Les Cahiers du GERAD ISSN: 0711–2440

Total Domination and the Caccetta-Häggkvist Conjecture

P. St-Louis, B. Gendron,
A. Hertz

G–2009–78

December 2009

Total Domination and the Caccetta-Häggkvist Conjecture

Patrick St-Louis

GIRO inc.
75, rue de Port-Royal est
Montréal (Québec) Canada, H3L 3T1
patrick.stlouis@giro.ca

Bernard Gendron

CIRRELT & Département d'informatique et de recherche opérationnelle
Université de Montréal
C.P. 6128, succ. Centre-ville
Montréal (Québec) Canada, H3C 3J7
bernard.gendron@cirrelt.ca

Alain Hertz

GERAD & Département de mathématiques et de génie industriel
École Polytechnique de Montréal
C.P. 6079, succ. Centre-ville
Montréal (Québec) Canada, H3C 3A7
alain.hertz@gerad.ca

December 2009

Les Cahiers du GERAD

G–2009–78

Abstract

A total dominating set in a digraph G is a subset W of its vertices such that every vertex of G has an immediate successor in W. The total domination number of G is the size of the smallest total dominating set. We consider several lower bounds on the total domination number and conjecture that these bounds are strictly larger than $g(G) - 1$, where $g(G)$ is the number of vertices of the smallest directed cycle contained in G. We prove that these new conjectures are equivalent to the Caccetta-Häggkvist conjecture which asserts that $g(G) - 1 < \frac{n}{r}$ in every digraph on n vertices with minimum outdegree at least $r > 0$.

Résumé

Un ensemble totalement dominant dans un graphe orienté G est un sous-ensemble W de ses sommets tel que chaque sommet dans G a un successeur immédiat dans W. Le nombre de domination totale de G est la taille du plus petit ensemble totalement dominant. Nous considérons plusieurs bornes inférieures sur le nombre de domination totale et formulons les conjectures que ces bornes sont strictement plus grandes que $g(G) - 1$, où $g(G)$ est le nombre de sommets dans le plus petit circuit orienté de G. Nous prouvons que ces nouvelles conjectures sont équivalentes à celle de Caccetta-Häggkvist qui stipule que $g(G) - 1 < \frac{n}{r}$ dans tout graphe orienté G ayant n sommets qui ont tous au moins $r > 0$ successeurs immédiats.

1 Introduction

Throughout this paper, we consider only digraphs without multiple arcs and without directed cycles of length 1 or 2. Let $G = (V, A)$ be a digraph with vertex set V and arc set A. The girth of G, denoted $g(G)$, is the number of vertices of the smallest directed cycle in G. Let $\delta^+(G)$ denote the minimum outdegree of G. In 1978, Caccetta and Häggkvist [1] proposed the following conjecture.

Conjecture 1 *Let G be a digraph with n vertices and $\delta^+(G) \geq r > 0$. Then $g(G) \leq \lceil \frac{n}{r} \rceil$.*

This conjecture has been verified for values of r up to 5 [1, 4, 5] and for $n \geq 2r^2 - 3r + 1$ [7]. Another approach is to show that if G is a digraph with n vertices and $\delta^+(G) \geq r$, then there is a directed cycle in G of length at most $\frac{n}{r} + c$ for some small c. This has been proved for $c = 2500$ [2], $c = 304$ [6] and $c = 73$ [8]. In 2006, a workshop was held in Palo Alto, California, with the Caccetta-Häggkvist conjecture as its central subject. A summary of the results (and much more) was published by Sullivan [9].

