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Empirical Rank Processes for Detecting Dependence

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Abstract

Exploiting an overlooked observation of Blum, Kiefer & Rosenblatt (1961), Dugué (1975) and Deheuvels (1981a) described a decomposition of empirical distribution processes into a finite sum of asymptotically mutually independent terms whose limiting distribution is simple under the hypothesis that a multivariate distribution is equal to the product of its marginals. This paper revisits this idea and proposes to test independence using a combination of Cramér-von Mises statistics arising from the decomposition of the empirical copula process, which involves only the ranks of the observations. Asymptotic and finite-sample tables of critical values are provided for carrying out the test, based on Fisher’s method of combining P -values. While the new statistic is inferior to the standard likelihood ratio test for multivariate normal data, Monte Carlo simulations show that it can be much more powerful than the latter when the marginal distributions of the data or their underlying dependence structure are non-normal. Using the canonical decomposition of Dugué and Deheuvels, a graphical device called a “dependogram” is also proposed which helps identify the dependence structure when the null hypothesis is rejected. The mathematical exposition, which is based on recent work of Ghoudi, Kulperger & Rémillard (2001), allows for a simultaneous treatment of the serial and non-serial case. It is shown, among other things, that the asymptotic distribution of rank statistics based on the empirical copula process is the same in both cases, thereby shedding new light on the theory of nonparametric tests of serial dependence initiated by Hallin, Ingenbleek & Puri (1985).

Keywords: Copula, Cramér-von Mises statistic, empirical process, Möbius inversion formula, pseudo-observations, semi-parametric models, serial dependence, tests of independence.

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Résumé

Exploitant une importante remarque de Blum, Kiefer & Rosenblatt (1961) qui a été négligée par la suite, Dugué (1975) and Deheuvels (1981a) décrivent une décomposition d'un processus empirique en une somme finie de termes qui sont asymptotiquement indépendants et dont la limite s'exprime simplement, sous l'hypothèse que la fonction de répartition multivariée est le produit de ses marginales. Cet article jette un nouveau regard sur cette idée et propose de tester l'hypothèse d'indépendance en utilisant une combinaison de statistiques du type Cramé-von Mises provenant de la décomposition de la copule empirique, qui s'écrit en fonction des rangs des observations. Des tableaux pour les valeurs critiques d'un test basé sur la méthode de Fisher pour la combinaison de probabilités critiques sont donnés, dans le cas de petits échantillons ainsi que dans le cas limite. Comme on pouvait s'y attendre, le nouveau test proposé est inférieur au test du rapport de vraisemblance dans le cas multivarié gaussien. Des simulations Monte Carlo montrent qu'il peut être beaucoup plus puissant lorsque les lois marginales ou la copule ne sont pas gaussiennes. Utilisant la décomposition canonique proposée par Dugué et Deheuvels, un outil graphique appelé "dépendogramme" est proposé afin de permettre l'identification de structures de dépendance lorsque l'hypothèse nulle est rejetée. La méthodologie mathématique, basée sur le récent article de Ghoudi, Kulperger & Rémillard (2001), permet le traitement simultané des cas sériels et non sériels. En outre, il est montré que la loi asymptotique de statistiques de rangs basées sur la copule empirique est la même dans les deux cas, expliquant ainsi les résultats de la théorie des tests non paramétriques dans le cas de séries chronologiques initiés par Hallin, Ingenbleek & Puri (1985).

1 Introduction

In a seminal paper concerned with testing the null hypothesis of independence between the $p \geq 2$ components of a multivariate vector with continuous distribution H and marginals F_1, \dots, F_p , Blum, Kiefer & Rosenblatt (1961) investigated the use of a Cramér-von Mises statistic derived from the process

$$\mathbb{H}_n(t) = \sqrt{n} \left\{ H_n(t) - \prod_{j=1}^p F_{nj}(t_j) \right\}, \quad t = (t_1, \dots, t_p) \in \mathbb{R}^p$$

that measures the difference between the empirical distribution function (EDF) H_n of H and the product of the marginal EDFs F_{jn} associated with the p components of the random vector. As Hoeffding (1948) had already noted, the asymptotic distribution of this test is generally not tractable, and hence tables of critical values are required for its use. Such tables were provided by Blum *et al.* (1961) themselves in the case $p = 2$, and were later expanded to $p \geq 3$ by Cotterill & Csörgő (1982, 1985), based on strong approximations of \mathbb{H}_n . See also Jing & Zhu (1996) for a bootstrap approach.

Despite this work and the anticipation that the Cramér-von Mises statistic

$$\int \mathbb{H}_n^2 dH_n$$

should be powerful, most subsequent research focussed on the case $p = 2$, where alternative tests (typically based on moment characterizations of independence) were proposed by Feuerverger (1993), Shih & Louis (1996), Gieser & Randles (1997) and Kallenberg & Ledwina (1999), among others.

Curiously, the literature seems to have largely ignored a suggestion of Blum *et al.* (1961) to circumvent the inconvenience caused by the complex nature of the limiting distribution of \mathbb{H}_n . To be specific, let $X_1 = (X_{11}, \dots, X_{1p}), \dots, X_n = (X_{n1}, \dots, X_{np})$ be a random sample from distribution H , and for arbitrary $A \subset S_p = \{1, \dots, p\}$ with $|A| > 1$, consider the empirical process

$$G_{A,n}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \prod_{j \in A} \{\mathbb{I}(X_{ij} \leq t_j) - F_{nj}(t_j)\}.$$

Using Möbius' inversion formula (cf., e.g., Spitzer 1974, p. 127), Blum, Kiefer and Rosenblatt showed that \mathbb{H}_n may be conveniently expressed as

$$\mathbb{H}_n(t) = \sum_{A \subset S_p, |A| > 1} G_{A,n}(t) \prod_{j \in S_p \setminus A} F_{n,j}(t_j).$$

Although their paper only discussed the case $p = 3$, these authors claimed (and this was later confirmed by Dugué 1975) that under the hypothesis of independence, $G_{A,n}$ converges weakly to a continuous centered Gaussian process with covariance function

$$\text{cov}_A(s, t) = \prod_{j \in A} [\min\{F_j(s_j), F_j(t_j)\} - F_j(s_j)F_j(t_j)]$$

whose eigenvalues, given by

$$\frac{1}{\pi^{2|A|}(i_1 \cdots i_{|A|})^2}, \quad i_1, \dots, i_{|A|} \in \mathbb{N} = \{1, 2, \dots\}$$

may be deduced from the Karhunen-Loève decomposition of the Brownian bridge. More importantly still, Blum *et al.* (1961) and Dugué (1975) pointed out that the processes $G_{A,n}$ and $G_{A',n}$ are mutually independent asymptotically whenever $A \neq A'$, so that Cramér-von Mises statistics based on the individual $G_{A,n}$'s could be combined to construct suitable statistics for testing against independence.

An obvious limitation of tests based on this approach, however, is the dependence of the asymptotic null distribution of the $G_{A,n}$'s on the marginals of H . To alleviate this problem, Deheuvels (1981a) suggested that the original observations X_1, \dots, X_n be replaced by their associated rank vectors $R_1 = (R_{11}, \dots, R_{1p}), \dots, R_n = (R_{n1}, \dots, R_{np})$. He then went on to characterize the asymptotic behavior of a Möbius decomposition of the copula process

$$\mathbb{C}_n(t) = \sqrt{n} \left\{ C_n(t) - \prod_{j=1}^p t_j \right\}, \quad (1)$$

where

$$C_n(t_1, \dots, t_p) = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^p \mathbb{I}(R_{ij} \leq nt_j)$$

is an estimation of the unique copula C defined implicitly by

$$C\{F_1(t_1), \dots, F_p(t_p)\} = H(t_1, \dots, t_p).$$

The latter reduces to $C(t_1, \dots, t_p) = t_1 \cdots t_p$ under the null hypothesis of independence. Although Cramér-von Mises and Kolmogorov-Smirnov statistics based on \mathbb{C}_n 's decomposition were then proposed by Deheuvels (1982a,b,c), he did not compute the quantiles for the asymptotic or finite-sample distribution of these statistics; nor did he mention how the $2^p - p - 1$ statistics derived from the rank analogues of the $G_{A,n}$'s could be combined to obtain a global statistic for testing independence.

The first objective of this paper is to fill this gap. In Section 2, unbiased variants of Deheuvels' rank-based processes $\mathbb{G}_{A,n}$ are introduced, and their asymptotic theory is briefly recapped. Two strategies for testing the null hypothesis of independence are then described in Section 3. Since the Cramér-von Mises functional of the $\mathbb{G}_{A,n}$ has a simple limiting distribution that depends only on the size of A , one possibility is to fix a global level for the test and to check for lack of independence in each subset A of variables at appropriate subsidiary levels. A table of 5% critical values is provided for this procedure, which allows for visual determination of specific violations of the independence assumption. A second option consists of combining the p -values of the $2^p - p - 1$ tests into a single statistic that is asymptotically distributed as a chi-square random variable with that many degrees of freedom. As shown in Section 6, this statistic turns out to be quite powerful in

a variety of settings, using as a benchmark the classical likelihood test statistic based on the multivariate Gaussian model.

