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Abstract

One critical difficulty in implementing Merton's (1974) credit risk model is that the underlying asset value cannot be directly observed. The model requires the unobserved asset value and the unknown volatility parameter as inputs. The estimation problem is further complicated by the fact that typical data samples are for the survived firms. This paper applies the maximum likelihood principle to develop an estimation procedure and study its properties. The maximum likelihood estimator for the mean and volatility parameters, asset value, credit spread and default probability are derived for Merton's model. A Monte Carlo study is conducted to examine the performance of this maximum likelihood method. An application to real data is also presented.

Key Words: Credit risk, maximum likelihood, option pricing, Monte Carlo simulation.

Résumé

Une des principales difficultés lors de l'implémentation du modèle de risque de crédit de Merton est que la valeur de la firme sous-jacente ne peut être observée. Le modèle exige cette valeur inobservée de la firme ainsi que le paramètre de volatilité. Le problème d'estimation est de plus compliqué par le fait que les données existantes sont pour les firmes qui n'ont pas fait faillite. Dans cet article, la méthode du maximum de vraisemblance est appliquée afin de développer une procédure d'estimation du modèle et ses propriétés sont étudiées. Les estimateurs du maximum de vraisemblance sont obtenus pour les paramètres de dérive et de volatilité, pour la valeur de la firme, pour l'écart de crédit et la probabilité de défaut du modèle de Merton. Une étude de Monte Carlo est menée afin d'étudier les performances de ces estimateurs. Une application à des données réelles est aussi présentée.

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1 Introduction

In Merton (1974), a pricing model for corporate liabilities was developed using an option valuation approach. In his setting, the unobserved asset value of the firm is governed by a geometric Brownian motion. Subsequently, many variants of this model have been proposed in the literature. Merton's and its extended models are typically referred to as structural credit risk (or risky bond) models. Examples abound; Longstaff and Schwartz (1995), Briys and de Varenne (1997), Madan and Unal (2000) and Collin-Dufresne and Goldstein (2001). This paper develops a maximum likelihood estimation method for the Merton (1974) model, and the same idea is applicable to other structural credit risk models.

As pointed out in Jarrow and Turnbull (2000) among others, there are several limitations associated with the implementation of structural credit spread models. First, the asset value is an unobserved quantity. This in turn creates problems with the estimation of the various required parameters such as the drift and volatility of the asset value process and the correlation among different asset value processes. In the academic literature, three approaches have been employed to deal with the estimation problem when the underlying asset value is unobserved.

The first approach was employed in Ronn and Verma (1984) to implement the deposit insurance pricing model of Merton (1978) and in Jones, Mason and Rosenfeld (1984) to conduct an empirical study of Merton's (1974) risky bond pricing model. We will refer to this approach as the RV-JMR estimation method, which uses some observed quantities and the corresponding restrictions derived from the theoretical model to extract point estimates for the asset volatility parameter and the unobserved asset value. The RV-JMR estimation method relies on two equations: one relating the equity value to the asset value and the other relating the equity volatility to the asset volatility. The two-equation system can then be solved for the two unknown variables: the asset value and volatility. The RV-JMR estimation method has been advocated in standard finance textbook such as Saunders (1999) and Hull (2003). A three-equation extension of the RV-JMR estimation method was used in Duan, Moreau and Sealey (1995) to implement their deposit insurance model with stochastic interest rate where the third equation relates the equity duration to the asset duration.

The second estimation approach was proposed by Duan (1994, 2000). A likelihood function based on the observed equity values is derived by employing the transformed data principle in conjunction with the equity pricing equation. With the likelihood function in place, maximum likelihood estimation and statistical inference become straightforward. The maximum likelihood method was applied to Merton's (1978) deposit insurance pricing model in Duan (1994), Duan and Yu (1994), and Laeven (2002). Later, Duan and Simonato (2002) extended the method to deposit insurance pricing under stochastic interest rate. For credit risk, the estimation method has been applied to a strategic corporate bond pricing model by Ericsson and Reneby (2001).

The third approach is based on an iterated scheme known in the financial industry as the KMV method. One starts with some initial guess of the asset volatility and then use it to obtain, through the equity pricing equation, the inverted asset values corresponding to the observed time series of equity prices. The inverted asset values are then used to update the volatility estimate. The process continues until convergence is achieved. The KMV method has been used, for example, by Vassalou and Xing (2003) to obtain a default likelihood indicator in their study of equity premiums.

Theoretically, the maximum likelihood estimation method has several advantages compared to the RV-JMR estimation method. First, the maximum likelihood method provides an estimate of the drift of the unobserved asset value process under the physical probability measure. This can in turn be used to obtain an estimate of the default probability of the firm. Such an estimate is not available within the context of the RV-JMR estimation method since the theoretical equity pricing equation does not contain the drift of the asset value process under the physical probability measure. The second advantage is associated with the asymptotic properties of the maximum likelihood estimator such as consistency and asymptotic normality, which in turn allows for statistical inference to assess the quality of parameter estimates and/or perform testing on the hypotheses of interest. In contrast, consistency is unattainable with the RV-JMR estimation method because it erroneously forces a stochastic variable to be a constant (see Duan (1994)).

Interestingly, the KMV method turns out to produce the point estimate identical to the maximum likelihood estimate. However, the KMV method cannot provide the sampling error of the estimate, which is critically important for statistical inferences. In short, the KMV method can be regarded an incomplete maximum likelihood method.

This paper follows the maximum likelihood approach introduced in Duan (1994). Different from the existing works, we explicitly take into account the survivorship issue. In the credit risk setting, it is imperative for analysts to recognize the fact that a firm in operation has by definition survived thus far. Estimating a credit risk model using the sample of equity prices needs to reflect this reality, or runs the risk of biasing the estimator. This contrasts interestingly with the deposit insurance setting for which survived banks may have actually failed but continue to stay afloat due to deposit insurance. To our knowledge, this paper is the first to address the survivorship issue as well as applying the maximum likelihood method to credit risk assessment in a portfolio context. For credit risk, it is important to work with a portfolio of firms because correctly assessing correlations is critical to the task of credit risk analysis; for example, standard credit risk management methods such as CreditMetrics and KMV are known to be highly sensitive to the correlation coefficients of asset returns (see Crouhy and Mark (1998)).

We develop the likelihood function and implement the maximum likelihood estimation procedure for Merton's (1974) model. We also perform a Monte Carlo study to ascertain the method's performance for a reasonable sample size. The bias caused by neglecting survivorship is examined. Finally, we apply the analysis to real data on a portfolio of two firms.

2 Merton's Credit Risk Model

In the Merton (1974) framework, firms have a very simple capital structure. It is assumed that the ith firm is financed by equity with a market value $S_{i,t}$ at time t and a zero-coupon debt instrument with a face value of F_i maturing at time T_i. Let $V_{i,t}$ be the asset value of the ith firm and $D_{i,t}(\sigma_{V_i})$ its risky zero-coupon bond value at time t. Let us consider m firms. Naturally, the following accounting identity holds for every time point and for every firm:

$$V_{i,t} = S_{i,t} + D_{i,t}(\sigma_{V_i}), \text{ for any } t \geq 0 \text{ and } i = 1, \cdots, m. \quad (1)$$

It is further assumed that the asset values follow geometric Brownian motions; that is, on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, P)$, we have

$$dV_{i,t} = \mu_i V_{i,t} dt + \sigma_{V_i} V_{i,t}\, dW_{i,t}, \; i \in \{1, \cdots, m\} \quad (2)$$

where μ_i and σ_{V_i} are, respectively, the drift and diffusion coefficients under the physical probability measure P and the m dimensional Brownian motion $W = \{(W_{1,t}, \cdots, W_{m,t}) : t \geq 0\}$ is such that

$$\mathrm{Cov}^P[W_{i,t}, W_{j,t}] = \rho_{ij} t \text{ for any } t \geq 0. \quad (3)$$

The default-free interest rate r is assumed to be a constant. The default of the ith firm occurs at time T_i if the asset value V_{i,T_i} is below the face value F_i of the debt. Using these assumptions, formulas for the bond value and default probability can be obtained. These formulas are provided in Appendix Appendix A. The credit spread formula follows immediately from the formula for $D_{i,t}(\sigma_{V_i})$. Because the default-free interest rate is a constant, the credit spread can be written as

$$C_{i,t}(\sigma_{V_i}) = -\frac{\ln[D_{i,t}(\sigma_{V_i})/F_i]}{T_i - t} - r. \quad (4)$$

To implement this model empirically, one needs values for the input variables. Specifically, for the ith firm, the asset value $V_{i,t}$, the drift μ_i and the diffusion coefficient σ_{V_i} are unknown. The correlation coefficient between any two asset values is also unknown. In the next section, we derive the likelihood function which serves as the basis for maximum likelihood estimation.

