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# The Metric Bridge Partition Problem

## Partitioning of a metric space into two subspaces linked by an edge in any optimal realization

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**Abstract**

Let $G = (V, E, w)$ be a graph with vertex and edge sets $V$ and $E$, respectively, and $w : E \rightarrow I\!R^+$ a function which assigns a positive weigth or length to each edge of $G$. $G$ is called a realization of a finite metric space $(M, d)$, with $M = \{1, ..., n\}$ if and only if $\{1, ..., n\} \subseteq V$ and $d(i, j)$ is equal to the length of the shortest chain linking $i$ and $j$ in $G$ $\forall i, j = 1, ..., n$. A realization $G$ of $(M, d)$, is said optimal if the sum of its weights is minimal among all the realizations of $(M, d)$. Consider a partition of $M$ into two nonempty subsets $K$ and $L$, and let $e$ be an edge in a realization $G$ of $(M, d)$; we say that $e$ is a bridge linking $K$ with $L$ if $e$ belongs to all chains in $G$ linking a vertex of $K$ with a vertex of $L$. The Metric Bridge Partition Problem is to determine if the elements of a finite metric space $(M, d)$ can be partitioned into two nonempty subsets $K$ and $L$ such that all optimal realizations of $(M, d)$ contain a bridge linking $K$ with $L$. We prove in this paper that this problem is polynomially solvable. We also describe an algorithm that constructs an optimal realization of $(M, d)$ from optimal realizations of $(K, d|_K)$ and $(L, d|_L)$.



**Résumé**

Soit $G = (V, E, w)$ un graphe ayant $V$ comme ensemble de sommets et $E$ comme ensemble d'arêtes, et soit $w : E \rightarrow I\!R^+$ une fonction qui attribue un poids positif, appelé longueur, à chaque arête de $G$. Le graphe $G$ est une réalisation d'un espace métrique fini $(M, d)$, avec $M = \{1, ..., n\}$, si et seulement si $\{1, ..., n\} \subseteq V$ et $d(i, j)$ est égal à la longueur de la chaîne la plus courte reliant $i$ et $j$ dans $G$ $\forall i, j = 1, ..., n$. Une réalisation $G$ de $(M, d)$ est dite optimale si la somme des poids dans $G$ est minimale parmi toutes les réalisations de $(M, d)$. Considérons une partition de $M$ en deux sous-ensembles non vides $K$ et $L$, et soit $e$ une arête dans une réalisation $G$ de $(M, d)$. Nous dirons que $e$ est un pont qui relie $K$ avec $L$ si $e$ appartient à toute chaîne de $G$ qui relie un sommet de $K$ à un sommet de $L$. Le problème de la *partition d'une métrique à l'aide d'un pont* consiste à déterminer s'il existe une partition des éléments d'un espace métrique fini $(M, d)$ en deux sous-ensembles $K$ et $L$ tel que toute réalisation de $(M, d)$ contienne un pont reliant $K$ avec $L$. Nous prouvons dans cet article que ce problème peut être résolu en temps polynomial. Nous décrivons également un algorithme qui construit une réalisation optimale de $(M, d)$ sur la base de réalisations optimales de $(K, d|_K)$ et $(L, d|_L)$.



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# 1 Introduction

A *metric space* is a couple $(M, d)$ such that $M$ is a set and $d$ is a function defined on $M \times M$ such that $d(x, y) = d(y, x) > 0 \ \forall x \neq y$, $d(x, x) = 0 \ \forall x$, and $d(x, z) \leq d(x, y) + d(y, z)$ $\forall x, y, z$. Moreover, $(M, d)$ is a finite metric space if $M$ has a finite number of elements.

Let $G = (V, E, w)$ be a graph, with vertex and edge sets $V$ and $E$, respectively, and $w : E \rightarrow I\!R^+$ a function which assigns a positive weight or length to each edge of $G$. Furthermore, let $d^G(i, j)$ denote the length of a shortest chain in $G$ linking vertices $i$ and $j$. We say that $G$ is a realization of a finite metric space $(M, d)$, with $M = \{1, ..., n\}$ if and only if $\{1, ..., n\} \subseteq V$ and $d^G(i, j) = d(i, j) \ \forall i, j = 1, ..., n$. The elements in $V \backslash M$ are called *auxiliary vertices*. A realization of $(M, d)$ is called optimal when the sum of its weights is minimal among all the realizations of $(M, d)$. For illustration, a metric space together with an optimal realization $G$ are shown in Figure 1. All edges of the graph have length one, and the black points $a, b, c, d, e$ are five auxiliary vertices while the white ones are the elements of $M$.

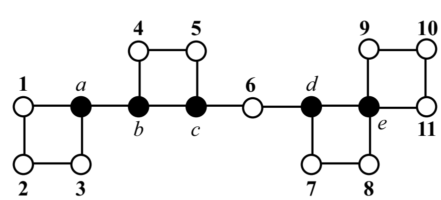
The embedding of finite metric spaces in graphs has applications in varied fields as computational biology [7, 9] (e.g., constructing phylogenetic trees from genetic distances among living species), electrical networks [4], coding techniques [3], psychology [2], internet tomography [1], and compression softwares [8].