The second goal of the paper, pursued in Section 4, is to show how these developments can be adapted easily to test against randomness in a time series context. Indeed, if vectors

$$X_i = (Y_i, \dots, Y_{i+p-1}), \quad 1 \leq i \leq n - p + 1 \quad (2)$$

are constructed from p consecutive values of a stationary univariate time series Y_1, Y_2, \dots , recent results of Ghoudi, Kulperger & Rémillard (2001) imply that under the null hypothesis of white noise, the serial and non-serial versions of the empirical rank processes $\mathbb{G}_{A,n}$ have the same asymptotic behavior when restricted to subsets A with $1 \in A$ and $|A| > 1$. In other words, the fact that two successive values of X_i have $p-1$ components in common (shifted by one) has no impact, in the limit. Given that many of the linear rank statistics traditionally used for testing independence are simple continuous functionals of the $\mathbb{G}_{A,n}$'s, it is not surprising, therefore, that serialized and non-serialized versions of Spearman's rho and other rank statistics (Laplace, van der Waerden, Wilcoxon, etc.) share the same limiting distribution. This point, which is made in Section 5, sheds new light on earlier results of Hallin, Ingenbleek & Puri (1985) obtained in the case $|A| = 2$.

For a general survey of the literature on rank tests of independence, see Hallin & Puri (1992) or Hallin & Werker (1999). For specific tests of serial dependence based on the Blum-Kiefer-Rosenblatt approach, refer to Skaug & Tjøstheim (1993) or Tjøstheim (1996). Other recent contributions are those of Kulperger & Lockhart (1998), Hong (1998, 2000), and Genest, Quessy & Rémillard (2001).

2 Set-indexed decomposition of the copula process

Let X_1, \dots, X_n be independent random vectors in \mathbb{R}^p with continuous distribution function H and marginals F_j , $1 \leq j \leq p$. Let also R_1, \dots, R_n be the associated ranks vectors. A simple application of Möbius' formula implies that the empirical copula process \mathbb{C}_n may be expressed in the form

$$\mathbb{C}_n(t) = \sum_{A \subset S_p, |A| > 1} \mathbb{G}_{A,n}(t) \prod_{j \in S_p \setminus A} t_j$$

in terms of set-indexed processes

$$\mathbb{G}_{A,n}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{ \prod_{j=1}^p \mathbb{I}(R_{ij} \leq nt_j) - \prod_{j=1}^p t_j \right\}$$

defined for arbitrary $A \subset S_p$ with $|A| > 1$. Following Blum *et al.* (1961) and Dugué (1975), Deheuvels (1981a,b,c) proved that under the null hypothesis of independence, $\mathbb{G}_{A,n}$ converges weakly to a continuous centered Gaussian process \mathbb{G}_A with covariance function

$$\Gamma_A(s, t) = \prod_{j \in A} \{\min(s_j, t_j) - s_j t_j\}. \quad (3)$$

Furthermore, he showed that $\mathbb{G}_{A,n}$ and $\mathbb{G}_{A',n}$ are asymptotically independent whenever $A \neq A'$, so that Cramér-von Mises or Kolmogorov-Smirnov statistics based on the $\mathbb{G}_{A,n}$'s could be combined into a global test of independence.

It is worth pointing out in passing that if

$$t^B = \begin{cases} t_j & \text{if } j \in B, \\ 1 & \text{if } j \notin B, \end{cases}$$

the asymptotic processes may actually be expressed as

$$\mathbb{G}_A(t) = \sum_{B \subset A} (-1)^{|A \setminus B|} \mathbb{C}(t^B) \prod_{j \in A \setminus B} t_j$$

in terms of the limit \mathbb{C} in $D([0, 1]^p)$ of the continuous functional

$$\mathbb{H}_n\{F_1^{-1}(t_1), \dots, F_p^{-1}(t_p)\}$$

of the Blum-Kiefer-Rosenblatt process that would be obtained by applying the probability integral transformation to the marginal distributions of the X_i 's, if they were known. Alternatively, \mathbb{C} in the above expression can be replaced by the limit in $D([0, 1]^p)$ of (1), viz.

$$\tilde{\mathbb{C}}(t) = \mathbb{C}(t) - \sum_{j=1}^p \beta_j(t_j) \prod_{k \neq j} t_k \quad (4)$$

which Deheuvels (1979, 1980), Stute (1984) and Gänssler & Stute (1987) showed to be a continuous centered Gaussian process involving p independent Brownian bridges β_1, \dots, β_p connected to \mathbb{C} through the relations

$$\beta_1(u) = \mathbb{C}(u, 1, \dots, 1), \dots, \beta_p(u) = \mathbb{C}(1, \dots, 1, u).$$

The occurrence of this linear combination of Brownian bridges in the limiting process (4) is a consequence of the fact that these marginals are not known.

Section 3 describes how it is possible to exploit the asymptotic independence of the $\mathbb{G}_{A,n}$'s to construct diagnostics and powerful tests of independence. In order to ensure the quality of the asymptotic approximation in small samples, however, it is actually preferable to work with zero-mean versions of the $\mathbb{G}_{A,n}$'s defined by

$$\mathbb{G}_{A,n}^*(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \prod_{j \in A} [\mathbb{I}\{R_{ij} \leq (n+1)t_j\} - U_n(t_j)],$$

where U_n is the distribution function of a discrete random variable U uniformly distributed on the set $\{1/(n+1), 2/(n+1), \dots, n/(n+1)\}$, i.e.,

$$U_n(u) = \min \left\{ \frac{[(n+1)u]}{n}, 1 \right\} \quad (5)$$

with $[(n+1)u]$ standing for the integer part of $(n+1)u$.

The limiting behavior of the processes $\mathbb{G}_{A,n}^*$ is given in the following extension of Theorem 2 in Deheuvels (1981a).

PROPOSITION 1. *Under the null hypothesis of independence, the processes $\mathbb{G}_{A,n}^*$ defined for arbitrary $A \subset S_p$ with $|A| > 1$ converge weakly to the mutually independent continuous centered Gaussian processes \mathbb{G}_A with covariance function (3).*

3 Tests statistics for independence

As is well known, p events E_1, \dots, E_p are mutually independent if and only if for all $A \subset S_p = \{1, \dots, p\}$,

$$P\left(\bigcap_{i \in A} E_i\right) = \prod_{i \in A} P(E_i).$$

Likewise, a distribution H is equal to the product of its marginals if and only if its associated copula C verifies the condition

$$\sum_{B \subset A} (-1)^{|A \setminus B|} C(t^B) \prod_{k \in A \setminus B} t_k = 0, \quad (6)$$

for all $t \in [0, 1]^p$ and $A \subset S_p$, as proved in Ghoudi *et al.* (2001, Proposition 2.1). Condition (6) is referred to as A -independence by Deheuvels (1981a). Now since it is generally true that

$$\prod_{i \in A} (x_i + y_i) = \sum_{B \subset A} \left(\prod_{i \in B} x_i \right) \prod_{j \in A \setminus B} y_j,$$

it follows at once from the ergodic theorem that for all $t \in [0, 1]^p$,

$$\begin{aligned} \mathbb{E} \left\{ \frac{\mathbb{G}_{A,n}^*(t)}{\sqrt{n}} \right\} &= \mathbb{E} \left[\prod_{j \in A} [\mathbb{I}\{R_{1j} \leq (n+1)t_j\} - U_n(t_j)] \right] \\ &= \sum_{B \subset A} (-1)^{|A \setminus B|} \mathbb{E} \left[\prod_{j \in A \setminus B} \mathbb{I}\{R_{1j} \leq (n+1)t_j\} \right] \prod_{k \in A \setminus B} U_n(t_k) \end{aligned}$$

tends to the left-hand term of (6). It seems reasonable, therefore, to reject the hypothesis of independence whenever the process $\mathbb{G}_{A,n}^*$ is significantly different from zero for some $A \subset S_p$.

This suggests basing a test of independence on a combination of statistics involving all the $\mathbb{G}_{A,n}^*$'s. For a fixed A , two obvious possibilities are the Cramér-von Mises statistic

$$T_{A,n} = \int_{[0,1]^p} \{\mathbb{G}_{A,n}^*(t)\}^2 dt, \quad (7)$$

and the Kolmogorov-Smirnov statistic

$$S_{A,n} = \sup_{t \in [0,1]^p} |\mathbb{G}_{A,n}^*(t)|. \quad (8)$$

Although the null distribution of these rank-based statistics can easily be tabulated for any sample size, this paper focuses on the use of the $T_{A,n}$'s because of their convenient asymptotic distribution, given below. See Deheuvels (1981a, Section 3) for a proof of this result.

PROPOSITION 2. *Under the null hypothesis of independence, the limiting distribution of $T_{A,n}$ is the same as that of $\xi_{|A|}$, where*

$$\xi_k = \sum_{(i_1, \dots, i_k) \in \mathbb{N}^k} \frac{1}{\pi^{2k} (i_1 \dots i_k)^2} Z_{i_1, \dots, i_k}^2,$$

and the Z_{i_1, \dots, i_k} 's are independent $N(0, 1)$ random variables.

In the light of the above discussion, a reasonable testing procedure consists of rejecting independence whenever the observed value $t_{A,n}$ of at least one of the $T_{A,n}$'s is too large. There are two natural ways to go about this: either critical values could be chosen for each of the statistics so as to achieve a predetermined global level, or the p -values of these statistics could be computed and then combined. The first option, considered in Section 3.1, leads naturally to a diagnostic test that makes it possible to identify failures of the independence assumption. As will be seen, however, the alternative test described in Section 3.2 turns out to be more powerful.

