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Optimal Retail Price Promotions over Time

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Abstract

The paper addresses the problem of determining a retailer's optimal price promotions of two brands in a product category. A dynamic model is constructed, taking into account interbrand substitution effects as well as a promotion's effects on the post-promotion demand for the brands. For the case of a myopic retailer who makes her marketing decisions on a period-by-period basis, optimal discounts and their durations can be determined simultaneously in forward time.

On the other hand, if the retailer is forward looking, the depths of optimal discounts as well as the timing and duration of promotions cannot be identified by such a procedure. As an approximation, we first determine optimal discounts, given that durations already are fixed. Next, we solve the problem of finding the optimal timing and durations of the promotions, given that discounts are fixed. This resembles the use in practice of a promotional menu that specifies regular prices as well as the discounts to be applied if a promotion is decided. If a discount is made, the retailer then decides the duration.

There is no general concensus in empirical marketing literature on what is net impact of a promotion on consumer response over time. The paper focuses on three main effects:

- (1) The immediate and positive impact of a price deal on the sales of a promoted brand during the promotional period.
- (2) Brand substitution within the category makes some consumers switch from a nonpromoted brand to a lower priced promoted brand.
- (3) Consumers stockpile a promoted brand during a deal period, which affects post-promotional demand of the brands in the category. This effect can last for a shorter or longer interval of time.

The main contribution of the paper is the characterization of optimal discounts, and the timing and duration of promotions, of a myopic and a forward looking retailer. We also provide a series of results that identify the dependence of discounts and durations upon key parameters of the model.

Keywords: Optimal Price Promotions; Retailing; Mathematical Programming.

Résumé

Cet article traite de la détermination optimale des prix de deux produits par un détaillant. Un modèle dynamique est construit et tient en compte les effets de substitution inter marques ainsi que les effets de la promotion sur les ventes futures des marques. Si le détaillant est myope, c'est-à-dire prend ses décisions période par période, alors la réduction du prix et la durée de la promotion optimales peuvent être déterminées simultanément.

Si le détaillant est non myope, alors on ne peut plus déterminer la solution optimale simultanément pour les deux décisions. Les deux cas où tour à tour on suppose qu'une décision est déjà prise et on optimise pour l'autre sont analysés. Cette approche semble correspondre à la pratique de la promotion.

Mots clés : Réduction optimale de prix; Ventes au détail; Programmation mathématique.

1 Introduction

A very popular practice in the marketing of consumer packaged goods is the use of temporary reductions of the retail price. A *retail price promotion* or *price deal* is a temporary reduction of a brand's regular price to consumers, lasting for a short period of time only. Price reductions can take various forms: A "shelf-price" reduction applies to any buyer of the brand, whereas price reductions based on coupons apply only to those who actually redeem a coupon. This paper is concerned with shelf-price reductions only.

Retail price promotions are accompanied by in-store signs and off-shelf displays, and will be featured in newspapers, television, and radio.

The success of a promotion depends primarily on its ability to draw customers to the store (i.e., to increase store traffic), the profit made on the promoted item, the cannibalization caused by customers who switch from regular-priced brands to a promoted item, increased sales of products that are complementary to the promoted product, and increased sales of items sold at regular prices (Mulhern and Leone (1991), Blattberg and Neslin (1993)).

Most retail promotions are triggered by manufacturers' *trade deals* that are temporary cuts in prices to retailers. The idea of giving trade deals is to push more merchandise through the supply chain, and the manufacturers' intention is that retailers increase their orders and reduce their prices toward consumers. In an ideal world one would expect a close connection between the availability and size of a trade deal and the subsequent retail price promotions. There is, however, an overwhelming amount of evidence that shows that retailers do not pass through the trade deals to the final consumers. In practice, retailers stockpile items bought at a discount and often they pocket part of the price savings offered by manufacturers.

When making strategic decisions about promotions, retailers must rely on their expectations of consumers' reactions (see also Kalwani and Yim (1992)). Here it is helpful to consider a categorization of reactions introduced by Pauwels et al. (2002). These authors distinguish the *immediate* effects of price promotions (short-term changes in sales), the *adjustment* (or transient) effects that refer to the transition period between the immediate response and a long-run equilibrium state, and the *permanent* effects that reflect the proportion of a promotion's impact that is carried forward and influences the long-run equilibrium state. In marketing literature there is, as we shall see, no general consensus about the signs and magnitudes of these three types of effects.

1.1 Literature Review

As to the *immediate* effects of a price promotion, it is widely agreed that a deal leads to an increase in sales of the promoted brand during the promotional period (Dodson et al. (1978), Guadagni and Little (1983), Moriarty (1985), Gupta (1988), Blattberg and Neslin (1990), Walters (1991), Mulhern and Leone (1991), Blattberg et al. (1981, 1995)). This is the primary impact of a promotion and it is caused by consumers who buy the brand earlier and in larger quantities than usually, consumers who switch from competing brands

and/or competing stores, and from consumers who only buy the category occasionally. Some empirical studies have found that brand choice (rather than category incidence and brand quantity) has the highest immediate promotional effect (Bell et al. (1999), Pauwels et al. (2002)).

Regarding the *adjustment* effects, Pauwels et al. (2002) note that these effects can be positive or negative, and their sign and size will affect significantly the profitability of the promotion. Adjustment effects include quite a wide range of specific influences:

- A promoted brand may suffer from a reduction of its postpromotion sales, a post-deal trough, due to consumers' stockpiling (forward buying, purchase acceleration). Consumers stockpile the promoted brand, having the rational expectation that the promotion only lasts for a short period of time after which the price returns to its regular level. This can make the interpurchase time for consumers buying on deal longer, unless these consumers increase their consumption of the product. There may also be a spillover effect such that postpromotion sales of other brands in the category are smaller than their regular levels. The empirical evidence of postdeal troughs is mixed. To illustrate, Neslin et al. (1985) report postdeal troughs, although not substantial ones; other studies (e.g., Grover and Srinivasan (1992)) find no such effects.
- The brand choice effect imply that a promotion causes a decrease in current sales of competing brands within the category (Dodson et al. (1978), Guadagni and Little (1983), Kumar and Leone (1988), Walters (1991)). Brand switching may account for the majority of promotional volume and can also persist beyond the promotional period. The empirical results are conflicting, although most studies seem to support a brand switching effect.
- Some studies advocate that when a brand is promoted, incremental sales mainly come from category expansion (rather than brand substitution), and could lead to increased category profits (e.g., Vilcassim and Chintagunta (1995)).
- Promotion of a brand may permanently increase its nonpromotion period sales by inducing its continued use among consumers who have switched from competing brands (Guadagni and Little (1983), Moriarty (1985)).
- Increased sales of a promoted brand during the promotional period may come from occasional or impulse buyers who were attracted by the price discount only. The exit from the market of this group of buyers, after the termination of the promotion, may affect postpromotional sales levels. Increased sales of a promoted item can also come from nonbuyers who choose to enter the market now (Moriarty (1985)).
- A promotion can stimulate purchases of nonpromoted complements (Mulhern and Leone (1991), Walters (1991)).
- Advertised promotions can generate more store traffic (Grover and Srinivasan (1992)). Some of this traffic may be due to store switching, where consumers take advantage of the discount in the promoting store (Kumar and Leone (1988)).

As to the *permanent* effects of a promotion, price deals may result in less loyalty toward the brand when retracted (Dodson et al. (1978)). Advertising agencies argue that promotions harm the long-term “brand image” or “brand equity”. One reason is that heavy and frequent promotions change the consumers’ reference prices (Kalwani and Yim (1992)). However, the empirical research on the long-term effects of promotions is far from conclusive. Thus, Pauwels et al. (2002) find that such effects are virtually absent for each of the three sales components, category incidence, brand choice, and purchase quantity.