3 The Likelihood Function

The specific idea for constructing the likelihood function is taken from Duan (1994, 2000) which treats the observed time series of equity prices as a sample of transformed data with the equity pricing equation defining the transformation. Loosely speaking, the resulting likelihood function becomes the likelihood function of the implied asset values multiplied by the Jacobian of the transformation evaluated at the implied asset values.

In general, one observes a time series of equity values for the ith firm corresponding to a known face value of debt over a sample period with a time step of length h. Denote the time series sample up to time t by $\{s_{i,0}, s_{i,h}, s_{i,2h}, \cdots, s_{i,Nh}\}$ with $t = Nh$ and $i \in \{1, \cdots, m\}$ The observation period is assumed to be the same for all firms to facilitate the estimation of the correlation coefficients. Let θ denote the vector containing all parameters associated with the m-dimensional geometric Brownian motion process; that is,

$$\theta = [\mu_1, \cdots, \mu_m, \sigma_{V_1}, \cdots, \sigma_{V_m}, \rho_{12}, \cdots, \rho_{1m}, \rho_{23}, \cdots, \rho_{2m}, \cdots]. \tag{5}$$

The function defining the critical transformation is the equity pricing equation which is $S_{i,t} = V_{i,t} - D_{i,t}(\sigma_{V_i})$. It can be easily shown that $S_{i,t}$ is an invertible function of $V_{i,t}$ for any σ_{V_i}. We denote it by $S_{i,t} = g_i(V_{i,t}; t, T, \sigma_{V_i})$.

The data sample may contain firms that have refinanced during the sample period. Assessing credit spreads is an exercise of attaching a premium to debt instruments of a firm that has survived. The fact that a firm survives needs to be incorporated into the likelihood function. Assume there are refinancing points in the sample, denoted by $n_1h < n_2h < \cdots < n_ch \leq Nh$. Refinancing occurs in the sample period whenever the zero-coupon debt of at least one firm reaches its maturity and new zero-coupon debts are issued. Survivorship is not the relevant issue in the case of Merton's (1974) model if the zero-coupon debt did not become due during the sample period because there would be no possibility of defaulting on the debt obligations. If the data sample contains points of refinancing, survivorship adjustment may be important. For purposes of addressing survivorship, we let $\mathcal{D}$ be the event "no default for the entire sample period". The relevant log-likelihood function is given in the following theorem.

Theorem 1 *Assume that refinancing took place but there was no default in the sample period. The log-likelihood function corresponding to the stock price sample is*

$$\begin{aligned} & L(\mathbf{s}_0, \mathbf{s}_h, \mathbf{s}_{2h}, \cdots, \mathbf{s}_{Nh}; \theta) \\ &= -\frac{mN}{2}\ln(2\pi) - \frac{N}{2}\ln(|\det \mathbf{\Sigma}|) - \frac{1}{2}\sum_{k=1kh}^{N} \mathbf{w}_{kh}^{*\prime}\Sigma^{-1}\mathbf{w}_{kh}^{*} - \sum_{k=1}^{N}\sum_{i=1}^{m}\ln v_{i,kh}^{*} \\ &- \sum_{k=1}^{N}\sum_{i=1}^{m}\ln\Phi\left(d\left(v_{i,kh}^{*}, kh, \sigma_{V_i}, F_{i,kh}, T_{i,kh}\right)\right) + \sum_{j=1}^{c}\ln \mathbf{1}_{\mathbf{v}_{n_jh}^{*}\in\mathcal{V}_j} - \ln P(\mathcal{D};\theta). \end{aligned} \tag{6}$$

where $\mathbf{v}_{kh}^{*}$ *is the* m*-dimensional vector of* $v_{i,kh}^{*}$ *(for* $i = 1, \cdots, m$*) with* $v_{i,kh}^{*} = g_i^{-1}(s_{i,kh}; kh, \sigma_{V_i}, F_{i,kh}, T_{i,kh})$ *being the asset value implied by the equity value,* $F_{i,t}$ *and* $T_{i,t}$ *are the face value and maturity date of the debt for the* i*th firm at time* t*,* $\Phi(\cdot)$ *is the standard normal distribution function,* $d(\cdot,\cdot,\cdot)$ *is defined in equation (10) of Appendix Appendix A,* $\mathbf{w}_{kh}^{*}$ *is an* m*-dimensional column vector defined as*

$$\mathbf{w}_{kh}^{*} = \left(\ln v_{i,kh}^{*} - \ln v_{i,(k-1)h}^{*} - \left(\mu_i - \frac{1}{2}\sigma_{V_i}^2\right)h\right)_{m\times 1}$$

and

$$\mathbf{\Sigma} = \left(\sigma_{V_i}\sigma_{Vj}\rho_{ij}h\right)_{i,j=1,\cdots,m} = \begin{bmatrix} \sigma^2_{V_1} & \cdots & \sigma_{V_1}\sigma_{Vm}\rho_{1m} \\ \vdots & \ddots & \vdots \\ \sigma_{V_1}\sigma_{Vm}\rho_{1m} & \cdots & \sigma^2_{V_m} \end{bmatrix} h$$

with $\det \mathbf{\Sigma}$ *being the determinant of* $\mathbf{\Sigma}$*. Moreover,* $\mathcal{V}_j = \bigcap_{i=1}^{m} \left\{ v_{i,n_j h} > F_{i,n_j h} \mathbf{1}_{T_{i,n_j h} = n_j h} \right\} \subseteq \mathbb{R}^m$ *is the subset of the sample space at time* $n_j h$ *corresponding to no default and* $P(\mathcal{D};\theta)$ *is the survival probability according to equation (13).*

The proof for this theorem is given in Appendix Appendix B. The first four terms on the right hand side of equation (6) constitute the log-likelihood function if the asset values were observed and no refinancing took place in the sample period. The fifth term in equation (6) corresponds to the Jacobian that accounts for the transformation from the observed equity values to the implied asset values. It is important to note that $\mathbf{v}^*_{kh}$ depends on the parameters of the model. If it were not, $\sum_{k=1}^{N}\sum_{i=1}^{m} \ln v^*_{i,kh}$ could be dropped from the likelihood function. This would be the case if the asset value could be directly observed. Finally, the survival probability $P(\mathcal{D};\theta)$ reflects the fact that the log-likelihood function is conditional on no default in the sample period whereas the term $\sum_{j=1}^{c} \ln \mathbf{1}_{\mathbf{v}^*_{n_j h} \in \mathcal{V}_j}$ simply assigns a zero likelihood (or log-likelihood equal to minus infinity) to the parameter value at which some implied asset value suggesting a default.

If the firms in the sample never faced refinancing during the sample period, survivorship is not an issue. In terms of the above theorem, both $\ln P(\mathcal{D};\theta) = 0$ and $\sum_{j=1}^{c} \ln \mathbf{1}_{\mathbf{v}^*_{n_j h} \in \mathcal{V}_j} = 0$ because the survival probability equals 1 and $\mathbf{1}_{\mathbf{v}^*_{n_j h} \in \mathcal{V}_j} = 1$ when there is no refinancing. This reduced case actually amounts to a straightforward generalization of Duan (1994, 2000) to a portfolio context. With the likelihood function in place, one can conduct the maximum likelihood estimation and statistical inference.

4 A Practical Estimation Procedure

Although directly maximizing the log-likelihood function seems a natural approach, it is actually not practical when many firms are in the data sample. The number of parameters involved increases rapidly and quickly becomes unmanageable. We therefore adopt the following practical three-step estimation procedure, knowing fully well that the true optimum may not be obtained this way:

- **Step 1:** Estimate the Merton (1974) model for each firm separately. For firm i, estimates $\hat{\mu}_i$ and $\hat{\sigma}_{V_i}$ are obtained using the log-likelihood function in (6) by imposing $m = 1$. The Monte Carlo study (see Section 5) indicates that the following standard asymptotic results are applicable to the typical sample size in applications.