3.1 Fixing a global level

One option consists of rejecting the null hypothesis of independence at approximate level α if $T_{A,n} = t_{A,n} > c_A$ for some $A \subset S_p$, where $c_A = c_{|A|}$ is chosen so that

$$\begin{aligned} \mathbb{P} \left(\bigcup_{A \subset S_p, |A| > 1} \{T_{A,n} > c_A\} \right) &\approx 1 - \prod_{A \subset S_p, |A| > 1} \mathbb{P}(\xi_{|A|} \leq c_{|A|}) \\ &= 1 - \prod_{k=2}^p \{\mathbb{P}(\xi_k \leq c_k)\}^{\binom{p}{k}} = \alpha, \end{aligned}$$

where the first identity is justified by the fact that the limiting distribution $\xi_{|A|}$ of $T_{A,n}$ depends only on the size of A and $T_{A,n} \equiv 0$ for $|A| \leq 1$.

TABLE 1. Critical values c_2, \dots, c_p for the global test of independence at level $\alpha = 5\%$ for a p -variate distribution, $p = 2, \dots, 5$, based on a random sample of size $n = 20, 50, 100, 250$ and $n \rightarrow \infty$.

$n = 20$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
c_2	0.052693	0.071797	0.085028	0.096548
c_3		0.007985	0.009174	0.010018
c_4			0.001184	0.001288
c_5				0.000189
<hr/>				
$n = 50$				
c_2	0.056104	0.078200	0.094521	0.109683
c_3		0.009339	0.010883	0.012303
c_4			0.001382	0.001516
c_5				0.000205
<hr/>				
$n = 100$				
c_2	0.057534	0.080452	0.097565	0.112749
c_3		0.009768	0.011405	0.012908
c_4			0.001450	0.001611
c_5				0.000216
<hr/>				
$n = 250$				
c_2	0.058256	0.081890	0.100115	0.115406
c_3		0.010071	0.012099	0.013757
c_4			0.001500	0.001668
c_5				0.000221
<hr/>				
$n = \infty$				
c_2	0.059279	0.084364	0.102630	0.117481
c_3		0.010358	0.012056	0.013358
c_4			0.001511	0.001628
c_5				0.000214

It is especially convenient to choose the c_k 's in such a way that $P(\xi_k \leq c_k) = \beta$ for all $2 \leq k \leq p$, whence

$$\alpha = 1 - \beta^{\sum_{k=2}^p \binom{p}{k}},$$

that is, $\beta = (1 - \alpha)^{1/(2^p - p - 1)}$. Table 1 gives the values of c_2, \dots, c_p for $p = 2, \dots, 5$ and $\alpha = 1, 5, 10\%$. These values were obtained from the Cornish-Fisher asymptotic expansion of the distribution of ξ_k using its first six cumulants given by

$$\kappa_m = \frac{2^{m-1}(m-1)!}{\pi^{2km}} \zeta(2m)^k, \quad 1 \leq m \leq 6,$$

and $\zeta(\cdot)$ denotes Riemann's zeta function.

Besides the fact that it gives the same weight to all the $T_{A,n}$'s, choosing $P(\xi_k \leq c_k) = \beta$ for all $2 \leq k \leq p$ allows for graphical representation of the test in the same spirit as the

classical correlogram. On the x -axis of what can be called a “dependogram,” the subsets A are ordered lexicographically by size, beginning with $|A| = 2$. The corresponding values of $T_{A,n}$ are represented by vertical bars, and horizontal lines placed at height $c_{|A|}$ make it easy to identify subsets leading of the rejection of the independence hypothesis.

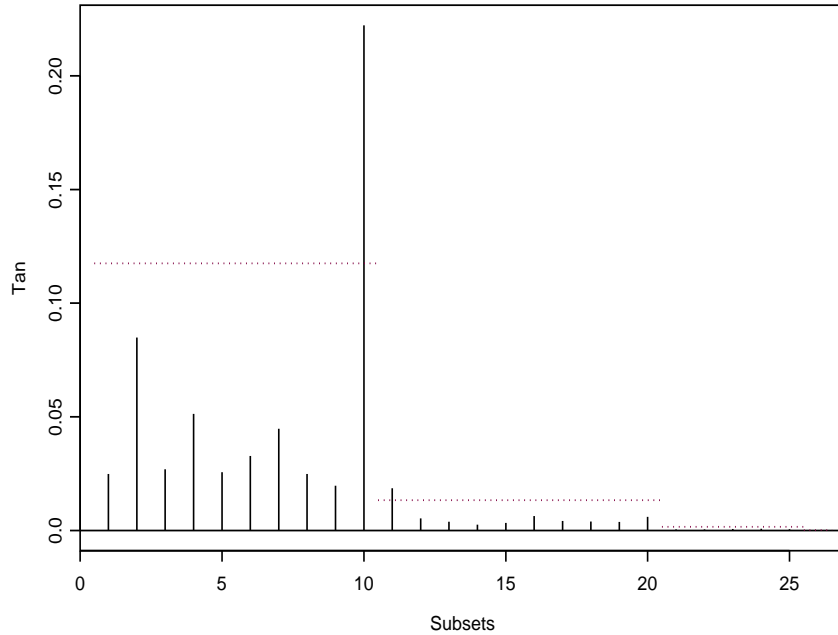


FIGURE 1. Dependogram of asymptotic level $\alpha = 5\%$ constructed for a random sample of size $n = 50$ from a vector (X_1, \dots, X_5) with standard normal marginals, pairwise but not joint independence between X_1, X_2, X_3 , and $\text{corr}(X_4, X_5) = 1/2$. See end of Section 3.1 for details of construction.

The construction of the dependogram is illustrated in Figure 1 for a random sample of size $n = 50$ from a vector (X_1, \dots, X_5) with standard normal marginals with $X_1 = |Z_1|\text{sign}(Z_2Z_3)$, $X_2 = Z_2$, $X_3 = Z_3$, $X_4 = Z_4$ and $X_5 = Z_4/2 + \sqrt{3}Z_5/2$, where the Z_i 's are mutually independent standard normal random variates. In this case, X_1, X_2, X_3 are pairwise but not jointly independent (Romano & Siegel 1985, p. 33); they are, however, independent from the pair (X_4, X_5) . As can be seen on Figure 1, the subset $A = \{4, 5\}$, which is tenth on the list, exhibits a clear dependence at approximate level $\alpha = 5\%$ between the two corresponding variables. In addition, the eleventh subset, $A = \{1, 2, 3\}$, makes apparent the joint dependence between variables X_1, X_2, X_3 which the pairwise statistics $T_{A,n}$ with $A = \{1, 2\}, \{1, 3\}$ and $\{2, 3\}$ could not possibly have detected.

3.2 Combining P -values

Under the null hypothesis of independence, the statistic $T_{A,n}$ has distribution function $F_{|A|,n}$ and the corresponding P -value

$$p_{A,n} = 1 - F_{|A|,n}(T_{A,n})$$

is asymptotically uniform on the interval $(0, 1)$. Since the $p_{A,n}$'s are also mutually independent in the limit, the combined statistic

$$T_{F,n} = -2 \sum_{A \subset S_p, |A| > 1} \log(p_{A,n}),$$

originally proposed by Fisher (1950, pp. 99–101), converges in law to a chi-square variable with $2(2^p - p - 1)$ degrees of freedom. As shown by Littell & Folks (1973), the statistic $T_{F,n}$ is asymptotically Bahadur optimal within a large class of reasonable combination procedures. Its good properties are confirmed in simulations reported in Section 6.

Table 2 below provides the $\alpha = 5\%$ critical values of $T_{F,n}$ when $n = 20, 50, 100$. Note the fairly quick convergence of the critical values to those of the chi-square with $2(2^p - p - 1)$ degrees of freedom, entered under $n = \infty$ in the table. Although it would be computationally more convenient to replace $F_{|A|,n}$ by the distribution function of $\xi_{|A|}$ identified in Proposition 2, the convergence of the statistic would then be much slower, as intensive simulations confirm.

TABLE 2. Critical points for testing independence at the $\alpha = 5\%$ level using Fisher's combined P -value test for a p -variate distribution, $p = 2, \dots, 5$, from a random sample of size $n = 20, 50, 100$ and $n \rightarrow \infty$.

	$p = 2$	$p = 3$	$p = 4$	$p = 5$
$2(2^p - p - 1)$	2	8	22	52
$n = 20$	6.00	15.08	33.79	74.23
$n = 50$	6.00	15.43	34.02	72.14
$n = 100$	5.85	15.30	33.6	70.30
$n = \infty$	5.99	15.51	33.92	69.83

4 The serial case

This section describes how the previous results must be adapted to test the white noise hypothesis for a stationary univariate time series Y_1, Y_2, \dots with continuous marginal distribution F . Tests of this hypothesis are commonly based on vectors $X_i = (Y_i, \dots, Y_{i+p-1})$ of successive data points constructed as in (2) for each $1 \leq i \leq n + 1 - p$. As noted by Delgado (1996), the empirical process

$$\mathbb{H}_n^s \{F^{-1}(t_1), \dots, F^{-1}(t_p)\} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-p+1} \left[\prod_{j=1}^p \mathbb{I}\{Y_{i+j-1} \leq F^{-1}(t_j)\} - \prod_{j=1}^p t_j \right]$$

and its rank-based analogue

$$\mathbb{C}_n^s(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-p+1} \left\{ \prod_{j=1}^p \mathbb{I}(R_{i+j-1} \leq nt_j) - \prod_{j=1}^p t_j \right\}$$

have unwieldy asymptotic limits in $D([0, 1]^p)$. The latter are given respectively by \mathbb{C}^s and $\tilde{\mathbb{C}}^s$, which are shown in Appendix A to be connected by the relation

$$\tilde{\mathbb{C}}^s(t) = \mathbb{C}^s(t) - \sum_{j=1}^p \beta(t_j) \prod_{k \neq j} t_k \quad (9)$$

with β a Brownian bridge satisfying $\beta(u) = \mathbb{C}^s(u, 1, \dots, 1)$.