The impacts of a promotion are not universally agreed upon, and cannot possibly be, since variations over product categories and store types are considerable. There may also be regional and cultural differences in the ways consumers react to deals. What the literature has done is to assess the impacts of promotions on a case-by-case basis, using a variety of empirical methods on data from specific product categories. The presence and strength of particularly the adjustment effects of a promotion vary quite considerably among the analyses; in some cases effects are predominantly positive, in others they are negative. Pauwels et al. (2002) noted that “...*the net impact of promotions on dynamic consumer response remains an empirical puzzle in marketing literature*” (Pauwels et al. (2002, p. 424)).

Empirical analyses of the impacts of promotions dominate the price promotions literature. There are a few prescriptive studies that we shall briefly review.

Vilcassim and Chintagunta (1995) studied a retailer’s optimal pricing decision in a period-by-period setup. Due to the essentially static setting, an optimal determination of duration and depth of consecutive price reductions is impossible. The authors suggested that a useful endeavor for future research would be to address this problem, noting that such an analysis requires the construction of a dynamic optimization model.

This approach was taken by Rao and Thomas (1973) who were probably the first to study the retail price promotion problem as a dynamic optimization problem. They suggested a dynamic programming model for determining simultaneously the optimal price-off and the number of times to promote a single brand during a fixed planning horizon. Optimal solutions were derived by numerical methods for a number of specific parameter sets.

Rao (1991) studied retail price promotions in a setup of a duopoly, consisting of a national brand and a private label. Rao argued that promotional decisions are part of a more general decision problem. Thus, a retailer first chooses a regular price for a brand, and then makes the promotional decisions. The latter have two elements: the depth and the frequency of promotions. The idea thus is to view a firm’s pricing problem as one being solved consecutively.

Tellis and Zufryden (1995) proposed an elaborate model for a retailer’s optimal timing and depth of discounts, combined with the optimal timing and quantity of the retailer’s orders from manufacturers. Trade deals are offered by manufacturers and the model accounts for inventories at both consumer and retail levels. The model incorporates multiple brands and deals and has a consumer response model based on brand choice, category incidence, and quantity events. Due to the complexity of the integer mathematical programming

model, no analytical results can be found. The authors used numerical simulations to characterize optimal solutions and their sensitivity to changing parameter values. Model parameters were estimated from scanner data for regular saltine crackers, a category that typifies consumer purchases in a market with frequent promotions.

Jørgensen et al. (2003), Jørgensen and Zaccour (2003) investigate the negative long-term impacts of promotions in a dynamic game setup. The level of feature advertising measures the intensity of a promotion.

1.2 Overview of the Model and Main Results

We assume that a retailer has two brands in a specific category, and one brand is promoted at a time. During the retailer's planning period there are two promotions only, one of brand 1 and one of brand 2. The assumption of no overlap of promotions is plausible and was employed by Rao and Thomas (1978) and most of the simulations in Tellis and Zufryden (1995) showed that brands should not be promoted simultaneously. An important reason for promoting brands one at a time is that the retailer incurs an opportunity cost: customers who are loyal to the discounted brand buy the product at the discounted price, but would have bought it at the regular price. When two brands are discounted simultaneously, this effect is reinforced. Tellis and Zufryden noted, however, that if one brand carries a very high margin and the other has a very high sensitivity to discounts, it may be profitable to discount both brands.

In view of the many, and sometimes conflicting and ambiguous, effects of a retail price promotion, we have chosen to focus upon the following:

- The positive influence of a price deal on the sales of the promoted brand during the promotional period.
- Brand substitution during the promotional period makes consumers switch from the nonpromoted brand to the lower priced, promoted brand.
- Consumers' stockpiling during the deal period affects postdeal demand of both brands, for a shorter or longer interval of time.

In this framework, the purpose of the paper is to study the problem of optimal duration, timing, and depth of discounts. We set up an intertemporal model of a retailer's promotional activities, supposing that the retailer has but two brands in the product category, one brand only is promoted at a time, and a brand is promoted at most once during the planning period. The solution of the resulting mathematical programming problem should be seen as a retailer decision support tool, but the issues of promotion frequency and depth of discounts can be interesting in other contexts, too. For example, frequency and depth of discounts may have a significant impact on consumers' price expectations (Kalwani and Yim (1992)).

Methodologically, the approach is *analytical*. The reason for this choice is twofold. First, we wished to introduce another line of research in the field of normative studies of retail price promotions; so far researchers have used numerical methods to compute optimal policies. Second, the analytical approach provides results that have more generality than

those obtained by numerical methods. The cost of our choice is that we cannot handle a more complex model as, for instance, that in Tellis and Zufryden (1995). In particular, we do not take into account manufacturers' trade deals and omit retailer and consumer inventories.

Among our findings are the following:

- Suppose that consumers stockpile during the promotion of brand 1. Then, the larger the discount on this brand, the larger the subsequent discount on brand 2. If consumers do not stockpile, the discount on brand 2 can be lighter.
- If brand switching effects are significant, a brand should be discounted lightly.
- A forward-looking retailer takes into account the impacts of promoting brand 1 on future consumer demand. She also takes into account the effects of promoting brand 1 on a subsequent promotion of brand 2.
- The discount on each brand decreases as regular prices of the brands increase.
- Promoting the two brands is not equally attractive.
- The brand which damages postpromotional category demand the most should be discounted the least.

The paper progresses as follows. In Section 2 we construct a dynamic model of a retailer's price promotion decisions. The purpose is to determine optimally the depth of discounts and the duration and timing of price deals. Section 3 determines an optimal solution for a myopic and a forward-looking retailer, respectively, and discusses the managerial implications of the model's recommendations. Section 4 concludes and offers some suggestions for future research in the challenging, but complicated area of optimal retail price promotions.

2 Dynamic Model of Retail Price Promotions

In reality "*Retailers face a complex problem with regard to optimizing promotions and the current environment. This is due to the large number of categories, the multiplicity of similar brands in each category and the numerous deals by manufacturers for each brand*" (Tellis and Zufryden (1995, p. 271)). However, our setup will be a simplistic one. A considerably more complex scenario is considered in Tellis and Zufryden (1995) who resorted to numerical methods to characterize optimal promotions. Nevertheless we believe that our framework is capable of illustrating a number of interesting issues in the optimal design of promotions, and provide insights that are usable beyond the limits of specific data sets.

Suppose that a retailer has two brands, 1 and 2, in a specific product category. Time t is measured continuously and the retailer's planning period starts at $t = 0$ and ends at $t = T$. The length T of the planning period is fixed. Typically, the planning of price promotion activities does not have a very long horizon, which can justify that we omit discounting of future revenues and costs.

The retailer considers a *master promotion plan* which is: First promote brand 1, then brand 2. The master plan includes three plans as special cases: (i) promote neither brand 1 nor 2, (ii) promote brand 1, but not 2, and (iii) promote brand 2, but not 1.

To characterize the retailer's plan, let time instants θ_1 [θ_2] denote the start [the end] of a promotion of brand 1, and let η_1 [η_2] denote the start [the end] of a promotion of brand 2. Thus, $\theta_2 - \theta_1$ [$\eta_2 - \eta_1$] is the duration of a promotion of brand 1 [2]. If $\theta_2 = \theta_1$ [$\eta_2 = \eta_1$], brand 1 [2] is not promoted. Since we have assumed that deals do not overlap, it must hold that $\theta_2 \leq \eta_1$.

There are five subintervals to consider:

- $[0, \theta_1)$: No brand is promoted
- $[\theta_1, \theta_2]$: Brand 1 is promoted
- (θ_2, η_1) : No brand is promoted
- $[\eta_1, \eta_2]$: Brand 2 is promoted
- $(\eta_2, T]$: No brand is promoted.

Notice that if $\theta_2 = \eta_1$, the promotion of brand 2 starts at the very moment where the promotion of brand 1 ends.

Our next task is to specify the demand conditions in the five subintervals. It would have been more satisfactory if these modeling choices could have been based on reasonably conclusive empirical evidence. In view of the many, and sometimes conflicting, derived effects of a retail price promotion, we focus upon the following:

- The immediate and positive influence of a price deal on the demand for a promoted brand during its promotion.
- During a promotional period, some consumers switch from the nonpromoted brand to the promoted one, causing a negative effect on the demand for the nonpromoted brand.
- Consumers' stockpiling affects negatively the postdeal demand rates of both brands, during shorter or longer intervals of time.