The problem of finding optimal realizations of metric spaces was first proposed by Hakimi and Yau [4] in 1964 who also gave a polynomial algorithm for the special case where the metric space has a realization as a tree. While every finite metric space has an optimal realization [5, 6], finding such realizations is an NP-hard problem [10].

Optimal realizations can be constructed using building blocks. More precisely, for a graph $G$, we recall that a *cutpoint*, respectively a *bridge*, is a vertex, respectively an *edge*, whose removal strictly increases the number of connected component of $G$; a *block* is a

| | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 1 | 2 | 3 | 4 | 4 | 6 | 7 | 7 | 8 | 7 |
| **2** | | 0 | 1 | 4 | 5 | 5 | 7 | 8 | 8 | 9 | 8 |
| **3** | | | 0 | 3 | 4 | 4 | 6 | 7 | 7 | 8 | 7 |
| **4** | | | | 0 | 1 | 3 | 5 | 6 | 6 | 7 | 6 |
| **5** | | | | | 0 | 2 | 4 | 5 | 5 | 6 | 5 |
| **6** | | | | | | 0 | 2 | 3 | 3 | 4 | 3 |
| **7** | | | | | | | 0 | 1 | 3 | 4 | 3 |
| **8** | | | | | | | | 0 | 2 | 3 | 2 |
| **9** | | | | | | | | | 0 | 1 | 2 |
| **10** | | | | | | | | | | 0 | 1 |
| **11** | | | | | | | | | | | 0 |

Upper diagonal of the distance matrix of a metric space $(M,d)$ with $M=\{1,...,11\}$




Optimal realization of $(M,d)$

Figure 1: A metric space with an optimal realization

maximal two-connected subgraph or a bridge in $G$. Imrich *et al.* [5] have proved the following theorem.

**Theorem 1** *[5] Let $G$ be an optimal realization of a finite metric space $(M,d)$, let $G_1, \cdots, G_k$ be the blocks of $G$, and let $M_i$ be the union of the points of $M$ in $G_i$ together with the cutpoints of $G$ in $G_i$. If every $G_i$ is an optimal realization of the metric space induced by $G$ on $M_i$, then $G$ is also optimal.*

For example an optimal realization of the metric space of Figure 1 can be obtained by putting together optimal realizations of the metric spaces induced on $\{1,2,3,a\}$, $\{a,b\}$, $\{4,5,b,c\}$, $\{6,c\}$, $\{6,d\}$, $\{7,8,d,e\}$, and $\{9,10,11,e\}$.

It is therefore interesting to be able to recognize metric spaces which contain at least one bridge in all optimal realizations. This is exactly the topic of our paper. More precisely, consider a partition of $M$ into two nonempty subsets $K$ and $L$, and let $e$ be an edge in a realization $G$ of $(M,d)$. We say that $e$ is a bridge linking $K$ with $L$ if $e$ belongs to all chains in $G$ linking a vertex of $K$ with a vertex of $L$. The *Metric Bridge Partition Problem* is to determine if the elements of a given finite metric space $(M,d)$ can be partitioned into two nonempty subsets $K$ and $L$ such that all optimal realizations of $(M,d)$ contain a bridge linking $K$ with $L$. For example, on the basis of the distance matrix of Figure 1 (and without any knowledge of the optimal realization), we would like to be able to state that all optimal realizations contain a bridge linking $K=\{1,2,3,4,5,6\}$ with $L=\{7,8,9,10,11\}$, or $K=\{1,2,3,4,5\}$ with $L=\{6,7,8,9,10,11\}$, or $K=\{1,2,3\}$ with $L=\{4,5,6,7,8,9,10,11\}$. We prove in this paper that the Metric Bridge Partition Problem is polynomially solvable.

## 2 Definitions and Known Results

It is well-known that the unique optimal realization of a metric space on three points $i,j,k$ is a tree $T$. The *hub* of $i,j,k$, denoted $h_{ijk}$, is the point in $T$ such that:

$$\begin{aligned}
d^T(h_{ijk},i) &= \tfrac{1}{2}(d(i,j)+d(i,k)-d(j,k)),\\
d^T(h_{ijk},j) &= \tfrac{1}{2}(d(j,i)+d(j,k)-d(i,k)),\\
d^T(h_{ijk},k) &= \tfrac{1}{2}(d(k,i)+d(k,j)-d(i,j)).
\end{aligned}$$

Assume that the distance $d(i,j)$ is larger than or equal to $d(i,k)$ and $d(j,k)$. If $d(i,j) < d(i,k)+d(j,k)$, then $T$ has three leaves $i$, $j$ and $k$, and one auxiliary vertex corresponding to the hub $h_{ijk}$, else $T$ is a chain linking $i$ and $j$ that traverses $k=h_{ijk}$ (see Figure 2).

Let $s_{ijk\ell}$ denote the sum $d(i,j)+d(k,\ell)$. It is also well-known that the optimal realization of a metric space on four points $i,j,k,\ell$ is a unique tree if and only if two of the sums $s_{ijk\ell}, s_{ikj\ell}, s_{i\ell jk}$ are equal and not smaller than the third. Moreover, if $s_{ijk\ell} < s_{ikj\ell} = s_{i\ell jk}$, then the tree has a bridge $(h_{ijk}, h_{ik\ell})$ of length $s_{ikj\ell} - s_{ijk\ell} > 0$ linking $\{i,j\}$ with $\{k,\ell\}$. The three possible configurations are represented in Figure 3 (the other cases are equivalent).