Applying the Möbius transformation to \mathbb{C}_n^s allows one to write it as a weighted linear combination of subprocesses

$$\mathbb{G}_{A,n}^s(t) = \sum_{B \subset A} (-1)^{|A \setminus B|} \mathbb{C}_n^s(t^B) \prod_{j \in A \setminus B} t_j$$

which converge weakly to

$$\mathbb{G}_A^s(t) = \sum_{B \subset A} (-1)^{|A \setminus B|} \tilde{\mathbb{C}}^s(t^B) \prod_{j \in A \setminus B} t_j = \sum_{B \subset A} (-1)^{|A \setminus B|} \mathbb{C}^s(t^B) \prod_{j \in A \setminus B} t_j,$$

where the second equality is justified by the chain of identities

$$\begin{aligned} \sum_{B \subset A} (-1)^{|A \setminus B|} \sum_{j \in B} a_j &= \sum_{j \in A} a_j \sum_{B \subset A, B \ni j} (-1)^{|A \setminus B|} \\ &= \sum_{j \in A} a_j (1 - 1)^{|A| - 1} = 0 \end{aligned} \quad (10)$$

with $a_j = \beta(t_j) \prod_{k \neq j} t_k$. Following Theorem 2.2 in Ghoudi *et al.* (2001), the \mathbb{G}_A^s have the same joint distribution as the \mathbb{G}_A 's for all sets A satisfying $1 \in A$ and $|A| > 1$.

Let

$$\mathbb{C}_n^{s*}(t_1, \dots, t_p) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-p+1} \left[\prod_{j=1}^p \mathbb{I}\{R_{i+j-1} \leq (n+1)t_j\} - \prod_{j=1}^p U_n(t_j) \right].$$

Since

$$\begin{aligned} \mathbb{G}_{A,n}^{s*}(t) &= \sum_{B \subset A} (-1)^{|A \setminus B|} \mathbb{C}_n^{s*}(t^B) \prod_{j \in A \setminus B} U_n(t_j) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-p+1} \prod_{j \in A} [\mathbb{I}\{R_{i+j-1} \leq (n+1)t_j\} - U_n(t_j)] \end{aligned}$$

obviously has the same limit as $\mathbb{G}_{A,n}^s$, the serial analogue of Proposition 1 reads as follows.

PROPOSITION 3. *Under the null hypothesis of independence, the processes $\mathbb{G}_{A,n}^{s*}$ defined for arbitrary $A \subset S_p$ such that $1 \in A$ and $|A| > 1$ converge weakly to the mutually independent continuous centered Gaussian processes \mathbb{G}_A with covariance function (3).*

TABLE 3. Critical values c_2^s, \dots, c_p^s for the global test of white noise at level $\alpha = 5\%$ against dependence of order $p = 2, \dots, 6$, based on a stationary univariate time series of length $n = 20, 50, 100, 250$ and $n \rightarrow \infty$.

$n = 20$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
c_2	0.049223	0.059356	0.065056	0.070118	0.072402
c_3		0.006363	0.006876	0.007412	0.007814
c_4			0.001009	0.001087	0.001172
c_5				0.000168	0.000180
c_6					0.000031
<hr/>					
$n = 50$					
c_2	0.054372	0.069768	0.081284	0.091476	0.100569
c_3		0.008237	0.009278	0.010056	0.010866
c_4			0.001221	0.001321	0.001429
c_5				0.000202	0.000216
c_6					0.000037
<hr/>					
$n = 100$					
c_2	0.056516	0.073182	0.085884	0.097123	0.108527
c_3		0.008938	0.010237	0.011310	0.012528
c_4			0.001327	0.001451	0.001586
c_5				0.000208	0.000221
c_6					0.000036
<hr/>					
$n = 250$					
c_2	0.057427	0.076014	0.090718	0.103241	0.116332
c_3		0.009314	0.010697	0.012089	0.013313
c_4			0.001402	0.001526	0.001625
c_5				0.000210	0.000225
c_6					0.000034
<hr/>					
$n = \infty$					
c_2	0.059279	0.079103	0.094541	0.108080	0.120404
c_3		0.009855	0.011315	0.012543	0.013603
c_4			0.001441	0.001555	0.001650
c_5				0.000207	0.000216
c_6					0.000030

Arguing as in Section 3.1, a serial version of the dependogram can be based on Proposition 3 by selecting critical values c_k^s so that for each k , $P(\xi_k \leq c_k^s) = \beta = (1 - \alpha)^{1/(2^{p-1} - 1)}$. The global level of the test is then

$$\begin{aligned}
 P\left(\bigcup_{A \subset S_p, |A| > 1, 1 \in A} \{T_{A,n}^s > c_A^s\}\right) &\approx 1 - \prod_{A \subset S_p, |A| > 1, 1 \in A} P\left(\xi_{|A|} \leq c_{|A|}^s\right) \\
 &= 1 - \prod_{k=2}^p \{P(\xi_k \leq c_k^s)\}^{\binom{p-1}{k-1}} = \alpha.
 \end{aligned}$$

Table 3 gives the values of c_2^s, \dots, c_p^s for $p = 2, \dots, 6$ and $\alpha = 1, 5, 10\%$. These critical points were again obtained from the Cornish-Fisher asymptotic expansion of the distribution of ξ_k using its first few cumulants.

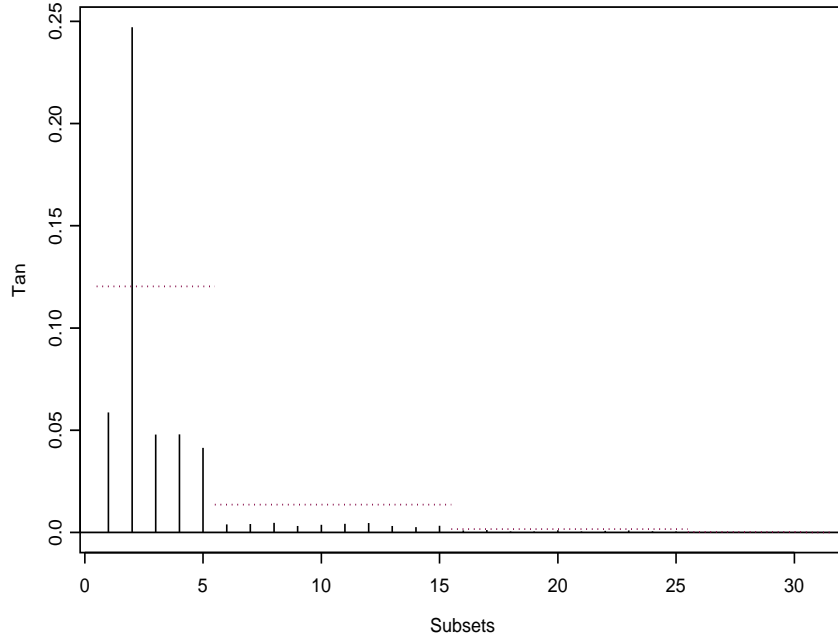


FIGURE 2. Serial dependogram of asymptotic level $\alpha = 5\%$ constructed for two intertwined AR(1) series, each of length $n = 50$ and having autocorrelation coefficient of $1/2$.

Figure 2 displays a serial dependogram with $p = 6$. This diagram allows both for a visual inspection and a formal test of the white noise hypothesis at an asymptotic nominal level $\alpha = 5\%$. The series analyzed here is composed of two intertwined AR(1) models with an autocorrelation coefficient of $1/2$ each, to ensure stationarity. The statistic $T_{A,n}$ corresponding to the set $A = \{1, 3\}$ is seen to be highly significant, as expected from the fact that observations x_t and x_{t+2} in the series are successive observations from the same AR(1) process. The fact that no other statistic exceeds its threshold is a combined effect of the quickly fading dependence within each series and the fact that the global level of the test is 5% .

As a second example, consider the deterministic “tent map” series defined by $X_{i+1} = f(X_i)$, where

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2(1-x) & \text{if } 1/2 < x \leq 1, \end{cases}$$

and X_0 is uniformly distributed on $[0, 1]$. As is well known (cf., e.g., Chatterjee & Yilmaz 1992), this series has the same autocorrelation function as white noise. In this situation, which is typical of chaotic time series, the dependence is usually stronger for small lags but slowly spreads to higher lags as the length of the series increases. This is clearly reflected in Figure 3, which depicts the serial dependogram with $p = 6$ and $\alpha = 5\%$ corresponding to the series with initial value $X_0 = 0.1789$.

