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Abstract

The proximity π of a graph G is the minimum average distance from a vertex of G to all others. Similarly, the remoteness of G is the maximum average distance from a vertex to all others. The girth g of a graph G is the length of its smallest cycle. In this paper, we provide and prove sharp lower and upper bounds, in terms of the order n of G, on the difference, the sum, the ratio and the product of the proximity and the girth. We do the same for the remoteness and the girth, except for the lower bound on ρ/g, for which a conjecture is given.

Key Words: Proximity, remoteness, girth, extremal graph.

Résumé

La proximité π d'un graphe G est le minimum de la distance moyenne d'un sommet de G à tous les autres. Semblablement, l'éloignement ρ de G est le maximum de la distance moyenne d'un sommet de G à tous les autres. La maille g de G est la longueur du plus petit cycle de G, s'il y en a. Dans cet article, nous donnons et prouvons des bornes inférieures et supérieures serrées pour la différence, la somme, le rapport et le produit de la proximité et de la maille. Nous faisons de même pour l'éloignement et la maille, à l'exception de la borne inférieure sur ρ/g, pour laquelle nous formulons une conjecture.

Mots clés : proximité, éloignement, maille, graphe extrême.

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1 Introduction

Let $G = (V, E)$ denote a simple and connected graph, with vertex set V and edge set E, containing $n = |V|$ vertices and $m = |E|$ edges. The distance between two vertices u and v in G, denoted by $d(u, v)$, is the length of a shortest path between u and v. The average distance between all pairs of vertices in G is denoted by $\bar{l}$. The eccentricity $e(v)$ of a vertex v in G is the largest distance from v to another vertex of G. The minimum eccentricity in G, denoted by r, is the radius of G. The maximum eccentricity of G, denoted by D, is the diameter of G. The average eccentricity of G is denoted ecc. That is

$$r = \min_{v\in V} e(v), \qquad D = \max_{v\in V} e(v) \qquad \text{and} \qquad ecc = \frac{1}{n}\sum_{v\in V} e(v).$$

The *girth* g of the graph G is the length of its smallest cycle, if any. The *proximity* π of G is the minimum average distance from a vertex of G to all others. Similarly, the *remoteness* of G is the maximum average distance from a vertex to all others. The two last concepts were recently introduced in [1, 3]. They are close to the concept of transmission $t(v)$ of a vertex v, which is the sum of the distances from v to all others. Indeed,

$$\pi = \min_{v\in V} \tilde{t}(v) = \min_{v\in V} \frac{t(v)}{n-1} \qquad \text{and} \qquad \rho = \max_{v\in V} \tilde{t}(v) = \max_{v\in V} \frac{t(v)}{n-1}.$$

Proximity and remoteness appear to be more convenient than minimum and maximum transmissions in comparisons with other metric invariants, such as the diameter, radius, average eccentricity and average distance, as they have the same order of magnitude when viewed as functions of the order n of G. Indeed, it follows from the definitions that

$$\pi \le r \le ecc \le D, \qquad \pi \le \bar{l} \le \rho \le D \qquad \text{and} \qquad \bar{l} = \frac{1}{n(n-1)}\sum_{v\in V} t(v).$$

In [5], π and ρ were compared with r, D, ecc and $\bar{l}$, as well as with the independence number α and the matching number μ. The purpose of the present paper is to compare π and ρ with another metric invariant, *i.e.*, the girth g of G. In the next section, we provide and prove the eight inequalities of the form:

$$l(n) \le \pi \oplus g \le u(n)$$

where $\oplus$ denotes one of the four operations $-, +, /, \times$, and $l(n)$ and $u(n)$ are best possible lower and upper bounding functions depending only on the order n of G. We also characterize extremal graphs, *i.e.*, those for which the bounds are attained.

In Section 3, we provide and prove seven of the eight inequalities of the form

$$l(n) \le \rho \oplus g \le u(n)$$

and propose a conjecture for the last one. Again, we characterize extremal graphs. For that prupose, we need the following two definitions.

