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Abstract

An infinite-horizon differential game between a manufacturer and a retailer is considered. The players control their marketing efforts and the sales share of the online channel is the state of the system. The manufacturer seeks to maximize her profit made on both the indirect and direct channels and faces, aside from her marketing effort, a logistics cost of selling online. The retailer seeks to keep consumers buying offline through her effort. A feedback Nash equilibrium is identified and results are discussed.

Key Words: Differential Game, Dual Distribution Channel, Electronic Commerce.

Résumé

Un jeu différentiel sur horizon infini entre un manufacturier et un détaillant est considéré. Les joueurs contrôlent leurs efforts marketing et l'état du système est la part de marché des ventes réalisées par le canal online. La manufacturier cherche à maximiser son profit sur le canal direct et le canal indirect. Il supporte des coûts logistiques des ventes online en sus des coûts associés à l'effort marketing. Le détaillant quant à lui, cherche à faire en sorte que les consommateurs continuent d'acheter dans ses magasins par le truchement de son effort marketing. Un équilibre de Nash en boucle fermée est identifié et nous discutons les résultats qui en découlent.

Mots clés : jeu différentiel, canal dual de distribution, commerce électronique.

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1 Introduction

During the last decade or so, manufacturers have increasingly been selling directly to consumers through the Internet. At the same time, retailers are competing directly with manufacturers of national brands by offering their private labels to the same pool of shoppers. Therefore, the manufacturers are also becoming retailers and *vice versa*.

The impact of this shift in traditional roles on the relationships between channel's partners and their performance have attracted the attention of marketing scholars. Regarding the introduction of private labels, Mills (1995, 1999), Raju et al. (1995) and Narasimhan and Wilcox (1998) have assessed the impact on manufacturers' and retailers' prices and profits. Generally speaking, it seems that retailers would benefit from offering their brands, along with the national ones, to consumers and eventually manufacturers would witness a decline in their profits. Karray and Zaccour (2004) identified the circumstances under which a cooperative advertising program allows a manufacturer to mitigate, at least partially, the profit losses due to the introduction by the retailer of the store brand. Morton and Zettelmeyer (2004) investigate whether it is beneficial for the retailer to replace an existing national brand with a private label, and which manufacturer's product should be eliminated from the existing retail assortment.

Turning now to the involvement of manufacturers in retailing, the natural questions are why manufacturers are going direct, and eventually how to manage a dual distribution channel. Although the general answer to the first question is to create a sustainable competitive advantage by increasing the market penetration of their products (Alba et al. (1997)), it seems that the strategy of selling goods and services online could be both performance-enhancing and performance-destroying (Geyskens et al. (2000)). The negative part is due to the cost of duplicating a distribution channel and the cost of conflict with the other intermediaries involved in the channel, retailers in particular (Sarkar et al. (1996)). As Chiang et al. (2003) point out, "*while more and more manufacturers are engaging in direct sales, their retailers partners voice the belief that ordered place through a manufacturer's direct channel are orders that should have been place through them*".

Although a lot has been said on the evolution in the role of manufacturers, the marketing science literature, and more specifically the game theoretic one, is rather sparse. Tsay and Agrawal (2004) survey contributions in multi-channel distribution systems, and touch the implications of manufacturers going direct on relationships in channels. The survey shows that the main decision variables used in the literature are the transfer and retail prices and/or inventories and that game models have been static. Balasubramanian (1998) considers a spatial model where one retailer distributes her product online and analyzes its effect on the conventional retail industry. Zettelmeyer (2000) studies the case of two integrated firms competing in both conventional and direct channels. Decision variables include prices and search costs for consumers. As a result, as the Internet gains ground among consumers, competition tends to shift from search cost towards price competition.

Chiang et al. (2003) consider the situation where a manufacturer is selling both directly (through the Internet) and indirectly (through an independent conventional retailer) to customers. The main result, while somewhat surprising, is that no sales are made online. The manufacturer has actually no interest in selling directly and her only purpose in establishing the new channel is to pressure the retailer to reduce her price and thus boost the demand for the manufacturer's brand. Fruchter and Tapiero (2004) extends the setting in Chiang and al. (2003) to a dynamic framework. They account for consumers' heterogeneity on the acceptance of the virtual channel, with prices as strategies in a Stackelberg game.

The purpose of this paper is to characterize equilibrium marketing strategies in a channel comprised of one manufacturer and one retailer where the former sells also directly to consumers. The model is fully dynamic with the online channel's share as the state variable. We assume that its evolution is governed by marketing expenditures of players and by consumers' channel switching behavior. The contribution of this paper lies in bringing the differential game methodology to this area and in the explicit analysis of the impact of marketing efforts and store switching on each distribution channel's share.

The remainder of the paper is organized as follows. In Section 2 we introduce the model and in Section 3 we characterize equilibrium strategies. In Section 4 we conduct some sensitivity analysis on strategies and steady state with respect to model's parameters. In Section 5 we offer our conclusions.

2 Model

Consider a manufacturer (player M) selling her brand through both a representative retailer (player R) and directly to consumers through, e.g., the Internet. Denote by $x(t), 0 \leq x(t) \leq 1$, the share of online total sales of the manufacturer's brand at time $t \in [0, \infty)$. The evolution over time of this state variable will be specified below.

Denote by $E_i(t)$ the marketing efforts of player $i, i \in \{M, R\}$ at time $t \in [0, \infty)$. Player i's marketing effort is aimed at keeping/attracting consumers to "her" preferred channel, i.e., the conventional (offline) channel for the retailer and the direct (online) channel for the manufacturer. Marketing efforts can be interpreted as means to provide relevant information (by advertising or other types of communication actions) to consumers so as to make their search processes less costly (see on this, e.g., Lal and Sarvary (1999), Zettelmeyer (2000)). They also intend to influence consumers' choice of purchasing option, as pointed out by Hauser et al. (1993), "*consumers should seek out formats that enable them to make selections that maximize consumption utility net of price and search costs (...) even if retail formats offer identical merchandise*".

Although the manufacturer is selling in both channels, the assumption here is that she prefers the direct channel. This preference stems from the fact that the manufacturer's margin, measured per point of share of each channel, on sales made directly to consumers (denoted m_D) is higher than the margin obtained by selling through the indirect channel (denoted m_I). Note that this ordering of manufacturer's margins, i.e., $m_I < m_D$, is a necessary condition to avoid having the retailer buying from the online market. Denote by m_R the retailer's margin per point of the offline's channel market share.