Let $N_G^+(v)$ denote the set of immediate successors of a vertex $v \in V$. A *total dominating set* in a digraph G is a subset W of its vertices such that $N_G^+(v) \cap W \neq \emptyset$ for every vertex $v \in V$. The total domination number of G, denoted $TD(G)$ is the size of the smallest total dominating set of G. We assume $\delta^+(G) > 0$, else G does not contain any total dominating set. Finding a total dominating set of size $TD(G)$ can be modeled as the assignment of a weight $\omega_v \in \{0, 1\}$ to every vertex $v \in V$ so that $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ for every $v \in V$ and $\sum_{v \in V} \omega_v$ is minimized. Note that $\sum_{u \in V} \omega_u \geq \sum_{u \in N_G^+(v)} \omega_u \geq 1$, for any $v \in V$, i.e., $TD(G)$ is at least 1. Better lower bounds on $TD(G)$ can be obtained by considering real values for the weights:

- We denote $TDF(G)$ the minimum total weight $\sum_{v \in V} \omega_v$ so that $\omega_v \in [0, 1]$ and $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ for every $v \in V$.
- By imposing $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ only for vertices with a strictly positive weight and by requiring that $\sum_{v \in V} \omega_v \geq 1$, one gets a lower bound on $TDF(G)$. More precisely, we denote $TDFR(G)$ the minimum total weight $\sum_{v \in V} \omega_v$ so that $\omega_v \in [0, 1]$ for every $v \in V$, $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ for every v with $\omega_v > 0$, and $\sum_{v \in V} \omega_v \geq 1$.

It follows from the above definitions that $TDFR(G) \leq TDF(G) \leq TD(G)$. We state the two following conjectures.

Conjecture 2 *The relation $g(G) - 1 < TDF(G)$ holds for all digraphs G with $\delta^+(G) > 0$.*

Conjecture 3 *The relation $g(G) - 1 < TDFR(G)$ holds for all digraphs G with $\delta^+(G) > 0$.*

We prove in this paper that the two new conjectures are equivalent to Conjecture 1 of Caccetta and Häggkvist. In Section 2, we present mathematical programming formulations that can be used to compute $TD(G)$ and its lower bounds $g(G)$, $TDF(G)$ and $TDFR(G)$. We use these formulations to prove the equivalence of the three conjectures. In Section 3, we show how to reformulate Conjecture 3 using Lagrangean relaxation techniques.

2 Mathematical Programming Formulations

The adjacency matrix A of a digraph G is the $n \times n$ matrix where $a_{ij} = 1$ if there is an arc from i to j, and $a_{ij} = 0$ otherwise. We denote e the vector with n entries equal to 1. The problem of determining $TD(G)$

will be denoted $P_{TD}(G)$ and can be modeled as an integer programming model as follows:

$$\begin{aligned} TD(G) = Min \quad & e^T\omega \\ s.t. \quad & A\omega \geq e, \\ & \omega \in \{0,1\}^n. \end{aligned} \tag{1}$$

Determining $g(G)$ can be viewed as the selection of the smallest subset W of vertices such that $N_G^+(v) \cap W \neq \emptyset$ for every vertex v in W. This problem, denoted $P_g(G)$, can be modeled with the following integer programming model, where constraints (3) ensure that at least one vertex is selected in W:

$$\begin{aligned} g(G) = Min \quad & e^T\omega \\ s.t. \quad & A\omega \geq \omega, & (2) \\ & e^T\omega \geq 1, & (3) \\ & \omega \in \{0,1\}^n. \end{aligned}$$

Property 4 *The relation $g(G) \leq TD(G)$ holds for all digraphs G with $\delta^+(G) > 0$.*

Proof. Since the inequalities $A\omega \geq e$ imply $e^T\omega \geq 1$, one can add constraints (3) to the computation of $TD(G)$ without modifying the optimal value of $P_{TD}(G)$. Since $\omega \leq e$, constraints (1) are stronger than (2), which proves that $g(G) \leq TD(G)$. ■

To prove the validity of Conjecture 1 it would have been sufficient to show that $TD(G) < \frac{n}{r} + 1$, since this would imply $g(G) - 1 \leq TD(G) - 1 < \frac{n}{r}$, which is equivalent to $g(G) \leq \lceil \frac{n}{r} \rceil$. There are however digraphs for which $TD(G) \geq \frac{n}{r} + 1$. For example, it is not difficult to verify that the digraph in Figure 1 satisfies $n = 10$, $r = \delta^+(G) = 2$ and $6 = TD(G) = \frac{n}{r} + 1$, the black vertices corresponding to a total dominating set of minimum size.