A *lollipop* $L_{n,g}$ is the graph obtained from a cycle C_g and a path P_{n-g} by adding an edge between an endpoint of P_{n-g} and a vertex of the cycle C_g. For a lollipop $L_{n,g}$, we have

$$\rho(L_{n,g}) = \begin{cases} \frac{n}{2} - \frac{g(g-2)}{4(n-1)} & \text{if } g \text{ is even} \\ \frac{n}{2} - \frac{g(g-2)+1}{4(n-1)} & \text{if } g \text{ is odd.} \end{cases}$$

A *turnip* $T_{n,g}$, with $n \ge g \ge 3$, is the graph obtained from a cycle C_g by attaching $n - p$ pending edges to one vertex from the cycle. If $g = n$, the turnip $T_{n,g} = T_{n,n}$ is the cycle C_n. For a turnip $T_{n,g}$, we have

$$\pi(T_{n,g}) = \begin{cases} \frac{g^2-4g+4n-1}{4(n-1)} & \text{if } g \text{ is odd} \\ \frac{g^2-4g+4n}{4(n-1)} & \text{if } g \text{ is even.} \end{cases}$$

Note that these two series of inequalities were first obtained with the system AGX [2, 7, 6]. Relations of Nordhaus–Gaddum type for proximity π and for remoteness ρ are given in [4].

2 The proximity and the girth

In this section, we prove all lower and upper bounds on $\pi - g$, $\pi + g$, π/g and $\pi \cdot g$ in terms of n, the order of G. We also characterize all the corresponding families of extremal graphs. To prove the lower bound on π/g, we need the following lemma.

Lemma 1 *Let G be a connected graph on $n \geq 3$ vertices with girth g.*

- *(i) If $g = 3$, then $\pi \geq 1$ with equality if and only if G contains at least n edges and a dominating vertex.*
- *(ii) If $g = 4$, then $\pi \geq 1 + 2/(n-1)$ with equality if and only if G contains the turnip $T_{n,4}$ as a spanning subgraph and is a spanning subgraph of the complete bipartite graph $K_{n-2,2}$.*
- *(iii) If $g \geq 5$, then $\pi \geq \pi(T_{n,g})$ with equality if and only if G is the turnip $T_{n,g}$.*

Proof.

(i) This case is trivial.

(ii) Let $u_1u_2u_3u_4u_1$ be a cycle in G. Assume without loss of generality that $\tilde{t}(u_1) \leq \tilde{t}(u_2) \leq \tilde{t}(u_3) \leq \tilde{t}(u_4)$, then

$$\tilde{t}(u_1) \geq \frac{(n-1)+2}{n-1} = 1 + \frac{2}{n-1}$$

with equality if and only if $u_1v \in E$ for all $v \in V \setminus \{u_1, u_2, u_3, u_4\}$.
If $u \in V \setminus \{u_1, u_2, u_3, u_4\}$, then u is adjacent to at most two vertices from $\{u_1, u_2, u_3, u_4\}$. Thus

$$\tilde{t}(u) \geq \frac{(n-2)+4}{n-1} = 1 + \frac{3}{n-1} > 1 + \frac{2}{n-1}.$$

Therefore

$$\pi = \tilde{t}(u_1) \geq 1 + \frac{2}{n-1}$$

with equality if and only if u_1 is adjacent to all vertices in $\{u_1, u_2, u_3, u_4\}$, *i.e.*, the turnip $T_{n,4}$ is a spanning subgraph of G, also G may contain any edges between u_3 and any vertex from $V \setminus \{u_1, u_2, u_3, u_4\}$ but none of the other possible edges.

(iii) Let $u_1u_2 \cdots u_gu_1$ be a cycle in G. Assume without loss of generality that $\tilde{t}(u_1) \leq \tilde{t}(u_2) \cdots \leq \tilde{t}(u_g)$, then

$$\tilde{t}(u_1) \geq \frac{(n-g) + d(u_1,u_2) + d(u_1,u_3) \cdots + d(u_1,u_g)}{n-1} = \pi(T_{n,g})$$

with equality if and only if $u_1v \in E$ for all $v \in V \setminus \{u_1, u_2, \cdots u_g\}$.