We assume that the evolution of the online market share depends on both players' marketing expenditures and on the market share of the online channel, i.e.,

$$\frac{dx(t)}{dt} = \dot{x}(t) = H(E_M(t), E_R(t), x(t)).$$

To keep things simple in this rather exploratary study, we assume that $H(.)$ is separable in the controls and state and is given by

$$\frac{dx}{dt} = \dot{x}(t) = (\mu_M E_M(t) - \mu_R E_R(t)) - (px(t) - q(1 - x(t))), \quad x(0) = x_0, \tag{1}$$

where μ_M and μ_R are strictly positive parameters, and p and q are parameters which satisfy $0 \leq p, q \leq 1$.

The first bracketed right-hand-side term of (1) states that the online market share evolution depends on the combined effect of players' marketing efforts; it increases (decreases) with manufacturer's (retailer's) marketing efforts. The parameters μ_M and μ_R can be interpreted as efficiency parameters in transforming marketing efforts into corresponding channel's market share. The second term captures the idea that consumers switch between the two channels, with parameters p and q measuring the intensity with which each channel's customers switch to the alternative one. Thus, the variation in the market share of the online channel depends on a differential in marketing efforts and a differential in the movement of consumers between the two channels. Note that these switches can be explained by a variety of reasons such as convinience, last buying experience, desire of change, etc.

Remark 1 *Note that if $p = q = 0$, then the market share evolution would be of the "excess-advertising" type, i.e., depending only on the difference in advertising efforts of both players (see Jørgensen and Zaccour (2004) for a recent review of differential games in advertising competition).*

Remark 2 *If the dynamics of the online channel share could not be at all controlled by the players, i.e.,*

$$\dot{x}(t) = -px(t) + q(1 - x(t)), \quad x(0) = x_0,$$

then equilibrium steady-state would be given by

$$0 < x_{ss} = \frac{q}{p+q} < 1.$$

Thus the higher the intensity of switching from the offline channel to the online one, the higher the steady-state share of the latter.

We assume that the total cost of marketing effort is convex increasing. For simplicity, we adopt, as in, e.g., Jørgensen et al. (2000, 2001a,b), a quadratic specification, i.e.,

$$C_i(E_i) = \frac{1}{2}a_i E_i^2, \qquad a_i > 0, \qquad i \in \{M, R\}.$$

Note that the above marketing effort cost functions are asymmetric, which reflects the idea that the two players are not necessarily using the same medias to advertise or communicate with their target markets.

The margins m_D and m_I defined above for the manufacturer are net of production cost. Furthermore, the margin m_I is also net of delivery cost to the retailer. In addition, the retailer's margin is net of retailing costs (e.g., inventories, replenishment of shelf space, merchandising, etc.). There is still one cost which has to be accounted for, i.e., the cost of selling online by the manufacturer. This cost may include treatment of orders, handling and shipping to consumers, etc. We will assume that these cost items can be captured by a function which depends on the share of the online channel and denoted $F(x)$. We assume that this function is positive and convex increasing and satisfies $F(0) = 0$. One way of justifying convexity is by stating that when the share of the online channel becomes larger, the manufacturer incurs an increasingly high (handling, shipping, etc.) cost because, e.g., of the extra working hours of the staff and the vehicles. Again for mathematical tractability, we will assume a quadratic specification, i.e., $F(x) = 1/2bx^2$, where $b > 0$.

Omitting from now on the time argument when no confusion may arise and assuming that both players discount their stream of profits at the same market rate r, their objective functionals are as follows:

$$\underset{E_M \geq 0}{Max} \Pi_M = \int_0^\infty e^{-rt}\{m_D x + m_I(1-x) - \frac{1}{2}a_M E_M^2 - \frac{1}{2}bx^2\}dt, \tag{2}$$

$$\underset{E_R \geq 0}{Max} \Pi_R = \int_0^\infty e^{-rt}\{m_R(1-x) - \frac{1}{2}a_R E_R^2\}dt. \tag{3}$$

To recapitulate, by (2) and (3)-(4) we have defined a two-player infinite-horizon differential game with one state variable $x, 0 \leq x(t) \leq 1$, and two controls E_M, E_R constrained to be non-negative. Note that the structure of our game is of the often used linear-quadratic variety.

To conclude, we wish to highlight the following features of the model:

- Since the manufacturer is selling in both channels, our model treats the issue of how to deal with a dual distribution channel. Indeed, by deciding on the optimal (in an equilibrium sense) marketing effort to attract customers to the online channel, the manufacturer is implicitly proceeding to an arbitrage between the share of sales in the two channels.

- We assume that given the intensities (p and q) at which consumers switch from one channel to another, their values can be different. Therefore, our framework accounts for the idea that these switches may be due to different reasons or underlying processes.
- The two players have clearly conflicting views on the online channel. For the manufacturer it is a complement to the conventional one, for the retailer it is a clear substitute. This raises the question, not answered here, whether coordination of the two players' marketing efforts can still be feasible.

3 Equilibrium

We assume that the players use feedback marketing effort strategies. Since the game is played over an infinite horizon, we shall confine our interest to stationary strategies. The following proposition characterizes the unique feedback Nash equilibrium.

Proposition 1 *Feedback Nash marketing equilibrium strategies are given by:*

$$
\begin{aligned}
E_M(x) &= \begin{cases} (Ax+B)\,\frac{\mu_M}{a_M}, & \text{for } x \le -\frac{B}{A} \\ 0, & \text{otherwise} \end{cases}\quad, &(4)\\
E_R(x) &= -D\frac{\mu_R}{a_R} = cst. &(5)
\end{aligned}
$$

Value functions are as follows

$$
\begin{aligned}
V_M(x) &= \frac{1}{2}Ax^2 + Bx + C, &(6)\\
V_R(x) &= Dx + E, &(7)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{a_M}{\mu_M^2}\left((p+q+\frac{r}{2}) - \sqrt{(p+q+\frac{r}{2})^2 + \frac{b\mu_M^2}{a_M}}\right) &(8)\\
B &= \frac{m_D - m_I + A\left(q + D\frac{\mu_R^2}{a_R}\right)}{\frac{r}{2} + \sqrt{(p+q+\frac{r}{2})^2 + \frac{b\mu_M^2}{a_M}}}, &(9)\\
C &= \frac{1}{r}\left(m_I + \frac{(B\mu_M)^2}{2a_M} + \frac{BD\mu_R^2}{a_R} + qB\right), &(10)
\end{aligned}
$$

$$D = \frac{-m_R}{\frac{r}{2}+\sqrt{(p+q+\frac{r}{2})^2+\frac{b\mu_M^2}{a_M}}}, \tag{11}$$

$$E = \frac{1}{r}\left(m_R + \frac{(D\mu_R)^2}{2a_R} + D\left(q + B\frac{\mu_M^2}{a_M}\right)\right). \tag{12}$$