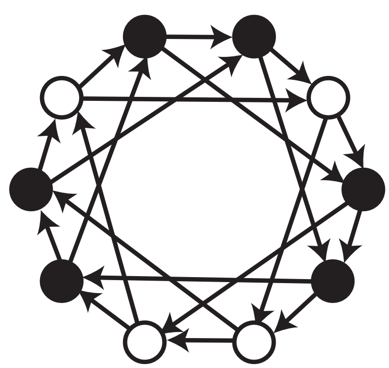


Figure 1: A digraph with $TD(G) = \frac{n}{r} + 1$

The problem of computing $TDF(G)$, denoted $P_{TDF}(G)$, can be modeled by relaxing the integrality constraints in $P_{TD}(G)$:

$$\begin{aligned} TDF(G) = Min \quad & e^T\omega \\ s.t. \quad & A\omega \geq e, \\ & \omega \geq 0. \end{aligned} \tag{1}$$

By imposing $A\omega \geq \lceil \omega \rceil$, we require $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ for every vertex v with $\omega_v > 0$. Hence, the problem $P_{TDFR}(G)$ of computing $TDFR(G)$ can be modeled as follows:

$$\begin{aligned} TDFR(G) = Min \quad & e^T\omega \\ s.t. \quad & A\omega \geq \lceil \omega \rceil, && (4) \\ & e^T\omega \geq 1, && (3) \\ & \omega \geq 0. \end{aligned}$$

Note that if Conjecture 3 is verified then $g(G)-1 < TDFR(G) \leq g(G)$ for all digraphs G with $\delta^+(G) > 0$, since by setting $\omega_v = 1$ for all vertices v in a smallest directed cycle in G and $\omega_v = 0$ for the other vertices, one gets a feasible solution to $P_{TDFR}(G)$ of value $g(G)$.

Theorem 5 *Conjectures 2 and 3 are equivalent.*

Proof. If Conjecture 3 is verified then $g(G) - 1 < TDFR(G) \leq TDF(G)$, which implies that Conjecture 2 is verified also.

So assume that Conjecture 3 is not verified and let G be a smallest counter-example (in terms of number of vertices). It remains to prove that Conjecture 2 is also not verified. Let ω^* denote an optimal solution to $P_{TDFR}(G)$, and let G' denote the sub-digraph of G induced by all vertices v with weight $\omega^*_v > 0$. Constraints (4) impose that each vertex in G' has at least one successor in G'. Hence G' contains at least one directed cycle and we obviously have $g(G') \geq g(G)$. Also, $TDFR(G') \leq TDFR(G)$ since the restriction of ω^* to G' is a feasible solution to $P_{TDFR}(G')$. In summary, $g(G')-1 \geq g(G)-1 \geq TDFR(G) \geq TDFR(G')$. Since G is the smallest counter-example to Conjecture 3, we necessarily have $G' = G$, which means that $\omega^*_v > 0$ for all vertices in G. Hence, ω^* is a feasible solution to $P_{TDF}(G)$, which means that $TDF(G) = TDFR(G)$ and G is therefore also a counter-example to Conjecture 2. ■

Theorem 6 *Conjectures 1 and 2 are equivalent.*

Proof. Let G be a digraph with n vertices and consider any real number r such that $0 < r \leq \delta^+(G)$. The vector ω defined by $\omega_v = \frac{1}{r}$ for all $v \in V$ is a feasible solution to $P_{TDF}(G)$, which means that $TDF(G) \leq e^T\omega = \frac{n}{r}$. Hence, if Conjecture 2 is verified, then $g(G) - 1 < TDF(G) \leq \frac{n}{r}$ for all digraphs G with n vertices and $\delta^+(G) \geq r > 0$, which implies that Conjecture 1 is verified also.

