the immediate drop in cash flow in the decision period.

The second region is composed of leverage values between 41% and 51%. For these values, the firm becomes financially constrained and its optimal investment level decreases as leverage increases. This means that in this region, the first-order derivative (27) is zero, and the firm is no longer financially capable of fully investing in the opportunity.

The third region encompasses large leverage values (52% and above) for which the cost of capital is so high that it is no longer optimal to invest in the opportunity. For these values, an increment in the leverage has a large negative effect on the firm's relative value (see Figure 4), which results in a negative first-order derivative as given by equation (27).

Result 3: *The presence of investment opportunities provides incentives for the firm to control the volatility of its profitability.*

The third result corresponds to the effect of the investment opportunity on the firm's hedging policy. The optimal hedging policy in Figure 4 shows that the presence of an investment opportunity in the decision period affects the magnitude of the optimal hedging ratio. When the opportunity is present, the firm hedges more in order to control the volatility of its future leverage ratio. The previous result goes in line with the familiar notion that firms with more growth options are more likely to hedge given that they can reduce the variance of their value (Myers (1977) and Nance et al. (1993)).

4.3.3 Optimal Risk Management Strategy

The optimal risk management strategy for a firm is a combination of two elements: a) the capital structure, specifically, the leverage necessary to attain the highest possible relative value, and b) the optimal hedge and investment level policies associated with the selected leverage. This section not only characterizes the optimal risk management strategy, but also analyzes how this strategy performs in different financial environments.

Panel A in Table 1 shows the optimal risk management strategy and the firm's relative value attained by this strategy for the base case. The optimal leverage for a firm that has no investment opportunities in the decision period is $\lambda = 40\%$, which corresponds to the leverage that minimizes the cost of capital. When an investment opportunity is present, the maximum is attained at $\lambda = 37\%$. The optimal investment policy is to fully invest in the opportunity when it is present. With respect to the optimal hedging policy, it is optimal to fully hedge when the investment opportunity is present; the opposite is true when there is no investment opportunity in the decision period. Finally, since the firm's relative value is greater than one, it is clear that the optimal risk management strategy allows the firm to create value out of the invested capital.

Result 4: *The need for financial flexibility makes the firm more conservative when it has an investment opportunity: it hedges more and borrows less*

The principal characteristic of this risk management strategy is that it provides the firm with the financial flexibility required to undertake the investment opportunity. In the base-case scenario, with an investment opportunity of 20%, the cash flow for a given period has a mean value of -10% with a standard deviation of 4%, so the firm is unlikely to finance the investment using exclusively the cash flow generated during a period. This means that the firm's financial flexibility depends on average on external sources to finance the investment opportunity, and so it becomes imperative to maintain a leverage that guarantees a low financing cost.

The previous trade-off between financial flexibility and leverage can be studied by working with the partial derivative of the firm's relative value with respect to the leverage, which, according to Appendix B, is:

$$\begin{aligned} \frac{\partial}{\partial \lambda_t} J_t^*(\lambda_t, i_t) &= -\frac{w'(\lambda_t)}{(1+w(\lambda_t))} J_t^*(\lambda_t, i_t) \\ &\quad - E_t \left\{ \frac{\partial J_{t+1}^*}{\partial \Lambda_{t+1}}(\Lambda_{t+1}, i_{t+1}) I_{\{a(\lambda_t) < z_{t+1} < b(\lambda_t)\}} \right\} \frac{(1+r(\lambda_t) + \lambda_t r'(\lambda_t))}{1+w(\lambda_t)}. \end{aligned} \quad (28)$$

Table 1: Optimal Risk Management Strategy

Panel A: $i_t = 20\%$

No Investment Opportunity				With Investment Opportunity			
J	λ_t	η_t	g_t	J	λ_t	η_t	g_t
1.1455	40%	0.85	0%	1.1787	37%	1.00	20%

Panel B: $i_t = 10\%$

No Investment Opportunity				With Investment Opportunity			
J	λ_t	η_t	g_t	J	λ_t	η_t	g_t
1.0307	40%	0.00	0%	1.0389	40%	1.00	10%

NOTES TO TABLE: The upper (bottom) panel represents a firm that faces an investment opportunity of 20% (10%) in the decision period. The cost of capital is minimized at $\lambda = 40\%$. The investment opportunity arrives each period with a probability of 40%. The expected value and standard deviation of the source of uncertainty z is 10% and 4% respectively. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$.

This equation demonstrates that the sensitivity of the firm's relative value to changes in leverage is governed by an immediate effect (first term) and an intertemporal effect (second term). Since the cost of capital's derivative is negative in the interval $[0\%, 40\%]$, the immediate effect has a positive sign in this interval. This means that to the left of the optimal leverage, the leading term is the first one, since the relative value's derivative is positive in this segment. To the right of the optimal leverage, the intertemporal effect is the one that determines the value and sign of the relative value's derivative, which is negative in this segment. This compromise between immediate benefits and intertemporal repercussions illustrates the way financial flexibility is created in a multi-period setting: the one-period tax shield provided by more debt is weighted against the future repercussions of this increment on the firm's relative value. When the immediate effect dominates the intertemporal effect, the firm is better off by increasing its leverage ratio; the inverse is true when the intertemporal effect dominates the immediate effect.

Contrary to the one-period model, in the multi-period model, the difference between the immediate and the intertemporal effect is marginal for leverage values to the left of the optimal leverage. In the one-period model, changes to the leverage have a permanent effect on the continuation value, as they affect the discounting rate for all the periods to come; in the multi-period model, this impact is attenuated by the fact that the firm can adjust its subsequent hedging and investment levels, and thus its future discounting rate.

To study the interplay between financial flexibility and the optimal risk management strategy in greater detail, consider a case where the firm is more likely to finance the investment from its expected cash flow than from external sources. Panel B in Table 1 presents the optimal risk management strategy for a lower magnitude of the investment opportunity $i_t = 10\%$. With an investment opportunity of this magnitude, the firm does not have to sacrifice the tax shield provided by more debt and the optimal leverage corresponds to the one that minimizes the cost-of-capital function. Regarding the optimal hedging policy when there is no investment opportunity, the underinvestment problem is less severe than for the base case, so the firm adjusts its optimal hedging ratio downwards.

4.3.4 Dependence between Investment Opportunities and Profitability

Since the optimal policy depends on the investment opportunities and the profitability, it is important to examine how the dependence between these two variables impacts the risk management strategy.

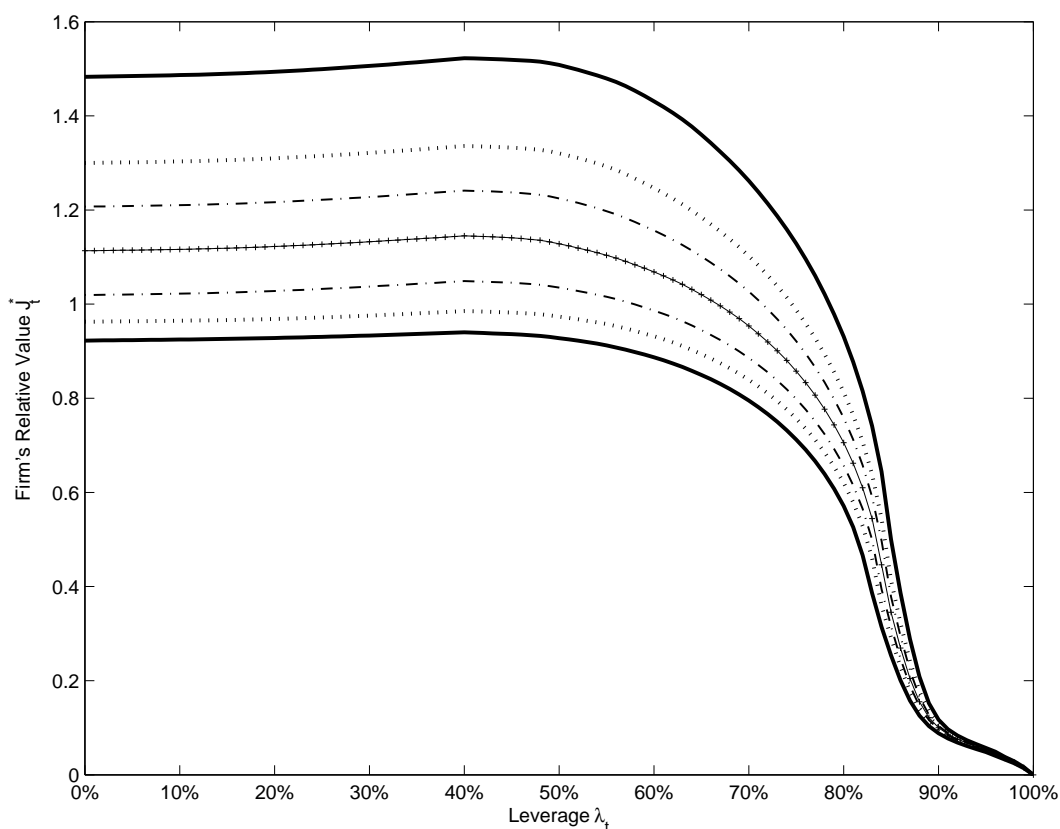
To study this impact, consider the following two cases. In the first one, suppose there is a negative dependence between investment opportunities i_t and the underlying source of volatility z_t ; that is, the

investment opportunity arises when the profitability is lower. In the second case, suppose that the investment opportunity arises when the profitability is higher; that is, there is a positive dependence between the two variables.