  Statement: *The parameter estimates* $(\hat{\mu}_i, \hat{\sigma}_{V_i})$ *are asymptotically normally distributed around the true parameter values with the covariance matrix being approxi-*

mated by $\hat{\mathbf{F}}_i^{-1}$ *where*

$$\hat{\mathbf{F}}_i = \begin{pmatrix} -\frac{1}{N}\frac{\partial^2 L\left(s_0,\ldots,s_{Nh};\hat{\mu}_i,\hat{\sigma}_{V_i}\right)}{\partial\mu_i^2} & -\frac{1}{N}\frac{\partial^2 L\left(s_0,\ldots,s_{Nh};\hat{\mu}_i,\hat{\sigma}_{V_i}\right)}{\partial\mu_i\partial\sigma_{V_i}} \\ -\frac{1}{N}\frac{\partial^2 L\left(s_0,\ldots,s_{Nh};\hat{\mu}_i,\hat{\sigma}_{V_i}\right)}{\partial\mu_i\partial\sigma_{V_i}} & -\frac{1}{N}\frac{\partial^2 L\left(s_0,\ldots,s_{Nh};\hat{\mu}_i,\hat{\sigma}_{V_i}\right)}{\partial^2\sigma_{V_i}} \end{pmatrix}.$$

- **Step 2:** Compute $V_{i,t}(\hat{\sigma}_{V_i}), C_{i,t}(\hat{\sigma}_{V_i})$ and $P_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})$ which represent, respectively, the point estimates for the asset value, credit spread and default probability. Because these quantities are continuously differentiable function of the parameter estimates, their distribution can be approximated by a normal distribution using the first derivatives of these quantities with respect to the parameter values and the covariance matrix estimate $\hat{\mathbf{F}}_i^{-1}$.(see Lo (1986) or Rao (1973), page 385). Appendix Appendix C provide the details of these computations.
- **Step 3:** Compute the sample correlation coefficient between $\ln\left(V_{i,kh}(\hat{\sigma}_{V_i}) / V_{i,(k-1)h}(\hat{\sigma}_{V_i})\right)$ and $\ln\left(V_{j,kh}(\hat{\sigma}_{V_i})/V_{j,(k-1)h}(\hat{\sigma}_{V_i})\right)$ and use it as the estimate for ρ_{ij}. The estimated correlation coefficient is expected to distribute normally around its true value with a variance taken from the corresponding diagonal entry of $\hat{\mathbf{F}}_{ij}^{-1}$ where $\hat{\mathbf{F}}_{ij} = -\frac{1}{N}\left.\frac{\partial^2 L(\bullet)}{\partial\theta_{ij}(k)\partial\theta_{ij}(l)}\right|_{\theta_{ij}=\hat{\theta}_{ij}}$. $\theta_{ij} = \left[\mu_i, \mu_j, \sigma_{V_i}, \sigma_{V_j}, \rho_{ij}\right]$, $\theta_{ij}(k)$ is the kth element of θ_{ij}, $\hat{\theta}_{ij}$ is the estimate for θ_{ij} and $L(\cdot)$ is the joint log-likelihood function of the data sample for the ith and jth firms. Note that the first four entries of $\hat{\theta}_{ij}$ are taken from the individual estimations for the ith and jth firms in Step 1 whereas the correlation coefficient estimate is obtained in this step. For any quantity that is a function of θ_{ij}, the variance of its distribution can be obtained using the whole matrix $\hat{\mathbf{F}}_{ij}^{-1}$ in a way similar to those in Step 2. If one is interested in any quantity that is a function of the parameters for more than two firms, the dimension of $\hat{\mathbf{F}}_{ij}$ can expanded to accommodate the new requirement.

The three-step estimation procedure can be completed fairly quickly; for example, on a standard desktop computer, the completion for two firms usually takes approximately 10 seconds. Parameter estimation for a large portfolio of firms is thus feasible with the three-step estimation procedure. The numerical optimization routine used here is the quadratic hill–climbing algorithm of Goldfeld, Quandt and Trotter (1966) with a convergence criterion based on the absolute values of the changes in parameter values and functional values between successive iterations. We consider convergence achieved when both of these changes are smaller than 10^{-5}. The three-step estimation procedure uses several simplifications. We need to ascertain its performance by a Monte Carlo study. Such a study is carried out in the next section.

5 A Monte Carlo Study

In order to assess the quality of the maximum likelihood procedure, we examine how well the asymptotic normal distribution suggested by the theory approximates the actual distribution for a reasonable sample size. In other words, we verify whether the parameter estimates for a sample size N is well approximated by the distribution given in the preceding section. Similarly, we check the distributions for the asset value $V_{i,t}(\hat{\theta}_N)$, credit spread $C_{i,t}(\hat{\theta}_N)$ and default probability $P_{i,t}(\hat{\theta}_N)$. Two Monte Carlo experiments are conducted. The first experiment focuses on the multivariate aspect of the proposed method whereas the second concentrates on the survivorship issue.

5.1 Description of the Two Experiments

In the first experiment, we consider two firms and simulate the data on a daily basis as follows:

- **Step 1:** Let $V_{i,kh}$ for $i = \{1, 2\}$ and $k = \{1, \cdots, N\}$ denote the simulated asset values in accordance with

$$V_{1,(k+1)h} = V_{1,kh} \exp\left(\mu_1 h - \frac{1}{2}\sigma_{V_1}^2 h + \sigma_{V_1}\sqrt{h}\epsilon_{1,k}\right) \tag{7}$$

$$V_{2,(k+1)h} = V_{2,kh} \exp\left(\mu_2 h - \frac{1}{2}\sigma_{V_2}^2 h + \sigma_{V_2}\sqrt{h}\epsilon_{2,k}\right) \tag{8}$$

  where $\left\{(\epsilon_{1,k}, \epsilon_{2,k})' : k \in \{1, \cdots, N\}\right\}$ is a sequence of independent and identically distributed vectors of standard normal random variables with a correlation coefficient of ρ_{12}. To be consistent with daily data, we set $h = 1/250$. The specific parameter values used in the simulation are given in the table.

- **Step 2:** Use the simulated asset values to compute equity values by the equity pricing equation: $S_{i,kh} = V_{i,kh} - D_{i,kh}(\sigma_{V_i})$ with the expression of $D_{i,kh}(\sigma_{V_i})$ given in Appendix Appendix A.

For each simulated data sample, we conduct maximum likelihood estimation and compute the point estimates and their associated variances using the simulated equity prices. We repeat the simulation run $5,000$ times to obtain the Monte Carlo estimates for the relevant quantities. Both firms have a debt maturity beyond the sample period. In other words, there will be no refinancing in the sample period and hence no need to consider the survivorship issue. Specifically, we have $T = 3, t = 2, h = 1/250$ and $N = 500$. This means that we simulate data daily for two years and at the beginning of the simulated sample, both firms have zero-coupon debts with three-year maturity.

In the second Monte Carlo experiment, we focus on the survivorship issue and consider one firm that faced refinancing. The data is also simulated on a daily basis in a way similar to the first experiment but with a major difference. If the zero-coupon debt of the firm comes due at any time point, its asset value may be less than the debt service requirement, i.e., the equity value becomes zero. When that occurs, the firm is deemed insolvent and that particular sample is discarded. In other words, we only keep a sample of equity prices if the firm survives the entire period.

In the second Monte Carlo experiment, we set $t = 2.5, h = 1/250$ and $N = 625$. The firm refinances twice with one-year zero coupon debt. More specifically, it has a zero-coupon debt with 1.0 year to maturity at the beginning of the data period and refinances twice in years 1 and 2. One needs to be particularly careful in dealing with refinancing. At a refinancing point, the market value of the new zero-coupon debt is lower than its face value. If one strives to maintain the same face value, the amount of the newly raised debt will be less than the payment required to retire the old debt. We thus adopt the following procedure at the refinancing points. Given the asset value at the refinancing point, we first find the face value of new debt that equates the market value of new debt with the face value of the old debt. To make sure that the ratio of face value of debt to asset value is the same at each refinancing point, we then adjust upward or downward the simulated asset value at this point. This adjustment can be regarded as a recapitalization to keep the target debt-to-asset-value ratio. Recapitalization creates a jump in the asset value and thus renders the one-period return to the refinancing point unusable. Thus, we drop from the sample such returns from the likelihood function.

5.2 Simulation Results

Table 1 reports the simulation results for the first experiment when survival is guaranteed. The following parameter values are used: $V_{i,0} = 10000, F_i = 9000, \mu_i = 0.1, \sigma_V = 0.3, r = 0.05$ and $\rho = 0.5$. These parameter values represent a firm with a high debt-to-asset-value ratio of $F_i/V_{i,0} = 0.9$ and a high noise to signal ratio of $\sigma_{V_i}/\mu_i = 3$. This case is interesting because high leverage represents a firm that is more likely to default whereas the high noise to signal ratio represents a case that is expected to more difficult to obtain good estimates. The results reveal that for all parameters and the inferred variables, the maximum likelihood estimators are unbiased (except in the case of the default probability). For the default probability, there appears to be an upward bias but the median is correctly located, indicating the presence of skewness. The coverage rates[1] indicate that the asymptotic distribution approximates well the small sample distribution.