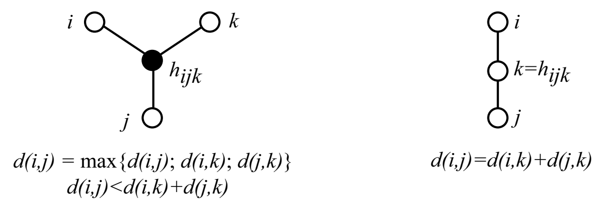



Figure 2: Optimal realizations of three points

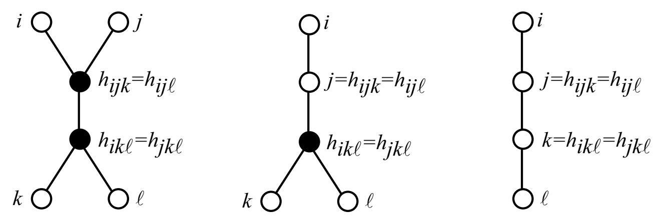



Figure 3: Optimal realizations of four points

**Definition 1** *A finite metric space $(M, d)$ is reducible if and only if all its optimal realizations contain a vertex of degree one (i.e, a vertex with exactly one neighbor).*

In other words (see for example [5]), a finite metric space $(M, d)$ is reducible if and only if $M$ contains an element $i$, called *endpoint*, such that $d(i, j) + d(i, k) - d(j, k) > 0$ for all $j, k \neq i$. An optimal realization of a reducible metric space $(M, d)$ can easily be obtained from an optimal realization of a metric space $(M', d')$ which has fewer endpoints or fewer elements than $M$. More precisely, consider an endpoint $i$ in a reducible metric space $(M, d)$, and define $\alpha$ =min$\{\frac{1}{2}(d(i, j) + d(i, k) - d(j, k))\}$, the minimum being taken over all $j, k \neq i$. There are two possible cases:

- If there is an element $j \in M$ with $d(i, j) = \alpha$, then set $M'$ equal to $M \backslash \{i\}$, and set $d' = d|_{M'}$ (i.e., $d'$ is the distance matrix induced by $d$ on $M'$). An optimal realization of $(M, d)$ can be obtained from an optimal realization of $(M', d')$ by adding a vertex $i$ and an edge of length $\alpha$ linking $i$ with $j$.
- If there is no element $j \in M$ with $d(i, j) = \alpha$, then set $M' = M \backslash \{i\} \cup \{a\}$ and define $d'(j, k) = d(j, k)$ for all $j, k \neq a$, $d'(a, j) = d(i, j) - \alpha$ for all $j \neq a$. An optimal realization of $(M, d)$ can be obtained from an optimal realization of $(M', d')$, by adding a vertex $i$ and an edge of length $\alpha$ linking $i$ with $a$.

**Definition 2** *Consider a finite metric space $(M, d)$, a partition of $M$ into two non-empty subsets $K, L$ and a mapping $f : M \rightarrow I\!R^+$. The triplet $(K, L, f)$ is said nice if*

- *$d(x,y) \leq f(x) + f(y)$ for all $x, y$ in $M$, equality holding whenever $x \in K$ and $y \in L$, and*
- *$f(x) > 0$ at least once in $K$ and once in $L$.*

The above definition is motivated by the following result proved in [5] and [6].

**Theorem 2** *[6, 5] Suppose $(M,d)$ is a finite metric space to which there exists a nice triplet $(K,L,f)$. Then every optimal realization $G$ of $(M,d)$ has a cut-point $c$ or a bridge with a point $c$ on it such that all chains linking $K$ with $L$ go through $c$, and $d^G(x,c) = f(x) \quad \forall x \in M$.*

## 3 New Results

We start with a sufficient condition for the existence of a bridge in all optimal realizations of a finite metric space $(M,d)$. It is a corollary of Theorem 2.

**Corollary 1** *Suppose $(M,d)$ is a finite metric space to which there exist a partition of $M$ into two non-empty subsets $K, L$ and two different mappings $f : M \to \mathbb{R}^+$ and $g : M \to \mathbb{R}^+$ such that both $(K,L,f)$ and $(K,L,g)$ are nice triplets. Then every optimal realization $G$ of $(M,d)$ has a bridge.*

**Proof.** Let $(K,L,f)$ and $(K,L,g)$ be two nice triplets with $f \neq g$, and let $G$ be any optimal realization of $(M,d)$. We know from Theorem 2 that all chains linking $K$ with $L$ go through two points $c$ and $c'$ such that $d^G(x,c) = f(x)$ and $d^G(x,c') = g(x) \ \forall x \in M$. Since $f \neq g$, we conclude that $c \neq c'$, which means that all chains linking $K$ with $L$ traverse a bridge containing points $c$ and $c'$. □

The next Theorem also provides a sufficient condition for the existence of a bridge in every optimal realization of a finite metric space $(M,d)$.

**Theorem 3** *Suppose $(M,d)$ is a finite metric space to which there exists a partition of $M$ into two non-empty subsets $K, L$ with $|K| > 1$ and $|L| > 1$, and assume the existence of four elements $x, y \in K$ and $z, t \in L$ such that*

(1) $s_{xzyt} - s_{xyzt} \leq s_{ikj\ell} - s_{ijk\ell} \quad \forall i,j \in K$ *and* $k, \ell \in L$

(2) $s_{ijk\ell} < s_{ikj\ell} = s_{i\ell jk} \quad \forall i,j \in K$ *and* $k, \ell \in L$.