This relationship is introduced into the model as follows: if there is no investment opportunity ($i_t = 0\%$), the distribution of z_t is normal with mean μ_z and variance σ_z^2 ; if there is an investment opportunity ($i_t \neq 0\%$), the distribution of z_t is normal with mean $\mu_z\% + \rho\%$ and variance σ_z^2 . The parameter ρ controls for the strength and sign of the relationship between the two sources of uncertainty. Appendix G provides new expressions for the dynamic program formulation in this case.

Figure 5 shows the effect of the dependence on the firm's relative value. As expected, positive values for ρ increase the relative value of the firm. This is due to the fact that high profitability is more likely to arrive with the investment opportunity; as such, more cash is available to finance the investment and to profit from the tax shield. On the contrary, a negative dependence implies that the firm could be cash poor when the investment opportunity arises, so more debt would be needed to finance the investment. This increment in leverage translates into higher cost of capital for the upcoming periods, and hence, lower firm values.

Figure 5: Firm's Relative Value for Different Dependence Levels



NOTES TO FIGURE: This figure represents the relative value of a firm that has no investment opportunities for the given period. The line with crosses represent the case without dependence (base case). Positive values of ρ are given for the curves above the base case line, while negative values of ρ are given by curves below. The continuous line represents the case where $\rho = \pm 2\%$. The dotted line represents the case where $\rho = \pm 1\%$. The dashed line represents the case where $\rho = \pm 0.5\%$. The cost of capital is minimized at $\lambda_t = 40\%$. The magnitude of the investment opportunity is equal to 20%, which arrives during a period with a probability of 40%. The expected value and standard deviation of the source of uncertainty z is 10% and 4% respectively. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$.

Result 5: *Positive dependence between investment opportunities and profitability acts as a natural hedge for the firm, while negative dependence exacerbates the underinvestment problem*

The next element to analyze is the effect of the dependence on the firm's optimal policies. Panel A of Table 2 shows the optimal relative value of the firm, the optimal leverage, and the associated optimal hedging and investment level policies for the base case parameters. Several observations can be drawn from this table.

Table 2: Impact of Dependence

Panel A: $i_t = 20\%$

ρ	No Investment Opportunity				With Investment Opportunity			
	J	λ_t	η_t	g_t	J	λ_t	η_t	g_t
-2%	0.9401	40%	0.11	0%	0.9401	40%	0.11	0%
-1%	0.9853	40%	1.00	0%	0.9897	40%	1.00	13%
0%	1.1455	40%	0.85	0%	1.1787	37%	1.00	20%
1%	1.3360	40%	0.69	0%	1.4108	37%	1.00	20%
2%	1.5226	40%	0.11	0%	1.6381	37%	1.00	20%

Panel B: $i_t = 10\%$

ρ	No Investment Opportunity				With Investment Opportunity			
	J	λ_t	η_t	g_t	J	λ_t	η_t	g_t
-2%	0.9401	40%	0.11	0%	0.9401	40%	0.11	0%
-1%	0.9811	40%	0.00	0%	0.9838	40%	1.00	10%
0%	1.0307	40%	0.00	0%	1.0389	40%	1.00	10%
1%	1.0951	40%	0.00	0%	1.1095	40%	0.94	10%
2%	1.1597	40%	0.00	0%	1.1802	40%	0.85	10%

NOTES TO TABLE: This table shows the impact that the dependence between the investment opportunities and the profitability have on the optimal risk management policy and capital structure. Positive values of ρ increase the dependence, while negative values have the opposite effect. In Panel A, the investment opportunity is 20%. In Panel B, the investment opportunity is 10%. The leverage that minimizes the cost of capital at $\lambda = 40\%$. The investment opportunity arrives each period with a probability of 40%. The standard deviation of the source of uncertainty is 4%. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$.

First, when no investment opportunity is present, increments in ρ decrease the hedging ratio. As argued in Froot et al. (1993) and ?, these results correspond to the fact that positive dependence offers a natural hedge for the firm, as investment opportunities arise precisely when the firm has more profitability. Nonetheless, if the firm has an investment opportunity in a given period, it is optimal for the firm to fully hedge. As argued before, with an investment opportunity of 20%, the firm has to guarantee its future financial flexibility and to control its leverage for the upcoming period by fully hedging.

Second, note that changing the magnitude and sign of the dependence only affects the optimal leverage when an investment opportunity is present. As ρ decreases, the firm finds itself lacking cash for financing the investment opportunity, so it has to use more debt in the long run to finance the investment. In other words, the firm is better off sacrificing the tax shield of debt in order to avoid the higher cost of capital associated with future high leverage values. On the contrary, in the case of a positive dependence, the firm is cash-rich when the opportunity is present, so it finances the investment opportunity directly from its cash flow.

Third, observe that the more the dependence becomes negative, the less the firm invests into the investment opportunity; it becomes financially constrained in such cases.

Finally, the only case in Panel A of Table 2 that does not correspond to the previous discussion is the one in which ρ is set to -2% . In this case, the relative value associated with the optimal risk management strategy is lower than one; the firm is not creating value. Moreover, the firm is not able to capture the investment opportunity. In such an extreme scenario, hedging only helps the firm to keep afloat.

Panel B of Table 2 shows the optimal policy when the investment opportunity is at $i_t = 10\%$. As previously discussed, firms facing an investment opportunity of this magnitude require less financial flexibility on average, as they can finance the invested opportunity with future cash flows. The need for less financial flexibility has clear impacts on the optimal risk management strategy. For the case without investment opportunities in the decision period, the firm does not hedge at all. When the investment opportunity is present, the hedging ratios for $\rho = 1\%$ and $\rho = 2\%$ are less than one, that is, the firm does not fully hedge. Also, less need for financial flexibility allows the firm to profit from the debt's tax shield by maintaining a higher leverage ratio when investment opportunities are present.

4.3.5 Changes in the Volatility of the Risk Factor

One of the main points discussed above is volatility management and how its proper implementation can help a firm accomplish its objectives without compromising its value. Hence, a natural question to analyze is how a firm would react in an environment where a higher volatility may compromise the firm's profitability, if not properly managed. This analysis is carried out in this section by keeping all parameters equal to those of the base case and increasing the risk factor's volatility from 4% to 12%. The results are summarized in Table 3.

Table 3: Impact of a Volatility Increment on the Optimal Risk Management Strategy

Panel A: No investment opportunity				
	J	λ_t	η_t	g_t
base case	1.1455	40%	0.85	0%
high vol.	1.2370	40%	0.94	0%
Panel B: With investment opportunity				
	J	λ_t	η_t	g_t
base case	1.1787	37%	1.00	20%
high vol.	1.2850	34%	1.00	20%

NOTES TO TABLE: This table summarizes the difference between the optimal policies for two firms that only differ on the volatility of the uncertainty source. The base case uses a standard deviation of 4%, while the high volatility case (high vol.) employs a standard deviation of 12%. The leverage that minimizes the cost of capital at $\lambda = 40\%$. The magnitude of the investment opportunity is equal to 20%, which arrives during a period with a probability of 40%. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$.

Result 6: *The firm adapts its capital structure and hedging policy to preserve its financial flexibility when exposed to more uncertainty about profitability*

From Table 3, observe first that the firm's relative value increases in this case, which goes in line with the intuition that a thorough risk management strategy can capture the value created by the profitability's volatility. Nonetheless, the riskier environment in which the firm is operating now implies a decline in the optimal leverage, as it becomes of paramount importance for the firm to preserve its financial flexibility. Observe, also, that the firm preserves its financial flexibility by increasing its hedging ratio when no investment opportunity is present.

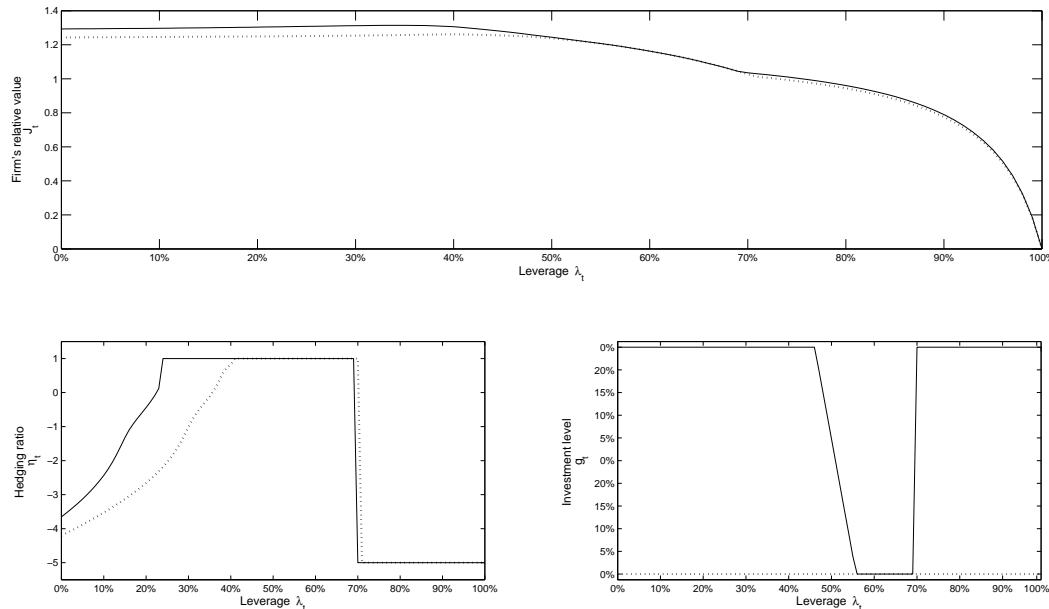
4.4 Hedging and Gambling

The study of the optimal hedging policy (Figure 4) showed that firms do not hedge for small and large leverage values. However, this could result from the constraint imposed on the hedging ratio: $0 \leq \eta \leq 1$. Would a firm increase its exposure through forward purchase (i.e. choose $\eta < 0$) if allowed? The analysis yields:

Result 7: *For low leverage, firms gamble by selling their risk short, and for high leverage, by fully investing.*

The combined result of this gambling behavior is summarized by the firm's relative value, as shown in the upper panel of Figure 6. In general, allowing the firm to gamble unrestrictedly increases the relative firm value for all leverage values, especially for the largest. This result comes as no surprise, as eliminating a constraint for the optimization problem increases the global optimum.