These modeling choices are crucial since they will be the main drivers of the results to follow.

2.1 Demand and Revenue Functions

This section describes the demand and revenue conditions during the five subintervals stated above. As to revenue, we adopt the standard assumption that the retailer is concerned with *category revenue*. Let \bar{p}_1 and \bar{p}_2 denote the *regular prices* of brand 1 and 2, respectively. The determination of these prices is not our concern here and we simply assume that regular prices already have been set at time $t = 0$. A regular price is constant over time and is valid during any period in which a brand is not promoted.

First time interval, $[0, \theta_1)$, where no brand is promoted.

A precise specification of the brands' demand functions is not needed here. Denote by $\bar{q}_1 > 0, \bar{q}_2 > 0$ the demand rates of the two brands during the time interval $[0, \theta_1)$. The category revenue rate is

$$\bar{K} = \bar{p}_1 \bar{q}_1 + \bar{p}_2 \bar{q}_2,$$

which can be viewed as the retailer's baseline revenue. The category revenue in the time interval $[0, \theta_1)$ equals $\theta_1 \bar{K}$.

Second time interval, $[\theta_1, \theta_2]$, where brand 1 is promoted.

The demand rates of both brands are affected by the depth of the discount of brand 1. Let p_1^* and d_θ denote the *promotion price* and the *discount* (cents-off the regular price), respectively, of brand 1 during its promotion. Thus

$$p_1^* = \bar{p}_1 - d_\theta \iff d_\theta = \bar{p}_1 - p_1^*.$$

In what follows, the discount will act as the retailer's decision variable.

Blattberg et al. (1995) state that *"little is known about the shape of the deal effect curve, though it determines the "optimal" dealing amounts"* (Blattberg et al. (1995, 127)). Here we assume that demand rates are given by the linear functions:

$$q_1(d_\theta) = \bar{q}_1 + \beta_1 d_\theta, \quad q_2(d_\theta) = \bar{q}_2 - \varepsilon_2 d_\theta, \quad (1)$$

in which $\beta_1 > 0$ and $\varepsilon_2 \geq 0$ are constants.

In (1), β_1 measures the marginal impact of the promotion of brand 1 on its own demand rate. The higher the discount on brand 1, the larger its demand rate during the promotion period. The parameter ε_2 reflects the effect of brand switching within the category, that is, the impact on the demand for brand 2 of a promotion of brand 1. The higher the discount on brand 1, the smaller the demand for brand 2. The hypothesis here is that more brand 2 buyers switch to brand 1 as its discount increases. We put $\varepsilon_2 = 0$ if brand 2 customers are extremely loyal to their brand. It is plausible to assume $\beta_1 > \varepsilon_2$, that is, a promotion has a stronger marginal impact on the demand for the promoted brand 1 than on the demand for brand 2 (see, e.g., Blattberg and Neslin (1990)). The reason is that brand 1 increases its sales both to its regular customers, to occasional buyers, and to consumers who switch from brand 2.

The category revenue rate is

$$K^*(d_\theta) = (\bar{p}_1 - d_\theta)[\bar{q}_1 + \beta_1 d_\theta] + \bar{p}_2 [\bar{q}_2 - \varepsilon_2 d_\theta] = \bar{K} + (\bar{p}_1 \beta_1 - \bar{p}_2 \varepsilon_2 - \bar{q}_1) d_\theta - \beta_1 (d_\theta)^2. \quad (2)$$

The retailer's category revenue in the time interval $[\theta_1, \theta_2]$ then equals $(\theta_2 - \theta_1) K^*$.

Third time interval, (θ_2, η_1) where no brand is promoted.

Here we shall account for the adjustment effect, i.e., demand rates have been affected by the promotion of brand 1. Postpromotion demand rates could in general be affected by the duration of the promotion, the depth of the discount, or both. (The latter was assumed in Rao and Thomas (1978)). The duration of the promotion can affect postpromotion demands, for instance, from the reason that the longer the duration, the more consumers will have the opportunity to buy and stockpile the promoted brand. We confine our interest to the case where postpromotion demand rates are affected by the depth of the discount only.

Suppose that the regular demand rates \bar{q}_1 and \bar{q}_2 are linearly affected by the depth of the discount. Demand rates during the time interval (θ_2, η_1) are given by

$$\tilde{q}_1(d_\theta) = \bar{q}_1 - \sigma_{1\theta}d_\theta, \quad \tilde{q}_2(d_\theta) = \bar{q}_2 - \sigma_{2\theta}d_\theta, \quad (3)$$

in which $\sigma_{1\theta} \geq 0$ and $\sigma_{2\theta} \geq 0$ are constants. The value of $\sigma_{1\theta}$ reflects the net effect of (i) consumers' stockpiling, (ii) consumers switch back to brand 2, and (iii) occasional buyers are no longer in the market. We would expect $\sigma_{1\theta} > \sigma_{2\theta}$, that is, the effect of the promotion of brand 1 is stronger on its own demand than on that of brand 2. It may happen that $\sigma_{1\theta} = 0$ and/or $\sigma_{2\theta} = 0$, which means that the postdeal demand of one or both brands is unaffected by the promotion of brand 1.

The postpromotion category revenue becomes

$$\begin{aligned} \tilde{K}(d_\theta) &= \bar{p}_1[\bar{q}_1 - \sigma_{1\theta}d_\theta] + \bar{p}_2[\bar{q}_2 - \sigma_{2\theta}d_\theta] = \\ &= \bar{K} - (\bar{p}_1\sigma_{1\theta} + \bar{p}_2\sigma_{2\theta})d_\theta. \end{aligned}$$

The retailer's category revenue in the time interval (θ_2, η_1) equals $(\eta_1 - \theta_2)\tilde{K}$. Note that $\tilde{K} \leq \bar{K}$, which means that the postpromotion category revenue will not exceed the baseline revenue.

Fourth time interval, $[\eta_1, \eta_2]$, where brand 2 is promoted.

Let $p_2^* = \bar{p}_2 - d_\eta$ and $d_\eta = \bar{p}_2 - p_2^*$ denote the promotion price and the discount, respectively, of brand 2. Demand functions during the promotion of brand 2 are given by

$$\begin{aligned} q_1(d_\theta, d_\eta) &= \tilde{q}_1(d_\theta) - \varepsilon_1 d_\eta = \bar{q}_1 - \sigma_{1\theta}d_\theta - \varepsilon_1 d_\eta \\ q_2(d_\theta, d_\eta) &= \tilde{q}_2(d_\theta) + \beta_2 d_\eta = \bar{q}_2 - \sigma_{2\theta}d_\theta + \beta_2 d_\eta, \end{aligned} \quad (4)$$

in which $\varepsilon_1 \geq 0$ and $\beta_2 > 0$ are constants. In (4), the parameter ε_1 measures the marginal impact of promoting brand 2 on the demand rate of brand 1. The parameter β_2 measures the direct impact of promoting brand 2 on its own demand. We assume $\beta_2 > \varepsilon_1$. Note that the ε -parameters in (1) and (4) may differ considerably, due to differences in brand equity (Blattberg et al. (1995)). The cross-promotional effects are most likely asymmetric (Walters (1991), Mulhern and Leone (1991), Grover and Srinivasan (1992)).

It is important to notice that the inclusion of the terms $\sigma_{1\theta}d_\theta$ and $\sigma_{2\theta}d_\theta$ in (4) reflects an assumption that the effects of the promotion of brand 1 are present not only in the immediate postdeal period (θ_2, η_1) , but also in the time interval (η_1, η_2) during which brand 2 is promoted. It may happen that the effects of the promotion of brand 1 have vanished before time η_1 ; then one must put $\sigma_{1\theta}d_\theta$ and $\sigma_{2\theta}d_\theta$ equal to zero in (4).