If $u \in V \setminus \{u_1, u_2, \cdots u_g\}$, let $p \in \{1, 2, \cdots g\}$ such that $\ell = d(u, u_p) = \min\{d(u, u_i); i = 1, 2, \cdots g\}$. We consider two cases according to the value of ℓ.
Case $\ell \geq \lceil g/2 \rceil$. The sum of distances from u to the vertices $u_1, \cdots u_g$ is at least g^2 and the sum of the distances to the other vertices larger that $n - g - 1$ (the number of such vertices). Thus

$$\tilde{t}(u) > \frac{(n-g-1) + \frac{g^2}{2}}{n-1} = \frac{2n-2g-2+g^2}{2(n-1)} > \pi(T_{n,g}).$$

Case $\ell \leq \lfloor g/2 \rfloor$. It is easy to see that for all $k \leq \lfloor g/2 \rfloor - \ell$,

$$d(u, u_{p\pm k}) = d(u, u_p) + d(u_p, u_{p\pm k}) = \ell + k.$$

Note that the sum and the difference in $u_{p\pm k}$ are taken modulo g.

For all $k > \lfloor g/2 \rfloor - \ell$, we have

$$d(u, u_{p\pm k}) \geq \ell.$$

The sum of the distances from u to the vertices on a shortest path from u to u_p, including u_p, is

$$\sum_{i=1}^{\ell} i = \frac{\ell(\ell+1)}{2}.$$

The sum of the distances from u to all the remaining vertices is at least $n - g - \ell$.

If g is even we have

$$\begin{aligned} t(u) &\geq 2\sum_{i=1}^{\frac{g}{2}-\ell}(\ell+i) + 2\ell^2 + \sum_{i=1}^{\ell} i + n - g - \ell \\ &= (g-2\ell)\ell + \frac{(g-2\ell)(g-2\ell+2)}{4} + 2\ell^2 + \frac{\ell(\ell+1)}{2} + n - g - \ell \\ &= \frac{g^2 - 2g + 6\ell^2 - 6\ell + 4n}{4} \geq \frac{g^2 - 2g + 4n}{4}. \end{aligned}$$

Thus $\tilde{t}(u) > \pi(T_{n,g})$.

If g is odd, we have

$$\begin{aligned} t(u) &\geq 2\sum_{i=1}^{\frac{g-1}{2}-\ell}(\ell+i) + \ell(2\ell+1) + \sum_{i=1}^{\ell} i + n - g - \ell \\ &= (g-1-2\ell)\ell + \frac{(g-1-2\ell)(g+1-2\ell)}{4} + 2\ell^2 + \frac{\ell^2+\ell}{2} + n - g \\ &= \frac{g^2 - 4g + 6\ell^2 - 2\ell + 4n - 1}{4} \geq \frac{g^2 - 4g + 4n + 3}{4}. \end{aligned}$$

In this case also $\tilde{t}(u) > \pi(T_{n,g})$.

In conclusion

$$\pi = \tilde{t}(u_1) \geq \pi(T_{n,g})$$

with equality if and only if u_1 is adjacent to all vertices in $V \setminus \{u_1, u_2, \cdots u_g\}$, *i.e.*, the turnip $T_{n,g}$ is a spanning subgraph of G; moreover adding any other edge would decrease the girth. Thus equality holds if and only if G is the turnip $T_{n,g}$. □