Figure 6: Firm's Relative Firm Value and Optimal Policies When Gambling is Allowed



NOTES TO FIGURE: The figure on the top shows the optimal relative firm value J^* for different leverage values. On the bottom, the left figure shows the optimal policy for the hedge ratio η_t (truncated in the figure to -5), and the right figure the optimal policy for the invested level g_t . The dotted line represents the case with no investment opportunity in the decision period ($i_t = 0$), while the continuous line represents the case when the investment opportunity is present ($i_t = 20\%$). The cost of capital is minimized at $\lambda_t = 40\%$. The magnitude of the investment opportunity is equal to 20%, which arrives during a period with a probability of 40%. The expected value and standard deviation of the source of uncertainty z is 10% and 4% respectively. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$.

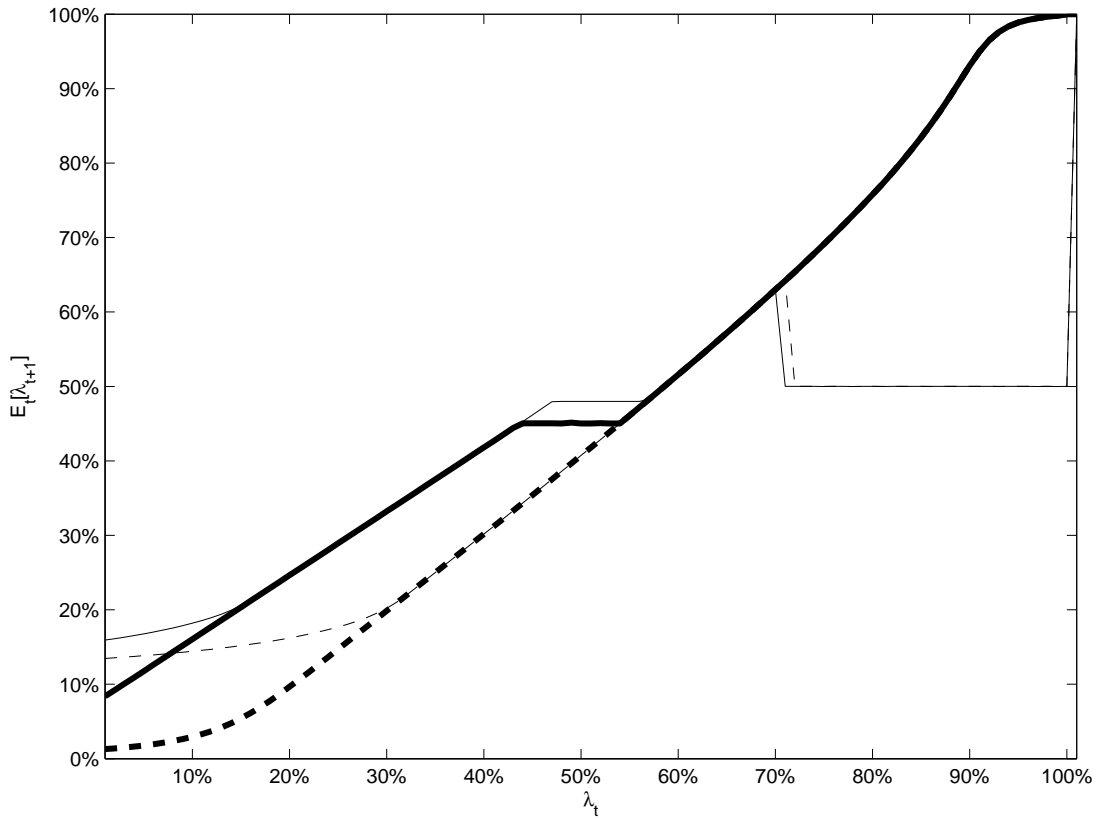
The lower left panel of Figure 6 shows the optimal policy for η_t as a function of λ_t and i_t . Several observations can be drawn from this figure. First, small and large leverage levels give incentives for the firm to gamble, which is evidenced by negative hedging ratios. As mentioned before, different factors induce the firm to increase its volatility. For small leverage ratios, the firm is financially strong to support larger expected leverages while benefiting from the debt's tax shield. For high leverage values, the firm increases its exposure to the source of uncertainty, and thus its chances of a higher profit. Finally, as in the constrained case, the optimal policy is to fully or nearly fully hedge for medium leverage levels.

The lower right panel of Figure 6 depicts the optimal investment policy g_t . Observe that, as in the constrained case, the investment opportunity, when present, is fully captured for small leverage values. After

a certain leverage value (around 45%), the optimal investment level decreases as the leverage increases, until the optimal decision is to not invest in the opportunity. However, in contrast to what was observed when gambling was not allowed, the optimal level of investment jumps to full investment for leverage values above 70%. For these leverage values, the firm increases its exposure by purchasing its risk forward while trying to increase its value by investing in the opportunity; in other words, the firm is gambling for resurrection.

One of the central themes in the previous discussion is the way a firm controls its expected leverage ratio. Thus, it is worth making some brief observations about this expected value. Figure 7 shows the expected leverage value for the case with and without gambling. First, note that, for small leverage values, gambling increases the expected leverage, especially when there are no investment opportunities. On the other hand, for large leverage values, gambling helps the firm keep the expected leverage at 50%. For these leverage values, the firm is gambling between only two results: either it experiences a positive shock that reduces its leverage to zero, or it experiences a negative shock that brings its leverage to one.

Figure 7: Impact of Gambling on the Expected Leverage at time $t + 1$



NOTES TO FIGURE: The figure shows the expected leverage at time $t + 1$ as a function of the leverage at time t . The lines in bold represent the case with constraints on the hedging ratio, while the other lines represent the case where gambling is allowed. The dotted line represents the case with no investment opportunity in the decision period ($i_t = 0$), while the continuous line represents the case when the investment opportunity is present ($i_t = 20\%$). The expected leverage is computed with equation (50) and the associated optimal values for η_t and g_t for both cases. The cost of capital is minimized at $\lambda_t = 40\%$. The magnitude of the investment opportunity is equal to 20%, which arrives during a period with a probability of 40%. The expected value and standard deviation of the source of uncertainty z is 10% and 4% respectively. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$.

Result 8: *Gambling requires the firm to increase its financial flexibility.*

Firms engaged in gambling not only increase their relative value, but also their cash flow volatility. As such, these firms are more exposed to adverse cash flows, which in turn may jeopardize the investment funding when profitable opportunities arise.

Panel A of Table 4 presents the optimal risk management strategy when gambling is allowed. When investment opportunities are present, the optimal leverage ratio is lower than the one observed when gambling is not allowed (34% vs. 37%). Since the optimal decision is to gamble for small and large leverage ratios, it adopts a more conservative initial capital structure that increases its financial flexibility.

Panel B of Table 4 shows the optimal risk management strategy when the investment opportunity i_t equals 10%. Similar to what is observed when gambling is not allowed, the firm does not require financial flexibility in this case, so the optimal leverage equals the one that minimizes the cost of capital. Moreover, with the given parameters, the firm is financially strong enough to invest in the opportunity without full hedging.

Table 4: Optimal Risk Management Strategy – Gambling is allowed

Panel A: $i_t = 20\%$							
No Investment Opportunity				With Investment Opportunity			
J	λ_t	η_t	g_t	J	λ_t	η_t	g_t
1.2619	40%	0.81	0%	1.3143	34%	1.00	20%

Panel B: $i_t = 10\%$							
No Investment Opportunity				With Investment Opportunity			
J	λ_t	η_t	g_t	J	λ_t	η_t	g_t
1.0720	40%	-0.07	0%	1.0822	40%	0.73	10%

NOTES TO TABLE: The upper panel represents a firm that faces an investment opportunity of 20% in the decision period, while the bottom panel represents a firm with an investment opportunities of 10% in the decision period. In both cases gambling is allowed. The cost of capital is minimized at $\lambda = 40\%$. The investment opportunity arrives each period with a probability of 40%. The expected value and standard deviation of the source of uncertainty z is 10% and 4% respectively. The market risk premium for the Miles-Ezzel model is set to 5%. The convexity of the cost of capital at the right of $\lambda = 40\%$ is set at $a = 1$.

5 Concluding remarks

This paper develops a dynamic risk management model to determine a firm's optimal risk management strategy. Firms engaging in hedging activities are not guaranteed to increase their value. Rather, successful hedging strategies are the result of a thorough risk management plan that allows firms to pursue core activities despite adverse cash flow shocks.

The impact of a multi-period setting on the decisions made by a firm in terms of its capital structure and hedging strategy is examined. The model shows that, when an investment opportunity is independent of profitability, it is not always optimal to fully hedge. This result contrasts with the standard result found in risk management literature (see Froot et al. (1993)), where the independence between investment opportunities and profitability implies fully hedging as the only optimal solution.