*Then every optimal realization of $(M,d)$ has a bridge $(h_{xyz}, h_{xzt})$ linking $K$ with $L$.*

**Proof.** Notice first that we know from (2) that the optimal realization of the metric space induced by four elements $i, j \in K$ and $k, \ell \in L$ is a tree $U$ with $d^U(h_{ijk}, h_{ik\ell}) = s_{ikj\ell} - s_{ijk\ell} = s_{i\ell jk} - s_{ijk\ell} > 0$ (see Section 2). Let $T$ be the optimal realization of the metric space induced by $x, y, z$ and $t$, and define

$$f(i) = \begin{cases} d(z,i) - d^T(z, h_{xyz}) & \text{if } i \in K \\ d(x,i) - d^T(x, h_{xyz}) & \text{if } i \in L \end{cases}$$

and

$$g(i) = \begin{cases} d(z,i) - d^T(z, h_{xzt}) & \text{if } i \in K \\ d(x,i) - d^T(x, h_{xzt}) & \text{if } i \in L \end{cases}$$

Consider any element $i \neq x$ in $K$, and let $U$ denote the optimal realization of the metric space induced on $x, z, t$ and $i$. By (1), we have $d^U(h_{xiz}, h_{xzt}) \geq d^T(h_{xyz}, h_{xzt})$, and since $d^U(h_{xzt}, z) = d^T(h_{xzt}, z)$, we have

$$\begin{aligned} f(i) &= d(z,i) - d^T(z, h_{xyz}) \\ &= d^U(z, h_{xzt}) + d^U(h_{xzt}, h_{xiz}) + d^U(h_{xiz}, i) - d^T(z, h_{xyz}) \\ &\geq d^T(z, h_{xzt}) + d^T(h_{xzt}, h_{xyz}) + d^U(h_{xiz}, i) - d^T(z, h_{xyz}) \\ &= d^U(h_{xiz}, i) \geq 0. \end{aligned}$$

Since $f(x) = d(z,x) - d^T(z, h_{xyz}) = d^T(x, h_{xyz}) \geq 0$, we have $f(i) \geq 0$ for all $i \in K$. Consider now any element $i \neq z$ in $L$, and let $U$ denote the optimal realization of the metric space induced on $x, y, z$ and $i$. Again, $d^U(h_{xyz}, h_{xzi}) \geq d^T(h_{xyz}, h_{xzt})$ and $d^U(x, h_{xyz}) = d^T(x, h_{xyz})$. Hence,

$$\begin{aligned} f(i) &= d(x,i) - d^T(x, h_{xyz}) \\ &= d^U(x, h_{xyz}) + d^U(h_{xyz}, h_{xzi}) + d^U(h_{xzi}, i) - d^T(x, h_{xyz}) \\ &\geq d^T(x, h_{xyz}) + d^T(h_{xyz}, h_{xzt}) + d^U(h_{xzi}, i) - d^T(x, h_{xyz}) \\ &= d^T(h_{xyz}, h_{xzt}) + d^U(h_{xzi}, i) > d^U(h_{xzi}, i) \geq 0. \end{aligned}$$

Since $f(z) = d(x,z) - d^T(x, h_{xyz}) = d^T(z, h_{xyz}) > 0$, we have $f(i) > 0$ for all $i \in L$. Consider now two elements $i \in K$ and $j \in L$. We have

$$\begin{aligned} f(i) + f(j) &= d(z,i) - d^T(z, h_{xyz}) + d(x,j) - d^T(x, h_{xyz}) \\ &= d(z,i) + d(x,j) - d(x,z). \end{aligned}$$

It follows that if $i = x$ or/and $j = z$ then $f(i) + f(j) = d(i,j)$. Otherwise, let $U$ denote the optimal realization of the metric space induced by $x, z, i$ and $j$. We have $d(z,i) + d(x,j) - d(x,z) = d^U(z,i) + d^U(x,j) - d^U(x,z) = d^U(i,j) = d(i,j)$. We conclude that $f(i) + f(j) = d(i,j)$ for all $i \in K$ and $j \in L$.

We know from (2) that $h_{xyz} = h_{xyi}$ for all $i \in K$, and $h_{xzt} = h_{izt}$ for all $i \in L$. Consider now two elements $i$ and $j$ in $L$, and let $U$ denote the optimal realization of the metric space induced by $x, y, i$ and $j$. We have

$$\begin{aligned} f(i) + f(j) &= d(x,i) + d(x,j) - 2d^T(x, h_{xyz}) \\ &= d^U(x,i) + d^U(x,j) - 2d^U(x, h_{xyi}) \\ &= d^U(i,j) + 2d^U(h_{xyi}, h_{xij}) > d^U(i,j) = d(i,j). \end{aligned}$$