Proof. See Appendix. □

The above proposition shows that the retailer's value function is linear and thus this player advertises at a constant rate. The manufacturer's marketing effort strategy is state-dependent. Note that since A is negative, for the manufacturer's strategy to make sense, B must be positive. Indeed, if B were negative, then the advertising strategy would read $E_M(x) = (Ax+B)\frac{\mu_M}{a_M} \geq 0$, for $x \leq -\frac{B}{A} < 0$, which contradicts the constraint $0 \leq x \leq 1$. We shall thus assume from now on that the parameters are such that B is positive. Since the denominator is always positive, then the necessary condition for having $B > 0$ is that the numerator be positive, i.e., $\left(m_D - m_I + A\left(q + D\frac{\mu_R^2}{a_R}\right)\right) > 0$. Moreover, if the online channel's share x is greater than the threshold level $-\frac{B}{A}$, assuming this threshold is lower than 1, the manufacturer will not invest in marketing efforts. Under such circumstances, the revenues from gaining an additional share will not counterbalance the cost.

Substituting for equilibrium marketing efforts from (4)-(5) in the state dynamics (1), we obtain, after some straightforward manipulations, the following value for the steady-state for the online channel

$$x_{ss} = \left(\frac{m_D - m_I}{Y(Y-r)}\right)\frac{\mu_M^2}{a_M} + \frac{(p+q+r)}{a_R Y^2 (Y-r)}\left(a_R Y q - m_R \mu_R^2\right), \tag{13}$$

where

$$Y = \frac{r}{2} + \sqrt{(p+q+\frac{r}{2})^2 + \frac{b\mu_M^2}{a_M}}.$$

As it is readily seen, the steady state depends on the model's parameters. Conditions on parameters' values can be derived to insure that the steady state is bounded between 0 and 1.

Remark 3 *In some papers dealing with conflicts and cooperation in marketing channels, the authors assume that the game is played à la Stackelberg, with the manufacturer often assuming the role of leader (e.g., Shugan (1985), Moorthy and Fader (1989), Jeuland and Shugan (1983) and Jørgensen et al. (2001b)). In our setting, irrespective of who the leader is, a Stackelberg equilibrium coincides with the Nash one derived above. The reason is that the follower's reaction function does not depend on the leader's marketing effort, i.e., the latter can not influence the former's choice.*

The following result applies when logistics cost of online deliveries is zero.

Corollary 1 *If the logistics cost for the online channel is zero, i.e.,* $b = 0$*, then value functions are linear*

$$V_M(x) = B_1 x + C_1, \qquad V_R(x) = D_1 x + E_1,$$

and feedback Nash equilibrium strategies are constant and given by

$$E_M = \frac{\mu_M}{a_M} B_1 \qquad E_R = -\frac{\mu_R}{a_R} D_1,$$

where

$$\begin{aligned} B_1 &= \frac{m_D - m_I}{r+p+q} > 0, \qquad C_1 = \frac{1}{r}[m_I + \frac{B_1 \mu_M^2}{2a_M} + \frac{B_1 D_1 \mu_R^2}{a_R} + qB_1], \\ D_1 &= \frac{-m_R}{r+p+q} < 0, \quad E_1 = \frac{1}{r}[m_R + \frac{(D_1 \mu_R)^2}{2a_R} + D_1(q + B_1 \frac{\mu_M^2}{a_M})]. \end{aligned}$$

The steady state is given by

$$x_{ss} = \frac{q}{(p+q)} + \frac{1}{(p+q+r)(p+q)} \left(\frac{(m_D - m_I)\mu_M^2}{a_M} - \frac{m_R \mu_R^2}{a_R} \right) \tag{14}$$

Proof. The results follow immediately from setting $b = 0$ in the previous proposition. □

The above scenario corresponds to a service or a product which can be delivered, for instance, via the Internet at (almost) zero cost for the service-provider or the manufacturer. Software which can be downloaded directly by the customer on her computer and electronic airline tickets are examples of such a context. Note that in this case, the players will constantly make marketing efforts at positive rates.

To interpret the steady-state value of the online channel, note that (14) can be rewritten as

$$x_{ss} = \frac{q}{(p+q)} + \frac{1}{(p+q)} (\mu_M E_M - \mu_R E_R).$$

Thus, the steady state of the online channel is equal to the steady state when the dynamics are not controlled, plus an “excess-advertising” term. The sign of the latter depends on relative advertising levels of the two players which in turn depend on all model’s parameters (margins, intensities of switching and cost parameters).

Comparing the retailer’s value functions under the two scenarios ($b \neq 0$ and $b = 0$) shows that the slope of the first one is bigger in absolute value than the value function with $b = 0$. Indeed,

$$D_1 = \frac{-m_R}{r+p+q} < D = \frac{-m_R}{\frac{r}{2} + \sqrt{(p+q+\frac{r}{2})^2 + \frac{b\mu_M^2}{a_M}}}.$$

Thus, the retailer's payoff decreases less sharply in x when the manufacturer faces a logistics cost. Although our model does not capture all competitive dimensions of e-commerce, this leads to conjecture that the competitive standing of offline retailing will heavily depend on the delivery cost of online purchases.

4 Sensitivity Analysis

To get more insight into the players' strategies and the steady state online market share, we study how they vary with the parameters' values. The following proposition provides the results for margins.

Proposition 2 *The manufacturer's marketing strategy is increasing in the online channel and retailer margins and decreasing in the offline channel margin.*

The retailer's marketing strategy is increasing in her margin and is independent of manufacturer's online and offline margins.

Proof. Straightforward derivations lead to

$$\begin{aligned} \frac{\partial E_M(x)}{\partial m_D} &= -\frac{\partial E_M(x)}{\partial m_I} = \frac{\mu_M}{a_M Y} > 0, \quad \frac{\partial E_M(x)}{\partial m_R} = -\frac{A\mu_M\mu_R^2}{a_M a_R Y^2} > 0, \\ \frac{\partial E_R}{\partial m_R} &= \frac{\mu_R}{a_R Y} > 0, \frac{\partial E_R}{\partial m_I} = \frac{\partial E_R}{\partial m_D} = 0. \end{aligned}$$

□

Increasing one's own margin (m_D for M or m_R for R) leads the concerned player to increase her marketing effort for the understandable reason that it becomes more profitable to attract customers to this channel. The independence of the retailer's strategy with respect to the manufacturer's margins is a by-product of the game structure, i.e., the retailer will buy from the manufacturer independently of the latter's margins and she does not intervene in the online market. Now, increasing the retailer's margin will eventually lead the manufacturer to increase her effort, as a reaction to the increase of the retailer's effort to attract customers to her offline channel. Note that although the retailer's margin does not appear explicitly in the manufacturer's problem, it affects nevertheless her profit via its impact on retailer's strategy and thus on the share of offline channel.