Despite its interest as a graphical diagnostic tool, the test based on the serial dependogram turns out to be somewhat less efficient than Fisher's combined P -value statistic

$$T_{F,n}^s = -2 \sum_{A \subset S_p, |A| > 1, 1 \in A} \log \{1 - F_{|A|,n}(T_{A,n})\},$$

which is asymptotically chi-square with $2(2^{p-1} - 1)$ degrees of freedom asymptotically. This convergence, which is an immediate consequence of Proposition 3, is illustrated in Table 4, where $\alpha = 5\%$ critical values of $T_{A,n}^s$ are given for testing against dependence of order $p = 2, \dots, 6$ from a univariate time series of length $n = 20, 50, 100$. Observe the somewhat slow convergence of these critical points to the limiting values, especially when $p = 6$. This is due to the relatively small number of (5-dependent!) X_i vectors; for instance, there are only $n + 1 - p = 15$ data points to work with when $n = 20$ and $p = 6$!

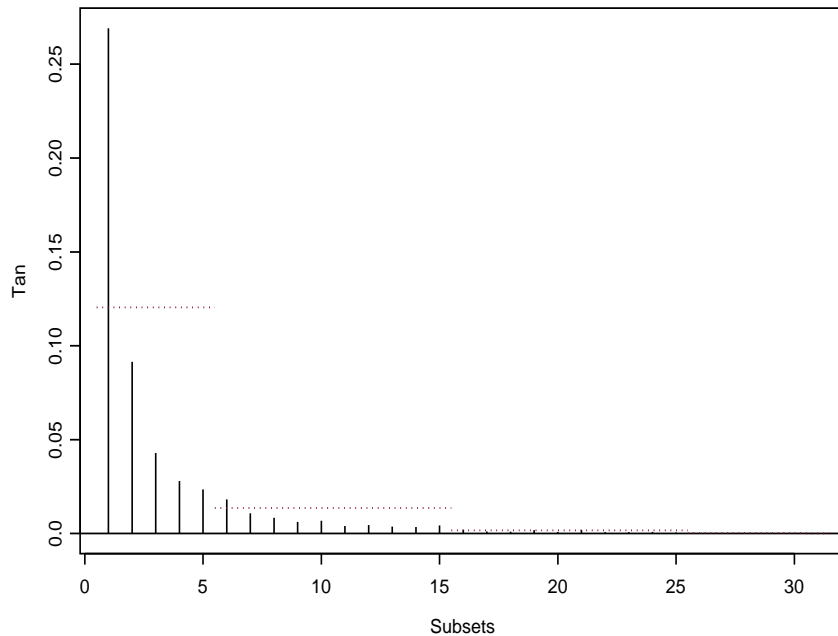


FIGURE 3. Serial dependogram of asymptotic level $\alpha = 5\%$ constructed from the deterministic tentmap series of length $n = 100$ with initial value $X_0 = 0.1789$.

TABLE 4. Critical points for Fisher's combined P -value test of white noise at the $\alpha = 5\%$ level against dependence of order $p = 2, \dots, 6$ from a univariate time series of length $n = 20, 50, 100$ and $n \rightarrow \infty$.

	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$n = 20$	8.09	16.96	35.5	73.5	145.4
$n = 50$	6.50	13.55	25.9	50.7	104.0
$n = 100$	6.28	13.05	24.9	47.3	95.6
$n = \infty$	5.99	12.59	23.69	43.77	81.38

5 Relations with standard linear rank statistics

The purpose of this section is to highlight the implications of results presented in Sections 2 and 4 for the asymptotic theory of linear rank statistics traditionally used to test independence, both in the serial and non-serial case. By a way of introduction, restrict initially to the non-serial case and for given $A = \{j, k\}$ of size 2, consider the statistic

$$\rho_{A,n} = \frac{1}{n} \sum_{i=1}^n \left(\frac{R_{ij}}{n+1} - \frac{1}{2} \right) \left(\frac{R_{ik}}{n+1} - \frac{1}{2} \right),$$

which is proportional to Spearman's rho statistic between two components X_j and X_k of a random vector (X_1, \dots, X_p) . A simple calculation shows that under the null hypothesis of independence,

$$\sqrt{n}\rho_{A,n} = \int_{[0,1]^p} \mathbb{G}_{A,n}^*(t) dt.$$

In view of Proposition 1, therefore, the $\sqrt{n}\rho_{A,n}$'s are asymptotically normally distributed with mean zero and variance $1/144 = 1/12^{|A|}$. In addition, $\rho_{A,n}$ and $\rho_{A',n}$ are asymptotically independent whenever $A \neq A'$.

As already observed by Deheuvels (1981a, Theorem 6), a distinct advantage of the above representation is that it generalizes readily to arbitrary subsets A of $S_p = \{1, \dots, p\}$, namely

$$\rho_{A,n} = \frac{1}{n} \sum_{i=1}^n \prod_{j \in A} \left(\frac{R_{ij}}{n+1} - \frac{1}{2} \right) = (-1)^{|A|} \int_{[0,1]^p} \mathbb{G}_{A,n}^*(t) dt.$$

Note in passing that the asymptotic distribution of this statistic is only equivalent to that of Kendall's tau when $|A| = 2$; cf., e.g., Barbe *et al.* (1996). A similar extension is also possible in the serial case, viz.

$$\rho_{A,n}^s = \frac{1}{n} \sum_{i=1}^{n-p+1} \prod_{j \in A} \left(\frac{R_{i+j-1}}{n+1} - \frac{1}{2} \right) = (-1)^{|A|} \int_{[0,1]^p} \mathbb{G}_{A,n}^{s*}(t) dt.$$

It follows from Propositions 1 and 3 that under the null hypothesis of independence (or randomness), $\rho_{A,n}$ and $\rho_{A,n}^s$ have the *same* asymptotic normal distribution with zero mean

and variance $1/12^{|A|}$. Furthermore, stochastic independence between $\rho_{A,n}$ and $\rho_{A',n}$ continues to hold asymptotically whenever $A \neq A'$, the same being true in the serial case (with the restriction that the element 1 must belong to the subsets considered).

The following proposition generalizes these findings to a much larger class of linear rank statistics which also includes as special cases the Laplace, van der Waerden and Wilcoxon rank statistics. This sheds new light on earlier results of Hallin, Ingenbleek & Puri (1985, 1987) for the case $|A| = 2$.

PROPOSITION 4. *Let $K = (K_1, \dots, K_p)$ be a vector whose j th component K_j is a distribution function with left-continuous inverses $L_j = K_j^{-1}$ and variance $\sigma_j^2 < \infty$. For arbitrary $A \subset S_p$ with $|A| > 1$, define*

$$\mathbb{R}^A = \left(\bigotimes_{j \in A} \mathbb{R} \right) \left(\bigotimes_{j \in A^c} [0, 1] \right)$$

and

$$J_{A,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \prod_{j \in A} \left\{ L_j \left(\frac{R_{ij}}{n+1} \right) - \bar{L}_j \right\} = (-1)^{|A|} \int_{\mathbb{R}^A} \mathbb{G}_{A,n}^* \{K(x)\} dx,$$

where

$$\bar{L}_j = \frac{1}{n} \sum_{i=1}^n L_j \left(\frac{i}{n+1} \right).$$

When $1 \in A$, let also

$$J_{A,n}^s = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-p+1} \prod_{j \in A} \left\{ L_j \left(\frac{R_{i+j-1}}{n+1} \right) - \bar{L}_j \right\} = (-1)^{|A|} \int_{\mathbb{R}^A} \mathbb{G}_{A,n}^{s*} \{K(x)\} dx.$$

Under the null hypothesis of independence, the statistics $J_{A,n}$ and $J_{A,n}^s$ converge weakly to mutually independent centered Gaussian variables

$$(-1)^{|A|} \int_{\mathbb{R}^A} \mathbb{G}_A \{K(x)\} dx$$

whose variances are equal to $\sigma_A^2 = \prod_{j \in A} \sigma_j^2$.

Taking $L_j(u) = u$ for $j = 1, \dots, p$ leads to the statistics $\rho_{A,n}$ discussed by Deheuvels (1981a). Similarly, the van der Waerden statistic obtains when $L_1(u) = \dots = L_p(u) = \Phi^{-1}(u)$, the quantile function of the standard normal distribution. In this case, it is easy to show that $\sigma_A = 1$ for all admissible subsets $A \subset S_p$. When $p = 2$, a third example is provided by the Wilcoxon statistic, which corresponds to the choices $L_1(u) = u$ and $L_2(u) = \log\{u/(1-u)\}$. Finally, taking $L_1(u) = \{1 + \text{sign}(u)\}/2$ and $L_2(u) = \log\{2 \min(u, 1-u)\}$ yields the Laplace statistic also referred to as the median test-score statistic.

Once again, the asymptotic independence allows for construction of Spearman-ogram, van der Waerden-ogram, etc.

REMARK. Proceeding as in the proof of Proposition 4, it can readily be seen that if $\tilde{J}_{A,n}$ and $\tilde{J}_{A',n}^s$ are defined as above in terms of distribution functions $\tilde{K}_1, \dots, \tilde{K}_p$, then for arbitrary $A, A' \in S_p$,

$$\lim_{n \rightarrow \infty} \text{cov} \left(J_{A,n}, \tilde{J}_{A',n} \right) = \lim_{n \rightarrow \infty} \text{cov} \left(J_{A,n}^s, \tilde{J}_{A',n}^s \right)$$

The latter equals

$$\prod_{j \in A} \int \left\{ K_j(x_j) \wedge \tilde{K}_j(y_j) - K_j(x_j) \tilde{K}_j(y_j) \right\} dx_j dy_j$$

if $A = A'$ and zero otherwise.