Theorem 1 *For any connected graph G on $n \geq 3$ vertices with a finite girth g and proximity π, we have*

$$\left.\begin{array}{ll} \text{if } n \text{ is odd,} & \frac{1-3n}{4} \\ \text{if } n \text{ is even,} & \frac{4n-3n^2}{4n-4} \end{array}\right\} \leq \pi - g \leq \left\{\begin{array}{ll} \frac{n-11}{4} - \frac{1}{n-1} & \text{if } n \text{ is odd,} \\ \frac{n-11}{4} - \frac{3}{4(n-1)} & \text{if } n \text{ is even;} \end{array}\right. \quad (1)$$

$$4 \leq \pi + g \leq \left\{\begin{array}{ll} \frac{5n+1}{4} & \text{if } n \text{ is odd,} \\ \frac{5n^2-4n}{4(n-1)} & \text{if } n \text{ is even;} \end{array}\right. \quad (2)$$

$$\frac{1}{2\lfloor\sqrt{n}\rfloor + 1} + \frac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 1)}{(2\lfloor\sqrt{n}\rfloor + 1)(n-1)} \leq \frac{\pi}{g} \leq \left\{\begin{array}{ll} \frac{n^2-4}{12n-12} & \text{if } n \text{ is even,} \\ \frac{n+1}{12} - \frac{1}{3n-3} & \text{if } n \text{ is odd;} \end{array}\right. \quad (3)$$

$$3 \leq \pi \cdot g \leq \left\{\begin{array}{ll} \frac{n^2+n}{4} & \text{if } n \text{ is odd,} \\ \frac{n^3}{4(n-1)} & \text{if } n \text{ is even;} \end{array}\right. \quad (4)$$

The lower bound in (1) and the upper bounds in (2) and (4) are reached if and only if G is the cycle C_n. The upper bounds in (1) and (3) are reached if and only if G is the lollipop $L_{n,3}$. The lower bounds in (2) and (4) are reached if and only if G contains a dominating vertex and at least n edges. The lower bound in (3) is reached if and only if G is the turnip $T_{n,s}$, where $s = 2\lfloor\sqrt{n}\rfloor + 1$ when $\sqrt{n}$ is not an integer, and if and only if G is any one of the turnips $T_{n,2\sqrt{n}-1}$, $T_{n,2\sqrt{n}}$ or $T_{n,2\sqrt{n}+1}$ when $\sqrt{n}$ is an integer.

Proof.

Lower bound on $\pi - g$.

It is easy to see that for the cycle C_n, equality holds. Since $\pi \geq 1$, to reach the bound, we must have $g \geq 3n/4$. Therefore, G must be a unicyclic graph. Thus, the proof is restricted to unicyclic graphs. Let G be a unicyclic graph. If G is the cycle C_n, we are done. Else, let C be the unique cycle in G and v a vertex on C such that $d(v) \geq 3$. Let v_1 and v_2 be two neighbors of v such that $v_1 \in C$ and $v_2 \in V \setminus C$. Consider the (unicyclic) graph G' obtained from G by deleting the edge vv_1 and adding the edge vv_2, *i.e.*, performing a rotation of an edge incident with v. This operation increases the girth by exactly one, while the distance between some, never all, pairs of vertices changes (decreases or increases) by one. In particular, the vertex corresponding to π which cannot be a pending vertex, hase at least two neighbors, the distance to at least one of which will not change. Thus the proximity changes by less then one, and the rotation decreases $\pi - g$. Iterating this operation $n - g$ times leads to a cycle.

Upper bound on $\pi - g$.

Since G contains at least a cycle, $D \leq n - 2$. Let v be a central vertex on a diametric path. Thus

$$\pi \leq \frac{t(v)}{n-1} \leq \begin{cases} \frac{1}{n-1}\left(\frac{n-1}{2} + \frac{n-3}{2} + 2\sum_{k=1}^{\frac{n-3}{2}} k\right) & \text{if } n \text{ is odd,} \\ \frac{1}{n-1}\left(\frac{n-2}{2} + 2\sum_{k=1}^{\frac{n-2}{2}} k\right) & \text{if } n \text{ is even.} \end{cases}$$
$$= \begin{cases} \frac{1}{n-1}\left(n-2 + \frac{(n-3)(n-1)}{4}\right) & \text{if } n \text{ is odd,} \\ \frac{1}{n-1}\left(\frac{n-2}{2} + \frac{n(n-2)}{4}\right) & \text{if } n \text{ is even.} \end{cases}$$
$$= \begin{cases} \frac{n+1}{4} - \frac{1}{n-1} & \text{if } n \text{ is odd,} \\ \frac{n+1}{4} - \frac{3}{4(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