The category revenue rate is

$$K^o(d_\theta, d_\eta) = \bar{p}_1[\tilde{q}_1(d_\theta) - \varepsilon_1 d_\eta] + (\bar{p}_2 - d_\eta)[\tilde{q}_2(d_\theta) + \beta_2 d_\eta] = \tilde{K} + (\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2 + \sigma_{2\theta}d_\theta)d_\eta - \beta_2(d_\eta)^2. \quad (5)$$

The retailer's category revenue during the interval $[\eta_1, \eta_2]$ equals $(\eta_2 - \eta_1)K^o$.

Fifth time interval, $(\eta_2, \mathbf{T}]$, where no brand is promoted.

As above we account for the possibility that demand rates were affected by the promotion. Hence, let postpromotion demand rates be given by

$$\hat{q}_1(d_\theta, d_\eta) = \bar{q}_1 - \sigma_{1\theta}d_\theta - \sigma_{1\eta}d_\eta, \quad \hat{q}_2(d_\theta, d_\eta) = \bar{q}_2 - \sigma_{2\theta}d_\theta - \sigma_{2\eta}d_\eta, \quad (6)$$

in which $\sigma_{1\eta} \geq 0$ and $\sigma_{2\eta} \geq 0$ are parameters that reflect the influence of the promotion of brand 2 on postdeal demand rates. It may happen that the effects of the promotion of brand 1 have vanished before time η_2 ; then one must put $\sigma_{1\theta}$ and $\sigma_{2\theta}$ equal to zero in (6).

The category revenue rate is

$$\hat{K} = \hat{K}(d_\theta, d_\eta) = \tilde{K} - (\bar{p}_1\sigma_{1\eta} + \bar{p}_2\sigma_{2\eta})d_\eta. \quad (7)$$

The retailer's category revenue over the time interval $(\eta_2, T]$ then equals $(T - \eta_2)\hat{K}$. Note that $\hat{K} \leq \tilde{K} \leq \bar{K}$, which means that the postpromotion category revenue, after the two promotions, will not exceed the baseline revenue.

This completes the description of demand functions and revenues in the five time periods. To save on notation, define the composite parameters

$$c_\theta \triangleq \bar{p}_1\sigma_{1\theta} + \bar{p}_2\sigma_{2\theta} \geq 0, \quad c_\eta \triangleq \bar{p}_1\sigma_{1\eta} + \bar{p}_2\sigma_{2\eta} \geq 0,$$

where $\sigma_{1\theta} + \sigma_{2\theta}$ [$\sigma_{1\eta} + \sigma_{2\eta}$] measures the marginal impact of a promotion of brand 1 [2] on postpromotion category demand.

Remark 1 A brief comment on notation is in order here. To avoid confusing the reader with a lot of sub- and superscripts, we have used the letter θ to refer to the brand 1 promotion, and η to the brand 2 promotion. Thus, these letters are used for the discount and duration of the respective promotions. With respect to the category revenues in the second and fourth time periods, a “star” and a “nought”, appearing as a superscript, signifies an optimal revenue during a discount period. A “tilde” and a “hat”, appearing as a superscript, refer to a postpromotion revenue. A “bar” signifies the benchmark revenue.

2.2 Profit Function

As already said, we disregard the possibility that promotions are triggered by trade deals and assume constant transfer prices of the two brands throughout the retailer's planning period. Furthermore, let the retailer's unit costs of processing the two products be constant. Defining the retail prices p_1 and p_2 as being net of purchase and processing costs, one can view p_1 and p_2 as margins.

The retailer incurs costs of feature advertising and of displaying a promoted brand. These costs are given by

$$C(t) = at,$$

where a is a positive constant. Thus, costs are independent of which brand is promoted and depend only on the length of the time interval during which a brand is promoted.

It is well known in practice that displays and feature advertising often influence item sales. However, the possible synergies between these activities and price discounts have only been sporadically researched in the empirical literature (cf. Blattberg et al. (1995)). Here we assume that advertising and displaying a brand do not affect the demand rates during a promotion. The implication is that advertising and display enter the model as a cost only, penalizing promotions with long durations and making it less attractive to promote all the time.

Advertising and display activities start at the same time where a price deal starts. When the brands are promoted over time periods $[\theta_1, \theta_2]$ and $[\eta_1, \eta_2]$, respectively, advertising costs amount to

$$\int_{\theta_1}^{\theta_2} at \, dt = \frac{a}{2}(\theta_2^2 - \theta_1^2) \triangleq A_\theta, \quad \int_{\eta_1}^{\eta_2} at \, dt = \frac{a}{2}(\eta_2^2 - \eta_1^2) \triangleq A_\eta. \quad (8)$$

Remark. One may argue that advertising should start before a promotion starts. To model this, let $\theta_1 - \Delta$ and $\eta_1 - \Delta$ be the instants at which advertising starts for the two brands (Δ is a positive constant). This case can easily be handled, but does not add much to the understanding of optimal promotions. In the sequel we set $\Delta = 0$.

The profit function of the retailer is category revenues minus advertising costs over the planning period $[0, T]$:

$$J(\theta_1, d_\theta, \theta_2, \eta_1, d_\eta, \eta_2) = \theta_1 \bar{K} + (\theta_2 - \theta_1)K^* - A_\theta + (\eta_1 - \theta_2)\tilde{K} + (\eta_2 - \eta_1)K^o - A_\eta + (T - \eta_2)\hat{K}.$$

3 Optimal Retail Price Promotions

The retailer's problem is a multi-stage decision problem and an optimal promotion plan is a set of decisions

$$\{\theta_1, d_\theta, \theta_2, \eta_1, d_\eta, \eta_2\},$$

determining the depths of the discounts and the durations and timing of the two promotions. We determine an optimal plan under two alternative assumptions concerning the retailer's optimizing behavior:

- The retailer is myopic and believes that a current decision influences the current state of the system only, or she does not care about the influence of the current decision upon future states (Vilcassim and Chintagunta (1995))
- The retailer is a dynamic optimizer who takes into account that the current decision affects the state of the system, now and in the future (Rao and Thomas (1978), Tellis and Zufryden (1995)).

3.1 Myopic Retailer

When the retailer is myopic, the problem of determining optimal discounts and duration and timing of promotions can be solved in forward time as a sequence of one-period optimization problems. Optimal discounts can be determined independently of the optimal duration of promotions. The problem is straightforward and we report the main results without proofs. Optimal discounts, denoted by δ_θ^* and δ_η^* , respectively, are given by

$$\begin{aligned}\delta_\theta^* &= \frac{1}{2\beta_1} [\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1] \\ \delta_\eta^* &= \frac{1}{4\beta_1\beta_2} [(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2)2\beta_1 + (\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1)\sigma_{2\theta}],\end{aligned}$$

and satisfy the following relationship:

$$\delta_\eta^* = \frac{1}{2\beta_2} [\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2 + \sigma_{2\theta}\delta_\theta^*]. \quad (9)$$

The optimal durations of promotions for a myopic retailer are the same as those that will be determined below for a forward-looking retailer. This is not generally true. The reason is that for a forward-looking retailer we shall need, for analytical tractability, to decompose the overall problem into two subproblems: one for the discount decisions (given durations) and one for the durations (given discounts). These problems are solved in Section 3.2.

3.2 Forward-looking Retailer

We need to solve a dynamic programming problem with five stages and six decision variables. However, this problem involves, at two of its stages, optimality conditions in the form of two interdependent nonlinear equations that are analytically intractable. Since we insist on analytical solutions, we modify the retailer's dynamic optimization problem in the following way.

- Determine optimally the discounts d_θ and d_η , given that the durations of the promotions, $\theta_2 - \theta_1$ and $\eta_2 - \eta_1$, respectively, have been fixed at time zero. This is done in Section 3.2.1.

- Determine optimally the durations of the promotions, *given* that the discounts have been fixed at time zero. This is done in Section 3.2.2.