We have also reported in Table 1 the results using the RV-JMR estimation method. A description of this method is given in Appendix Appendix D. It is clear from this table that the RV-JMR estimation produces poor estimates. The estimates for volatility are significantly biased downward. For asset value, they are biased upward substantially. For the inference on biases, the standard deviations reported in the tables should be divided by a factor of $\sqrt{5000}$ to reflect the fact that 5000 simulation runs are used to produce the

[1]The coverage rate represents the percentage of the parameter estimates for which the true parameter value is contained in the α confidence interval implied by the asymptotic distribution.

means. Additional simulation results with different parameter values (not reported here) show that the performance of the RV-JMR estimation method is related to the noise-to-signal ratio of the firm. For high noise-to-signal firms, the biases are very pronounced whereas for low noise-to-signal firms, the estimates may be regarded as acceptable. Also worth noting is that the RV-JMR estimation method cannot yield an estimate for the drift coefficient and consequently it cannot provide an estimate for the default probability. On the efficiency side, the maximum likelihood estimators always show a smaller standard deviation when compared to the RV-JMR estimators. For all parameter estimates and inferred variables, except for the correlation coefficient, the standard deviations of the RV-JMR estimators are approximately 10 times larger than their maximum likelihood counterparts.

The correlation estimate based on the RV-JMR estimation method is very good in terms of bias and standard deviation. This indicates that, in the context of Merton's (1974) model, the correlation between the stock price return is a very good estimate of the true correlation between the asset returns. This is perhaps not too surprising because the cross variation process between $\ln S_i$ and $\ln S_j$ is a function of the correlation between the two Wiener processes that drive the asset values. Equity correlation amounts to normalizing this expression by the two sample standard deviations of the stock returns, and thus yields an estimate that is close to the asset return correlation.

Table 2 reports the simulation results for the second experiment where survival is not guaranteed. The parameter values are identical to those used in the previous experiment, i.e., we consider a firm with a high debt-to-asset-value ratio and a high noise to signal ratio. In the two panels, the parameter values are obtained by ignoring the last two terms of equation (6) and incorporating the last two terms, respectively, to assess the impact of survivorship. The results reveal that ignoring survivorship does not really distort the estimates except for μ and the default probability.

Ignoring survivorship has a non-trivial effect on the estimate of μ because both the mean and median become twice the value of the true parameter value. This is not surprising because only samples corresponding to the survived firm are used in estimation. In other words, ignoring survivorship will naturally bias upward the estimate for μ. The default probability, which critically depends on μ, is affected as a result. In fact, one should expect a downward bias in the default probability estimate, which is confirmed by the results reported in the first panel. The fact that the volatility estimate does not exhibit any bias is not surprising. It is well known that the diffusion model's rapid movement is a sole manifestation of the diffusion term of the process. Every sample path, whether it drifts upward or downward, is equally informative about the volatility parameter.

The second panel of Table 2 provides the results when survivorship has been properly considered. The estimate for μ becomes much closer to the true value viewing from either the mean or median value. Clearly, properly incorporating survivorship has a meaningful effect on the drift parameter. Unfortunately, the coverage rates do not improve, suggesting that the asymptotic distribution does not adequately match the finite sample distribution as far as μ is concerned. We have also checked to see whether the coverage rates improve with the sample size. Indeed, it is the case as expected, but the improvement is rather slow.

6 Empirical Analysis

In this section, we implement Merton's (1974) model using real data. Although this model assumes a zero-coupon debt, most corporations have much more complex liability structures. Liabilities with different properties such as maturity, seniority and coupon rate must be aggregated into one quantity to implement the model. Obviously, there is no clear-cut solution to these problems. One possible approach to determining debt maturity is to find a "theoretical" zero-coupon bond that has the same duration as the aggregated debt. Doing so will, however, fundamentally change the pattern of cash flows. Another way of addressing this issue is to argue that the annual report on profits and losses is perceived by equity holders as the maturity date of their option, which then leads to a pseudo debt maturity of one year. At the time of the public reporting of the annual profits and losses, debt holders may decide to take control of the firm in case of insolvency.

Determining the amount of debt for Merton's (1994) model is also not an obvious matter. The simplest approach would be to set the face value of debt equal to the total amount of short- and long-term liabilities. However, as argued in Crouhy et al. (2000), the probability of the asset value falling below the total face value of liabilities may not be an accurate measure of the actual default probability. Default tends to occur when the asset value reaches a level somewhere between the face value of total liabilities and the face value of the short-term debt. Moreover, there are unknown undrawn commitments (lines of credit) which can be used in case of financial distress.

Two companies from the Canadian retailing sector are examined for years 1999-2001: Hudson's Bay Company and Sears Canada Inc. Their stock prices are taken from DataStream and the information regarding the long- and short-term debts is extracted from the Financial Post Historical Reports and Reuters. The short-term debt is defined as the "Current Liabilities" reported yearly in the consolidated balance sheet. The long-term debt is the account "Long-term Debt, net" which represents the long-term debt net of long-term debt maturing during the current year. For Hudson's Bay, three major new issues of debt were found while five new issues were recorded for Sears. These new debt issues are considered as refinancing points.

The first panel of Table 3 reports the results when maturity is set equal to the duration of the aggregated debt and the face value of debt is set equal to the sum of the short-term and long-term liabilities. For the time period examined in the table, the average duration was 2.25 years for Hudson's Bay and 1.65 years for Sears. The estimates of the drift coefficients are close to zero for Hudson's Bay whereas the estimates are large and positive for Sears. Most of these estimates are, however, statistically insignificant from zero. The estimates for the asset return volatility are similar in magnitude and significantly different from zero. The estimated credit spreads are small and far less than the observed spreads for their bonds. For example, in May 1999, the credit spread for Hudson's Bay was in the order of 200 basis points for a newly issued five year bond while the estimated spread is in the order of 20 basis points. For Sears in the beginning of year 2000, the estimated spread is around 1.6×10^{-8} while the spread on a newly issued 5 year coupon bond was around 60 basis points.

The estimated default probabilities are much higher for Hudson's Bay with a probability of 8.5% for 1999 whereas the estimated default probability is 1.6×10^{-5} for Sears. Finally, the estimated correlation between the asset values of these two firms is close to 0.2 for each year of three years studied, and they are statistically significant.

The second and third panel of the table provide the estimation results when the debt maturity is set to 5 and 10 years, respectively. When maturity is increased, the estimated drift becomes smaller for Hudson's Bay but larger for Sears. The estimated asset volatility increases for both companies, however. The estimated spreads and default probabilities for Hudson's Bay increase to the more sensible levels. The estimates for Sears, however, remain to be low by a common sense yardstick. The correlation estimates remains stable with respect to this change in the maturity assumption.

7 Conclusion

We have developed a maximum likelihood estimation method for Merton's (1974) credit spread model in a multiple-firm setting. Our method also explicitly takes into account the survivorship issue inherent to the credit analysis problem. Through two simulation studies, we show that maximum likelihood method performs very well for all parameters/variables of interest when refinancing does not occur in the sample period but the quality of the asymptotic inference related to the drift parameter drops when refinancing enters into the picture even with the proper consideration of survivorship. This paper also reveals that the RV-JMR estimation method produces very poor estimates for the asset value and volatility parameter. The simulation results, however, suggest that approximating the correlation between asset returns with the correlation between stock returns seem to produce a good correlation estimate.

Finally, the results of our empirical analysis on two firms show that the parameter estimates are highly sensitive to changes in the assumptions necessary for performing the model estimation. Although Merton's (1974) stylized assumptions are not incompatible with reality, one hopes that these simplification assumptions do not distort results too much. The empirical results suggest, however, that these assumptions have non-trivial effects.