Numerical examples show how a firm's capital structure and risk management policies are affected by changes to the business environment. For instance, the capital structure is driven to a large extent by the investment opportunities that a firm faces. Conservative management practices are observed when firms face

investment opportunities. On the contrary, when firms do not have investment opportunities, they tend to have higher leverage ratios.

This paper also documents how demand for hedging is affected by the relationship between investment opportunities and profitability. As in previous studies, a positive dependence acts as a natural hedge, so hedging ratios tend to be smaller. This result is accentuated when the firm has the ability to finance the investment opportunity from its internal cash flow.

This analysis can be extended in several directions. First, a more complex capital structure, including equity issuance and optimal dividend distribution, could be examined. Currently, financing needs are met by debt issuance, and excess cash is used to pay down debt, until all debt is repaid. While this approach is internally consistent, introducing equity would allow for a richer set of tradeoffs. Second, multiple risk factors could be included. Except for commodity producers, very few firms face a single risk factor. Determining the overall hedging strategy and, in particular, the tradeoffs between hedging the different risks would be very valuable. Finally, non-linear hedging strategies could be examined. Senior managers and Boards are often reluctant to hedge their risk using forward contracts, as it deprives them of the potential upside, should the output price go up (or the input price go down). Investing in options might prove more acceptable to them, if the value was clearly identified. All these extensions would render the analysis closer to the reality and choices faced by firms, hence would provide clear and practical guidance as firms strive to define their risk management strategy.

A Derivation of the Bellman equations

This section deduces the Bellman equations (18) and (19).

For $t = 0, \dots, T - 1$, let $J_t^*(\lambda_t, i_t)$ be the optimal value for the $T - t$ stage problem that starts at date t in state (λ_t, i_t) , as defined in equation (16). First, consider the case for $t = T$. Using the continuation value obtained in (17), one has:

$$\begin{aligned} J_T^*(\lambda_T, i_T) &= \max_{\substack{\{g_T, \eta_T\} \\ 0 \leq g_T \leq i_T, 0 \leq \eta_T \leq 1}} \mathbf{E}_T \left\{ \sum_{s=T+1}^{\infty} \frac{x_s - g_{s-1}}{\prod_{k=T+1}^s (1 + w(\lambda_{k-1}))} \prod_{u=T}^{s-2} (1 + g_u) \right\} \\ &= \max_{\substack{\{g_T, \eta_T\} \\ 0 \leq g_T \leq i_T, 0 \leq \eta_T \leq 1}} \mathbf{E}_T \left\{ \frac{(x_{T+1} - g_T)}{1 + w(\lambda_T)} + \frac{(1 + g_T)}{1 + w(\lambda_T)} \frac{\mathbf{E}[z] - g}{w(\lambda_{T+1}) - g} \right\} \\ &= \max_{\substack{\{g_T, \eta_T\} \\ 0 \leq g_T \leq i_T, 0 \leq \eta_T \leq 1}} \frac{1}{1 + w(\lambda_T)} \left(\mathbf{E}[z] - g_T + (1 + g_T) \mathbf{E}_T \left\{ \frac{\mathbf{E}[z] - g}{w(\lambda_{T+1}) - g} \right\} \right), \end{aligned}$$

which gives equation (18). Now, from (16), the problem at t can be written as

$$\begin{aligned} J_t^*(\lambda_t, i_t) &= \max_{\substack{\{g_s, \eta_s\}_{s \geq t} \\ 0 \leq g_s \leq i_s, 0 \leq \eta_s \leq 1}} \mathbf{E}_t \left\{ \sum_{s=t+1}^{\infty} \frac{x_s - g_{s-1}}{\prod_{k=t+1}^s (1 + w(\lambda_{k-1}))} \prod_{u=t}^{s-2} (1 + g_u) \right\} \\ &= \max_{\substack{\{g_s, \eta_s\}_{s \geq t} \\ 0 \leq g_s \leq i_s, 0 \leq \eta_s \leq 1}} \mathbf{E}_t \left\{ \frac{(x_{t+1} - g_t)}{1 + w(\lambda_t)} + \frac{1 + g_t}{1 + w(\lambda_t)} \sum_{s=t+2}^{\infty} \frac{x_s - g_{s-1}}{\prod_{k=t+2}^s (1 + w(\lambda_{k-1}))} \prod_{u=t+1}^{s-2} (1 + g_u) \right\}. \quad (29) \end{aligned}$$

Let $\pi_{t+1}^* = \{(g_{t+1}^*, \eta_{t+1}^*), (g_{t+2}^*, \eta_{t+2}^*), \dots, (g_T^*, \eta_T^*)\}$ be an optimal policy for the subproblem $J_{t+1}^*(\lambda_{t+1}, i_{t+1})$ that starts at date $t + 1$ and has final decision at stage T . Using the optimal value the subproblem

$J_{t+1}^*(\lambda_{t+1}, i_{t+1})$, one can rewrite (29) as follows:

$$\begin{aligned} J_t^*(\lambda_t, i_t) &= \max_{\substack{\{g_t, \eta_t\} \\ 0 \leq g_t \leq i_t, 0 \leq \eta_t \leq 1}} \mathbf{E}_t \left\{ \frac{(x_{t+1} - g_t)}{1 + w(\lambda_t)} + \frac{(1 + g_t)}{1 + w(\lambda_t)} J_{t+1}^*(\lambda_{t+1}, i_{t+1}) \right\} \\ &= \max_{\substack{\{g_t, \eta_t\} \\ 0 \leq g_t \leq i_t, 0 \leq \eta_t \leq 1}} \frac{1}{1 + w(\lambda_t)} (\mathbf{E}[z] - g_t + (1 + g_t) \mathbf{E}_t \{ J_{t+1}^*(\lambda_{t+1}, i_{t+1}) \}), \end{aligned}$$

which is equation (19).

B Analysis of the Solution

B.1 Hedging and Investment Level Policies

To gain more insight into the solution of the dynamic programming problem, one can analyze the behavior of the Bellman equations. To simplify the notation, define:

$$\Theta_t(g_t, \eta_t) = \frac{1}{1 + w(\lambda_t)} (\mathbf{E}[z] - g_t + (1 + g_t) \mathbf{E}_t \{ J_{t+1}^*(\lambda_{t+1}, i_{t+1}) \}) \quad (30)$$

The Lagrangian function associated to the maximization problem in equation (19) is:

$$\mathcal{L}(g_t, \eta_t, \psi) = \Theta_t(g_t, \eta_t) - \psi \cdot (h(g_t, \eta_t) - c), \quad (31)$$

where

$$\begin{aligned} h(g_t, \eta_t) &= [g_t, g_t, \eta_t, \eta_t], \\ c &= [i_t, 0, 1, 0], \\ \psi &= [\psi_1, \psi_2, \psi_3, \psi_4]. \end{aligned}$$

The first-order necessary conditions for a point (g_t, η_t, ψ) to be a local maximum are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \eta_t}(g_t, \eta_t, \psi) &= \frac{\partial \Theta_t}{\partial \eta_t}(g_t, \eta_t) - \psi_3 - \psi_4 = 0, \\ \frac{\partial \mathcal{L}}{\partial g_t}(g_t, \eta_t, \psi) &= \frac{\partial \Theta_t}{\partial g_t}(g_t, \eta_t) - \psi_3 - \psi_4 = 0, \\ \frac{\partial \mathcal{L}}{\partial \psi_i}(g_t, \eta_t, \psi) &= h_i(g_t, \eta_t) = c_i, \quad i = 1, 2, 3, 4. \end{aligned}$$

The previous equations shows that $\eta_t^* = 1$, $\eta_t^* = 0$, $g_t^* = i_t$, and $g_t^* = 0$ are trivial candidates for local maxima. When the constraints are inactive, the other feasible points come from the inner conditions

$$\begin{aligned} \frac{\partial \Theta_t}{\partial \eta_t}(g_t, \eta_t) &= 0, \\ \frac{\partial \Theta_t}{\partial g_t}(g_t, \eta_t) &= 0. \end{aligned}$$

B.1.1 First-Order Condition for the Hedging Ratio

The first-order condition for the hedge ratio η_t comes from the first derivative of (30) with respect to η_t :

$$\frac{\partial \Theta_t}{\partial \eta_t}(g_t, \eta_t) = \frac{1 + g_t}{1 + w(\lambda_t)} \frac{\partial}{\partial \eta_t} \mathbf{E}_t [J_{t+1}^*(\lambda_{t+1}, i_{t+1})] = 0. \quad (32)$$

Appendix C shows that:

$$\frac{\partial}{\partial \eta_t} \mathbf{E}_t [J_{t+1}^*(\lambda_{t+1}, i_{t+1})] = \mathbf{E}_t \left[\frac{\partial J_{t+1}^*}{\partial \Lambda_{t+1}}(\Lambda_{t+1}, i_{t+1}) I_{\{a(\eta_t) < z_{t+1} < b(\eta_t)\}} \frac{z_{t+1} - \mathbf{E}[z]}{1 + g_t} \right], \quad (33)$$

with $a(\eta_t)$ and $b(\eta_t)$ defined as

$$a(\eta_t) = \frac{-\eta_t \mathbf{E}[z] + \lambda_t(1+r(\lambda_t)) - 1}{1 - \eta_t}, \quad (34)$$

$$b(\eta_t) = \frac{g_t - \eta_t \mathbf{E}[z] + \lambda_t(1+r(\lambda_t))}{1 - \eta_t}. \quad (35)$$

From (32) and (33), the first-order condition becomes

$$0 = \frac{\mathbf{E}_t \left[\frac{\partial J_{t+1}^*}{\partial \Lambda_{t+1}}(\Lambda_{t+1}, i_{t+1}) I_{\{a(\eta_t) < z_{t+1} < b(\eta_t)\}} (z_{t+1} - \mathbf{E}[z]) \right]}{1 + w(\lambda_t)}. \quad (36)$$

Appendix D shows that this condition can be written as

$$\begin{aligned} \mathbf{Cov}(f(\mathbf{E}[z], i_{t+1}), z_{t+1}) &= \frac{1 - \eta_t}{1 + g_t} \left(\mathbf{E}_t \left[f_1(\mathbf{E}[z], i_{t+1}) (z_{t+1} - \mathbf{E}[z])^2 \right] - \right. \\ &\quad \left. \frac{1}{2} \mathbf{E}_t \left[f_2(z_{t+1}, i_{t+1}) (\xi_{t+1} - \mathbf{E}[z])^2 (z_{t+1} - \mathbf{E}[z]) \right] \frac{1 - \eta_t}{1 + g_t} \right), \end{aligned} \quad (37)$$

with f_1 , f_2 , and ξ_{t+1} defined in (24), (25), and (26) respectively.