We now turn to sensitivity analysis of strategies with respect to the switching parameters.

Proposition 3 *The retailer's marketing effort strategy is decreasing in both channels' switching intensities parameters.*

Proof. Straightforward computations lead to the result.

$$\frac{\partial E_R}{\partial p} = \frac{\partial E_R}{\partial q} = -\frac{\mu_R}{a_R}\frac{m_R}{Y^2}\frac{(p+q+\frac{r}{2})}{(Y-r)} \leq 0,$$

where

$$Y = \frac{r}{2} + \sqrt{(p+q+\frac{r}{2})^2 + \frac{b\mu_M^2}{a_M}}.$$

□

The above proposition states that a retailer's response to a change in either p or q leads to the same impact on marketing efforts, i.e., $\frac{\partial E_R}{\partial p} = \frac{\partial E_R}{\partial q} \leq 0$. The first result $\frac{\partial E_R}{\partial p} \leq 0$ is rather intuitive; indeed the higher the (uncontrollable) propensity of the switch from the online channel to the offline one, the less the retailer needs to spend on marketing efforts to attract customers to her outlet. Thus p and E_R are substitutes. The second result is surprising; it says that the higher the intensity of customers leaving the offline channel to the online one, the less the retailer invests in marketing efforts.

To shed a light on this result, we examine the impact of these parameters on manufacturer's strategy. We could not determine analytically the signs of $\frac{\partial E_M}{\partial q} \leq 0$ and $\frac{\partial E_M}{\partial p} \leq 0$ (the expressions are very long, and do not admit any apparent interpretation). We shall thus assess numerically the impact of these parameters on the manufacturer's strategy. We computed these derivatives for different values of p and q.[1]

Figure 1 shows that increasing p (or q) also has a decreasing effect on manufacturer's strategy. The result for q is intuitive; indeed, q and E_M are substitutes for the manufacturer. The result for p, namely saying that the higher the intensity of consumers leaving her "own" channel, the lower the marketing effort, is also a surprise. One possible explanation of $\frac{\partial E_R}{\partial q} \leq 0$ and $\frac{\partial E_M}{\partial p} \leq 0$, may lie in the dynamics of the model. Indeed, the equation describing the evolution of the online channel market share states, implicitly, that the higher the switch from one channel to another at any given instant in time, the higher the potential for a come-back at a later instant in time. Put differently, given that the flows of switchers are uncontrollable, the best a player can do is adopt a *surf-the-wave strategy.*[2]

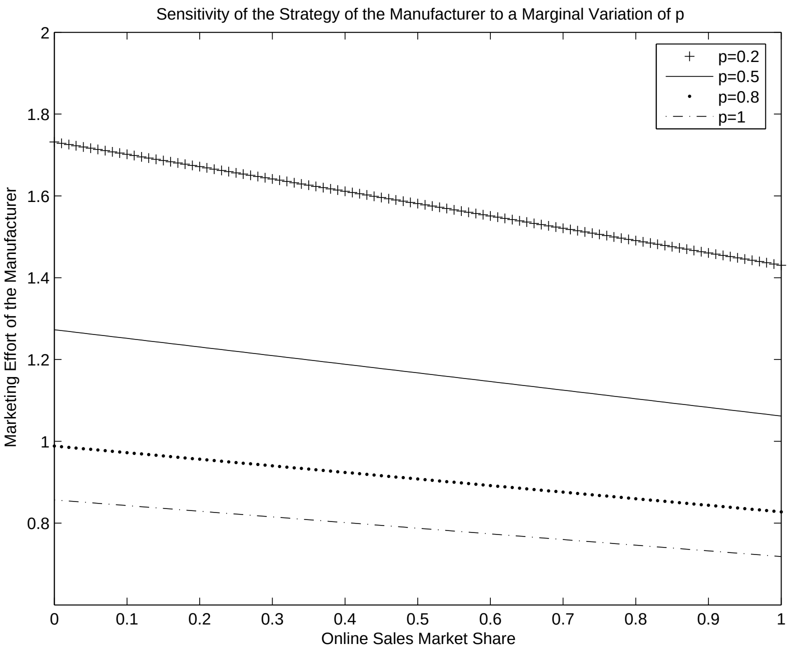
The following proposition gives the results of the sensitivity analysis of retailer's strategy with respect to the cost parameters.

Proposition 4 *Increasing any cost parameter leads the retailer to reduce her marketing effort.*

---

[1] Although we conducted numerical experiments for all values of p and q lying between 0.1 and 0.9, by step of 0.1, we only print few curves for the sake of clarity.

[2] This type of phenomenon is also present in the Lanchester model, one of the most celebrated differential game model of advertising competition (see, e.g., Jarrar et al. (2004) or Jørgensen and Zaccour (2004)).



	m_D	m_I	m_R	a_M	a_R	b	r	μ_M	μ_R	p	q
q	4.5	1.6	1.3	1.4	1.1	0.1	0.1	0.6	0.9		0.3

Figure 1: Sensitivity of the manufacturer's strategy to p.

Proof. Straightforward computations give

$$\frac{\partial E_R}{\partial a_R} = \frac{D\mu_R}{a_R^2} \leq 0, \qquad \frac{\partial E_R}{\partial a_M} = -\frac{m_R \mu_R b \mu_M^2}{2a_R a_M^2 Y^2 \left(Y - \frac{r}{2}\right)} \leq 0,$$

$$\frac{\partial E_R}{\partial b} = -\frac{\mu_M^2}{a_M}\frac{\mu_R}{a_R}\frac{m_R}{(Y-r)\,Y^2} \leq 0.$$

□

The impact of a_R is intuitive; indeed, the marketing effort cost is convex increasing and thus an upward shift in the value of the parameter leads, quite naturally, to a decrease in this activity. To attempt to interpret the other results, it is insightful to consider the impact of varying these parameters on the manufacturer's strategy. We claim the following:

Claim 1 *Increasing any cost parameter, i.e., a_M, a_R or b, leads to a decrease in manufacturer's marketing effort.*

This claim is based on the following observations. First, note that

$$\frac{\partial E_M(x)}{\partial a_R} = \frac{-AD\mu_R^2}{a_R^2 Y}\frac{\mu_M}{a_M} \leq 0.$$