So assume that Conjecture 2 is not verified. It remains to show that Conjecture 1 is also not verified. Let G be a smallest counter-example to Conjecture 2 (in terms of number of vertices), and let ω^* be any optimal basic solution to $P_{TDF}(G)$. We necessarily have $\omega^*_v > 0$ for all $v \in V$, otherwise by using the same arguments as in the proof of the previous theorem, we can show that the sub-digraph G' induced by the vertices with $\omega^*_v > 0$ verifies $g(G') - 1 \geq g(G) - 1 \geq TDF(G) \geq TDF(G')$, which contradicts the minimality of G.

Constraints (1) can be rewritten as $A\omega - s = e$ by using slack variables $s \geq 0$. Consider the values $s^* = A\omega^* - e$ of the slack variables associated with ω^*. Since $P_{TDF}(G)$ contains n constraints, we necessarily have $s^* = 0$, else there would be at least one vertex $v \in V$ with $\omega^*_v = 0$. We therefore have $A\omega^* = e$. In other words, if we denote $P^=_{TDF}(G)$ the linear program obtained from $P_{TDF}(G)$ by replacing inequalities (1) by equalities, we have shown that $P^=_{TDF}(G)$ and $P_{TDF}(G)$ have the same set of optimal solutions.

We now show that the determinant $det(A)$ of matrix A is not equal to 0. If $det(A) = 0$, then at least one of the n constraints in $P^=_{TDF}(G)$ is redundant. By removing such a constraint, the optimal value remains unchanged, while there are now $n-1$ constraints for n variables. This means that $P^{=}_{TDF}(G)$ has an optimal solution ω^* (which is also optimal for $P_{TDF}(G)$) with at least one variable $\omega^*_v = 0$, a contradiction. We therefore have $det(A) \neq 0$.

Let A_v denote the matrix obtained from A by replacing the $v-th$ column by vector e. Cramer's rule [3] states that $\omega^*_v = \frac{\det(A_v)}{\det(A)}$. Since $\omega^*_v > 0$, we can write

$$\omega^*_v = \frac{|\det(A_v)|}{|\det(A)|}.$$

We now construct a new graph $\tilde{G}$ from G by replacing every vertex v by a set S_v of $|\det(A_v)|$ non-adjacent vertices. We put an arc from a vertex in S_u to a vertex in S_v if and only if there is an arc from u to v in G. Let $\tilde{V} = \bigcup_{v\in V} S_v$ denote the vertex set of $\tilde{G}$ and define $\tilde{\omega}_{\tilde{v}} = \frac{1}{|\det(A)|}$ for all $\tilde{v} \in \tilde{V}$. In other words, $\tilde{G}$ is obtained from G by replacing every vertex v of weight ω^*_v by $|\det(A_v)|$ non-adjacent copies of v of weight $\frac{1}{|\det(A)|}$. This means that the following equalities hold for every vertex $\tilde{v} \in S_v$:

$$\sum_{\tilde{u}\in N^+_{\tilde{G}}(\tilde{v})} \tilde{\omega}_{\tilde{u}} = \sum_{u\in N^+_G(v)} \sum_{\tilde{u}\in S_u} \frac{1}{|\det(A)|} = \sum_{u\in N^+_G(v)} \frac{|\det(A_u)|}{|\det(A)|} = \sum_{u\in N^+_G(v)} \omega^*_u = 1.$$

In addition, we have:

$$\sum_{\tilde{v}\in\tilde{V}} \tilde{\omega}_{\tilde{v}} = \sum_{v\in V}\sum_{\tilde{v}\in S_v} \tilde{\omega}_{\tilde{v}} = \sum_{v\in V} \frac{|\det(A_v)|}{|\det(A)|} = \sum_{v\in V} \omega^*_v = TDF(G).$$

Hence, $\tilde{\omega}$ is a feasible solution to $P_{TDF}(\tilde{G})$ of value $TDF(G)$, which means that $TDF(\tilde{G}) \leq TDF(G)$.