The reader should be aware that this approach does not provide an optimal overall plan. Nevertheless, the approach has a certain practical flavor and resembles the one proposed by Rao (1991). Rao viewed promotional decision making as a three-stage problem. In stage one, the retailer fixes the regular prices of the brands. In stage two, the depths of promotions are determined, and in stage three the frequencies of promotions are decided. Such sequential reasoning may resemble actual promotional decision making behavior. In practice, a retailer can define a *promotion menu*, being a rule that sets the regular prices and the depths of discounts. The menu is an initial decision that is called upon when it comes to the decision whether or not to discount, and, if the answer is affirmative, for how long a period?

In our setup, the menu consists of the depths of discounts only since we assumed that regular prices are fixed. The decision whether or not to discount, and for how long a period, amounts in our setup to the determination of the duration and timing of the two promotions.

3.2.1 Optimal Discounts When Durations are Fixed This subsection determines optimal values for the discounts d_θ and d_η , under the assumption that $\theta_1, \theta_2, \eta_1$, and η_2 are fixed. Then the advertising costs in (8) are constant and can be disregarded.

Remark. The linearity of the demand functions in (1), (3), (4), and (6) imposes a number of constraints on the discounts d_θ and d_η . In principle, this does not pose a problem, but it is very cumbersome to handle all these restrictions in the optimizations. The relative merits of considering all corner solutions are not clear and we confine our main interest to “interior solutions” where discounts are positive and have values for which demand rates are positive. Occasionally, we shall identify circumstances under which it is worthwhile not to discount, i.e., $d_\theta = 0$ and/or $d_\eta = 0$.

A main result is the following proposition.

Proposition 1 *Whenever positive, the optimal discount of brand 1 is given by*

$$d_\theta^* = \frac{\Gamma_\theta}{(\eta_2 - \eta_1)(\sigma_{2\theta})^2 - (\theta_2 - \theta_1)4\beta_1\beta_2}, \quad (10)$$

in which Γ_θ is a constant, given by

$$\begin{aligned} \Gamma_\theta = & (T - \eta_2)\sigma_{2\theta}c_\eta + (T - \theta_2)2\beta_2c_\theta - \\ & (\theta_2 - \theta_1)(\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1)2\beta_2 - \\ & (\eta_2 - \eta_1)(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2)\sigma_{2\theta}. \end{aligned} \quad (11)$$

Whenever positive, the optimal discount of brand 2 is given by

$$d_\eta^* = \frac{1}{2\beta_2} \left[\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2 + \sigma_{2\theta}d_\theta^* - c_\eta \frac{T - \eta_2}{\eta_2 - \eta_1} \right] \quad (12)$$

or, equivalently,

$$d_\eta^* = \frac{\Gamma_\eta}{(\eta_2 - \eta_1)(\sigma_{2\theta})^2 - (\theta_2 - \theta_1)4\beta_1\beta_2}, \quad (13)$$

in which Γ_η is a constant, given by

$$\begin{aligned} \Gamma_\eta = & (\theta_2 - \theta_1) \frac{(T - \eta_2)2\beta_1c_\eta}{\eta_2 - \eta_1} + (T - \theta_2)\sigma_{2\theta}c_\theta - \\ & (\theta_2 - \theta_1)(\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1)\sigma_{2\theta} - \\ & (\theta_2 - \theta_1)(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2)2\beta_1. \end{aligned} \quad (14)$$

The optimal profit over the planning period $[0, T]$ is

$$\begin{aligned} J(d_\theta^*, d_\eta^*) = & -[A_\theta + A_\eta] + T\bar{K} - (T - \eta_2)c_\eta d_\eta^* - (T - \theta_2)c_\theta d_\theta^* + \\ & (\eta_2 - \eta_1)[(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2 + \sigma_{2\theta}d_\theta^*)d_\eta^* - \beta_2(d_\eta^*)^2] + \\ & (\theta_2 - \theta_1)[(\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1)d_\theta^* - \beta_1(d_\theta^*)^2]. \end{aligned} \quad (15)$$

In (12) it holds, by (3), that $-\bar{q}_2 + \sigma_{2\theta}d_\theta^* = -\tilde{q}_2(d_\theta^*)$. Thus, the optimal discount can be seen as a feedback, $d_\eta^* = f(\tilde{q}_2)$, making the discount of brand 2 a function of postpromotional demand (after the promotion of brand 1). The larger the postpromotional demand \tilde{q}_2 , the smaller the discount on brand 2. Put in another way, the larger the discount on brand 1, the larger the subsequent discount on brand 2. To interpret this result, recall that $\sigma_{2\theta} > 0$ means that a promotion of brand 1 decreases postpromotion demand for brand 2. Thus, if promoting brand 1 seriously damages postpromotion demand of brand 2, and the discount on brand 1 was large, then brand 2 should have a large discount when promoted later on, in order to stimulate its demand. On the other hand, if there is no postpromotional effect ($\sigma_{2\theta} = 0$), the discount on brand 2 will be smaller.

Eq. (15) shows that the optimal profit has the components:

- $A_\theta + A_\eta$: Total advertising cost
- $T\bar{K}$: Aggregate category revenue if there were no discounts at all
- $-(T - \eta_2)c_\eta d_\eta^* + (T - \theta_2)c_\theta d_\theta^*$: Aggregate loss of postpromotional revenues, caused by the two promotions (that end at time θ_2 and η_2 , respectively)
- The last two terms represent the extra revenues, generated by the two promotions. In the first term we have

$$(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2 + \sigma_{2\theta}d_\theta^*)d_\eta^* - \beta_2(d_\eta^*)^2 = K^o - \tilde{K}, \quad (16)$$

which is the difference between the category revenue rate during the promotion of brand 2 and the prepromotion category revenue rate. In the second term we have

$$(\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1)d_\theta^* - \beta_1(d_\theta^*)^2 = K^* - \bar{K}, \quad (17)$$

which has a similar interpretation. By the optimality of the discounts, the differences on the right-hand sides of (16) and (17) are positive; otherwise there should be no promotions.

3.2.2 Postoptimality Analysis First we make a number of sensitivity analyses of the optimal discounts, given by (10) and (13). We need to assume

$$(\eta_2 - \eta_1)(\sigma_{2\theta})^2 < (\theta_2 - \theta_1)4\beta_1\beta_2. \quad (18)$$

To motivate the assumption in (18), suppose that promotions have approximately the same duration. A sufficient (but not necessary) condition for the inequality in (18) to be satisfied then is

$$\beta_i > \sigma_{2\theta}, i \in \{1, 2\}. \quad (19)$$

This means that the marginal effect β_i of a discount on brand i on its own current demand dominates the marginal effect $\sigma_{2\theta}$ of the discount on the postpromotion demand for brand 2. According to the empirical studies quoted above, this assumption is not implausible.

Using (10) and (13) provides the following results, stated in terms of the partial derivatives of discounts with respect to a parameter.

- *Effect of promoting brand i on current demand for brand j .* It holds, for example, that $\frac{\partial d_i^*}{\partial \varepsilon_2} < 0$. This means that the discount on brand 1 should be smaller, the more damage a promotion of this brand will do to the current demand for brand 2 (cf. (1)). Thus, if brand substitution effects are significant (ε_2 large), the retailer should discount brand 1 moderately - or even refrain from promoting this brand. Another explanation lies in brand loyalty related to consumer responses to price cuts (Guadagni and Little (1983)). Thus, if brand 2 customers are very loyal to their brand, i.e., $\varepsilon_2 \simeq 0$, then brand 1 can have a deep discount without affecting demand for brand 2 very much.
- *Effect of promoting brand i on its demand during a promotion of brand j .* It holds, for example, that $\frac{\partial d_i^*}{\partial \varepsilon_1} < 0$. This means that the discount on brand 1 should be smaller, the more damage a promotion of brand 2 will do to the demand for brand 1 during the promotion of brand 2 (cf. (4)). If ε_1 is large, and when deciding on the depth of discount on brand 1, the retailer foresees that a subsequent promotion of brand 2 will do considerable, instantaneous harm to the demand for brand 1. Thus, brand 1 is discounted less, in order not to decrease too much its demand after its own promotion (cf. (3)). Taking ε_1 as a measure of consumer loyalty to brand 1, a small value of this parameter means the consumers are very loyal to their brand. Hence, the depth of the discount of brand 1 can be substantial. Although the relationship between the discounts is positive, brand 1 can be discounted deeply since a subsequent deep discount of brand 2 will not affect demand for brand 1 very much.