6 Simulation study

In order to illustrate the above results and investigate the power of the two tests of independence proposed in this paper, extensive simulations were run, both in serial and non-serial contexts, and for a variety of alternatives. As the test based on Fisher's combination of P -values turned out to be generally the more powerful of the two for the models considered, only those results are presented here. Since the conclusions were substantively the same for time series and multivariate data, this section concentrates mostly on the latter.

Given a random sample from some p -variate distribution, the most common procedure used for checking independence is the likelihood ratio test (LRT), derived under the assumption of multivariate normality. The statistic is defined as

$$\text{LRT} = - \left(n - 1 - \frac{2p + 5}{6} \right) \log \{ \det(R_{p,n}) \},$$

in terms of the empirical $p \times p$ correlation matrix $R_{p,n}$. Because

$$\det(R_{p,n}) = 1 - \frac{1}{n} \sum_{1 \leq i < j \leq p} (\sqrt{n} r_{ij,n})^2 + o_P(1/n),$$

it is well known that, under the null hypothesis of independence, the test statistic converges in law to

$$\sum_{1 \leq i < j \leq p} Z_{ij}^2,$$

where the Z_{ij} are independent standard Gaussian random variables. Note that this remains true whether the underlying population is multivariate normal or not. Therefore, the LRT statistic is asymptotically chi-square with $p(p-1)/2$ degrees of freedom, so long as the second moments exist.

The following pages show comparative graphs of the power function of the LRT and Fisher's statistic $T_{F,n}$ for

- two dimensions, viz. $p = 2$ and 5 ;
- three sample sizes, viz. $n = 20, 50$ and 100 ;
- three dependence structures, viz., the equicorrelated normal and two copula models, i.e., those of Clayton (1978) and Gumbel (1960);
- four marginal distributions, viz., standard normal, exponential, and Cauchy, as well as the Pareto distribution with distribution function $1 - x^{-4}$ for $x \geq 1$.

Here, the equicorrelated normal model refers to a p -variate Gaussian vector with components $\sqrt{r}Z_0 + \sqrt{1-r}Z_i$, $1 \leq i \leq p$, where Z_0, \dots, Z_p form a random sample from the standard univariate normal distribution.

The p -variate model of Clayton (1978), with arbitrary marginals F_1, \dots, F_p , is defined as

$$C_\theta \{F_1(x_1), \dots, F_p(x_p)\} \quad (11)$$

in terms of the copula

$$C_\theta(u_1, \dots, u_p) = \left(\sum_{i=1}^p u_i^{-\theta} + p - 1 \right)^{-1/\theta}, \quad \theta > 0$$

with $C_0(u_1, \dots, u_p) = \lim_{\theta \rightarrow 0} C_\theta(u_1, \dots, u_p) = u_1 \times \dots \times u_p$ for all $0 \leq u_1, \dots, u_p \leq 1$. This Archimedean copula model is quite popular in survival analysis, where it provides a natural multivariate extension of Cox's proportional hazards model; cf., e.g., Oakes (2001, Section 7.3).

The p -variate copula model of Gumbel (1960), which arises in extreme value theory, is also of the form (11), but with

$$C_\theta(u_1, \dots, u_p) = \exp \left[- \left\{ \sum_{i=1}^p |\log(u_i)|^{1/\theta} \right\}^\theta \right], \quad 0 < \theta \leq 1$$

with C_1 being the independence copula.

Figures 4–6 pertain to the case $n = 20$, Figures 7–9 to the case $n = 50$, and Figures 10–12 to the case $n = 100$. In each figure, eight graphs are displayed in two columns, corresponding to dimension $p = 2$ on the left, and $p = 5$ on the right. Each graph compares the power curve of Fisher's test $T_{F,n}$ (solid line) to that of the LRT (dotted line). In order to facilitate comparisons between dependence structures, all curves are plotted as a function of the population value of Kendall's tau for the various bivariate alternatives. Thus

$$\tau = \frac{2}{\pi} \arcsin(r)$$

for the equicorrelated normal model, while

$$\tau = \frac{\theta}{\theta + 2} \quad \text{and} \quad \tau = 1 - \theta$$

for Clayton's and Gumbel's model, respectively; cf., e.g., Genest & MacKay (1986) or Nelsen (1999, Chapter 5). All graphs were smoothed from 10,000 replicates obtained for each value of $\tau = j/20$ for $0 \leq j \leq 19$, with $j = 0$ corresponding to the null hypothesis of stochastic independence.

Detailed inspection of these 72 graphs leads to the following general observations:

- a) both the dependence structure and the choice of marginals influence the relative performance of the tests;
- b) while the nonparametric test always maintains its nominal level, such is not the case for the LRT, particularly when the marginals are Cauchy;
- c) Fisher's test is far superior to the LRT for a Clayton type of dependence, except when the marginals are exponential;
- d) for normal or Gumbel type dependences, the results are more mitigated when $n = 20$ or 50, but the new test outperforms the classical one, except in the case of multivariate normality (for which the LRT is optimal) and either when Pareto marginals are coupled with an Gumbel extreme value dependence structure, or when exponential marginals are mixed with Clayton's copula;
- e) as a general rule, the LRT loses ground to the test based on $T_{F,n}$ when sample size increases, so that $T_{F,n}$ gradually catches up with the LRT or leaves it further behind as $n \rightarrow \infty$;
- f) differences between the two tests are more noticeable for small to moderate degrees of dependence (i.e., $\tau < 1/2$) and far less perceivable with $p = 5$ than with $p = 2$.

As a numerical illustration of the final point, consider the ratio $Q = \Pi(T_{F,n}) / \Pi(\text{LRT})$ comparing the power of the two tests for a random sample of size $n = 100$ from either one of the nine p -variate families obtained by crossing the three dependence structures and either normal, exponential or Pareto marginals. (The data for the Cauchy were ignored, given the inappropriate level of the LRT). Looking at the range of Q reported in Table 5 for the case $p = 2$, one can see that except in the cases along the main diagonal, one has nothing to lose — and potentially much to gain — by preferring the Fisher test $T_{F,n}$ to the standard likelihood ratio test statistic. Judging from Table 6, however, larger sample sizes are required to reach the same conclusion in a multivariate population of dimension 5.

TABLE 5. Range of the ratio $\Pi(T_{F,n}) / \Pi(\text{LRT})$ measuring the power of Fisher's test $T_{F,n}$ relative to the LRT for random samples of size $n = 100$ from the specified bivariate models ($p = 2$).

Copula/Marginals	Normal	Exponential	Pareto
Normal	0.86 – 1.00	0.98 – 1.10	1.00 – 1.45
Clayton	1.00 – 1.46	0.78 – 1.00	1.00 – 3.67
Gumbel	1.07 – 2.36	1.00 – 3.79	0.48 – 1.04

TABLE 6. Range of the ratio $\Pi(T_{F,n})/\Pi(\text{LRT})$ measuring the power of Fisher's test $T_{F,n}$ relative to the LRT for random samples of size $n = 100$ from the specified p -variate models with $p = 5$.

Copula/Marginals	Normal	Exponential	Pareto
Normal	0.60 – 1.00	0.67 – 1.00	0.62 – 1.07
Clayton	0.60 – 1.09	0.67 – 1.00	1.00 – 2.90
Gumbel	0.71 – 1.49	1.00 – 2.94	0.94 – 1.14

Much similar conclusions were obtained in simulations involving time series. In the serial case, comparisons were made with the Ljung-Box statistic

$$(n-p) \sum_{j=1}^{2^{p-1}-1} r_{j,n}^2,$$

where $r_{j,n}$ is the empirical autocorrelation coefficient of order j . Under the null hypothesis of randomness, the latter is known to be asymptotically distributed as a chi-square with $2^{p-1} - 1$ degrees of freedom, as long as the second moment exists.

Figure 13 offers an extreme illustration of the gains in power associated with the serial version $T_{F,n}^s$ of Fisher's test, based on 10,000 independent series of length $n = 50$ from the deterministic tentmap series of Section 4, with initial value X_0 drawn from the uniform distribution on the unit interval. While the Ljung-Box test is incapable of detecting this strong dependence because of the vanishing autocorrelation function, $T_{F,n}^s$ picks it up rather easily, though much less quickly as p increases from 2 to 6.

In this case, the poor performance of the Ljung-Box statistic is easy to explain. Indeed, a simple application of the central limit theorem for reverse martingales (cf., e.g., Genest *et al.* 2001) shows that when the data arise from the tentmap series, the Ljung-Box statistic converges in law to

$$\sum_{j=1}^{2^{p-1}-1} Z_j^2,$$

where the Z_j 's are jointly centered Gaussian variables with variance $1 + 4^{2-k}/5$ and $\text{cov}(Z_j, Z_k) = 3/4^{k-1}$ for $1 \leq j < k$.