Therefore the upper bound on $\pi - g$ follows. Equality holds only if $D = n-2$, thus G contains an induced path on $n - 1$ vertices. The n^{th} vertex must be attached to an endpoint of the path and its neighbor, otherwise, the proximity would decrease and G would contain no cycle.

The lower and upper bounds on $\pi + g$, as well as those on $\pi \cdot g$, and the characterization of the corresponding extremal graphs are immediate consequences of the respective bounds on π and g.

Lower bound on π/g.

If $g = 3$, using Lemma 1, we have

$$\frac{\pi}{g} \geq \frac{1}{3} > \frac{1}{2\lfloor\sqrt{n}\rfloor + 1} + \frac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 1)}{(2\lfloor\sqrt{n}\rfloor + 1)(n-1)}$$

for all $n \geq 5$.

If $g = 4$, again using Lemma 1, we have

$$\frac{\pi}{g} \geq \frac{1}{4} + \frac{1}{2(n-1)} > \frac{1}{2\lfloor\sqrt{n}\rfloor + 1} + \frac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 1)}{(2\lfloor\sqrt{n}\rfloor + 1)(n-1)}$$

for all $n \geq 5$.

If $g \geq 5$, according to Lemma 1 the minimum of π/g is necessary reached for a turnip. So it remains to compute the value of the girth g, in terms of n, for which the minimum is reached. For this purpose, we need to minimize the following function.

$$f(g) = \begin{cases} \frac{g-4}{4(n-1)} + \frac{4n-1}{4(n-1)g} & \text{if } g \text{ is odd} \\ \frac{g-4}{4(n-1)} + \frac{1}{(n-1)} & \text{if } g \text{ is even.} \end{cases}$$

Using the derivative of $f(g)$, if g is considered as a continuous variable, the minimum is reached at $g = \sqrt{4n-1}$ for the expression corresponding to g odd, and at $g = 2\sqrt{n}$ for the expression corresponding to g even.

If $\sqrt{n} = k$ is an integer, then the minimum is reached for some $g \in \{2k-1, 2k, 2k+1\}$. However, we have

$$f(2k-1) = f(2k) = f(2k+1) = \frac{1}{k+1}.$$

Thus, in this case the minimum is reached for $T_{n,2k-1}$, $T_{n,2k}$ and $T_{n,2k+1}$.

If $\sqrt{n}$ is not an integer, let $n = k^2 + l$ with $k = \lfloor \sqrt{n} \rfloor$ and $1 \leq l \leq 2k$. In this case the minimum is reached for some $g \in \{2k-1, 2k, 2k+1, 2k+2\}$, for which the values of $f(g)$ are

$$\begin{aligned} f(2k-1) &= \frac{k-1}{n-1} + \frac{l}{(2k-1)(n-1)}; \\ f(2k) &= \frac{k-1}{n-1} + \frac{l}{2k(n-1)}; \\ f(2k+1) &= \frac{k-1}{n-1} + \frac{l}{(2k+1)(n-1)}; \\ f(2k+2) &= \frac{k-1}{n-1} + \frac{l}{(2k+2)(n-1)}. \end{aligned}$$

It is easy to see that the minimum is reached for and only for $g = 2k+1$, *i.e.*, the graph G is the turnip $T_{n,2\lfloor\sqrt{n}\rfloor+1}$.

The proof of the upper bound on π/g, as well as the characterization of the corresponding extremal graphs, is similar to that on $\pi - g$ and omitted here. □

3 The remoteness and the girth

In this section, we prove the lower bounds on $\rho - g$, $\rho + g$ and $\pi \cdot g$, as well as the upper bounds on $\rho - g$, $\rho + g$, ρ/g and $\pi \cdot g$. We also characterize the corresponding families of extremal graphs for these bounds. To prove all the upper bounds, we need the following lemma.