Table 1: Simulation results for two firms with no refinancing

Maximum Likelihood

	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\sigma}_{V_1}$	$\hat{\sigma}_{V_2}$	$\hat{\rho}_{12}$	$\hat{V}_{1t_0}$-V_{1t_0}	$\hat{V}_{2t_0}$-V_{2t_0}	$\hat{C}_{1t_0}$-C_{1t_0}	$\hat{C}_{2t_0}$-C_{2t_0}	$\hat{P}_{1t_0}$-P_{1t_0}	$\hat{P}_{2t_0}$-P_{2t_0}
True	0.100	0.100	0.300	0.300	0.500	0.000	0.000	0.000	0.000	0.000	0.000
Mean	0.101	0.095	0.300	0.300	0.500	-0.784	1.853	0.000	-0.000	0.048	0.049
Median	0.102	0.098	0.299	0.299	0.501	0.155	0.118	-0.000	-0.000	-0.000	0.001
Std	0.209	0.208	0.018	0.018	0.033	110.522	116.660	0.020	0.021	0.080	0.080
25 % cvr	0.258	0.251	0.250	0.255	0.244	0.252	0.255	0.252	0.255	0.260	0.259
50 % cvr	0.514	0.516	0.506	0.504	0.497	0.506	0.509	0.507	0.509	0.512	0.512
75 % cvr	0.751	0.756	0.754	0.749	0.757	0.752	0.750	0.753	0.750	0.747	0.759
95 % cvr	0.951	0.955	0.947	0.942	0.953	0.934	0.933	0.934	0.932	0.952	0.955

RV-JMR

	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\sigma}_{V_1}$	$\hat{\sigma}_{V_2}$	$\hat{\rho}_{12}$	$\hat{V}_{1t_0}$-V_{1t_0}	$\hat{V}_{2t_0}$-V_{2t_0}	$\hat{C}_{1t_0}$-C_{1t_0}	$\hat{C}_{2t_0}$-C_{2t_0}	$\hat{P}_{1t_0}$-P_{1t_0}	$\hat{P}_{2t_0}$-P_{2t_0}
True	–	–	0.300	0.300	0.500	0.000	0.000	0.000	0.000	–	–
Mean	–	–	0.230	0.228	0.492	612.955	632.409	-0.086	0.023	–	–
Median	–	–	0.243	0.240	0.493	134.507	141.436	-0.016	-0.122	–	–
Std	–	–	0.119	0.120	0.036	1003.317	1022.175	0.153	0.755	–	–

True is the parameter value used in the Monte Carlo simulation; **Mean**, **Median** and **Std** are the sample statistics computed with the 5000 estimated parameter values; **cvr** is the coverage rate defined as the percentage of the 5000 parameter estimates for which the true parameter value is contained in the α confidence interval implied by the asymptotic distribution; $V_{0,1} = 10000$, $V_{0,2} = 10000$, $F_1 = 9000$, $F_2 = 9000$, $T = 3.00$, $t_0 = 2.00$, $r = 0.05$ and $N = 500$.

Table 2: Simulation results for one firm with refinancing

Incorrect Maximum Likelihood

	$\hat{\mu}$	$\hat{\sigma}_V$	$\hat{V}_{t_0}$-V_{t_0}	$\hat{C}_{t_0}$-C_{t_0}	$\hat{P}_{t_0}$-P_{t_0}
True	0.100	0.300	0.000	0.000	0.000
Mean	0.205	0.299	1.510	-0.000	-0.033
Median	0.201	0.299	0.749	-0.000	-0.025
Std	0.151	0.013	53.946	0.013	0.074
25 % cvr	0.231	0.234	0.234	0.234	0.226
50 % cvr	0.442	0.485	0.482	0.483	0.443
75 % cvr	0.677	0.739	0.740	0.740	0.682
95 % cvr	0.908	0.940	0.931	0.931	0.913

Maximum Likelihood

	$\hat{\mu}$	$\hat{\sigma}_V$	$\hat{V}_{t_0}$-V_{t_0}	$\hat{C}_{t_0}$-C_{t_0}	$\hat{P}_{t_0}$-P_{t_0}
True	0.100	0.300	0.000	0.000	0.000
Mean	0.080	0.300	-0.721	0.000	0.037
Median	0.108	0.299	0.286	-0.000	-0.001
Std	0.241	0.013	55.750	0.013	0.124
25 % cvr	0.191	0.236	0.237	0.237	0.193
50 % cvr	0.369	0.486	0.485	0.485	0.372
75 % cvr	0.624	0.739	0.742	0.742	0.631
95 % cvr	0.904	0.937	0.929	0.930	0.911

True is the parameter value used in the Monte Carlo simulation; **Mean**, **Median** and **Std** are the sample statistics computed with the 5000 estimated parameter values; **cvr** is the coverage rate defined as the percentage of the 5000 parameter estimates for which the true parameter value is contained in the α confidence interval implied by the asymptotic distribution; $V_0 = 10000$, $F = 9000$, $r = 0.05$ and $N = 625$.

Table 3: Maximum likelihood estimation results on two firms

			Hudson's Bay					**Sears Canada**			
Year	$\hat{\mu}$	$\hat{\sigma}$	$\hat{V}_{t_0}$ \$000's	$\hat{C}_{t_0}$	$\hat{P}_{t_0}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{V}_{t_0}$ \$000's	$\hat{C}_{t_0}$	$\hat{P}_{t_0}$	$\hat{\rho}$
					Maturity equal to duration of total debt						
1999	0.007	0.195	3445267	2.219e-003	8.463e-002	0.190	0.186	3486694	2.910e-005	1.579e-005	0.211
	(0.014)	(0.006)	(1917)	(4.240e-004)	(1.928e-002)	(0.132)	(0.006)	(53)	(1.967e-005)	(6.481e-005)	(0.045)
2000	0.005	0.200	3638239	4.520e-003	1.403e-001	0.285	0.207	6255902	1.601e-008	5.247e-010	0.206
	(0.011)	(0.007)	(3347)	(7.800e-004)	(2.048e-002)	(0.135)	(0.006)	(0)	(5.968e-009)	(3.595e-009)	(0.043)
2001	-0.094	0.231	2680738	1.348e-003	6.680e-002	0.097	0.252	3842943	4.925e-004	4.532e-003	0.171
	(0.244)	(0.008)	(570)	(3.688e-004)	(1.340e-001)	(0.178)	(0.008)	(234)	(7.548e-005)	(1.325e-002)	(0.047)
					Maturity equal to 5 years						
1999	0.003	0.209	3153189	6.514e-003	2.986e-001	0.205	0.199	3252416	1.223e-003	2.070e-004	0.203
	(0.014)	(0.008)	(7812)	(9.110e-004)	(5.474e-002)	(0.150)	(0.007)	(2051)	(3.065e-004)	(1.315e-003)	(0.044)
2000	-0.038	0.230	3127037	1.328e-002	6.185e-001	0.289	0.219	5851614	2.439e-004	1.115e-006	0.207
	(0.099)	(0.009)	(15271)	(1.653e-003)	(3.659e-001)	(0.143)	(0.007)	(827)	(1.027e-004)	(8.077e-006)	(0.043)
2001	-0.119	0.282	2260213	1.796e-002	7.909e-001	0.105	0.267	3590032	3.845e-003	3.600e-002	0.170
	(0.202)	(0.012)	(13612)	(2.279e-003)	(4.608e-001)	(0.333)	(0.009)	(4978)	(7.625e-004)	(2.203e-001)	(0.048)
					Maturity equal to 10 years						
1999	-0.019	0.240	2684806	1.155e-002	6.524e-001	0.223	0.221	2938729	3.716e-003	1.491e-004	0.203
	(0.020)	(0.009)	(16408)	(1.316e-003)	(9.743e-002)	(0.278)	(0.008)	(6969)	(6.800e-004)	(2.287e-003)	(0.044)
2000	-0.136	0.279	2477402	2.072e-002	9.800e-001	0.305	0.239	5428619	1.323e-003	5.551e-007	0.191
	(0.153)	(0.012)	(26277)	(2.193e-003)	(8.356e-002)	(0.175)	(0.008)	(5814)	(4.892e-004)	(6.523e-006)	(0.042)
2001	-0.162	0.312	1865562	2.273e-002	9.794e-001	0.109	0.289	3230527	7.781e-003	7.847e-002	0.155
	(0.298)	(0.013)	(19433)	(2.429e-003)	(1.493e-001)	(0.130)	(0.010)	(10953)	(1.158e-003)	(2.094e-001)	(0.047)

The maximum likelihood estimates at the beginning of Year are computed using the previous two years of daily time series data; $r =$ is set to the yield to maturity of a representative one year Canadian government bond; For each firm, the face value of debt is set equal to the long term debt plus the short term debt obtained from the yearly annual report.