Equation (37) tells us how the optimal hedging ratio controls for the covariance between the marginal value of firm leverage $\left(\frac{\partial J_{t+1}^*}{\partial \Lambda_{t+1}} \right)$ and the source of uncertainty z_{t+1} .

B.1.2 First-Order Derivative of the Investment Level

The first-order derivative of (30) with respect to g_t is:

$$\frac{\partial \Theta_t}{\partial g_t} = \frac{1}{1 + w(\lambda_t)} \left(-1 + \mathbf{E}_t \{ J_{t+1}^* \} + (1 + g_t) \frac{\partial}{\partial g_t} \mathbf{E}_t \{ J_{t+1}^*(\lambda_{t+1}, i_{t+1}) \} \right).$$

Using the result from Appendix C, equation (27) is obtained, with a and b defined in (34) and (35), respectively.

The analysis carried out above has been done without taking into account the opportunity constraint (2). If the optimum turns out to be an interior critical point, the firm does not fully capture the investment opportunity i_t in order to preserve its financial flexibility. In that case, the firm is said to be financially constrained. On the other hand, if the constraint is active at the optimum and the investment opportunity is fully taken, the firm is considered to be opportunity constrained.

B.2 Impact of the Leverage on the Firm's Value

Although leverage is not a control variable in the present framework, one can still assess how marginal changes of the leverage ratio affect the firm's relative value. Consider first how these changes affect the continuing value at $T + 1$. From (17):

$$\frac{\partial J_{T+1}}{\partial \lambda_{T+1}}(\lambda_{T+1}) = - \frac{E[z] - g}{(w(\lambda_{T+1}) - g)^2} \frac{\partial w}{\partial \lambda_{T+1}}(\lambda_{T+1}). \quad (38)$$

Because of the assumption on the WACC curve, the derivative of $w(\lambda)$ with respect to λ is negative if $\lambda < \bar{\lambda}$ and positive otherwise. Consequently, as $E[z] - g$ is positive, equation (38) is positive if $\lambda_{T+1} < \bar{\lambda}$ and negative otherwise, which means that to the left of $\bar{\lambda}$, an increase in the leverage leads to a reduction in the cost of capital, hence an increase in the continuation value.

The impact of leverage on the value of the firm at time t can be studied by applying the envelope theorem to $J_t^*(\lambda_t, i_t)$. Given that the constraints of the problem do not depend on λ_t , and using the result from Appendix C, one has:

$$\begin{aligned} & \frac{\partial J_t^*}{\partial \lambda_t}(\lambda_t, i_t) \\ &= -\frac{\frac{\partial w}{\partial \lambda_t}(\lambda_t)}{(1+w(\lambda_t))} (\Theta_t(\lambda_t, g_t^*, \eta_t^*)) + \frac{(1+g_t)}{1+w(\lambda_t)} E_t \left\{ \frac{\partial J_{t+1}^*}{\partial \Lambda_{t+1}}(\Lambda_{t+1}, i_{t+1}) \frac{\partial \Lambda_{t+1}}{\partial \lambda_t} I_{\{a(\lambda_t) < z_{t+1} < b(\lambda_t)\}} \right\}, \end{aligned}$$

which gives rise to equation (28).

C Derivative of Expected Value $\frac{\partial}{\partial y} \mathbf{E}_t [J_{t+1}^* (\lambda_{t+1}, i_{t+1})]$

This appendix show how to compute derivatives of $\mathbf{E}_t [J_{t+1}^* (\lambda_{t+1}, i_{t+1})]$ with respect to a variable y given that λ_{t+1} is a function of y . The variable y represents either η_t , g_t , or λ_t

Notice first that the expected value can be viewed as a weighted integration using the density function as a weight. As such, the derivative of this integral can be computed using Leibniz integration rule provided that the integrand is differentiable. Nonetheless, the derivative of J_{t+1}^* , which is obtained by the chain derivative $\frac{\partial J_{t+1}^*}{\partial \lambda_{t+1}} \frac{\partial \lambda_{t+1}}{\partial y}$, is not differentiable everywhere because of the non-differentiability of λ_{t+1} at some points. To overcome this issue, the integral can be rewritten piecewise so that λ_{t+1} is differentiable within each region. Defining the regions where λ_{t+1} is differentiable in terms of z_{t+1} , one has:

$$\lambda_{t+1}(z_{t+1}) = \begin{cases} 0 & z_{t+1} \geq b(y) \\ 1 & z_{t+1} \leq a(y) \\ \Lambda_{t+1} & a(y) < z_{t+1} < b(y) \end{cases},$$

where

$$\begin{aligned} a(y) &= \frac{-\eta_t \mathbf{E}[z] + \lambda_t (1+r(\lambda_t)) - 1}{1-\eta_t}, \\ b(y) &= \frac{g_t - \eta_t \mathbf{E}[z] + \lambda_t (1+r(\lambda_t))}{1-\eta_t}, \end{aligned}$$

Working with the integral form of the expected value and using the Leibniz rule gives:

$$\begin{aligned} & \frac{\partial}{\partial \eta_t} \mathbf{E}_t [J_{t+1}^* (\lambda_{t+1}, i_{t+1})] \\ &= \sum_{i \in \Upsilon} \frac{\partial}{\partial \eta_t} \mathbf{E}_t [J_{t+1}^* (\lambda_{t+1}, i_{t+1}) | i_{t+1} = i] P[i_{t+1} = i] \\ &= \sum_{i \in \Upsilon} P[i_{t+1} = i] \frac{\partial}{\partial \eta_t} \int_{-\infty}^{\infty} J_{t+1}^* (\lambda_{t+1}(z_{t+1}), i) \phi_{z_{t+1}|i_{t+1}=i}(z_{t+1}) dz_{t+1} \\ &= \sum_{i \in \Upsilon} P[i_{t+1} = i] \left(\int_{-\infty}^{a(y)} \frac{\partial J_{t+1}^*}{\partial \eta_t} (\lambda_{t+1}(z_{t+1}), i) \phi_{z_{t+1}|i_{t+1}=i}(z_{t+1}) dz_{t+1} \right. \\ & \quad + \int_{a(y)}^{b(y)} \frac{\partial J_{t+1}^*}{\partial \eta_t} (\lambda_{t+1}(z_{t+1}), i) \phi_{z_{t+1}|i_{t+1}=i}(z_{t+1}) dz_{t+1} \\ & \quad \left. + \int_{b(y)}^{\infty} \frac{\partial J_{t+1}^*}{\partial \eta_t} (\lambda_{t+1}(z_{t+1}), i) \phi_{z_{t+1}|i_{t+1}=i}(z_{t+1}) dz_{t+1} \right), \end{aligned}$$

where $\phi_{z_{t+1}|i_{t+1}}$ is the density distribution of z_{t+1} conditional on i_{t+1} and Υ represents the countable set of all possible values that the random variable i_{t+1} can take. Notice that Leibniz rule also requires the terms $J_{t+1}^* (\lambda_{t+1}(b(y)), i_{t+1}) \phi_{z_{t+1}|i_{t+1}}(b(y)) \frac{\partial b}{\partial y}(y)$ and $J_{t+1}^* (\lambda_{t+1}(a(y)), i_{t+1}) \phi_{z_{t+1}|i_{t+1}}(a(y)) \frac{\partial a}{\partial y}(y)$; however,

these terms cancel out when they are summed over the three integrals. Applying the chain derivative to $J_{t+1}^*(\lambda_{t+1}, i_{t+1})$, and taking into account that $\frac{\partial \lambda_{t+1}}{\partial y} = 0$ outside of $a(y) < z_{t+1} < b(y)$, one has:

$$\begin{aligned} & \frac{\partial}{\partial y} \mathbf{E}_t [J_{t+1}^*(\lambda_{t+1}, i_{t+1})] \\ &= \sum_{i \in \Upsilon} P[i_{t+1} = i] \int_{a(y)}^{b(y)} \frac{\partial J_{t+1}^*}{\partial \Lambda_{t+1}}(\Lambda_{t+1}(z_{t+1}), i) \frac{\partial \Lambda_{t+1}}{\partial y}(z_{t+1}) \phi_{z_{t+1}|i_{t+1}=i}(z_{t+1}) dz_{t+1} \\ &= \mathbf{E}_t \left[I_{\{a(\eta_t) < z_{t+1} < b(\eta_t)\}} \frac{\partial}{\partial \Lambda_{t+1}} J_{t+1}^*(\lambda_{t+1}, i_{t+1}) \frac{\partial \Lambda_{t+1}}{\partial y}(z_{t+1}) \right], \end{aligned}$$

where

$$\frac{\partial \Lambda_{t+1}}{\partial y}(z_{t+1}) = \begin{cases} \frac{(z_{t+1} - \mathbf{E}[z])}{1+g_t} & y = \eta_t \\ \frac{1 - \Lambda_{t+1}}{1+g_t} & y = g_t \\ \frac{1+r(\lambda_t) + \lambda_t r'(\lambda_t)}{1+g_t} & y = \lambda_t \end{cases}.$$