Lemma 2 *Let G be a connected graph on $n \geq 4$ vertices with a girth $g \leq n-1$ and remoteness ρ. Then*

$$\rho \leq \rho(L_{n,g})$$

with equality if and only if G is the lollipop $L_{n,g}$.

Proof. Let v be a vertex and C a cycle of length g in G.

If $v \in C$, we have

$$t(v) \leq \begin{cases} 2\left(1 + \cdots \frac{g-2}{2}\right) + \frac{g}{2} + (\frac{g}{2}+1) + \cdots (\frac{g}{2} + n - g) & \text{if } g \text{ is even} \\ 2\left(1 + \cdots \frac{g-1}{2}\right) + (\frac{g-1}{2}+1) + \cdots (\frac{g-1}{2} + n - g) & \text{if } g \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{n(n-1)}{2} - \frac{n(g-2)}{2} + \frac{g(2g-1)}{2} & \text{if } g \text{ is even} \\ \frac{(g-1)(g+1)}{4} + \frac{n(n-g)}{2} & \text{if } g \text{ is odd} \end{cases}$$

Easy algebraic manipulations show that $\tilde{t}(v) < \rho(L_{n,g})$.

If $v \in V \setminus C$, we have

$$\begin{aligned} t(v) &\leq 1 + 2 + \cdots (n-g) + \\ &+ \begin{cases} (g-1)(n-g) + 2\left(1 + \cdots \frac{g-2}{2}\right) + \frac{g}{2} & \text{if } g \text{ is even} \\ (g-1)(n-g) + 2\left(1 + \cdots \frac{g-1}{2}\right) & \text{if } g \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{n(n-1)}{2} - \frac{g(g-2)}{4} & \text{if } n \text{ is even} \\ \frac{n(n-1)}{2} - \frac{g(g-2)+1}{4} & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{5}$$

Therefore

$$\tilde{t}(v) \leq \rho(L_{n,g})$$

with equality if and only if G is $L_{n,g}$ and v is its pending vertex. □

Theorem 2 *For any connected graph G on $n \geq 3$ vertices with remoteness ρ and girth g, we have*

$$\left.\begin{array}{ll} \textit{if } n \textit{ is even,} & \frac{4n-3n^2}{4n-4} \\ \textit{if } n \textit{ is odd,} & \frac{1-3n}{4} \end{array}\right\} \leq \rho - g \leq \frac{(n+1)(n-2)}{2n-2} - 3; \tag{6}$$

$$4 \leq \rho + g \leq \begin{cases} \frac{5n^2-4n}{4n-4} & \textit{if } n \textit{ is even,} \\ \frac{5n+1}{4} & \textit{if } n \textit{ is odd;} \end{cases} \tag{7}$$

$$\frac{\rho}{g} \leq \frac{(n+1)(n-2)}{6n-6}; \tag{8}$$

$$3 \leq \rho \cdot g \leq \rho(L_{n,g^*}) \cdot g^* \tag{9}$$

where g^* is the girth for which $\rho(L_{n,g_i}) \cdot g_i$, $i = 1, \cdots 4$, is maximum with $g_1 = \left\lfloor \frac{2+\sqrt{6n^2-6n+4}}{3} \right\rfloor$, $g_2 = \left\lceil \frac{2+\sqrt{6n^2-6n+4}}{3} \right\rceil$, $g_3 = \left\lfloor \frac{2+\sqrt{6n^2-6n+7}}{3} \right\rfloor$ and $g_4 = \left\lceil \frac{2+\sqrt{6n^2-6n+7}}{3} \right\rceil$.

The lower bound in (6) and the upper bound in (7) are reached if and only if G is the cycle C_n. The upper bounds in (6) and (8) are reached if and only if G is the lollipop $L_{n,3}$. The lower bounds in (7) and (4) are reached if and only if G is the complete graph K_n. The upper bound in (9) is reached if and only if G is the lollipop L_{n,g^*}.