Appendix A Formulas for debt and default probability in Merton's (1974) model

In Merton (1974), $D_{i,T_i}(\sigma_{V_i}) = \min\{V_{i,T_i}, F_i\}$. For valuation, it is well-known that we can use the risk-neutral asset price dynamic to evaluate the discounted expected payout, where the risk-neutral dynamic has the risk-free rate as the drift term but the same diffusion term. Consequently, the bond value at time t is

$$D_{i,t}(\sigma_{V_i}) = F_i e^{-r(T_i-t)} \left(\frac{V_{i,t}}{F_i e^{-r(T_i-t)}} \Phi\left(-d(V_{i,t}, t, \sigma_{V_i})\right) + \Phi\left(d(V_{i,t}, t, \sigma_{V_i}) - \sigma_{V_i}\sqrt{T_i - t}\right)\right) \tag{9}$$

where $\Phi(\bullet)$ is the standard normal distribution function and

$$d(V_{i,t}, t, \sigma_{V_i}) = \frac{\ln(V_{i,t}) - \ln(F_i) + \left(r + \frac{1}{2}\sigma_{V_i}^2\right)(T_i - t)}{\sigma_{V_i}\sqrt{T_i - t}}. \tag{10}$$

Furthermore, Merton's model implies the following default probability under measure P at time t:

$$P_{i,t}(\mu_i, \sigma_{V_i}) = P\left[V_i(T_i) < F_i \,|\mathcal{F}_t\right] = \Phi\left(\frac{\ln(F_i) - \ln(V_{i,t}) - \left(\mu_i - \frac{1}{2}\sigma_{V_i}^2\right)(T_i - t)}{\sigma_{V_i}\sqrt{T_i - t}}\right). \tag{11}$$

where $P[\bullet\,|\mathcal{F}_t]$ denotes for the conditional probability taken at time t under the measure P. The joint probability of default for several firms can be expressed using the multivariate normal cumulative distribution function $N_{\mathbf{0},\rho} : \mathbb{R}^m \to [0,1]$ with mean $0_{m\times 1}$ and covariance matrix $\rho \equiv (\rho_{ij})_{i,j\in\{1,\cdots,m\}}$ and relying on the following quantity:

$$P\left[V_{1,T_i} < \alpha_1, \cdots, V_{m,T_i} < \alpha_m \,|\mathcal{F}_t\right] = \mathbf{N}_{\mathbf{0},\rho}(\beta_1, \cdots, \beta_m) \tag{12}$$

where

$$\beta_i = \frac{\ln(\alpha_i) - \ln(V_{i,t}) - \left(\mu_i - \frac{1}{2}\sigma_{V_i}^2\right)(T_i - t)}{\sigma_{V_i}\sqrt{T_i - t}}.$$

This gives rise to the joint default probability of the firms $i_1, \cdots, i_k$ where $k \le m$ by setting

$$\alpha_i = F_i \text{ for any } i \in \{i_1, \cdots, i_k\} \text{ and } \alpha_i \to \infty \text{ for all } i \notin \{i_1, \cdots, i_k\}$$

in equation (12).

Appendix B The likelihood function for Merton's (1974) model with refinancing during the sample period

In this section we derive the likelihood function for the multivariate version of Merton's (1974) model with refinancing during the sampling period. More precisely, at times $n_1 h <$

$n_2h < ... < n_ch \leq Nh$, the zero-coupon debt of at least one firm reaches its maturity and is refinanced with the new zero-coupon debt. Let $\mathcal{D}_j$ be the event "no default at time n_jh" and $\mathcal{D}$ is the event "no default for the whole sample period". That is,

$$\mathcal{D}_j = \bigcap_{i=1}^{m} \left\{ V_{i,n_jh} > F_{i,t}\mathbf{1}_{T_{i,t}=n_jh} \right\}, \; j \in \{1, 2, \cdots, c\},$$
$$\mathcal{D} = \bigcap_{j=1}^{c} \mathcal{D}_j$$

Recall that $F_{i,t}$ and $T_{i,t}$ are the face value and maturity date of the debt for the ith firm at time t. $\mathbf{1}_A$ is the indicator function that is equal one if the event A is realized and zero otherwise. If the firm i, for example, have a zero coupon debt with a face value of F_i and a maturity date of t_jh and refinances at time t_jh with a zero coupon debt with a face value of F_i^* and a maturity date of $T_i > Nh$, then $F_{i,t} = F_i\mathbf{1}_{t\leq t_jh} + F_i^*\mathbf{1}_{t>t_jh}$ and $T_{i,t} = t_jh\mathbf{1}_{t\leq t_jh} + T_i\mathbf{1}_{t>t_jh}$.

The conditional density function is

$$f_{\mathbf{V}_0,\mathbf{V}_h,\mathbf{V}_{2h},\ldots,\mathbf{V}_{Nh}|\mathcal{D}} \left(\mathbf{v}_0, \mathbf{v}_h, \mathbf{v}_{2h}, \cdots, \mathbf{v}_{Nh}; \theta\right) = \frac{f_{\mathbf{V}_0,\mathbf{V}_h,\mathbf{V}_{2h},\ldots,\mathbf{V}_{Nh},\mathcal{D}} \left(\mathbf{v}_0, \mathbf{v}_h, \mathbf{v}_{2h}, \cdots, \mathbf{v}_{Nh}; \theta\right)}{P\left(\mathcal{D}; \theta\right)}$$

provided that $P(\mathcal{D}; \theta) > 0$. By the Markov property of the geometrical Brownian motion, the survival probability can be expressed in terms of the multivariate normal cumulative distribution function $N_{\mathbf{0},\rho} : \mathbb{R}^m \to [0,1]$ with mean $0_{m\times 1}$ and covariance matrix $\rho \equiv (\rho_{ij})_{i,j\in\{1,\cdots,m\}}$:

$$\begin{aligned} P\left(\mathcal{D}; \theta\right) &= \prod_{j=1}^{c} P\left(V_{1,n_jh} > F_{1,t}\mathbf{1}_{T_{1,n_jh}=n_jh}, \cdots, V_{m,n_jh} > F_{m,t}\mathbf{1}_{T_{m,n_jh}=n_jh} \left| \mathcal{F}_{n_{j-1}h} \right.\right) \\ &= \prod_{j=1}^{c} N_{\mathbf{0},\rho}\left(\beta_{1,j}^*, \cdots, \beta_{m,j}^*\right) \end{aligned} \tag{13}$$

where

$$\beta_{i,j}^* = -\left(\frac{\ln\left(F_{i,n_jh}\right) - \ln\left(v_{i,n_{j-1}h}\right) - \left(\mu_i - \frac{1}{2}\sigma_{V_i}^2\right)\left(n_j - n_{j-1}\right)h}{\sigma_{V_i}\sqrt{\left(n_j - n_{j-1}\right)h}} \mathbf{1}_{T_{i,n_jh}=n_jh} - \infty \mathbf{1}_{T_{i,n_jh}\neq n_jh} \right).$$

Let $\mathcal{V}_j = \bigcap_{i=1}^{m} \left\{ v_{i,n_j h} > F_{i,n_j h} \mathbf{1}_{T_{i,n_j h} = n_j h} \right\} \subseteq \mathbb{R}^m$ be the subset of the sample space at time $n_j h$ corresponding to no default. The joint density function of the firm values and the survival event is

$$
\begin{aligned}
& f_{\mathbf{V}_0, \mathbf{V}_h, \mathbf{V}_{2h}, \ldots, \mathbf{V}_{Nh}, \mathcal{D}} \left(\mathbf{v}_0, \mathbf{v}_h, \mathbf{v}_{2h}, \cdots, \mathbf{v}_{Nh}; \theta \right) \\
&= \prod_{j=1}^{c} \left(\prod_{k=n_{j-1}+1}^{n_j} f_{\mathbf{V}_{kh} | \mathbf{V}_{(k-1)h}} \left(\mathbf{v}_{kh} \left| \mathbf{v}_{(k-1)h} \right. ; \theta \right) \mathbf{1}_{\mathbf{v}_{n_j h} \in \mathcal{V}_j} \right) \\
&\qquad \left(\prod_{k=n_c+1}^{N} f_{\mathbf{V}_{kh} | \mathbf{V}_{(k-1)h}} \left(\mathbf{v}_{kh} \left| \mathbf{v}_{(k-1)h} \right. ; \theta \right) \right) \\
&= \left(\prod_{j=1}^{c} \mathbf{1}_{\mathbf{v}_{n_j h} \in \mathcal{V}_j} \right) \left(\prod_{k=1}^{N} f_{\mathbf{V}_{kh} | \mathbf{V}_{(k-1)h}} \left(\mathbf{v}_{kh} \left| \mathbf{v}_{(k-1)h} \right. ; \theta \right) \right)
\end{aligned}
$$

where $n_0 = 0$ and $f_{\mathbf{V}_{kh} | \mathbf{V}_{(k-1)h}} \left(\mathbf{v}_{kh} \left| \mathbf{v}_{(k-1)h} \right. \right)$ denotes the conditional density function of V_{kh} given $V_{(k-1)h}$. Note that, the value of $\prod_{k=N+1}^{N} f_{\mathbf{V}_{kh} | \mathbf{V}_{(k-1)h}}$ is set to 1 if such a case occurs.