Since, for every $\tilde{v} \in S_v$, $\sum_{\tilde{u}\in N^+_{\tilde{G}}(\tilde{v})} \tilde{\omega}_{\tilde{u}} = \sum_{\tilde{u}\in N^+_{\tilde{G}}(\tilde{v})} \frac{1}{|\det(A)|} = 1$, we deduce that $|N^+_{\tilde{G}}(\tilde{v})| = |\det(A)|$, which means that $\delta^+(\tilde{G}) = |\det(A)|$. Also, since $\sum_{\tilde{v}\in\tilde{V}} \tilde{\omega}_{\tilde{v}} = \sum_{\tilde{v}\in\tilde{V}} \frac{1}{|\det(A)|} = TDF(G)$, we have $|\tilde{V}| = |\det(A)| \cdot TDF(G)$. Moreover, we clearly have $g(G) = g(\tilde{G})$.

To conclude, let $n' = |\tilde{V}|$ denote the number of vertices in $\tilde{G}$ and let $r' = \delta^+(\tilde{G}) = |\det(A)|$. We have

$$g(\tilde{G}) - 1 = g(G) - 1 \geq TDF(G) = \frac{|\det(A)| \cdot TDF(G)}{|\det(A)|} = \frac{n'}{r'}$$

which means that $\tilde{G}$ is a counter-example to Conjecture 1. ■

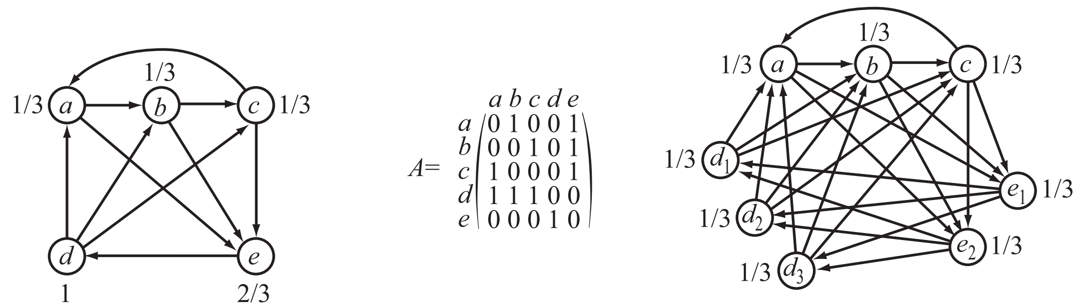
The construction described in the proof of Theorem 6 is illustrated in Figure 2 for a digraph G with $TDF(G) = \frac{8}{3}$ and $g(G) = 3$.

3 Reformulation of Conjecture 3

Problem $P_{TDFR}(G)$ can be written as an integer linear programming model by replacing $A\omega \geq \lceil\omega\rceil$ with $A\omega \geq y \geq \omega$, $y \in \{0,1\}^n$. Hence, an equivalent model for $P_{TDFR}(G)$ reads as follows:

$$\begin{aligned} TDFR(G) = Min \quad & e^T\omega \\ s.t. \quad & A\omega \geq y, && (5)\\ & y \geq \omega, && (6)\\ & e^T\omega \geq 1, && (3)\\ & \omega \geq 0, y \in \{0,1\}^n. \end{aligned}$$

If we now replace constraints (6) by $y = \omega$, we obtain an equivalent model for $P_g(G)$ since



A digraph G with optimal weights ω^*_v on the vertices.

The adjacency matrix A with *det(A)*=3.

The corresponding digraph $\tilde{G}$ with weights $\tilde{\omega}_{\tilde{v}}$ on the vertices.

Figure 2: A digraph G and the corresponding digraph $\tilde{G}$

- constraints (5) are then equivalent to constraints (2);
- constraints $\omega \geq 0, y \in \{0,1\}^n$ are then equivalent to $\omega \in \{0,1\}^n$.