These two results are quite intuitive and are related to a main result of Raju et al. (1990) who found that brands with a strong brand loyalty are less often promoted than brands with weaker loyalty.

- *Dependence of a discount on the timing of the promotion.* It holds, for instance, that $\frac{\partial d_i^*}{\partial \eta_1} < 0$. This inequality means that the sooner the second promotion starts (η_1),

given the end of the first promotion (θ_2) and given its own ending time (η_2), the deeper the discount on brand 2. Hence, extending the promotion period of brand 2 (making the time interval $\eta_1 - \theta_2$ shorter), increases the discount on brand 2.

It also holds that $\frac{\partial d_\eta^*}{\partial \eta_2} > 0$ which has the same interpretation: a promotion with a longer duration should have a more significant discount.

- *Effects of regular demand on discounts.* It holds that $\frac{\partial d_\theta^*}{\partial \bar{q}_i} < 0$, $\frac{\partial d_\eta^*}{\partial \bar{q}_i} < 0$, $i \in \{1, 2\}$, which means that the discount on each brand decreases as both regular demand rates increase. For sufficiently high regular demand rates, no brand should not be promoted. The intuition is simple: when demand already is high, there is less need for a promotion.

In the sequel we wish to examine the impact on optimal discounts of changing the demand specifications, in particular in the postpromotional periods. We denote by *post-promotional effects* the net impact of consumer stockpiling, brand switching in the time period following a promotion, as well as the exit of occasional buyers. The postpromotional effects are reflected in the σ 's appearing in the demand functions in (3) and (6).

- *Postpromotional Effects 1:* It holds that $\frac{\partial d_\theta^*}{\partial \sigma_{1\theta}} < 0$, $\frac{\partial d_\eta^*}{\partial \sigma_{1\theta}} < 0$. This means that the discount on *each* brand is lower, the more a promotion of brand 1 will decrease that brand's postpromotion demand. Thus, if discounting brand 1 deeply during its promotion seriously damages its postpromotion demand, the retailer should apply a lighter discount on brand 1 and hence, due to the positive relationship between discounts, she should also discount brand 2 lightly. A similar result is valid for the parameters $\sigma_{i\eta}$, $i \in \{1, 2\}$ in (6).
- *Postpromotional Effects 2:* The analysis in item 1 can be supplemented, by changing the demand functions such that postpromotional effects last for one or two periods only. Note that the results so far developed have assumed that the postpromotional effects of the first promotion last three periods. (Due to the finite horizon date T , the postpromotional effects of the second promotion last one period only). If the effects of the first promotion last two periods, the only change is in the postpromotion revenue \hat{K} , cf. (7), which obviously increases. The interesting case is when the effects of the first promotion last one period only. Then three things change: the revenue K^o during the promotion of brand 2, the optimal discount d_η^* , and the postpromotion revenue \hat{K} . Using (12) with $\sigma_{2\theta} = 0$ shows that the optimal discount of brand 2 is *less* than the one which applies when the effects of promoting brand 1 last for two or more periods. To see the intuition of this result, note that when the effects of the first promotion last one period only, the depth of the first discount does not influence that of the second discount (cf. (12)). Hence, after a deep discount of brand 1 demand has recovered when it comes to the promotion of brand 2 and the discount of this brand can be lighter. Clearly, when postpromotional effects are completely absent ($\sigma_{i\theta} = \sigma_{i\eta} = 0$, $i \in \{1, 2\}$), the retailer can act myopically.

- *Postpromotional Effects 3:* To compare the magnitudes of the discounts, one needs a symmetry assumption

$$\begin{aligned} \bar{p}_1 &= \bar{p}_2 = \bar{p}, & \bar{q}_1 &= \bar{q}_2 = \bar{q} \\ \beta_1 &= \beta_2 = \beta, & \varepsilon_1 &= \varepsilon_2 = \varepsilon \\ \theta_2 - \theta_1 &= \eta_2 - \eta_1, \end{aligned} \quad (20)$$

which means that the two brands have the same regular price and demand rate. Moreover, the impact of a promotion of brand i on that brand's own demand is the same for both brands, as is the impact of a promotion on the demand for the substitute brand. Finally, the durations of the promotions are equal. Using (10) and (13) then yields

$$d_\theta^* \left\{ \begin{array}{l} > \\ = \\ < \end{array} \right\} d_\eta^* \iff [(T - \theta_2)c_\theta - (T - \eta_2)c_\eta](2\beta - \sigma_{2\theta}) \left\{ \begin{array}{l} < \\ = \\ > \end{array} \right\} 0, \quad (21)$$

in which it holds that $T - \theta_2 > T - \eta_2$. By (19), the term $2\beta - \sigma_{2\theta}$ is positive. Using (21) then shows that if

$$c_\theta \geq c_\eta \iff \bar{p}[\sigma_{1\theta} + \sigma_{2\theta}] \geq \bar{p}[\sigma_{1\eta} + \sigma_{2\eta}]$$

then $d_\theta^* < d_\eta^*$. Hence, if a promotion of brand 1 causes more damage to postpromotion *category* demand than a promotion of brand 2, then brand 1 should have a smaller discount than brand 2. The intuition is as above, with the exception that now we evaluate a promotion's damage to postpromotion category demand, not only demand for the brand itself.

In the limiting cases

$$(i): c_\theta = 0, c_\eta > 0, \quad (ii): c_\theta > 0, c_\eta = 0, \quad (iii): c_\theta = c_\eta = 0$$

it holds in (i) that $d_\theta^* > d_\eta^*$ and $d_\theta^* < d_\eta^*$ in (ii). A c -parameter being zero means that there are no effects of a promotion on future category revenues. Using (10) and (13) shows that the larger of the two discounts, i.e., d_θ^* in (i) and d_η^* in (ii), is smaller than the corresponding discount prescribed for the general case ($c_\theta > 0, c_\eta > 0$). In (iii), the brands are symmetric with respect to their impacts on demand and get the same discount, $d_\theta^* = d_\eta^*$. All these results are as expected.

3.2.3 Optimal Durations with Fixed Discounts This subsection deals with the following problem: Given that the retailer has fixed both discounts, what is the optimal timing and duration of the promotion periods? Hence, let D_θ and D_η denote the fixed discounts

on brands 1 and 2, respectively. Then the prices charged during promotions are fixed, too, and equal $p_1^* = \bar{p}_1 - D_\theta$, $p_2^* = \bar{p}_2 - D_\eta$. The category revenue rates \bar{K} , K^* , \tilde{K} , K^o , \hat{K} during the five subintervals are constant.

The problem is to determine $\theta_1, \theta_2, \eta_1, \eta_2$ such that the objective

$$J(\theta_1, \theta_2, \eta_1, \eta_2) = \theta_1 \bar{K} + (\theta_2 - \theta_1) K^* + (\eta_1 - \theta_2) \tilde{K} + (\eta_2 - \eta_1) K^o + (T - \eta_2) \hat{K} - \left[\frac{a}{2} (\theta_2^2 - \theta_1^2) + \frac{a}{2} (\eta_2^2 - \eta_1^2) \right]$$

is maximized, subject to the constraints

$$\theta_1 \geq 0, \theta_2 - \theta_1 \geq 0, \eta_1 - \theta_2 \geq 0, \eta_2 - \eta_1 \geq 0, T - \eta_2 \geq 0. \quad (22)$$

This is a quadratic programming problem and the constraints in (22) have the following interpretation. The first and fifth are obvious, recalling that the planning period is $[0, T]$. The second and fourth state that a promotional period cannot be negative, and the third one reflects the assumption that promotions must not overlap. Our main results for the quadratic programming problem are stated in Proposition 2, in which the assumption is that no constraints are binding. The proof of the proposition is straightforward and omitted.