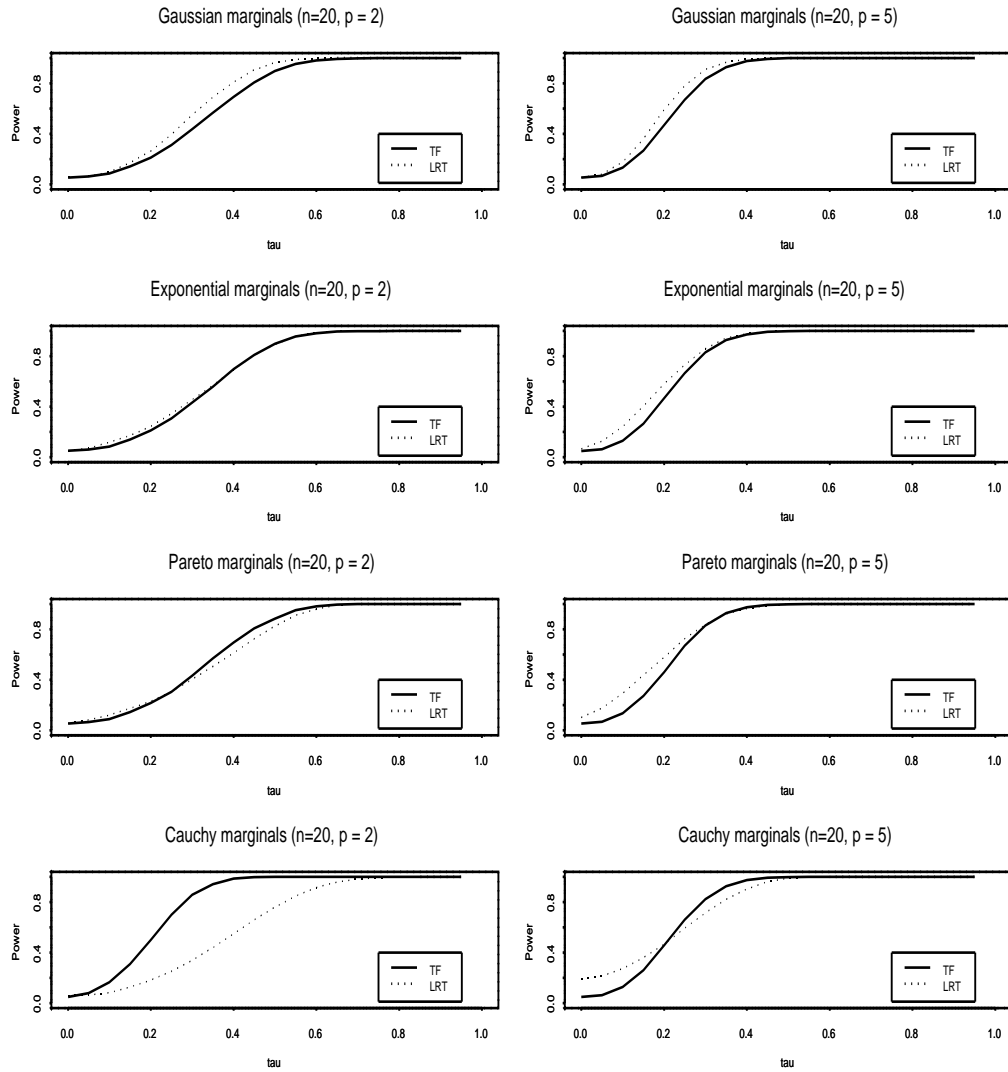


FIGURE 4. Power curves of the LRT and Fisher's $T_{F,n}$ test at the 5% level, based on 10,000 independent pseudo-random samples of size $n = 20$ from a p -variate distribution, $p = 2, 5$, with **normal** copula and marginal distributions that are either Gaussian, exponential, Pareto or Cauchy.

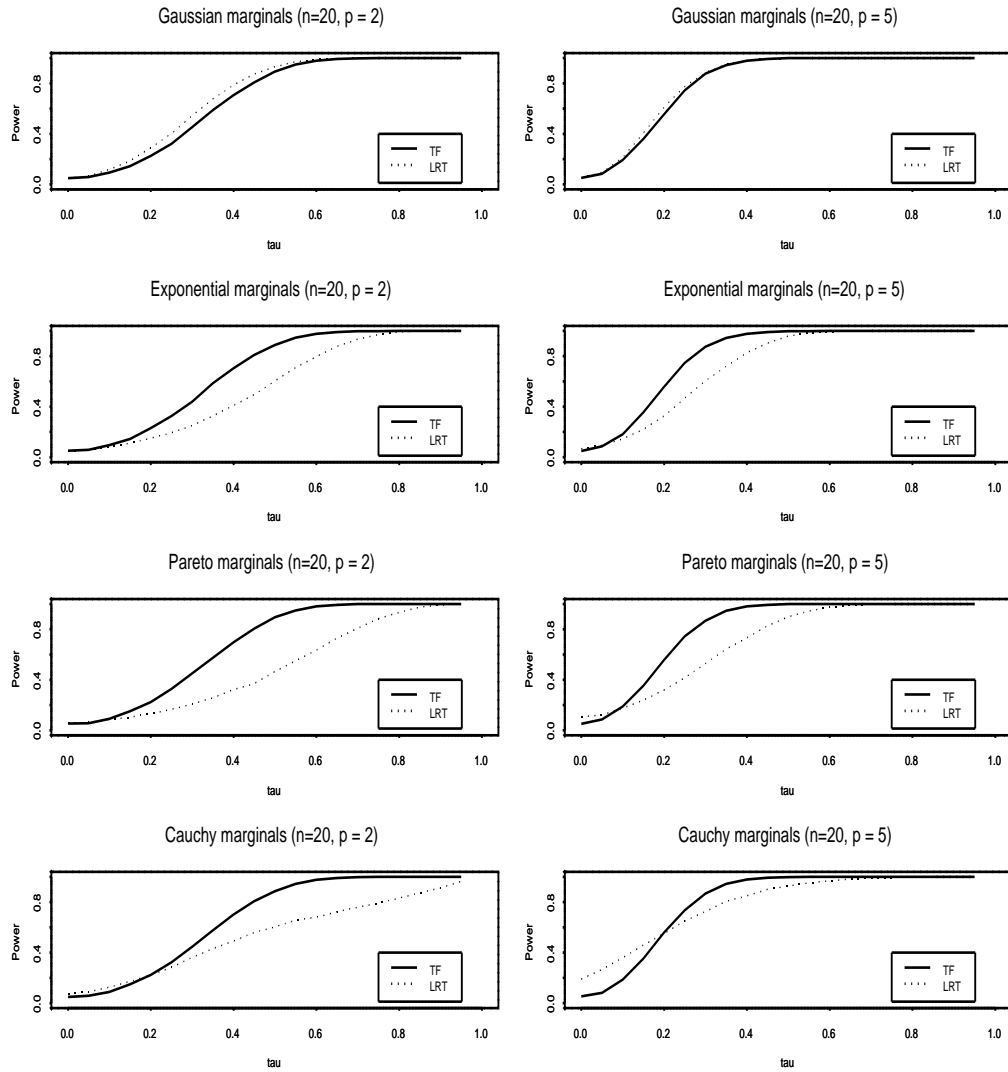


FIGURE 5. Power curves of the LRT and Fisher's $T_{F,n}$ test at the 5% level, based on 10,000 independent pseudo-random samples of size $n = 20$ from a p -variate distribution, $p = 2, 5$, with **Clayton** copula and marginal distributions that are either Gaussian, exponential, Pareto or Cauchy.