In other words, a model for $P_g(G)$ can be obtained from the above model for $P_{TDFR}(G)$ by adding the constraints $y \leq \omega$. Consider now the Lagrangian relaxation of this model for $P_g(G)$ obtained by relaxing constraints $y \leq \omega$ and by penalizing their violation in the objective function. More formally, given a penalty vector $\lambda \geq 0$ with n entries, we consider the problem $P_{g_\lambda}(G)$ of computing $g_\lambda(G)$ defined as follows:

$$\begin{aligned} g_\lambda(G) = Min \quad & e^T\omega + \lambda^T(y-\omega) \\ s.t. \quad & A\omega \geq y, && (5) \\ & y \geq \omega, && (6) \\ & e^T\omega \geq 1, && (3) \\ & \omega \geq 0, y \in \{0,1\}^n. \end{aligned}$$

Property 7 *The relations $TDFR(G) \leq g_\lambda(G) \leq g(G)$, for any $\lambda \geq 0$, $TDFR(G) = g_0(G)$ and $g(G) = g_\lambda(G)$, for any $\lambda \geq e$, hold for all digraphs G with $\delta^+(G) > 0$.*

Proof. For any $\lambda \geq 0$, $P_{g_\lambda}(G)$ and $P_{TDFR}(G)$ have the same set of feasible solutions. Since $\lambda^T(y-\omega) \geq 0$ because of constraints (6), we have $TDFR(G) \leq g_\lambda(G)$. Let ω^* be an optimal solution to $P_g(G)$ and define $y^* = \omega^*$. Since (ω^*, y^*) is feasible for $P_{g_\lambda}(G)$ and $\lambda^T(y^*-\omega^*) = 0$, we have $g_\lambda(G) \leq g(G)$. For $\lambda = 0$, $P_{g_\lambda}(G)$ corresponds to $P_{TDFR}(G)$, which means that $TDFR(G) = g_0(G)$. For any $\lambda \geq e$, let (ω^*, y^*) be an optimal solution to $P_{g_\lambda}(G)$. If $\omega^* < y^*$, we can replace ω^* by y^* and remain feasible, but also optimal, since the objective would then vary by the quantity $(e-\lambda)^T(y^*-\omega^*) \leq 0$. Therefore, for any $\lambda \geq e$, there exists an optimal solution to $P_{g_\lambda}(G)$ that satisfies $\omega^* = y^*$, which means that $g(G) = g_\lambda(G)$. ∎

From this result, it follows directly that $TDFR(G) = \min_{\lambda \geq 0} g_\lambda(G)$ and $g(G) = \max_{\lambda \geq 0} g_\lambda(G)$. Conjecture 3 can therefore be rewritten in the following way:

Reformulation of Conjecture 3

The relation $\max_{\lambda \geq 0} g_\lambda(G) - \min_{\lambda \geq 0} g_\lambda(G) < 1$ holds for all digraphs G with $\delta^+(G) > 0$.

References

[1] L. Caccetta and R. Häggkvist. On minimal digraphs with given girth. *Congressus Numerantium* 21:181–187, 1978.

[2] V. Chvátal and E. Szemerédi. Short cycles in directed graphs. *Journal of Combinatorial Theory, Series B*, 35:323–327, 1983.

[3] G. Cramer. Introduction à l'analyse des lignes courbes algébriques. Geneva, 657-659, 1750.

[4] Y. O. Hamidoune. A note on minimal directed graphs with given girth. *Journal of Combinatorial Theory, Series B*, 43:343–348, 1987.

[5] C. Hoáng and B. Reed. A note on short cycles in digraphs. *Discrete Mathemathics*, 66:103–107, 1987.

[6] T. Nishimura. Short cycles in digraphs. *Discrete Mathematics*, 72:295–298, 1988.

[7] J. Shen. On the girth of digraphs. *Discrete Mathematics*, 211:167–181, 2000.

[8] J. Shen. On the Caccetta-Häggkvist Conjecture. *Graphs and Combinatorics*, 18:645–654, 2002.

[9] B. D. Sullivan. A summary of results and problems related to the Caccetta-Häggkvist conjecture. 2008, arXiv:math/0605646v1