Consider finally two elements $i$ and $j$ in $K$, and let $U$ denote the optimal realization of the metric space induced by $i, j, z$ and $t$. Since $d^U(h_{ijz}, h_{izt}) \geq d^T(h_{xyz}, h_{xzt})$ and $d^U(h_{izt}, z) = d^T(h_{xzt}, z)$, we have

$$\begin{aligned} f(i) + f(j) &= d(z,i) + d(z,j) - 2d^T(z, h_{xyz}) \\ &= d^U(i,j) + 2d^U(h_{ijz}, h_{izt}) + 2d^U(h_{izt}, z) - 2d^T(z, h_{xyz}) \\ &\geq d(i,j) + 2d^T(h_{xyz}, h_{xzt}) + 2d^T(h_{xzt}, z) - 2d^T(z, h_{xyz}) = d(i,j). \end{aligned}$$

Since $0 < d(x,y) \leq f(x) + f(y)$ we know that $f(x)$ or/and $f(y)$ is strictly positive. We can therefore conclude that $(K, L, f)$ is a nice triplet. The proof that $(K, L, g)$ is a nice triplet is similar and can be obtained by permuting the roles of $x, y$ and $K$ with those of $z, t$ and $L$.

Notice that $f \neq g$ since

$$\begin{aligned} g(i) &= f(i) + d^T(z, h_{xyz}) - d^T(z, h_{xzt}) = f(i) + d^T(h_{xyz}, h_{xzt}) > f(i) \quad \forall i \in K \\ g(i) &= f(i) + d^T(x, h_{xyz}) - d^T(x, h_{xzt}) = f(i) - d^T(h_{xyz}, h_{xzt}) < f(i) \quad \forall i \in L. \end{aligned}$$

By Corollary 1, we know that each realization $G$ of $(M, d)$ has a bridge $(u, v)$ linking $K$ with $L$. It follows from Theorem 2 that $d^G(i, u) = f(i)$ and $d^G(i, v) = g(i)$ for all $i \in M$. Since $f(i) = d^T(i, h_{xyz})$ and $g(i) = d^T(i, h_{xzt})$ for $i = x, y, z$, we conclude that $u = h_{xyz}$ and $v = h_{xzt}$. □

We now give a necessary condition for the existence of a bridge.

**Theorem 4** *Suppose $(M, d)$ is an irreducible finite metric space. If there is a partition of $M$ into two non-empty subsets $K, L$ such that all optimal realizations of $(M, d)$ contain a bridge $(u, v)$ linking $K$ with $L$, then*

(1) $|K| > 1$ *and* $|L| > 1$,
(2) $s_{ijk\ell} < s_{ikj\ell} = s_{i\ell jk} \quad \forall i, j \in K$ *and* $k, \ell \in L$,
(3) $\exists x, y \in K$ *and* $z, t \in L$ *such that*
- $s_{xzyt} - s_{xyzt} \leq s_{ikj\ell} - s_{ijk\ell} \quad \forall i, j \in K$ *and* $k, \ell \in L$
- $i \in K \Leftrightarrow d(x, i) - d(z, i) \leq d(x, y) - d(z, y)$.

**Proof.** Consider a partition of $M$ into two non-empty subsets $K, L$ such that all optimal realizations of $(M, d)$ contain a bridge $(u, v)$ linking $K$ with $L$. If $|K| = 1$ then the unique element in $K$ is a vertex of degree 1 in all optimal realizations of $(M, d)$. But this is impossible since $(M, d)$ is irreducible. Hence $|K| > 1$, and $|L| > 1$ by symmetry.

Consider now any four elements $i, j \in K$ and $k, \ell \in L$ and let $G$ be any optimal realization of $(M, d)$. Since all chains linking $K$ with $L$ in $G$ traverse the bridge $(u, v)$, we have

$$\begin{aligned} s_{ikj\ell} &= d(i,k)+d(j,\ell) \\ &= d^G(i,u)+d^G(u,v)+d^G(v,k)+d^G(j,u)+d^G(u,v)+d^G(v,\ell) \\ &= d(i,\ell)+d(j,k)=s_{i\ell jk} \\ &> d^G(i,u)+d^G(j,u)+d^G(v,k)+d^G(v,\ell) \\ &\geq d^G(i,j)+d^G(k,\ell)=d(i,j)+d(k,\ell)=s_{ijk\ell}. \end{aligned}$$

Consider now four elements $x, y$ in $K$ and $z, t$ in $L$ such that $s_{xzyt} - s_{xyzt} \leq s_{ikj\ell} - s_{ijk\ell}$ for all $i, j$ in $K$ and $k, \ell$ in $L$, and let $T$ be the optimal realization of the metric space induced on $x, y, z$ and $t$. Also, consider any $i \in M$. If $i = x$, then $d(x,i) - d(z,i) = -d(z,x) \leq d(x,y) - d(z,y)$, and if $i = z$, then $d(x,i) - d(z,i) = d(x,z) > d(x,y) - d(z,y)$. So assume $i \neq x, z$, and let $W$ be the optimal realization of the metric space induced on $x, z, i$.