Proof.

The lower bound on $\rho - g$ follows from the lower bound on $\pi - g$ proved in Theorem 1 above.

The lower bound on $\rho + g$ and $\rho \cdot g$, as well as the characterizations of the corresponding extremal graphs, are trivial.

Using Lemma 2, the proofs of the upper bounds may be restricted to the set of lollipops. Then it suffices to maximize $\rho - g$, ρ/g, $\rho + g$ and $\rho \cdot g$ as functions of g on the set of lollipops $L_{n,g}$, considering n as a parameter.

Upper bound on $\rho - g$:

If g is even, we have

$$\rho(L_{n,g}) - g = \frac{n}{2} - \frac{g(g-2)}{4(n-1)} - g,$$

which is a decreasing function in g, and thus reaches its maximum for $g = 4$.
If g is odd, we have

$$\rho(L_{n,g}) - g = \frac{n}{2} - \frac{g(g-2)+1}{4(n-1)} - g,$$

which is a decreasing function in g, and thus reaches its maximum for $g = 3$.
In addition,

$$\rho(L_{n,4}) - 4 = \frac{n}{2} - \frac{2}{n-1} - 4 < \frac{n}{2} - \frac{1}{n-1} - 3 = \rho(L_{n,3}) - 3.$$

Then

$$\rho - g \leq \frac{(n+1)(n-2)}{2n-2} - 3 \quad \text{and} \quad \frac{\rho}{g} \leq \frac{(n+1)(n-2)}{6n-6}$$

with equality in both cases if and only if G is $L_{n,3}$.

The upper bound on ρ/g is proved as the previous bound.

Upper bound on $\rho + g$:

If g is even, we have

$$\rho(L_{n,g}) + g = \frac{n}{2} - \frac{g(g-2)}{4(n-1)} + g,$$

which is an increasing function in g, and thus reaches its maximum for $g = n$ if n is even and for $g = n - 1$ if n odd.

If g is odd, we have

$$\rho(L_{n,g}) - g = \frac{n}{2} - \frac{g(g-2)+1}{4(n-1)} + g,$$

which is an increasing function in g, and thus reaches its maximum for $g = n - 1$ if n is even and for $g = n$ if n odd.

A comparison between $\rho(L_{n,n}) - n$ and $\rho(L_{n,n-1}) - (n-1)$ is both cases, n even and n odd, leads to the result.

Upper bound on $\rho \cdot g$:

If g is even, we have

$$\rho(L_{n,g}) \cdot g = \frac{ng}{2} - \frac{g^2(g-2)}{4(n-1)}.$$

Using the derivative of this expression, it is easy to show that if g is considered as a continuous variable, the maximum is reached for $g = \frac{2+\sqrt{6n^2-6n+4}}{3}$. Thus the bound is $\rho(L_{n,g_1}) \cdot g_1$ or $\rho(L_{n,g_2}) \cdot g_2$.

If g is odd, we have

$$\rho(L_{n,g}) \cdot g = \frac{ng}{2} - \frac{g^2(g-2)}{4(n-1)} + \frac{g}{4(n-1)}.$$

Using the derivative of this expression, it is easy to show that if g is considered as a continuous variable, the maximum is reached for $g = \frac{2+\sqrt{6n^2-6n+7}}{3}$. Thus the bound is $\rho(L_{n,g_3}) \cdot g_3$ or $\rho(L_{n,g_4}) \cdot g_4$. □

The lower bound on ρ/g remains an open conjecture. It is as follows.

Conjecture 1 *For any connected graph G on $n \geq 3$ vertices with remoteness ρ and girth g,*

$$\frac{\rho}{g} \geq \begin{cases} \frac{n}{4n-4} & \text{if } n \text{ is even,} \\ \frac{n+1}{4n} & \text{if } n \text{ is odd.} \end{cases}$$

with equality if and only if G is a cycle C_n.

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