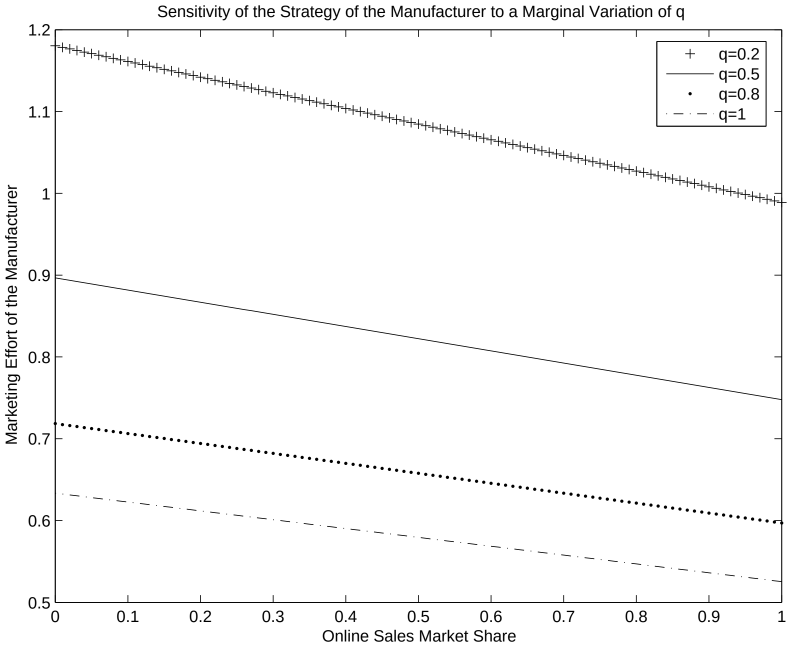
Next, although we could not assess analytically the signs of $\frac{\partial E_M(x)}{\partial a_M}$ and $\frac{\partial E_M(x)}{\partial b}$, numerical and other evidence seem to show that these derivatives are negative. Indeed, in the simple case where there is no logistics cost ($b = 0$), the manufacturer's equilibrium strategy is given by

$$E_M = \frac{\mu_M}{a_M}\frac{m_D - m_I}{r + p + q},$$

and hence

$$\frac{\partial E_M}{\partial a_M} = -\frac{\mu_M}{a_M^2}\frac{m_D - m_I}{r + p + q} \leq 0.$$

Furthermore, the results of the numerical essays (see Figure 2) also show that the manufacturer's strategy is, in the general case ($b > 0$), a decreasing function in her cost parameters a_M and b. Hence, we believe that the above proposition and the other indications allow us to state that increasing the cost of either player leads to a decrease in both players'



	m_D	m_I	m_R	a_M	a_R	b	r	μ_M	μ_R	p	q
q	4.5	1.6	1.3	1.4	1.1	0.1	0.1	0.6	0.9	0.7	

Figure 2: Sensitivity of the manufacturer's strategy to q.

marketing effort. A simple explanation is as follows. It is straightforward to admit that increasing any cost parameter implies a decrease in the activity by the concerned player. This means a reduction in the competitive pressure on the other player to defend "her" channel, and find it optimal to decrease her marketing expenditures. This result can also be interpreted in terms of strategic dependency. Marketing efforts are strategic complements (substitutes) when an increase in the expenditures by one player leads the other player to increase (decrease) her expenditures. Here, we have strategic complementarity, which is due to the "excess-advertising" term in the dynamics.

Proposition 5 *Increasing either marketing effort efficiency parameter leads to an increase in retailer's marketing effort.*

Proof. It suffices to note that

$$\frac{\partial E_R}{\partial \mu_R} = -\frac{D}{a_R} \geq 0, \quad \frac{\partial E_R}{\partial \mu_M} = \frac{m_R \mu_R b \mu_M}{a_R a_M Y^2 \left(Y - \frac{r}{2}\right)} \geq 0.$$

□

The above proposition is stated for the retailer, but actually the result can also be claimed for the manufacturer. Indeed, differentiating the manufacturer's marketing effort equilibrium strategy with respect to retailer's efficiency parameter gives

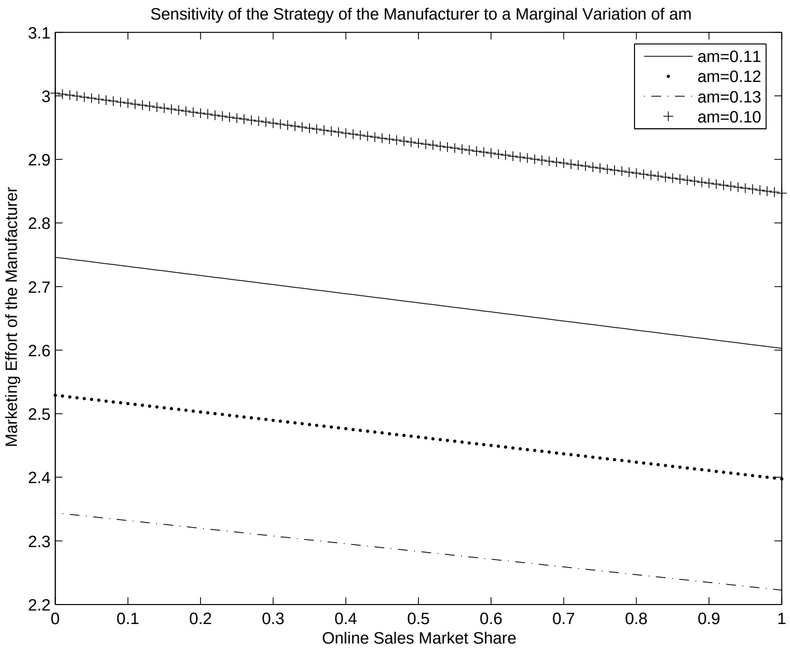
$$\frac{\partial E_M(x)}{\partial \mu_R} = \frac{2AD\mu_R}{a_R Y}\frac{\mu_M}{a_M} \geq 0.$$

We cannot characterize analytically the sign of $\frac{\partial E_M(x)}{\partial \mu_M}$. Again, looking at the special case where the logistics cost is zero, we have

$$\frac{\partial E_M}{\partial \mu_M} = \mu_M \left(\frac{m_D - m_I}{r+p+q}\right) \geq 0.$$

Further, the numerical results reported in Figure 3 for the general case (i.e., $b > 0$) show that manufacturer's marketing effort decreases with μ_M. These results and the ones in the above propositions provide sufficient evidence to claim that increasing the value of any efficiency parameter positively affects both players' marketing efforts. Actually, this claim mirrors the claim stated for the cost parameters. Indeed, if one player finds it advantageous to increase her marketing effort because she is more efficient, then the other player follows in order to defend her channel's market share. Here again we see that marketing efforts are strategic complements.