- If $i \in K$, then let $U$ be the optimal realization of the metric space induced on $x, z, t$ and $i$. Since $d^U(h_{xiz}, h_{xzt}) \geq d^T(h_{xyz}, h_{xzt})$ and $d^U(h_{xzt}, z) = d^T(h_{xzt}, z)$, we have

$$\begin{aligned} d^W(x,h_{xiz}) &= d^U(x,h_{xiz}) = d(x,z) - d^U(h_{xiz},h_{xzt}) - d^U(h_{xzt},z) \\ &\leq d(x,z) - d^T(h_{xyz},h_{xzt}) - d^T(h_{xzt},z) = d^T(x,h_{xyz}). \end{aligned}$$

- If $i \in L$, then let $U$ be the optimal realization of the metric induced by $x, y, z$ and $i$. We have

$$\begin{aligned} d^W(x,h_{xiz}) &= d^U(x,h_{xiz}) = d^U(x,h_{xyz}) + d^U(h_{xyz},h_{xzi}) \\ &= d^T(x,h_{xyz}) + d^U(h_{xyz},h_{xzi}) > d^T(x,h_{xyz}). \end{aligned}$$

We therefore conclude that

$$\begin{aligned} i \in K &\Leftrightarrow d^W(x,h_{xiz}) \leq d^T(x,h_{xyz}) \\ &\Leftrightarrow \tfrac{1}{2}(d(x,z)+d(x,i)-d(z,i)) \leq \tfrac{1}{2}(d(x,z)+d(x,y)-d(z,y)) \\ &\Leftrightarrow d(x,i)-d(z,i) \leq d(x,y)-d(z,y). \end{aligned}$$

□

# 4 Algorithms

The following algorithm determines if a given finite irreducible metric space $(M, d)$ contains a bridge.

**Theorem 5** *The MetricBridgePartition algorithm works correctly and is polynomial.*

**Proof.** Correctness of the algorithm follows from the results of the previous section. Indeed, if the algorithm stops with four elements $x, y, z, t$ and a partition of $M$ into two sets $K$ and $M \backslash K$, then properties (1) and (2) of Theorem 3 are satisfied, and we conclude that

**Algorithm 1** *MetricBridgePartition*

**Require:** A finite irreducible metric space $(M, d)$;

**Ensure:** Four elements $x, y, z, t \in M$ and a set $K$ such that there is a bridge linking $K$ with $M \backslash K$, or a message indicating that no optimal realization of $(M, d)$ has a bridge;

**for all** $x, y, z, t \in M$ such that $s_{xyzt} < s_{xzyt} = s_{xtyz}$ **do**
  $K \leftarrow \{x, y\}$ and $L \leftarrow \{z, t\}$;
  **for all** $i \in M \backslash \{x, y, z, t\}$ **do**
    **if** $d(x, i) - d(z, i) \leq d(x, y) - d(z, y)$ **then**
      $K \leftarrow K \cup \{i\}$
    **else**
      $L \leftarrow L \cup \{i\}$
    **end if**
  **end for**
  **if** $s_{xzyt} - s_{xyzt} \leq s_{ikj\ell} - s_{ijk\ell}$ and $s_{ijk\ell} < s_{ikj\ell} = s_{i\ell jk}$ $\forall i, j \in K$ $k, \ell \in L$ **then**
    STOP: return $x, y, z, t$ and $K$.
  **end if**
**end for**
return a message indicating that no optimal realization of $(M, d)$ has a bridge.

every optimal realization of $(M, d)$ has a bridge linking $K$ with $L$. Moreover, if there exists a partition of $M$ into two sets $K$ and $L$ such that every optimal realization of $(M, d)$ has a bridge, then we know from Theorem 4 that such a partition will be found. The algorithm is polynomial since its complexity is $O(|M|^8)$. $\square$

The *MetricBridgePartition* algorithm can be used to decompose a given finite metric space $(M, d)$ into metric spaces $(M_1, d_1), \cdots, (M_r, d_r)$ such that no optimal realization of $(M_i, d_i)$ $(i = 1, \cdots, r)$ has a bridge. According to Theorem 1, an optimal realization of $(M, d)$ can then be obtained by connecting optimal realizations of $(M_1, d_1), \cdots, (M_r, d_r)$ with bridges. More precisely, assume the existence of the three following algorithms:

- algorithm *NoBridge* constructs an optimal realization of a finite metric space if such a realization has no bridge;
- algorithm *Reduce* transforms any finite reducible metric space $(M, d)$ into an irreducible metric space $(M', d')$;
- given a finite reducible metric space $(M, d)$ and an irreducible metric space $(M', d')$ obtained by applying *Reduce* on $(M, d)$, and given also an optimal realization $G'$ of $(M', d')$, algorithm *Extend* constructs an optimal realization $G$ of $(M, d)$.

As explained in Section 2, algorithms *Reduce* and *Extend* are easy to implement. Assume now that algorithm *MetricBridgePartition* produces an output $x, y, z, t, K$ when applied on a metric space $(M, d)$. This means that there is a bridge $(h_{xyz}, h_{xzt})$ linking $K$ with $L = M \backslash K$ in all optimal realizations of $(M, d)$. According to the proof of Theorem 3, such an optimal realization $G$ can be obtained as follows.