Since the conditional distribution of V_{kh} given $V_{(k-1)h}$ is lognormal, we have the following conditional log-likelihood function:

$$
\begin{aligned}
& L \left(\mathbf{v}_0, \mathbf{v}_h, \mathbf{v}_{2h}, \cdots, \mathbf{v}_{Nh}; \theta \right) \\
&= \ln f_{\mathbf{V}_0, \mathbf{V}_h, \mathbf{V}_{2h}, \cdots, \mathbf{V}_{Nh} | \mathcal{D}} \left(\mathbf{v}_0, \mathbf{v}_h, \mathbf{v}_{2h}, \cdots, \mathbf{v}_{Nh}; \theta \right) \\
&= \sum_{k=1}^{N} \left(\mathbf{v}_{kh} \left| \mathbf{v}_{(k-1)h} \right. ; \theta \right) + \sum_{j=1}^{c} \ln \mathbf{1}_{\mathbf{v}_{n_j h} \in \mathcal{V}_j} - \ln P \left(\mathcal{D}; \theta \right) \\
&= -\frac{mN}{2} \ln (2\pi) - \frac{N}{2} \ln \left(|\det \mathbf{\Sigma}| \right) - \frac{1}{2} \sum_{k=1}^{N} \mathbf{w}_{kh}^{'} \mathbf{\Sigma}^{-1} \mathbf{w}_{kh} - \sum_{k=1}^{N} \sum_{i=1}^{m} \ln v_{i,kh} \\
&\quad + \sum_{j=1}^{c} \ln \mathbf{1}_{\mathbf{v}_{n_j h} \in \mathcal{V}_j} - \ln P \left(\mathcal{D}; \theta \right)
\end{aligned}
\tag{14}
$$

where $\mathbf{\Sigma} \equiv \left(h \sigma_{V_i} \sigma_{V_j} \rho_{ij} \right)_{i,j \in \{1,\ldots,m\}}$ and $\mathbf{w}_{kh}$ is the column vector

$$
\mathbf{w}_{kh} \equiv \left(\ln v_{i,kh} - \ln v_{i,(k-1)h} - \left(\mu_i - \frac{1}{2} \sigma_{V_i} \right) h \right)_{m \times 1}.
$$

We of course do not observe $\mathbf{v}_0, \mathbf{v}_h, \mathbf{v}_{2h}, \cdots, \mathbf{v}_{Nh}$. Instead, we have a time series of the equity values $\mathbf{s}_0, \mathbf{s}_h, \mathbf{s}_{2h}, \cdots, \mathbf{s}_{Nh}$. The equity pricing equation is

$$
S_{i,t} = V_{i,t} \Phi \left(d \left(V_{i,t}, t, \sigma_{V_i}; F_{i,t}, T_{i,t} \right) \right) - F_i e^{-r(T_i - t)} \Phi \left(d \left(V_{i,t}, t, \sigma_{V_i}; F_{i,t}, T_{i,t} \right) - \sigma_{V_i} \sqrt{T_{i,t} - t} \right)
\tag{15}
$$

where

$$d(V_{i,t}, t, \sigma_{V_i}; F_{i,t}, T_{i,t}) = \frac{\ln(V_{i,t}) - \ln(F_{i,t}) + \left(r + \frac{1}{2}\sigma_{V_i}^2\right)(T_{i,t} - t)}{\sigma_{V_i}\sqrt{T_{i,t} - t}} \tag{16}$$

and $\Phi(\bullet)$ is the cumulative standard normal distribution function. It defines the transformation between $S_{i,t}$ and $V_{i,t}$. We denote it by $S_{i,t} = g_i(V_{i,t}; t, \sigma_{V_i}, F_{i,t}, T_{i,t})$. This function is invertible in the sense that for any fixed t and σ_{V_i}, $V_{i,t} = g_i^{-1}(S_{i,t}; t, \sigma_{V_i}, F_{i,t}, T_{i,t})$.[2] Defining v_{kh}^* as the implied asset prices at time kh obtained from the observed equity prices and a given set of parameters $(\sigma_{V_1}, \cdots, \sigma_{V_m})$, that is

$$\mathbf{v}_{kh}^* = \begin{bmatrix} v_{1,kh}^* \\ \vdots \\ v_{m,kh}^* \end{bmatrix} = \begin{bmatrix} g_1^{-1}(s_{1,kh}; kh, \sigma_{V_1}, F_{1,kh}, T_{1,kh}) \\ \vdots \\ g_m^{-1}(s_{m,kh}; kh, \sigma_{V_m}, F_{m,kh}, T_{m,kh}) \end{bmatrix}.$$

We can express the log-likelihood function for the sample of equity values as

$$\ln f_{\mathbf{S}_0,\mathbf{S}_h,\mathbf{S}_{2h},\cdots,\mathbf{S}_{Nh}|\mathcal{D}}(\mathbf{s}_0, \mathbf{s}_h, \mathbf{s}_{2h}, \cdots, \mathbf{s}_{Nh}; \theta)$$
$$= \ln f_{\mathbf{V}_0,\mathbf{V}_h,\mathbf{V}_{2h},\cdots,\mathbf{V}_{Nh}|\mathcal{D}}(\mathbf{v}_0^*, \mathbf{v}_h^*, \cdots, \mathbf{v}_{Nh}^*; \theta) + \ln|\det \mathcal{J}|$$

where the Jacobian $\mathcal{J}$ of the transformation is a block diagonal matrix $\mathcal{J} = (\mathcal{J}_{nh})_{n\in\{1,\ldots,N\}}$ and the sub-matrix $\mathcal{J}_{kh}$ is the $m \times m$ matrix

$$\mathcal{J}_{kh} = \left(\frac{\partial v_{i,kh}^*}{\partial s_{j,kh}}\right)_{i,j\in\{1,\cdots,m\}}.$$

One can show

$$\frac{\partial v_{i,kh}^*}{\partial s_{j,kh}} = \begin{cases} \frac{1}{\Phi\left(d\left(v_{i,kh}^*, kh, \sigma_{V_i}, F_{i,kh}, T_{i,kh}\right)\right)} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Therefore

$$\ln|\det \mathcal{J}| = \ln \prod_{k=1}^{N}\prod_{i=1}^{m} \frac{1}{\Phi\left(v_{i,kh}^*\right)} = -\sum_{k=1}^{N}\sum_{i=1}^{m} \ln \Phi\left(d\left(v_{i,kh}^*, kh, \sigma_{V_i}, F_{i,kh}, T_{i,kh}\right)\right).$$

Thus, the log-likelihood function based on the sample of observed equity values is

[2] In the case where we are at a refinancing date for the ith firm, that is $T_{i,t} = t$, then

$$S_{i,t} = \max(V_{i,t} - F_{i,t}; 0) = V_{i,t} - F_{i,t}$$

if we have conditioned upon the survival of the firm. This case is also invertible.

$$
\begin{aligned}
&L\left(\mathbf{s}_0, \mathbf{s}_h, \mathbf{s}_{2h}, \cdots, \mathbf{s}_{Nh}; \theta\right) \\
&= -\frac{mN}{2}\ln\left(2\pi\right) - \frac{N}{2}\ln\left(|\mathbf{\Sigma}|\right) - \frac{1}{2}\sum_{k=1kh}^{N}\mathbf{w}_{kh}^{*\prime}\Sigma^{-1}\mathbf{w}_{kh}^{*} - \sum_{k=1}^{N}\sum_{i=1}^{m}\ln v_{i,kh}^{*} \\
&+ \sum_{j=1}^{c}\ln \mathbf{1}_{\mathbf{v}_{n_j h}^{*}\in\mathcal{V}_j} - \ln P\left(\mathcal{D};\theta\right) \\
&- \sum_{k=1}^{N}\sum_{i=1}^{m}\ln\Phi\left(d\left(v_{i,kh}^{*}, kh, \sigma_{V_i}, F_{i,kh}, T_{i,kh}\right)\right)
\end{aligned}
$$

where w_{kh}^{*} is an m-dimensional column vector defined as

$$
\mathbf{w}_{kh}^{*} = \left(\ln v_{i,kh}^{*} - \ln v_{i,(k-1)h}^{*} - \left(\mu_i - \frac{1}{2}\sigma_{V_i}\right)h\right)_{m\times 1}.
$$

Appendix C Point estimates and standard errors for functions of the parameter values