D First-Order Condition for the Hedging Ratio η_t

With Λ_{t+1} a function of z_{t+1} , define

$$f(z_{t+1}, i_{t+1}) = \frac{\partial J_{t+1}^*}{\partial \Lambda_{t+1}}(\Lambda_{t+1}(z_{t+1}), i_{t+1}) I_{\{a(\eta_t) < z_{t+1} < b(\eta_t)\}},$$

where the notation $\Lambda_{t+1}(u)$ stands for the replacement of the occurrence of z_{t+1} by u in equation (9). Since f as a function of z_{t+1} is almost surely differentiable, the representation of the function f in terms of a Taylor series at $z_{t+1} = \mathbf{E}[z]$ is given by

$$\begin{aligned} & \frac{\partial J_{t+1}^*}{\partial \Lambda_{t+1}}(\Lambda_{t+1}, i_{t+1}) I_{\{a(\eta_t) < z_{t+1} < b(\eta_t)\}} \\ & \stackrel{a.s.}{=} f(\mathbf{E}[z], i_{t+1}) + \frac{\partial f}{\partial z_{t+1}}(\mathbf{E}[z], i_{t+1})(z_{t+1} - \mathbf{E}[z]) + \frac{1}{2} \frac{\partial^2 f}{\partial z_{t+1}^2}(\xi_{t+1}, i_{t+1})(\xi_{t+1} - \mathbf{E}[z])^2 \\ & \stackrel{a.s.}{=} f(\mathbf{E}[z], i_{t+1}) - f_1(\mathbf{E}[z], i_{t+1})(z_{t+1} - \mathbf{E}[z]) \frac{1 - \eta_t}{1 + g_t} \\ & + \frac{1}{2} f_2(\xi_{t+1}, i_{t+1})(\xi_{t+1} - \mathbf{E}[z])^2 \left(\frac{1 - \eta_t}{1 + g_t} \right)^2, \end{aligned} \quad (39)$$

where

$$\begin{aligned} f_1(z_{t+1}, i_{t+1}) &= \frac{\partial^2 J_{t+1}^*}{\partial \Lambda_{t+1}^2}(\Lambda_{t+1}(z_{t+1}), i_{t+1}) I_{\{a(\eta_t) < z_{t+1} < b(\eta_t)\}}, \\ f_2(z_{t+1}, i_{t+1}) &= \frac{\partial^3 J_{t+1}^*}{\partial \Lambda_{t+1}^3}(\Lambda_{t+1}(z_{t+1}), i_{t+1}) I_{\{a(\eta_t) < z_{t+1} < b(\eta_t)\}}, \end{aligned}$$

and ξ_{t+1} a random variable satisfying equation (26).

Using (39), the first-order condition (36) simplifies to

$$\begin{aligned} \mathbf{E}_t [f(\mathbf{E}[z], i_{t+1})(z_{t+1} - \mathbf{E}[z])] &= \mathbf{E}_t \left[f_1(\mathbf{E}[z], i_{t+1})(z_{t+1} - \mathbf{E}[z])^2 \frac{1 - \eta_t}{1 + g_t} \right. \\ & \quad \left. - \frac{1}{2} \mathbf{E}_t \left[f_2(\xi_{t+1}, i_{t+1})(\xi_{t+1} - \mathbf{E}[z])^2 (z_{t+1} - \mathbf{E}[z]) \right] \left(\frac{1 - \eta_t}{1 + g_t} \right)^2 \right], \end{aligned}$$

which can be written as

$$\begin{aligned} \text{cov}(f(\mathbf{E}[z], i_{t+1}), z_{t+1}) &= \frac{1 - \eta_t}{1 + g_t} \left(\mathbf{E}_t \left[f_1(\mathbf{E}[z], i_{t+1}) (z_{t+1} - \mathbf{E}[z])^2 \right] - \right. \\ &\quad \left. \frac{1}{2} \mathbf{E}_t \left[f_2(\xi_{t+1}, i_{t+1}) (\xi_{t+1} - \mathbf{E}[z])^2 (z_{t+1} - \mathbf{E}[z]) \right] \frac{1 - \eta_t}{1 + g_t} \right) \end{aligned}$$

since $\mathbf{E}_t [z_{t+1} - \mathbf{E}[z]]$ equals zero.

E Numerical Implementation for Multi-period Model

To implement the dynamic programming model (19), one has to rely on a suboptimal approximation. Consider a finite partition $[\lambda_l, \lambda_{l+1})$ of the λ dimension such that $0 = \lambda_0 < \lambda_1 < \dots < \lambda_N = 1$. Given the discrete nature of the variable i , the state space takes values on the grid:

$$\mathcal{G} = \{(\lambda_l, i_m) | l = 0, \dots, N \text{ and } m = 1, \dots, |M|\},$$

where $|\cdot|$ denotes the number of elements in the set.

Since the state space has been approximated by the grid \mathcal{G} , \tilde{J}_{t+1} denotes the approximation of J_{t+1} that is available at each point of \mathcal{G} . This means that the suboptimal dynamic programming problem (19) to solve for $t = 1, \dots, T - 1$ is the following:

$$\tilde{J}_t(\lambda^l, i^m) = \max_{\substack{g_t, \eta_t \\ 0 \leq g_t \leq i_t}} \left\{ \frac{1}{1 + w(\lambda_t)} \left(\mathbf{E}[z] - g_t + (1 + g_t) \mathbf{E}_{lmt} \left[\tilde{J}_{t+1}(\lambda_{t+1}, i_{t+1}) \right] \right) \right\}, \quad (40)$$

where $\mathbf{E}_{lmt}[\cdot]$ stands for the conditional expectation at time t given that the state variables were at (λ^l, i^m) .

Given that the state space is continuous on the λ dimension, and in order to compute the recursion step in the Bellman equation, the cost-to-go function $\tilde{J}_{t+1}(\lambda_{t+1}, i_{t+1})$ has to be approximated with some function so that the problem can be implemented numerically for $t = 1, \dots, T - 1$.

The function \tilde{J}_{t+1} is approximated with a piecewise quadratic interpolation on the λ dimension, which will be denoted by \hat{J}_{t+1} .⁶ Other types of interpolation schemes can be used, such as higher order polynomials, spline functions, or Laguerre approximations; however, in the present case, these approximations will not contribute much to the precision of the results but will render the implementation more complicated.

To solve (40), one needs to approximate the term $\mathbf{E}_{lmt}[\cdot]$. Suppose that at time t the current state is $(\lambda_l, i_m) \in \mathcal{G}$. Now, consider the following regions that partition the space state:

$$\mathcal{A}_j^k = \{(\lambda, i) | \lambda_j \leq \lambda \leq \lambda_{j+1} \text{ and } i = i_k\}, \quad j = 0, \dots, N - 1, k = 1, \dots, |M|.$$

The value of $\tilde{J}_{t+1}(\lambda_{t+1}, i_{t+1})$ is approximated within each region by:

$$\hat{J}_{j,k,t+1}(\lambda_{t+1}, i_k) = \psi_{j,k,t} + \theta_{j,k,t}(\lambda_{t+1} - \lambda_j) + \gamma_{j,k,t}(\lambda_{t+1} - \lambda_j)(\lambda_{t+1} - \lambda_{j+1}), \quad (41)$$

where $\psi_{j,k,t}$, $\theta_{j,k,t}$, and $\gamma_{j,k,t}$ are defined in (47), (48), and (49) respectively. Then, one can write the expected value of \tilde{J}_{t+1} in terms of the regions \mathcal{A}_j^k as:

$$\mathbf{E}_{lmt} \left[\tilde{J}_{t+1}(\lambda_{t+1}, i_{t+1}) \right] \simeq \sum_{j=0}^{N-1} \sum_{k=0}^{|M|} \mathbf{E}_{lmt} \left[\hat{J}_{j,k,t+1}(\lambda_{t+1}, i_{t+1}) I \{ \mathcal{A}_j^k \} \right], \quad (42)$$

where $I \{ \mathcal{A}_j^k \}$ is the indicator function for the event $(\lambda_{t+1}, i_{t+1}) \in \mathcal{A}_j^k$.