- Compute $f(i) = d(z,i) - \frac{1}{2}(d(x,z) + d(y,z) - d(x,y))$ for all $i \in K$, and $g(i) = d(x,i) - \frac{1}{2}(d(x,z) + d(x,t) - d(z,t))$ for all $i \in L$.
- Construct a metric space $(K', d_{K'})$ as follows: if there is an element $i \in K$ with $f(i) = 0$ then set $K' = K$ and $u = i$, and define $d_{K'} = d|_K$; else build $K'$ by adding an auxiliary element $u$ to $K$, and define $d_{K'}(i,j) = d(i,j)$ for all $i, j \in K$ and $d_{K'}(i,u) = f(i)$ for all $i \in K$.
- Construct a metric space $(L', d_{L'})$ as follows: if there is an element $i \in L$ with $g(i) = 0$ then set $L' = L$ and $v = i$, and define $d_{L'} = d|_L$; else build $L'$ by adding an auxiliary element $v$ to $L$, and define $d_{L'}(i,j) = d(i,j)$ for all $i, j \in L$ and $d_{L'}(i,v) = g(i)$ for all $i \in L$.
- Construct two optimal realizations $G_{K'}$ and $G_{L'}$ of $(K', d_{K'})$ and $(L', d_{L'})$.
- Construct an optimal realization $G$ of $(M, d)$ by linking $G_{K'}$ and $G_{L'}$ with an edge $(u, v)$ of length $s_{xzyt} - s_{xyzt}$.

Algorithm *OptimalRealization* uses *MetricBridgePartition* recursively to build an optimal realization of any finite metric space $(M, d)$. Figure 4 illustrates its use on the example of Figure 1. The possible outputs (up to symmetry) of *MetricBridgePartition* applied on $(M, d)$ are

- $x = 1, y = 3, z = 4, t \in \{6, 7, 8, 9, 10, 11\}, K = \{1, 2, 3\}$;
- $x \in \{1, 2, 3\}, y = 5, z = 6, t \in \{7, 8, 9, 10, 11\}, K = \{1, 2, 3, 4, 5\}$;
- $x \in \{1, 2, 3, 4, 5\}, y = 6, z = 7, t \in \{9, 10, 11\}, K = \{1, 2, 3, 4, 5, 6\}$.

Assume the algorithm produces the output $x = 1, y = 3, z = 4, t = 6, K = \{1, 2, 3\}$. Since $f(1) = 1, f(2) = 2$, and $f(3) = 1$, we construct a metric space $\mathbf{M}_1$ on $\{1, 2, 3, u\}$. Algorithm *MetricBridgePartition* applied on $\mathbf{M}_1$ produces a message indicating that no optimal realization of $\mathbf{M}_1$ contains a bridge. An optimal realization $G_1$ of $\mathbf{M}_1$ is therefore obtained by applying the *NoBridge* algorithm. Since $g(4) = 1, g(5) = g(6) = 2, g(7) = 4, g(8) = g(9) = g(11) = 5, g(10) = 6$, we construct a metric space $\mathbf{M}_2$ on $\{4, 5, 6, 7, 8, 9, 10, 11, v\}$. Then, the possible outputs (up to symmetry) of *MetricBridgePartition* applied on $\mathbf{M}_2$ are

- $x = v, y = 5, z = 6, t \in \{7, 8, 9, 10, 11\}, K = \{4, 5, v\}$;
- $x \in \{4, 5, v\}, y = 6, z = 7, t \in \{9, 10, 11\}, K = \{4, 5, 6, v\}$.

Assume the output is $x = v, y = 5, z = 6, t = 7, K = \{4, 5, v\}$.

**Algorithm 2** *OptimalRealization*

**Require:** A finite metric space $(M, d)$;
**Ensure:** An optimal realization $G$ of $(M, d)$;

**if** $(M, d)$ is reducible **then**
  Apply *Reduce* on $(M, d)$ to build an irreducible metric space $(M', d')$;
**else**
  $(M', d') \leftarrow (M, d)$;
**end if**

Apply $MetricBridgePartition$ on $(M', d')$;
**if** the output indicates that no optimal realization of $(M', d')$ has a bridge **then**
  Apply $NoBridge$ on $(M', d')$ to build an optimal realization $G'$ of $(M', d')$;
**else**
  Let $x, y, z, t, K$ be the output of $MetricBridgePartition$;
  Build the metric spaces $(K', d_{K'})$ and $(L', d_{L'})$ as explained above;
  Get $G_{K'}$ and $G_{L'}$ by applying *OptimalRealization* on $(K', d_{K'})$ and $(L', d_{L'})$;
  Build $G'$ by linking $G_{K'}$ and $G_{L'}$ with an edge $(u, v)$ of length $s_{xzyt} - s_{xyzt}$;
**end if**

**if** $(M, d) \neq (M', d')$ **then**
  Apply *Extend* to $G'$ to build an optimal realization $G$ of $(M, d)$;
**else**
  $G \leftarrow G'$.
**end if**

- Since $f(4) = 2, f(5) = 1$, and $f(v) = 1$, we construct a metric space $\mathbf{M}_3$ on $\{4, 5, v, u'\}$. Since $MetricBridgePartition$ detects that no optimal realization of $\mathbf{M}_3$ has a bridge, we apply $NoBridge$ on $\mathbf{M}_3$ to get an optimal realization $G_3$.
- Since $g(6) = 0$, we consider the metric space $\mathbf{M}_4$ induced on $\{6, 7, \cdots, 11\}$ and set $v' = 6$. $\mathbf{M}_4$ is first reduced to a metric space $\mathbf{M}_5$, where an auxiliary element $a$ replaces element 6. An optimal realization $G_5$ of $\mathbf{M}_5$ is then obtained by applying $NoBridge$ (since $G_5$ has no bridge), and an optimal realization of $\mathbf{M}_4$ is then obtained by applying *Extend* on $G_5$.