The following proposition provides the results of the sensitivity analysis of the steady state with respect to the different parameters.



	m_D	m_I	m_R	a_M	a_R	b	r	μ_M	μ_R	p	q
a_M	5	4	1		0.4	0.1	0.1	0.5	1.3	0.8	0.7

Figure 3: Sensitivity of the manufacturer's strategy to a_M.

Proposition 6 *The steady state increases in online margin and decreases in offline ones.*

Proof. Recall that the steady state is given by

$$\begin{aligned} x_{ss} &= \left(\frac{m_D - m_I}{Y\,(Y-r)}\right)\frac{\mu_M^2}{a_M} + \frac{(p+q+r)}{a_R Y^2\,(Y-r)}\left(a_R Y q - m_R \mu_R^2\right), \\ \text{with } Y &= \frac{r}{2} + \sqrt{(p+q+\frac{r}{2})^2 + \frac{b\mu_M^2}{a_M}}, \end{aligned}$$

It suffices to compute the following derivatives to get the result.

$$\frac{\partial x_{ss}}{\partial m_D} = -\frac{\partial x_{ss}}{\partial m_I} = \frac{\mu_M^2}{Y\,(Y-r)\,a_M} \geq 0, \quad \frac{\partial x_{ss}}{\partial m_R} = \frac{-\mu_R^2\,(p+q+r)}{Y^2\,(Y-r)\,a_R} \leq 0.$$

□

The steady state of the online market share increases in the margin that the manufacturer gets in this channel and decreases in the margin of the alternative channel. This result is expected. Note that the sum of variations of the steady state with respect to the manufacturer's margins is zero, i.e., $\left(\frac{\partial x_{ss}}{\partial m_D} + \frac{\partial x_{ss}}{\partial m_I} = 0\right)$. This "trade-off" rule is due to the linearity in x of the revenues' terms in the manufacturer's optimization problem.

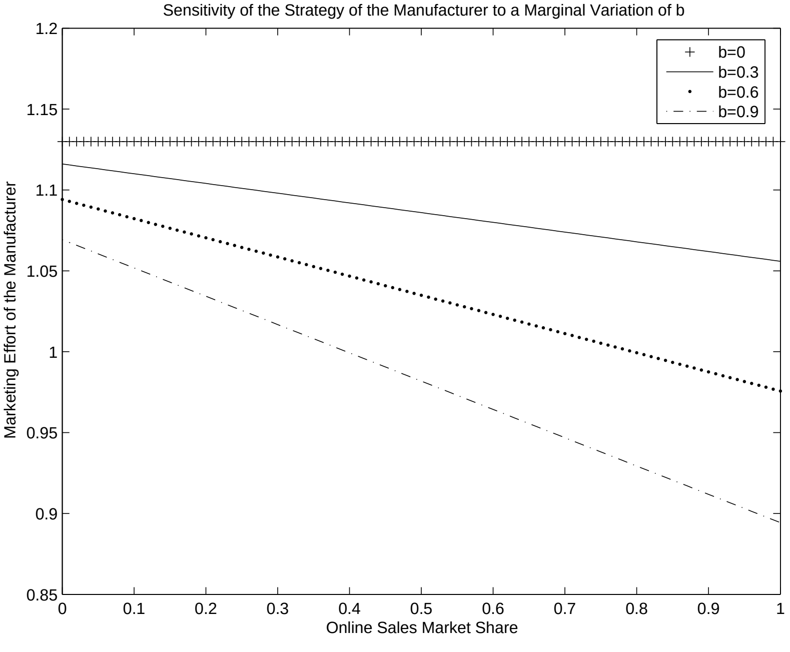
The impact of p and q on the steady state cannot be assessed analytically. It is however easy to establish that $\frac{\partial x_{ss}}{\partial p} < \frac{\partial x_{ss}}{\partial q}$. Indeed, straitghtforward computations lead to

$$\frac{\partial x_{ss}}{\partial p} - \frac{\partial x_{ss}}{\partial q} = -\frac{(p+q+r)}{Y\,(Y-r)} < 0 \Leftrightarrow \frac{\partial x_{ss}}{\partial p} < \frac{\partial x_{ss}}{\partial q}. \tag{15}$$

Since we expect p and q to play opposite roles, the inequality in (15) provides a basis to conjecture that increasing the propensity of consumers leaving the online channel hurts its steady state value, i.e., $\frac{\partial x_{ss}}{\partial p} < 0$, and that it is the other way around for q. We conducted a series of numerical simulations for $p, q \in [0.1, 0.9]$,and in all cases the results confirm this rather intuitive conjecture (see Figure 4).

Turning to the impact of cost parameters, the results are summarized in the following:

Proposition 7 *The online channel steady state market share increases in the retailer's marketing effort cost parameter and decreases in the efficiency of marketing effort.*



	m_D	m_I	m_R	a_M	a_R	b	r	μ_M	μ_R	p	q
b	4.5	1.6	1.3	1.4	1.1		0.1	0.6	0.9	0.7	0.3

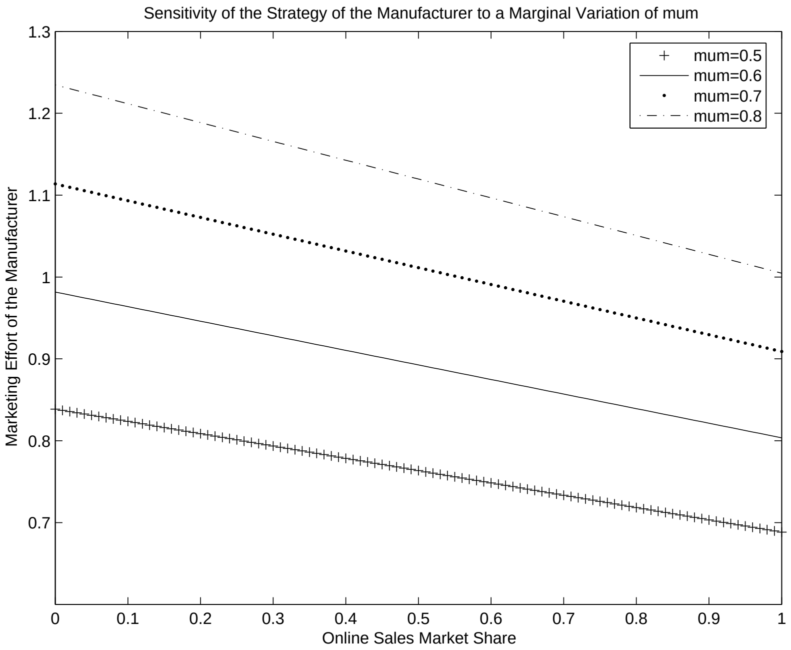
Figure 4: Sensitivity of the manufacturer's strategy to b.