Proposition 2 *Optimal time instants and durations are given by*

$$\begin{aligned} \theta_1 &= \frac{1}{a} [K^* - \bar{K}], & \theta_2 &= \frac{1}{a} [K^* - \tilde{K}] \\ \eta_1 &= \frac{1}{a} [K^o - \tilde{K}], & \eta_2 &= \frac{1}{a} [K^o - \hat{K}] \end{aligned} \quad (23)$$

and

$$\begin{aligned} \theta_2 - \theta_1 &= \frac{1}{a} [\bar{K} - \tilde{K}] = \frac{c_\theta D_\theta}{a} \geq 0 \\ \eta_2 - \eta_1 &= \frac{1}{a} [\tilde{K} - \hat{K}] = \frac{c_\eta D_\eta}{a} \geq 0, \end{aligned} \quad (24)$$

respectively. The durations are related in the following way:

$$\eta_2 - \eta_1 = \frac{c_\theta D_\theta + c_\eta D_\eta}{a} - (\theta_2 - \theta_1). \quad (25)$$

3.2.4 Postoptimality Analysis

- *Relationship between durations.* By (25), the duration of the second promotion is a linearly decreasing function of the duration of the first promotion (and vice versa). Essentially, the problem of determining the duration of the promotions is one of allocating a fixed amount of time, $\kappa \triangleq (c_\theta D_\theta + c_\eta D_\eta)/a > 0$, among the two

promotions. The quantity κ has the following interpretation. Its numerator equals $\bar{K} - \hat{K}$ and represents the loss in category revenue per unit of time, caused by the two promotions. The denominator is the advertising cost per unit of time. Hence κ is the decrease in postpromotion category revenue per advertising dollar. If the advertising cost parameter a increases, the duration of at least one of the promotions must be reduced. If the loss of category revenue diminishes, at least one promotion can be extended in time.

- *Relationship between depth of discount and duration.* The results in (24) show, for any fixed pair (c_θ, c_η) , that the duration of a promotion increases (linearly) with the depth of the discount. Thus, a promotion with a deep discount should last longer than one involving a more modest discount. This confirms a result in Rao and Thomas (1973), and has already been noted in Section 3.2.2 - although it certainly disagrees with the practice of having short campaigns with deep discounts.
- *Postpromotional effect.* Recall that a c -parameter measures the impact of a promotion on postpromotional category demand. The result in (24) states that the more damaging the postpromotional effects, the longer the duration of a promotion. The intuition here could be that a promotion is prolonged in order to avoid the harmful postpromotional effects. Further, from the postoptimality analysis in Section 3.2.2 we know that discounts are lower, the more damage a promotion does to postpromotional demands. Thus, if postpromotional effects are significant, the duration of a promotion is longer and the discount is smaller. Yet another explanation is obtained from the term in middle in (24). The (nonnegative) difference in brackets is the difference between category revenue in the period before and in the period after a promotion. The revenue before the promotion can be viewed as a benchmark and hence, the closer the category revenue after a promotion is to the benchmark, the shorter the duration of the promotion.
- *Relationship between durations.* To compare the durations of the promotions, consider the inequalities

$$\theta_2 - \theta_1 \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \eta_2 - \eta_1 \iff c_\theta D_\theta \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} c_\eta D_\eta, \quad (26)$$

derived from (25). Suppose that $\bar{p}_1 = \bar{p}_2$. The upper inequalities, for instance, in (26) then show that brand 1 is promoted for a longer period of time than brand 2 if $c_\theta > c_\eta$ and $D_\eta = D_\theta$. Thus, when discounts are equal, the brand with the largest impact on postpromotion category demand should be promoted for the longer period of time. The intuition here is derived from a previous one, that the larger the postpromotional effect, the longer the duration of a promotion. On the other hand, if $D_\theta > D_\eta$ and $c_\theta = c_\eta$, the impacts of promotions on postpromotion demands are the same and the retailer should promote for the longer period of time the brand that has the deepest discount.

- *Time interval between promotions.* Using (2) and (5) yields

$$\frac{\partial(\eta_1 - \theta_2)}{\partial\sigma_{2\theta}} = -\frac{1}{a}D_\theta(\bar{p}_2 - D_\eta) < 0.$$

Thus, time interval between the two promotions becomes shorter when the parameter $\sigma_{2\theta}$ increases. Hence, if the impact of a promotion of brand 1 on postpromotion demand for brand 2 is sizeable, the promotion of brand 2 should start shortly after the end of the promotion of brand 1, in order to counterbalance the postpromotional effects of the first promotion.

Recalling that K^* and K^o are the category revenues during the promotion of brands 1 and 2, respectively, we get by using (23)

$$\eta_1 - \theta_2 = \frac{1}{a}(K^o - K^*),$$

and hence

$$K^o \left\{ \begin{array}{l} > \\ = \\ < \end{array} \right\} K^* \implies \eta_1 \left\{ \begin{array}{l} > \\ = \\ = \end{array} \right\} \theta_2. \quad (27)$$

The inequalities in (27) state the following. If $K^o > K^*$, that is, the category revenue earned by promoting brand 2 exceeds that of brand 1, then there is no promotion in the time interval $[\theta_2, \eta_1]$. If $K^o \leq K^*$, the category revenue obtained from promoting brand 2 falls short of that of brand 1. Then the retailer should start promoting brand 2 at the instant of time where the promotion of brand 1 ends.

3.3 Myopic vs. Forward-looking Retailer

In Proposition 1 we derived a relationship between the optimal discounts d_θ^* and d_η^* of a forward-looking retailer:

$$d_\eta^* = \frac{1}{2\beta_2} \left[\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2 + \sigma_{2\theta}d_\theta^* - c_\eta \frac{T - \eta_2}{\eta_2 - \eta_1} \right]. \quad (28)$$

From (9), we have a relationship between the myopic discounts:

$$\delta_\eta^* = \frac{1}{2\beta_2} [\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2 + \sigma_{2\theta}\delta_\theta^*]. \quad (29)$$

The two relationships in (28) and (29) are similar, with the exception that the term $-c_\eta \frac{T - \eta_2}{\eta_2 - \eta_1}$ is missing in (29). The presence of this term in (28) is due to the fact that the retailer is forward-looking; she takes into account that the second discount decision has ramifications beyond the time instant η_2 .

If $T = \eta_2$, then d_η^* equals δ_η^* , which is intuitive: if the second promotion ends at the horizon date, a forward-looking retailer does not need to account for postpromotional effects and hence can behave myopically. (Clearly, this is an end-game phenomenon).

Using (29) and (28) shows that for both types of retailer behavior the following is true. If the first discount is increased by one dollar, the second discount is increased by $\frac{\sigma_2\theta}{2\beta_2}$ dollars. Under the assumption $\beta_2 > \sigma_2\theta$, introduced in (19), the increase of the second promotion will be less than one dollar.

If, for any reason, it happens that the first discount has the same size in the two cases ($\delta_\theta = d_\theta$), then the second discount is lowest for a forward-looking retailer. This illustrates a main tenet of dynamic behavior that being forward-looking is being cautious. To account for the postpromotion impacts of the second promotion, a forward-looking retailer modifies the myopic decision rule and sets a smaller discount in the second promotion.

4 Conclusion, Limitations and Future Research

The paper has suggested a dynamic planning model for a retailer who wishes to determine optimal discounts, timing, and duration of promotions for two brands in a specific category. The model also reports a key performance measure, category profits, in each time period as well as payoffs-to-go from the start of each time period. We have calculated optimal discounts (if any) on each brand, and their timing and duration. Another main contribution of our work is the postoptimality analysis, where the dependence of optimal promotional decisions on demand characteristics was studied.