Working with $\mathbf{E}_{lmt} \left[\hat{J}_{j,k,t+1}(\lambda_{t+1}, i_{t+1}) I \{ \mathcal{A}_j^k \} \right]$ for $j \in \{0, \dots, N - 2\}$, $k \in \{0, \dots, |M|\}$ and using (41), one has:

$$\mathbf{E}_{lmt} \left[\hat{J}_{j,k,t+1}(\lambda_{t+1}, i_{t+1}) I \{ \mathcal{A}_j^k \} \right] = (\psi_{j,k,t} - \theta_{j,k,t} \lambda_j + \gamma_{j,k,t} \lambda_j \lambda_{j+1}) \mathbf{E}_{lmt} [I \{ \mathcal{A}_j^k \}] +$$

⁶This kind of approximation has been used in the context of option pricing with a Brownian process in Ben-Ameur et al. (2002) and for GARCH process in Ben-Ameur et al. (2008).

$$\begin{aligned}
& (\theta_{j,k,t} - \gamma_{j,k,t} (\lambda_j + \lambda_{j+1})) \mathbf{E}_{lmt} [\lambda_{t+1} I \{\mathcal{A}_j^k\}] \\
& + \gamma_{j,k,t} \mathbf{E}_{lmt} [(\lambda_{t+1})^2 I \{\mathcal{A}_j^k\}].
\end{aligned} \tag{43}$$

Combining (40), (42), and (43) gives:

$$\begin{aligned}
\tilde{J}_t (\lambda^l, i^m) \simeq & \max_{\substack{g_t, \eta_t \\ st: 0 \leq g_t \leq i_t}} \left\{ \frac{1}{1 + w(\lambda_t)} \left(\mathbf{E} [z_t] - g_t + (1 + g_t) \sum_{j=0}^{N-1} \sum_{k=0}^{|M|} \left(\tilde{\psi}_{j,k,t} \mathbf{E}_{lmt} [I \{\mathcal{A}_j^k\}] \right. \right. \right. \\
& \left. \left. \left. + \tilde{\theta}_{j,k,t} \mathbf{E}_{lmt} [\lambda_{t+1} I \{\mathcal{A}_j^k\}] + \gamma_{j,k,t} \mathbf{E}_{lmt} [(\lambda_{t+1})^2 I \{\mathcal{A}_j^k\}] \right) \right) \right\},
\end{aligned} \tag{44}$$

where $\tilde{\psi}_{j,k,t}$ and $\tilde{\theta}_{j,k,t}$ are defined in as

$$\tilde{\psi}_{j,k,t} = \psi_{j,k,t} - \theta_{j,k,t} \lambda_j + \gamma_{j,k,t} \lambda_j \lambda_{j+1}, \tag{45}$$

$$\tilde{\theta}_{j,k,t} = \theta_{j,k,t} - \gamma_{j,k,t} (\lambda_j + \lambda_{j+1}), \tag{46}$$

$$\psi_{j,k,t} = \tilde{J}_{t+1} (\lambda_j, i_k), \tag{47}$$

$$\theta_{j,k,t} = \frac{\tilde{J}_{t+1} (\lambda_{j+1}, i_k) - \tilde{J}_{t+1} (\lambda_j, i_k)}{\lambda_{j+1} - \lambda_j}, \tag{48}$$

$$\gamma_{j,k,t} = \begin{cases} \frac{(\tilde{J}_{t+1} (\lambda_j, i_k) (\lambda_{j+2} - \lambda_{j+1}) - \tilde{J}_{t+1} (\lambda_{j+1}, i_k) (\lambda_{j+2} - \lambda_j) + \tilde{J}_{t+1} (\lambda_{j+2}, i_k) (\lambda_{j+1} - \lambda_j))}{(\lambda_{j+2} - \lambda_{j+1}) (\lambda_{j+2} - \lambda_j) (\lambda_{j+1} - \lambda_j)} & j < N - 1 \\ 0 & j = N - 1 \end{cases}, \tag{49}$$

respectively.

Given that the underlying source of volatility of the profitability is assumed Gaussian, closed-form expressions for $\mathbf{E}_{lmt} [I \{\mathcal{A}_j^k\}]$, $\mathbf{E}_{lmt} [\lambda_{t+1} I \{\mathcal{A}_j^k\}]$, and $\mathbf{E}_{lmt} [(\lambda_{t+1})^2 I \{\mathcal{A}_j^k\}]$ can be computed. These expressions are presented in Appendix F.

F Expressions for expected values

This appendix presents closed-form expressions for the expected values $\mathbf{E}_{lmt} [I \{\mathcal{A}_j^k\}]$, $\mathbf{E}_{lmt} [\lambda_{t+1} I \{\mathcal{A}_j^k\}]$, and $\mathbf{E}_{lmt} [(\lambda_{t+1})^2 I \{\mathcal{A}_j^k\}]$.

First, equation (8) implies that λ_{t+1} is a truncated normal variable over the interval $[0, 1]$ with expected value and variance conditional on t given by

$$\mathbf{E} [\lambda_{t+1} | \lambda_t, g_t, \eta_t] = \mu_{\Lambda_{t+1}} - \sigma_{\Lambda_{t+1}} \frac{\phi(\bar{\lambda}_{t+1}^1) - \phi(\bar{\lambda}_{t+1}^0)}{\Phi(\bar{\lambda}_{t+1}^1) - \Phi(\bar{\lambda}_{t+1}^0)}, \tag{50}$$

$$\begin{aligned}
\mathbf{Var} [\lambda_{t+1} | \lambda_t, g_t, \eta_t] &= \sigma_{\Lambda_{t+1}}^2 \left(1 - \frac{\bar{\lambda}_{t+1}^1 \phi(\bar{\lambda}_{t+1}^1) - \bar{\lambda}_{t+1}^0 \phi(\bar{\lambda}_{t+1}^0)}{\Phi(\bar{\lambda}_{t+1}^1) - \Phi(\bar{\lambda}_{t+1}^0)} \right. \\
&\quad \left. - \left(\frac{\phi(\bar{\lambda}_{t+1}^1) - \phi(\bar{\lambda}_{t+1}^0)}{\Phi(\bar{\lambda}_{t+1}^1) - \Phi(\bar{\lambda}_{t+1}^0)} \right)^2 \right),
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
\mu_{\Lambda_{t+1}} &= \frac{g_t - \mu_z + \lambda_t (1 + r(\lambda_t))}{1 + g_t}, \\
\sigma_{\Lambda_{t+1}}^2 &= \left(\frac{1 - \eta_t}{1 + g_t} \right)^2 \sigma_z^2,
\end{aligned}$$

$\bar{\lambda}_{t+1}^1 = \frac{1 - \mu_{\Lambda_{t+1}}}{\sigma_{\Lambda_{t+1}}}$, $\bar{\lambda}_{t+1}^0 = \frac{-\mu_{\Lambda_{t+1}}}{\sigma_{\Lambda_{t+1}}}$, and ϕ and Φ are the density and cumulative distribution of the normal variable.

Given the independence between z and i , the probability of being in a certain region is equivalent to the multiplication of being in the region defined on the dimension of λ , multiplied by the probability of being in the state $i_{t+1} = i^k$, which is P_k . Consequently, one has:

$$\mathbf{E}_{lmt} [I \{\mathcal{A}_j^k\}] = P(\lambda_j \leq \lambda_{t+1} \leq \lambda_{j+1}) P_k = \frac{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)}{\Phi(\bar{\lambda}_{t+1}^1) - \Phi(\bar{\lambda}_{t+1}^0)} P_k,$$

with $\tilde{\lambda}^j = \frac{\lambda_j - \mu_{\lambda_{t+1}}}{\sigma_{\lambda_{t+1}}}$, $\bar{\lambda}_{t+1}^1 = \frac{1 - \mu_{\lambda_{t+1}}}{\sigma_{\lambda_{t+1}}}$, and $\bar{\lambda}_{t+1}^0 = \frac{-\mu_{\lambda_{t+1}}}{\sigma_{\lambda_{t+1}}}$.

To compute the terms $\mathbf{E}_{lmt} [\lambda_{t+1} I \{\mathcal{A}_j^k\}]$ and $\mathbf{E}_{lmt} [(\lambda_{t+1})^2 I \{\mathcal{A}_j^k\}]$, the moment generating function of the truncated normal is used:

$$\begin{aligned} M(t) &= \mathbf{E} [e^{Xt} | X \in [\alpha_1, \alpha_2]] \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}} \frac{\Phi\left(\frac{\alpha_2 - \mu}{\sigma} - \sigma t\right) - \Phi\left(\frac{\alpha_1 - \mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{\alpha_2 - \mu}{\sigma}\right) - \Phi\left(\frac{\alpha_1 - \mu}{\sigma}\right)}, \end{aligned}$$

where μ and σ are the expected value and standard deviation of the normal distribution. Expressions for $\mathbf{E}_{lmt} [\lambda_{t+1} I \{\mathcal{A}_j^k\}]$ with $j \in \{0, \dots, N-1\}$, can be computed as follows:

$$\begin{aligned} \mathbf{E}_{lmt} [\lambda_{t+1} I \{\mathcal{A}_j^k\}] &= P_k \int_{\lambda_j}^{\lambda_{j+1}} \lambda_{t+1} \frac{\frac{1}{\sigma_{\lambda_{t+1}}} \varphi\left(\frac{\lambda_{t+1} - \mu_{\lambda_{t+1}}}{\sigma_{\lambda_{t+1}}}\right)}{\Phi(\bar{\lambda}_{t+1}^1) - \Phi(\bar{\lambda}_{t+1}^0)} d\lambda_{t+1} \\ &= P_k \frac{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)}{\Phi(\bar{\lambda}_{t+1}^1) - \Phi(\bar{\lambda}_{t+1}^0)} \int_{\lambda_j}^{\lambda_{j+1}} \lambda_{t+1} \frac{\frac{1}{\sigma_{\lambda_{t+1}}} \varphi\left(\frac{\lambda_{t+1} - \mu_{\lambda_{t+1}}}{\sigma_{\lambda_{t+1}}}\right)}{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)} d\lambda_{t+1} \\ &= P_k \frac{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)}{\Phi(\bar{\lambda}_{t+1}^1) - \Phi(\bar{\lambda}_{t+1}^0)} \mathbf{E}[\lambda_{t+1} | \lambda_{t+1} \in [\lambda_j, \lambda_{j+1}]] \\ &= P_k \frac{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)}{\Phi(\bar{\lambda}_{t+1}^1) - \Phi(\bar{\lambda}_{t+1}^0)} M'(t)|_{t=0} \\ &= P_k \frac{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)}{\Phi(\bar{\lambda}_{t+1}^1) - \Phi(\bar{\lambda}_{t+1}^0)} \left(\mu_{\lambda_{t+1}} - \sigma_{\lambda_{t+1}} \frac{\varphi(\tilde{\lambda}_{j+1}) - \varphi(\tilde{\lambda}_j)}{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)} \right). \end{aligned}$$