The point estimate for the asset value can be obtained with

$$
\hat{V}_{i,t} \equiv V_{i,t}(\hat{\sigma}_{V_i}) = g_i^{-1}\left(S_{i,t}; t, \hat{\sigma}_{V_i}\right). \tag{18}
$$

Because $V_{i,t}(\hat{\sigma}_{V_i})$ is a continuously differentiable function of $\hat{\sigma}_{V_i}$, the distribution for the firm's asset value can be approximated by a normal distribution (see Lo (1986) or Rao (1973), page 385). Let $\hat{\nabla}_{V_i} = \left(\frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \mu_i}, \frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}}\right)$, then[3]

$$
V_{i,t}(\hat{\sigma}_{V_i}) - V_{i,t} \sim N\left\{0, \hat{\nabla}_{V_i}\hat{\mathbf{F}}_i^{-1}\hat{\nabla}_{V_i}'\right\}. \tag{19}
$$

For the credit spread, recall equation (4). Its point estimate can be computed by

$$
C_{i,t}(\hat{\sigma}_{V_i}) = -\frac{\ln\left(\frac{\hat{V}_{i,t}(\hat{\sigma}_{V_i}) - S_{i,t}}{F_i}\right)}{T_i - t} - r \tag{20}
$$

[3]Note that the equity pricing formula does not depend on μ_i. Thus, $\frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \mu_i} = 0$. Moreover,

$$
\frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}} = \frac{1}{\hat{V}_i\left(t\right)\sqrt{T-t}\phi\left(d\left(\hat{V}_{i,t}, t, \hat{\sigma}_{V_i}\right)\right)}
$$

where $\phi(\cdot)$ denotes the standard normal density function.

and its distribution can be approximated by

$$C_{i,t}(\hat{\sigma}_{V_i}) - C_{i,t}(\sigma_{V_i}) \sim N\left\{0, \hat{\nabla}_{C_i}\hat{\mathbf{F}}_i^{-1}\hat{\nabla}'_{C_i}\right\} \tag{21}$$

where[4] $\hat{\nabla}_{C_i} = \left(\frac{\partial C_{i,t}(\hat{\sigma}_{V_i})}{\partial \mu_i}, \frac{\partial C_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}}\right)$. The default probability is a function of both μ_i and σ_{V_i} and its expression is given in Appendix Appendix B. The point estimate can thus be expressed as

$$P_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) = \Phi\left(x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}, \hat{V}_{i,t})\right) \tag{22}$$

where Φ is the standard normal distribution function and

$$x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) = \frac{\ln(F_i) - \ln(V_{i,t}(\hat{\sigma}_{V_i})) - \left(\hat{\mu}_i - \frac{1}{2}\hat{\sigma}^2_{V_i}\right)(T_i - t)}{\hat{\sigma}_{V_i}\sqrt{T_i - t}}.$$

The distribution for the default probability estimate should be treated with care. A direct application of the first-order Taylor approximation as the typical asymptotic theory calls for is not an advisable approach to this particular estimator. Because the sampling error associated with $\hat{\mu}_i$ is quite large and the mapping from $(\hat{\mu}_i, \hat{\sigma}_{V_i})$ to $P_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})$ is highly non-linear, we thus have adopted a two-step construction and found it work well. First, we use the first-order Taylor approximation for $x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})$; that is,

$$x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) - x_{i,t}(\mu_i, \sigma_{V_i}) \sim N\left\{0, \hat{\nabla}_{x_i}\hat{\mathbf{F}}_i^{-1}\hat{\nabla}'_{x_i}\right\} \tag{23}$$

where $\hat{\nabla}_{x_i} = \left(\frac{\partial x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})}{\partial \mu_i}, \frac{\partial x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})}{\partial \sigma_{V_i}}\right)$. The $1-\alpha$ confidence interval for the default probability $P_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) = \Phi\left(x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})\right)$ is then constructed as:

$$\left[\Phi\left(\underline{b}\left(\hat{\mu}_i, \hat{\sigma}_{V_i}, \hat{V}_{i,t}\right)\right); \Phi\left(\overline{b}\left(\hat{\mu}_i, \hat{\sigma}_{V_i}, \hat{V}_{i,t}\right)\right)\right]$$

where $\underline{b}(\bullet)$ and $\overline{b}(\bullet)$ are the lower and upper bounds of the $1-\alpha$ confidence interval for $x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})$. The validity of this confidence interval can be easily verified as follows:

$$\begin{aligned}
&1-\alpha \\
&= P\left[\underline{b}(\hat{\mu}_i, \hat{\sigma}_{V_i}) \leq x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) \leq \overline{b}(\hat{\mu}_i, \hat{\sigma}_{V_i})\right] \\
&= P\left[\Phi\left(\underline{b}(\hat{\mu}_i, \hat{\sigma}_{V_i})\right) \leq \Phi\left(x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})\right) \leq \Phi\left(\overline{b}(\hat{\mu}_i, \hat{\sigma}_{V_i})\right)\right].
\end{aligned}$$

[4] Again, the first component of $\hat{\nabla}_{C_i}$ is zero. The second one is

$$\frac{\partial C_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}} = -\frac{1}{T_i - t}\frac{1}{V_{i,t}(\sigma_{V_i}) - S_{i,t}}\frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}},$$

where $\frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}}$ is given in an earlier footnote.

Appendix D The RV-JMR estimation method

Following Ronn and Verma (1984) and Jones et al. (1984), an equation relating the diffusion coefficient of the stock price process to that of the asset value process can be obtained because stock price is a function of the asset value. Formally, $S_{i,t} = g_i(V_{i,t}; t, \sigma_{V_i})$. Applying Itô's lemma gives rise to

$$d \ln S_{i,t} = \left(\frac{\mu_i V_{i,t} \Delta_i(t) + \theta_i(t) + \frac{1}{2}\sigma_{V_i}^2 V_i^2(t) \Gamma_i(t)}{S_{i,t}} - \frac{\sigma_{V_i}^2 V_i^2(t) \Delta_i^2(t)}{2S_i^2(t)} \right) dt + \frac{\sigma_{V_i} V_{i,t} \Delta_i(t)}{S_{i,t}} dW_{i,t}. \tag{24}$$

where

$$\Delta_i(t) \equiv \frac{\partial g_i(V_{i,t}; t, \sigma_{V_i})}{\partial v} = \Phi(d(V_{i,t}, \sigma_{V_i})),$$
$$\Gamma_i(t) \equiv \frac{\partial^2 g_i(V_{i,t}; t, \sigma_{V_i})}{\partial v^2} = \frac{\phi(d(V_{i,t}, \sigma_{V_i}))}{\sigma_{V_i} V_{i,t} (T_i - t)^{1/2}},$$
$$\theta_i(t) \equiv \frac{\partial g_i(V_{i,t}; t, \sigma_{V_i})}{\partial t}$$

and ϕ and Φ denotes respectively the density function and the cumulative function of a standard normal random variable. The diffusion coefficient of the stock return process can thus be written as:

$$\sigma_{S_i}(t) = \frac{\sigma_{V_i} V_{i,t} \Delta_i(t)}{S_{i,t}}.$$

This coefficient is time dependent and stochastic. Although it is inconsistent with the model, the RV-JMR estimation method assumes that the sample standard deviation of stock returns sampled over a time period prior to time t is a good estimate of $\sigma_{S_{i,t}}$. This relationship in conjunction with $S_{i,t} = g_i(V_{i,t}; t, \sigma_{V_i})$ can be used to solve for two unknowns - $V_{i,t}$ and σ_{V_i} - using the observed value for $S_{i,t}$ and the estimate for $\sigma_{S_{i,t}}$.

In the multi-firm context, the quadratic variation between two stock returns is $\sigma_{S_i}(t)$ $\sigma_{S_j}(t) \rho_{ij}$ because

$$\begin{aligned} d \langle \ln S_i, \ln S_j \rangle_t &= \frac{\sigma_{V_i} V_{i,t} \Delta_i(t)}{S_{i,t}} \frac{\sigma_{V_j} V_j(t) \Delta_j(t)}{S_j(t)} d \langle W_i, W_j \rangle_t \\ &= \frac{\sigma_{V_i} V_{i,t} \Delta_i(t)}{S_{i,t}} \frac{\sigma_{V_j} V_j(t) \Delta_j(t)}{S_j(t)} \rho_{ij}\, dt \\ &= \sigma_{S_i}(t) \sigma_{S_j}(t) \rho_{ij}\, dt. \end{aligned}$$

Using the sample correlation may yields to an estimate for ρ_{ij}. Note that this estimation procedure is again inconsistent with the model because the quadratic variation process between $\ln S_i$ and $\ln S_j$ is also time dependent and stochastic.

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