In order to focus on the depths and timing of optimal discounts, and to make an analytical solution possible, a number of simplifications were made in the modeling process. Among the simplifying assumptions are the following:

- The driving force behind retail price promotions, trade deals, has been omitted. Clearly, the availability, the duration, and the magnitude of a trade deal influences the design of the retailer's price promotions. An interesting modification of the model would be to superimpose a (parsimonious) trade deal pattern on the five-period planning horizon, to see how the design of such a scheme would impact the retailer's promotional decisions. Clearly, such an extension is complicated - also from the reason that the issue of the retailer's pass-through of trade deals needs to be modeled
- Demand functions are linear. Since the shape of the deal effect curve has no empirical support (Blattberg et al. (1995)), it seems worthwhile to extend the model to include nonlinear demand specifications, and to start empirical work to try to assess the shape of demand functions during promotional periods
- Feature advertising and display activities do not influence demand during a promotion. To remedy this, one would need to incorporate the levels of these activities as decision variables that affect demand levels during promotions. A serious obstacle here is the lack of knowledge of the simultaneous effects of advertising, displays, and discounts on demand. Also in this area there is a clear need for empirical research

- The setup only allows for at most one promotion of each of the brands. A more elaborate framework would involve a longer planning period such that multiple promotions of each brand are possible
- The solution was derived by partitioning the overall problem into two subproblems: one of finding optimal discounts, given the durations, and another of finding optimal durations, given the discounts. Although this may resemble the planning procedure of retailers in real life, it could be interesting to confront this solution with one obtained by solving for the six decision variables simultaneously. As already said, such a solution cannot be found by analytical methods. It is, however, possible to solve the problem by numerical methods. This is a topic of ongoing research of the authors.

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5 Appendix

The appendix presents the solution of the dynamic programming problem in Section 3.2.1.

Denote by J_i the value function at stage $i \in \{1, 2, \dots, 5\}$. The value function measures the optimal profit-to-go as of stage i and will be calculated in backward time.

Stage 5 starts at time $t = \eta_2$ and we have

$$J_5 = (T - \eta_2)\hat{K} = (T - \eta_2)(\bar{K} - c_\theta d_\theta - c_\eta d_\eta).$$

The profit J_5 simply is the length of the last time interval times the revenue rate \hat{K} that prevails after the second promotion.

At **stage 4** we find J_4 from

$$\begin{aligned} J_4 = & -A_\eta + \max_{d_\eta \geq 0} \{(\eta_2 - \eta_1)K^o + J_5\} = -A_\eta + \\ & \max_{d_\eta \geq 0} \{(\eta_2 - \eta_1)[\bar{p}_1(\bar{q}_1 - \sigma_{1\theta}d_\theta - \varepsilon_1 d_\eta) + \\ & (\bar{p}_2 - d_\eta)(\bar{q}_2 - \sigma_{2\theta}d_\theta + \beta_2 d_\eta)] + J_5\}. \end{aligned}$$

Note that K^o is a strictly concave function of d_η . Performing the maximization on the right-hand side then yields the optimal discount of brand 2. Whenever positive, it is given by

$$d_\eta^* = \frac{1}{2\beta_2} \left[\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2 + \sigma_{2\theta}d_\theta - c_\eta \frac{T - \eta_2}{\eta_2 - \eta_1} \right],$$

and using d_η^* one can calculate J_4 , the optimal profit-to-go as of the start η_1 of the second promotion:

$$\begin{aligned} J_4 = & -A_\eta - (T - \eta_2)c_\eta d_\eta^* + (T - \eta_1)\bar{K} + \\ & (\eta_2 - \eta_1)[(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2)d_\eta^* - c_\theta d_\theta + \sigma_{2\theta}d_\theta d_\eta^* - \beta_2 d_\eta^{*2}], \end{aligned}$$

At **stage 3** we determine J_3 , the profit-to-go as of time θ_2 :

$$\begin{aligned} J_3 &= (\eta_1 - \theta_2)\tilde{K} + J_4 = \\ & -A_\eta - (T - \eta_2)c_\eta d_\eta^* + \\ & (\eta_2 - \eta_1)[(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2)d_\eta^* + \sigma_{2\theta}d_\theta d_\eta^* - \beta_2(d_\eta^*)^2] + \\ & (T - \theta_2)[\bar{K} - c_\theta d_\theta]. \end{aligned}$$

At **stage 2** we determine J_2 , the optimal profit-to-go as of time θ_1 :

$$\begin{aligned} J_2 &= -A_\theta + \max_{d_\theta \geq 0} \{(\theta_2 - \theta_1)K^* + J_3\} = -A_\theta + \\ & \max_{d_\theta \geq 0} \{(\theta_2 - \theta_1)[(\bar{p}_1 - d_\theta)(\bar{q}_1 + \beta_1 d_\theta) + \bar{p}_2(\bar{q}_2 - \varepsilon_2 d_\theta)] + J_3\}. \end{aligned}$$

Note that K^* is strictly concave in d_θ . Performing the indicated maximization provides the optimal discount for brand 1:

$$d_\theta^* = \frac{\Gamma_\theta}{(\eta_2 - \eta_1)(\sigma_{2\theta})^2 - (\theta_2 - \theta_1)4\beta_1\beta_2},$$

in which the numerator is a constant, given by

$$\begin{aligned} \Gamma_\theta &= (T - \eta_2)\sigma_{2\theta}c_\eta + (T - \theta_2)2\beta_2c_\theta - \\ & (\theta_2 - \theta_1)(\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1)2\beta_2 - \\ & (\eta_2 - \eta_1)(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2)\sigma_{2\theta}. \end{aligned}$$

Then J_2 can be calculated:

$$\begin{aligned} J_2 &= -A_\theta - A_\eta - (T - \eta_2)c_\eta d_\eta^* + \\ & (\eta_2 - \eta_1)[(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2)d_\eta^* + \sigma_{2\theta}d_\theta^* d_\eta^* - \beta_2(d_\eta^*)^2] - \\ & (T - \theta_2)c_\theta d_\theta^* + \\ & (\theta_2 - \theta_1)[(\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1)d_\theta^* - \beta_1(d_\theta^*)^2] + \\ & (T - \theta_1)\bar{K}. \end{aligned}$$

Finally, at **stage 1** we have J , the profit for the entire planning period:

$$\begin{aligned} J &= \theta_1\bar{K} + J_2 = \\ & T\bar{K} - [A_\theta + A_\eta] - [(T - \eta_2)c_\eta d_\eta^* + (T - \theta_2)c_\theta d_\theta^*] + \\ & (\eta_2 - \eta_1)[(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2)d_\eta^* + \sigma_{2\theta}d_\theta^* d_\eta^* - \beta_2(d_\eta^*)^2] + \\ & (\theta_2 - \theta_1)[(\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1)d_\theta^* - \beta_1(d_\theta^*)^2]. \end{aligned}$$

Note that the optimal discount d_η^* on brand 2 was expressed as a function of the discount d_θ on brand 1. To have d_η^* in terms of the parameters only, insert d_θ^* into d_η^* , and use (11), to obtain

$$d_\eta^* = \frac{\Gamma_\eta}{(\eta_2 - \eta_1)(\sigma_{2\theta})^2 - (\theta_2 - \theta_1)4\beta_1\beta_2}$$

in which Γ_η is a constant, given by

$$\begin{aligned} \Gamma_\eta = & (\theta_2 - \theta_1) \frac{(T - \eta_2)2\beta_1 c_\eta}{\eta_2 - \eta_1} + (T - \theta_2)\sigma_{2\theta} c_\theta - \\ & (\theta_2 - \theta_1)(\bar{p}_1\beta_1 - \bar{p}_2\varepsilon_2 - \bar{q}_1)\sigma_{2\theta} - \\ & (\theta_2 - \theta_1)(\bar{p}_2\beta_2 - \bar{p}_1\varepsilon_1 - \bar{q}_2)2\beta_1. \end{aligned}$$

Finally, inserting the optimal discounts into J yields the optimal value of the retailer's objective function.

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