In a similar way, for $j \in \{0, \dots, N-2\}$ one has:

$$\begin{aligned} &\mathbf{E}_{lmt} [(\lambda_{t+1})^2 I \{\mathcal{A}_j^k\}] \\ &= P_k \frac{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)}{\Phi_{t+1}(\tilde{\lambda}) - \Phi_{t+1}(\tilde{\lambda}_{t+1})} M''(t)|_{t=0} \\ &= P_k \frac{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)}{\Phi_{t+1}(\tilde{\lambda}) - \Phi_{t+1}(\tilde{\lambda}_{t+1})} \left(\sigma_{\lambda_{t+1}}^2 + \mu_{\lambda_{t+1}}^2 \right. \\ &\quad \left. - \sigma_{\lambda_{t+1}}^2 \frac{\tilde{\lambda}_{j+1} \varphi(\tilde{\lambda}_{j+1}) - \tilde{\lambda}_j \varphi(\tilde{\lambda}_j)}{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)} \right. \\ &\quad \left. - 2\mu_{\lambda_{t+1}} \sigma_{\lambda_{t+1}} \frac{\varphi(\tilde{\lambda}_{j+1}) - \varphi(\tilde{\lambda}_j)}{\Phi(\tilde{\lambda}_{j+1}) - \Phi(\tilde{\lambda}_j)} \right). \end{aligned}$$

G Dependence between the Investment Opportunity and the Source of Volatility

This section presents the formulation of the problem when the investment opportunity i is correlated with the source of uncertainty z .

The relative value of the firm at time T can be rewritten as:

$$\begin{aligned}
& J_T^*(\lambda_T, i_T) \\
&= \max_{\substack{g_T, \eta_T \\ st: 0 \leq g_T \leq i_T}} \left(\mathbf{E}_T \left\{ \frac{(x_{T+1} - g_T)}{1 + w(\lambda_T)} + \frac{(1 + g_T)}{1 + w(\lambda_T)} \frac{\mathbf{E}[z] - g}{w(\lambda_{T+1}) - g} \middle| i_{T+1} \neq 0 \right\} P(i_{T+1} \neq 0) \right. \\
&\quad \left. + \mathbf{E}_T \left\{ \frac{(x_{T+1} - g_T)}{1 + w(\lambda_T)} + \frac{(1 + g_T)}{1 + w(\lambda_T)} \frac{\mathbf{E}[z] - g}{w(\lambda_{T+1}) - g} \middle| i_{T+1} = 0 \right\} P(i_{T+1} = 0) \right) \\
&= \max_{\substack{g_T, \eta_T \\ st: 0 \leq g_T \leq i_T}} \frac{1}{1 + w(\lambda_T)} \left((\mathbf{E}[z] + \rho_\mu) P(i_{T+1} \neq 0) + \mathbf{E}[z] P(i_{T+1} = 0) - g_T \right. \\
&\quad \left. + \mathbf{E}_{\{\tilde{z}_{T+1}\}} \left\{ \frac{(\mathbf{E}[z] - g)(1 + g_T)}{w(\lambda_{T+1}) - g} \right\} P(i_{T+1} \neq 0) + \mathbf{E}_{\{z_{T+1}\}} \left\{ \frac{(\mathbf{E}[z] - g)(1 + g_T)}{w(\lambda_{T+1}) - g} \right\} P(i_{T+1} = 0) \right),
\end{aligned}$$

where ρ_μ is a shock to the mean of the profitability coming from the presence of the investment opportunity, that is, it controls for the sign and magnitude of the dependence between these two variables. The variable \tilde{z}_{T+1} has the same distribution as z_{T+1} but with mean $\mathbf{E}[z] + \rho_\mu$. The previous equation is equivalent to (18) but adjusted to take into account the dependence. In a similar way, the dependence can be incorporated into (19) to obtain:

$$\begin{aligned}
& J_t^*(\lambda_t, i_t) \\
&= \max_{\{g_t, \eta_t\}} \frac{1}{1 + w(\lambda_t)} \left(\mathbf{E}[z + \rho_\mu] P(i_{t+1} \neq 0) + \mathbf{E}[z] P(i_{t+1} = 0) - g_t \right. \\
&\quad \left. + (1 + g_t) \left(\mathbf{E}_{\{\tilde{z}_{t+1}\}} \{J^*(\lambda_{t+1}, i_{t+1})\} P(i_{t+1} \neq 0) + \mathbf{E}_{\{z_{t+1}\}} \{J^*(\lambda_{t+1}, i_{t+1})\} P(i_{t+1} = 0) \right) \right).
\end{aligned}$$

Expressions for $\mathbf{E}_{lmt} [I \{\mathcal{A}_j^k\}]$, $\mathbf{E}_{lmt} [\lambda_{t+1} I \{\mathcal{A}_j^k\}]$, and $\mathbf{E}_{lmt} [(\lambda_{t+1})^2 I \{\mathcal{A}_j^k\}]$ when investment opportunity i depends on the source of volatility z are given by:

$$\begin{aligned}
\mathbf{E}_{lmt} [I \{\mathcal{A}_j^k\}] &= P(\lambda_j \leq \lambda_{t+1} \leq \lambda_{j+1}, i_{t+1} = i^k) \\
&= P_k P(\lambda_j \leq \lambda_{t+1} \leq \lambda_{j+1} | i_{t+1} = i^k) \\
&= P_k \frac{\Phi(\check{\lambda}_{j+1}) - \Phi(\check{\lambda}_j)}{\Phi(\check{\lambda}_{t+1}^1) - \Phi(\check{\lambda}_{t+1}^0)},
\end{aligned}$$

with $\check{\lambda}^j = \frac{\lambda_j - \mu_{\Lambda_{t+1}}^i}{\sigma_{\Lambda_{t+1}}^i}$, $\check{\lambda}_{t+1}^1 = \frac{1 - \mu_{\Lambda_{t+1}}^i}{\sigma_{\Lambda_{t+1}}^i}$, $\check{\lambda}_{t+1}^0 = \frac{-\mu_{\Lambda_{t+1}}^i}{\sigma_{\Lambda_{t+1}}^i}$, and $\mu_{\Lambda_{t+1}}^i$ and $\sigma_{\Lambda_{t+1}}^i$ being defined as:

$$\begin{aligned}
\mu_{\Lambda_{t+1}}^i &= \begin{cases} \frac{g_t - \mathbf{E}[z + \rho_\mu] + \lambda_t(1 + r(\lambda_t))}{1 + g_t} & i \neq 0 \\ \frac{g_t - \mathbf{E}[z] + \lambda_t(1 + r(\lambda_t))}{1 + g_t} & i = 0 \end{cases}, \\
\sigma_{\Lambda_{t+1}}^i &= \begin{cases} \left(\frac{1 - \eta_t}{1 + g_t} \right)^2 (\sigma_z \rho_\sigma)^2 & i \neq 0 \\ \left(\frac{1 - \eta_t}{1 + g_t} \right)^2 \sigma_z^2 & i = 0 \end{cases}.
\end{aligned}$$

The computation of the next two expected values is done in a similar way to the base case. For $j \in \{0, \dots, N - 1\}$ one has:

$$\mathbf{E}_{lmt} [\lambda_{t+1} I \{\mathcal{A}_j^k\}] = P_k \frac{(\Phi(\check{\lambda}_{j+1}) - \Phi(\check{\lambda}_j))}{\Phi(\check{\lambda}_{t+1}^1) - \Phi(\check{\lambda}_{t+1}^0)} \left(\mu_{\Lambda_{t+1}}^k - \frac{\sigma_{\Lambda_{t+1}}^k (\varphi(\check{\lambda}_{j+1}) - \varphi(\check{\lambda}_j))}{\Phi(\check{\lambda}_{j+1}) - \Phi(\check{\lambda}_j)} \right),$$

and for $j \in \{0, \dots, N-2\}$ one has:

$$\begin{aligned} \mathbf{E}_{lmt} \left[(\lambda_{t+1})^2 I \{ \mathcal{A}_j^k \} \right] &= P_k \frac{\Phi(\check{\lambda}_{j+1}) - \Phi(\check{\lambda}_j)}{\Phi_{t+1}(\check{\lambda}) - \Phi_{t+1}(\check{\lambda}_{t+1})} \left((\sigma_{\Lambda_{t+1}}^k)^2 + (\mu_{\Lambda_{t+1}}^k)^2 \right. \\ &\quad - (\sigma_{\Lambda_{t+1}}^k)^2 \frac{\check{\lambda}_{j+1} \varphi(\check{\lambda}_{j+1}) - \check{\lambda}_j \varphi(\check{\lambda}_j)}{\Phi(\check{\lambda}_{j+1}) - \Phi(\check{\lambda}_j)} \\ &\quad \left. - 2\mu_{\Lambda_{t+1}}^k \sigma_{\Lambda_{t+1}}^k \frac{\varphi(\check{\lambda}_{j+1}) - \varphi(\check{\lambda}_j)}{\Phi(\check{\lambda}_{j+1}) - \Phi(\check{\lambda}_j)} \right). \end{aligned}$$

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