Proof. The derivatives are given by

$$\begin{aligned}\frac{\partial x_{ss}}{\partial a_R} &= \frac{(p+q+r)}{Y^2\,(Y-r)}\left(\frac{m_R\mu_R^2}{a_R^2}\right) \geq 0,\\ \frac{\partial x_{ss}}{\partial \mu_R} &= \frac{-2m_R\mu_R\,(p+q+r)}{a_RY^2\,(Y-r)} \leq 0.\end{aligned}$$

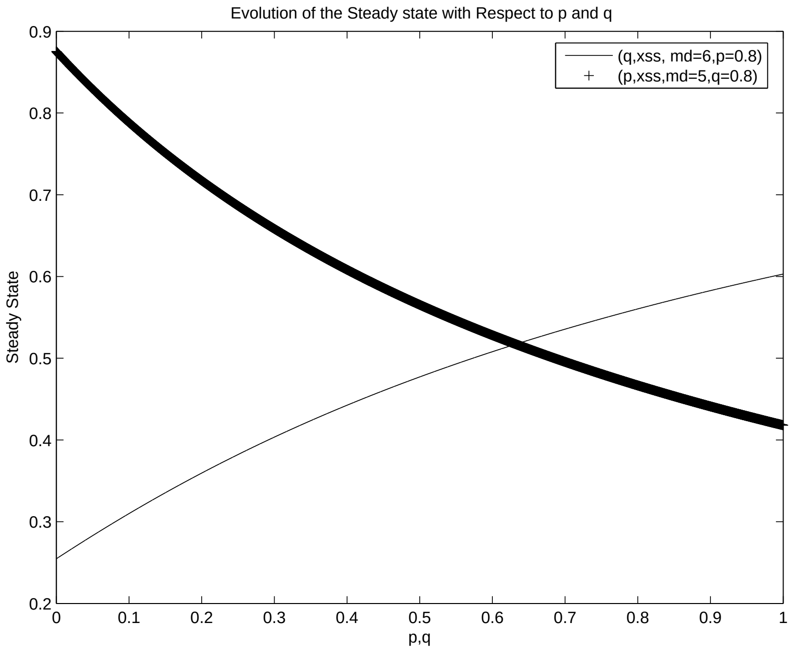
□

Regarding the manufacturer, numerical simulations show that the online channel steady state market share decreases in her marketing effort and logistics cost parameters and increases in the efficiency parameter μ_M (see Figures 5 and 6). These results are not surprising. If the cost parameter of player is increased, then the level of the marketing effort is reduced and so is the share of this player channel. The efficiency parameter plays an opposite role.



	m_D	m_I	m_R	a_M	a_R	b	r	μ_M	μ_R	p	q
μ_M	4.5	1.6	1.3	1.4	1.1	0.1	0.1		0.9	0.7	0.7

Figure 5: Sensitivity of the manufacturer's strategy to μ_M



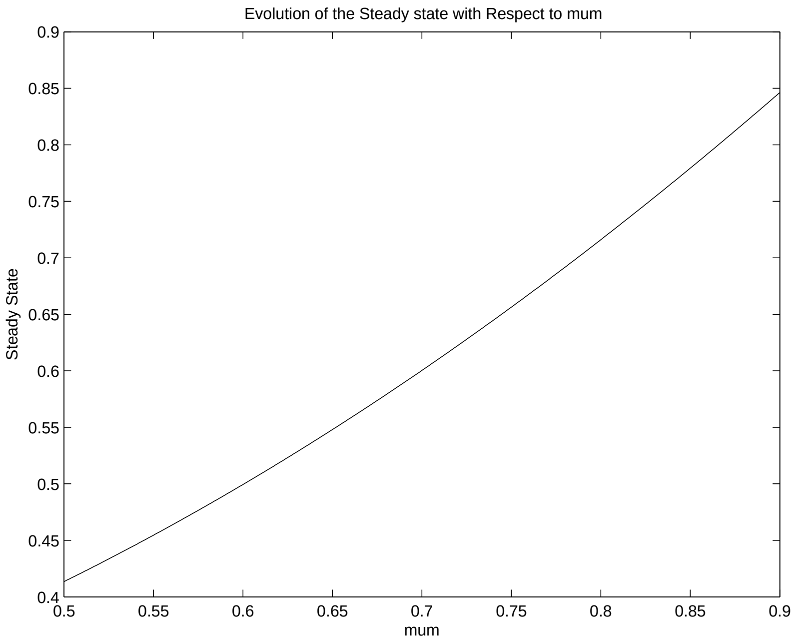
	m_D	m_I	m_R	a_M	a_R	b	r	μ_M	μ_R	p	q
p	5	1.6	1.3	1.4	1.1	0.1	0.1	0.6	0.9		0.8
q	6	1.6	1.3	1.4	1.1	0.1	0.1	0.6	0.9	0.8	

Figure 6: Sensitivity of the steady state to p and q.

5 Conclusion

This rather exploratory work on marketing efforts in a dual distribution channel game could be extended in numerous ways. We have considered margins as given. Since pricing issues are of crucial importance in e-commerce, introducing them in the model would lead to a more realistic one and provide valuable insight on competitive issues in dual channels. This is an ongoing research by the authors. A second restriction in our model is the assumption that the switching intensities between the two channels are exogenous. One possible extension is to let them be a function of marketing effort or prices. In any event, this would lead to a structure which is not of the tractable linear-quadratic variety. Here, one would have to resort to numerical methods to obtain an equilibrium. A third limitation in our model is in the number of players. Introducing up-stream and down-stream competitions will certainly have an impact on the results.

We wish to conclude by raising the question of coordination/cooperation in the case of e-commerce between a manufacturer and her retailer. The manufacturer's decision to compete with her retailers by adding a new distribution channel is likely to generate



	m_D	m_I	m_R	a_M	a_R	b	r	μ_M	μ_R	p	q
μ_M	5	1.6	1.3	1.4	1.1	0.1	0.1		0.9	0.8	0.8

Figure 7: Sensitivity of the steady state to μ_M.