Finally, $G_3$ and $G_4$ are linked together with an edge $(u', v' = 6)$ of length 1 to produce an optimal realization $G_2$ of $\mathbf{M}_2$; $G_1$ and $G_2$ are linked together with an edge $(u, v)$ of length 1 to produce an optimal realization $G$ of the original metric space $(M, d)$.

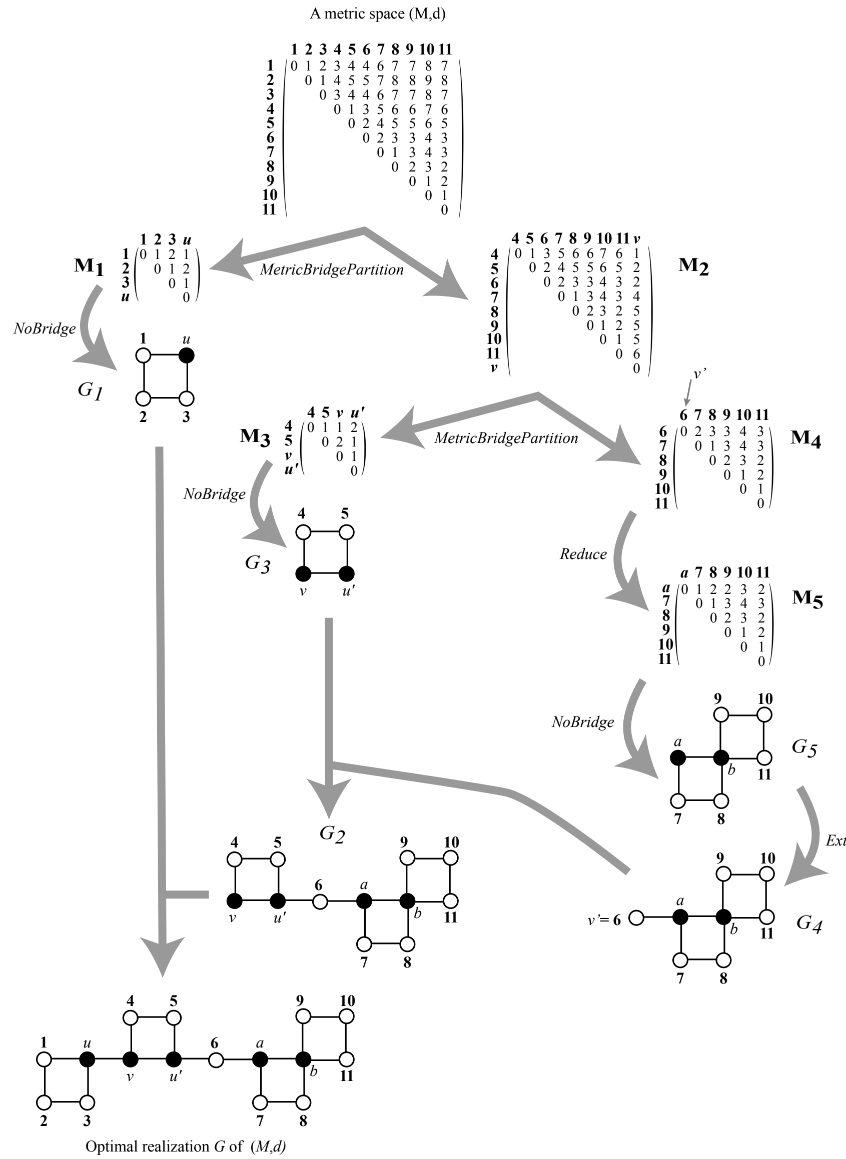



Figure 4: Construction of an optimal realization

## 5 Final Remarks and Conclusion

We have proved that the Metric Bridge Partition Problem is polynomially solvable. The proposed algorithm can be used to decompose any metric space $(M, d)$ into metric spaces $(M_1, d_1), \cdots, (M_r, d_r)$ such that no optimal realization of $(M_i, d_i)$ $(i = 1, \cdots, r)$ has a bridge. An optimal realization of $(M, d)$ can then easily be obtained by adding some edges linking optimal realizations of $(M_1, d_1), \cdots, (M_r, d_r)$.

An ideal algorithm, as indicated in Theorem 1, should decompose a metric space into blocks (i.e., maximal two-connected subgraphs or bridges). The proposed algorithm is not able to detect cutpoints that do not belong to a bridge. For example, we have not been able to further decompose $\mathbf{M}_5$ in the example of Figure 4, while its optimal realization $G_5$ has two blocks sharing the cutpoint $b$. Our algorithm for the solution of the Metric Bridge Partition Problem relies on the fact that if there is a bridge $(u, v)$ linking $K$ and $L$, it is possible to decide if an element of $M$ belongs to $K$ or $L$ by computing its distance to $u$ and $v$. We do not know how to make such a partition using only a cutpoint $u$. Future work will consist in studying the more general Metric Cutpoint Partition Problem, which is to determine if the elements of a metric space $(M, d)$ can be partitioned into two nonempty subsets $K$ and $L$ such that all optimal realizations of $(M, d)$ contain a cutpoint linking $K$ with $L$. The complexity of this problem is still unknown.

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