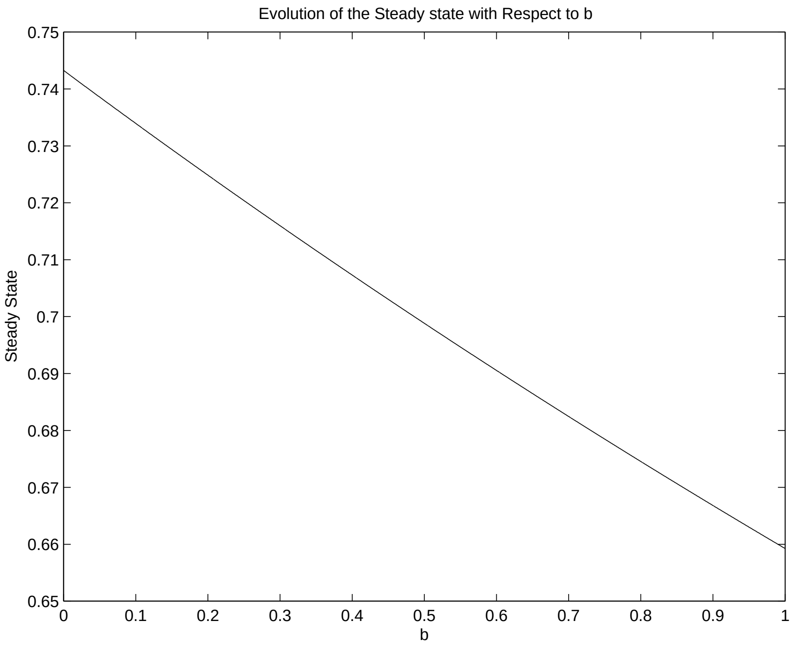
some conflicts between the partners, and possibly also lead to inefficiencies (managerial, economic and strategic). The natural question is thus what kind of mechanisms one can think of that could improve channel's profitability as well as individual profits. This is an interesting open ended question.

6 Appendix

We apply a standard sufficient condition for a stationary Markov perfect Nash equilibrium and wish to find bounded and continuously differentiable functions $V_i(x), i \in \{M, R\}$, satisfying, for all $0 \leq x(t) \leq 1$, the Hamilton-Jacobi-Bellman (HJB) equations

$$\begin{aligned} rV_M(x) &= \underset{E_M \geq 0}{Max}\{m_D x + m_I(1-x) - \frac{1}{2}a_M E_M^2 - \frac{1}{2}bx^2 \\ &\quad + \frac{dV_M}{dx}(\mu_M E_M - \mu_R E_R - px + q(1-x))\} \end{aligned} \tag{16}$$

$$rV_R(x) = \underset{E_R \geq 0}{Max}\{m_R(1-x) - \frac{1}{2}a_R E_R^2 \tag{17}$$



	m_D	m_I	m_R	a_M	a_R	b	r	μ_M	μ_R	p	q
b	5	1.6	1.3	1.4	1.1		0.1	0.7	0.9	0.8	0.8

Figure 8: Sensitivity of the steady state to b.

$$+\frac{dV_R}{dx}\left(\mu_M E_M - \mu_R E_R - px + q(1-x)\right)\}$$

Differentiating the right-hand-sides and equating to zero gives

$$E_M = V_M' \frac{\mu_M}{a_M}, \tag{18}$$

$$E_R = -V_R' \frac{\mu_R}{a_R}. \tag{19}$$

Note that in (16) and (17), the maximands are concave in E_M and E_R respectively, yielding thus a unique stationary feedback marketing effort rates.

Substituting for E_M and E_R from (18)-(19) in (16)-(17) leads to

$$\begin{aligned} rV_M(x) &= m_D x + m_I(1-x) + \frac{1}{2a_M}\left(V_M'\mu_M\right)^2 \qquad (20)\\ &\quad -\frac{1}{2}bx^2 + V_M'\left(V_R'\frac{\mu_R^2}{a_R} - px + q(1-x)\right), \\ rV_R(x) &= m_R(1-x) + \frac{1}{2a_R}\left(V_R'\mu_R\right)^2 + \qquad (21) \end{aligned}$$

$$V_R^{'}\left(V_M^{'}\frac{\mu_M^2}{a_M}-px+q(1-x)\right). \tag{22}$$

We conjecture that solutions to (16) and (17) will be quadratic:

$$\begin{aligned} V_M(x) &= \frac{1}{2}Ax^2+Bx+C \qquad (23)\\ V_R(x) &= Dx+E \qquad (24)\end{aligned}$$

in which A, B, C, D, E are constants. Substitute $V_M\left(x\right)$ and $V_R\left(x\right)$ from (20) and (21), as well as their derivatives $V_M'\left(x\right)=Ax+B$, $V_R'\left(x\right)=D$ into (18) and (19) and collect terms to obtain

$$\begin{aligned} r\left(\frac{1}{2}Ax^2+Bx+C\right) &= x^2\left(\frac{\mu_M^2A^2}{2a_M}-\frac{b}{2}-A\left(p+q\right)\right)+\\ & x\left(m_D-m_I+\left(\frac{A\mu_M^2}{a_M}-p-q\right)B+A\left(q+D\frac{\mu_R^2}{a_R}\right)\right)\\ & +m_I+\frac{\mu_M^2B^2}{2a_M}+\frac{DB\mu_R^2}{a_R}+qB,\\ r\left(Dx+E\right) &= x\left(-m_R+D\left(\frac{A\mu_M^2}{a_M}-p-q\right)\right)+m_R\\ & +\frac{D^2\mu_R^2}{2a_R}+\frac{BD\mu_M^2}{a_M}+Dq.\end{aligned}$$

By identification, we obtain

$$\begin{aligned} A &= \frac{a_M}{\mu_M^2}\left((p+q+\frac{r}{2})-\sqrt{(p+q+\frac{r}{2})^2+\frac{b\mu_M^2}{a_M}}\right)\\ B &= \frac{m_D-m_I+A\left(q+D\frac{\mu_R^2}{a_R}\right)}{\frac{r}{2}+\sqrt{(p+q+\frac{r}{2})^2+\frac{b\mu_M^2}{a_M}}},\\ C &= \frac{1}{r}\left(m_I+\frac{\left(B\mu_M\right)^2}{2a_M}+\frac{BD\mu_R^2}{a_R}+qB\right),\\ D &= \frac{-m_R}{\frac{r}{2}+\sqrt{(p+q+\frac{r}{2})^2+\frac{b\mu_M^2}{a_M}}},\\ E &= \frac{1}{r}\left(m_R+\frac{\left(D\mu_R\right)^2}{2a_R}+D\left(q+B\frac{\mu_M^2}{a_M}\right)\right).\end{aligned}$$

To obtain an asymptotically stable steady state, choose for A the negative solution.

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