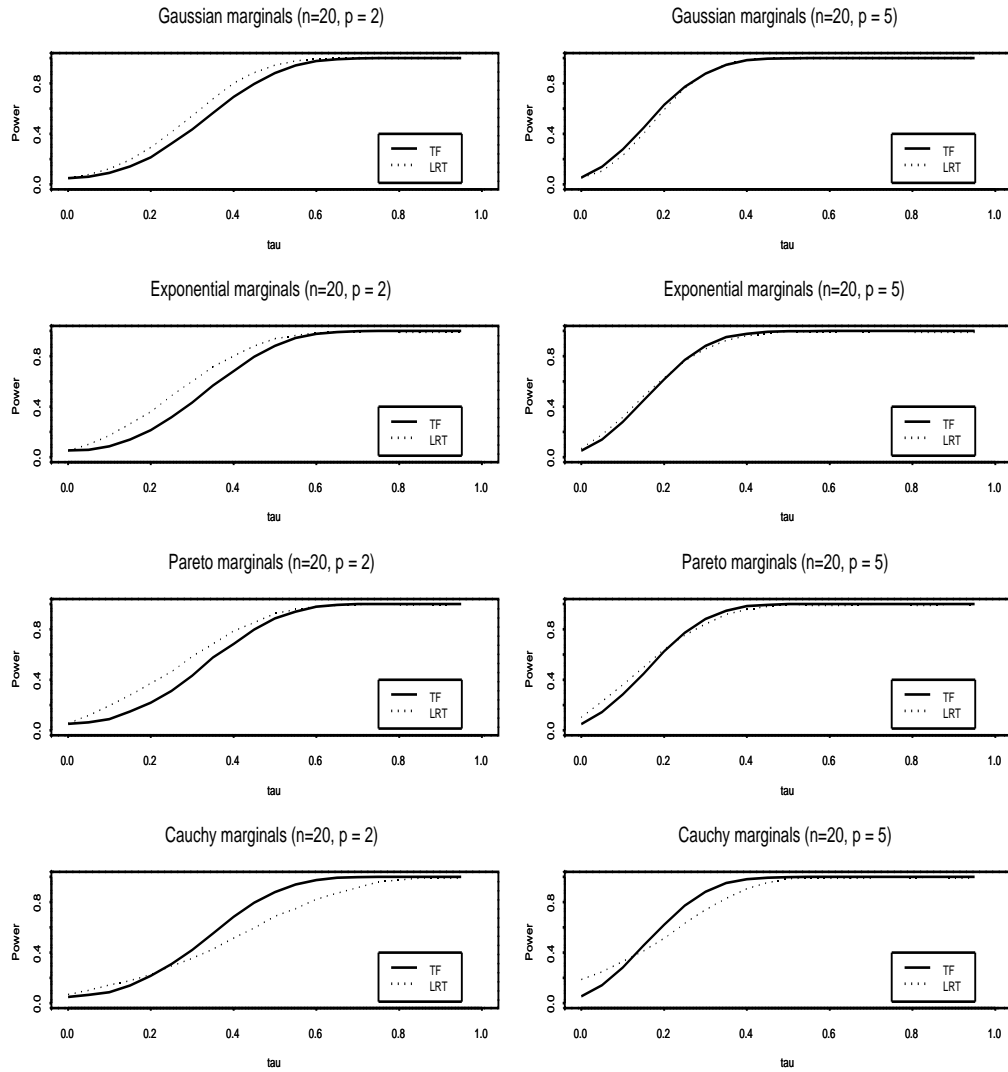


FIGURE 6. Power curves of the LRT and Fisher's $T_{F,n}$ test at the 5% level, based on 10,000 independent pseudo-random samples of size $n = 20$ from a p -variate distribution, $p = 2, 5$, with **Gumbel** copula and marginal distributions that are either Gaussian, exponential, Pareto or Cauchy.

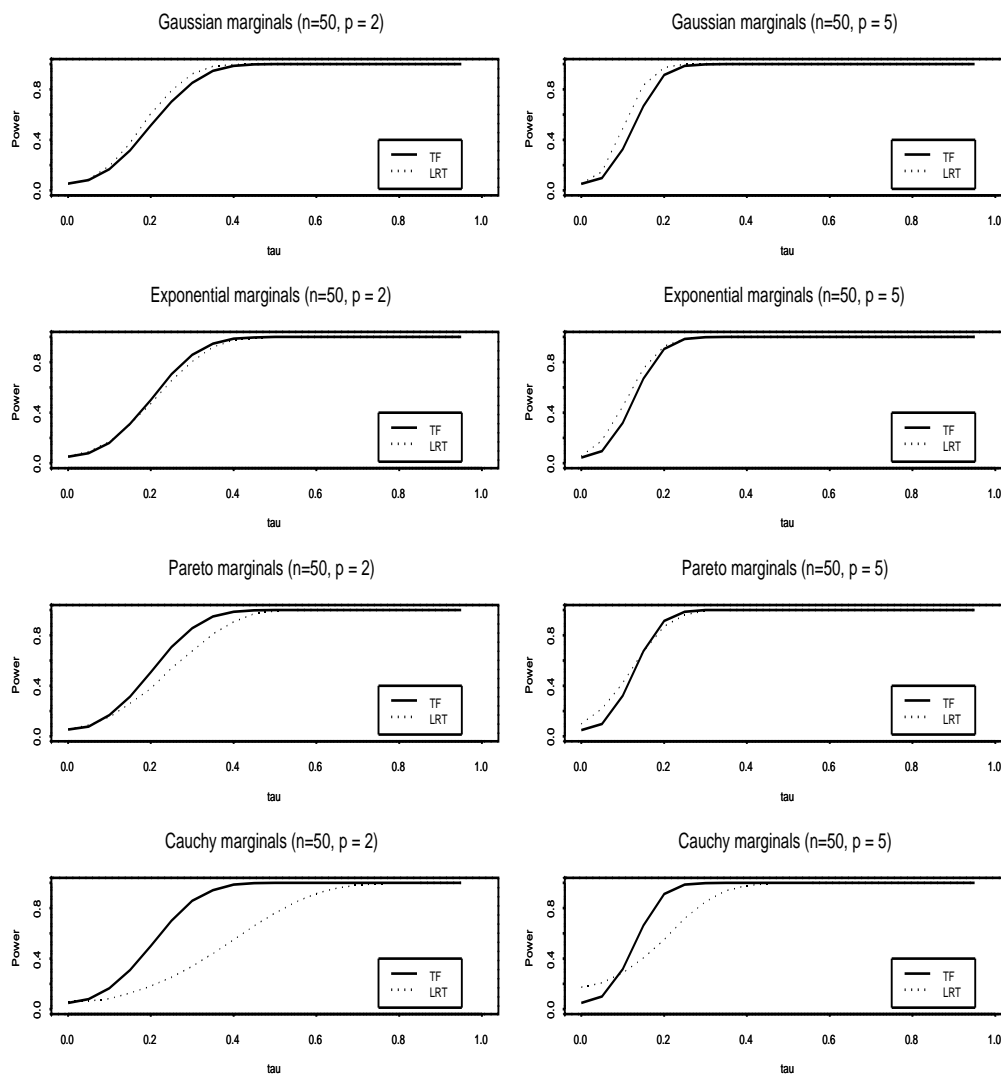


FIGURE 7. Power curves of the LRT and Fisher's $T_{F,n}$ test at the 5% level, based on 10,000 independent pseudo-random samples of size $n = 50$ from a p -variate distribution, $p = 2, 5$, with **normal** copula and marginal distributions that are either Gaussian, exponential, Pareto or Cauchy.

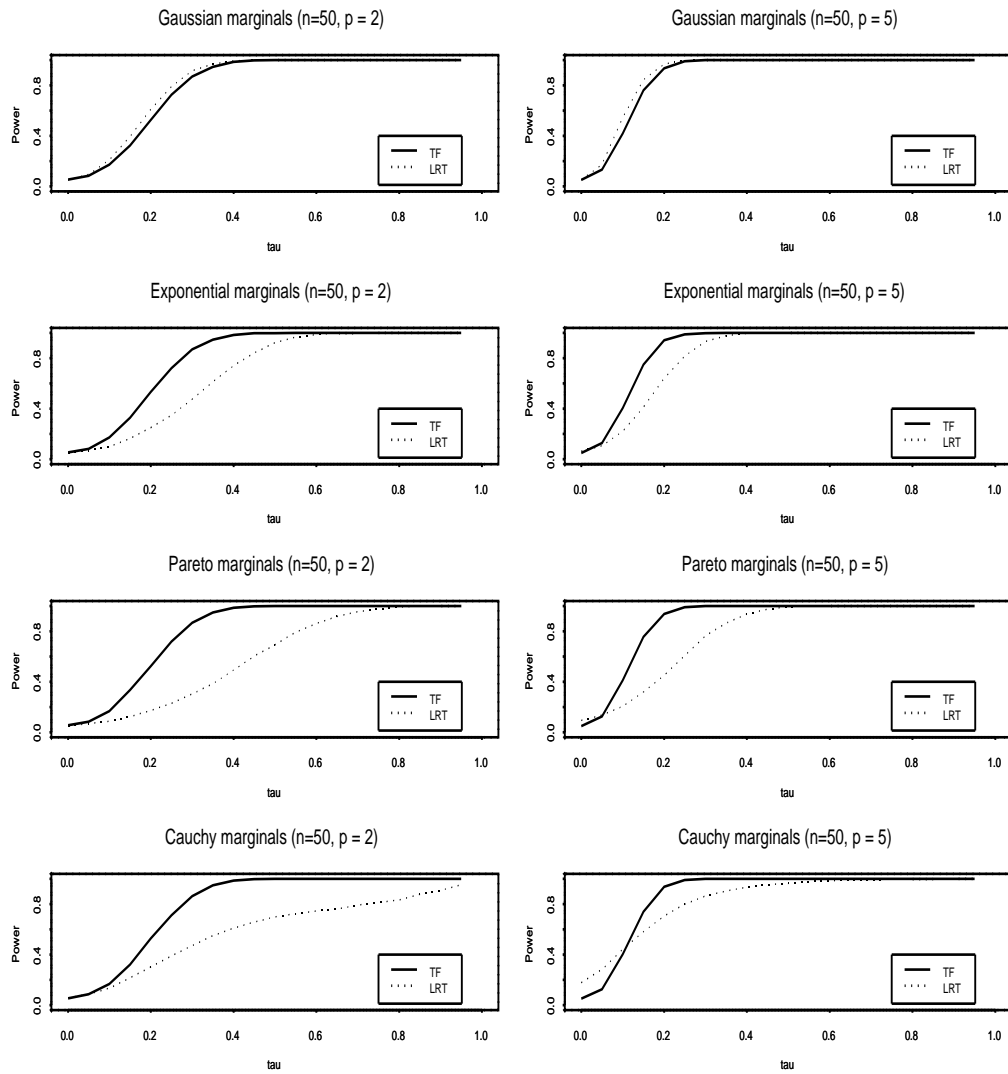


FIGURE 8. Power curves of the LRT and Fisher's $T_{F,n}$ test at the 5% level, based on 10,000 independent pseudo-random samples of size $n = 50$ from a p -variate distribution, $p = 2, 5$, with **Clayton** copula and marginal distributions that are either Gaussian, exponential, Pareto or Cauchy.

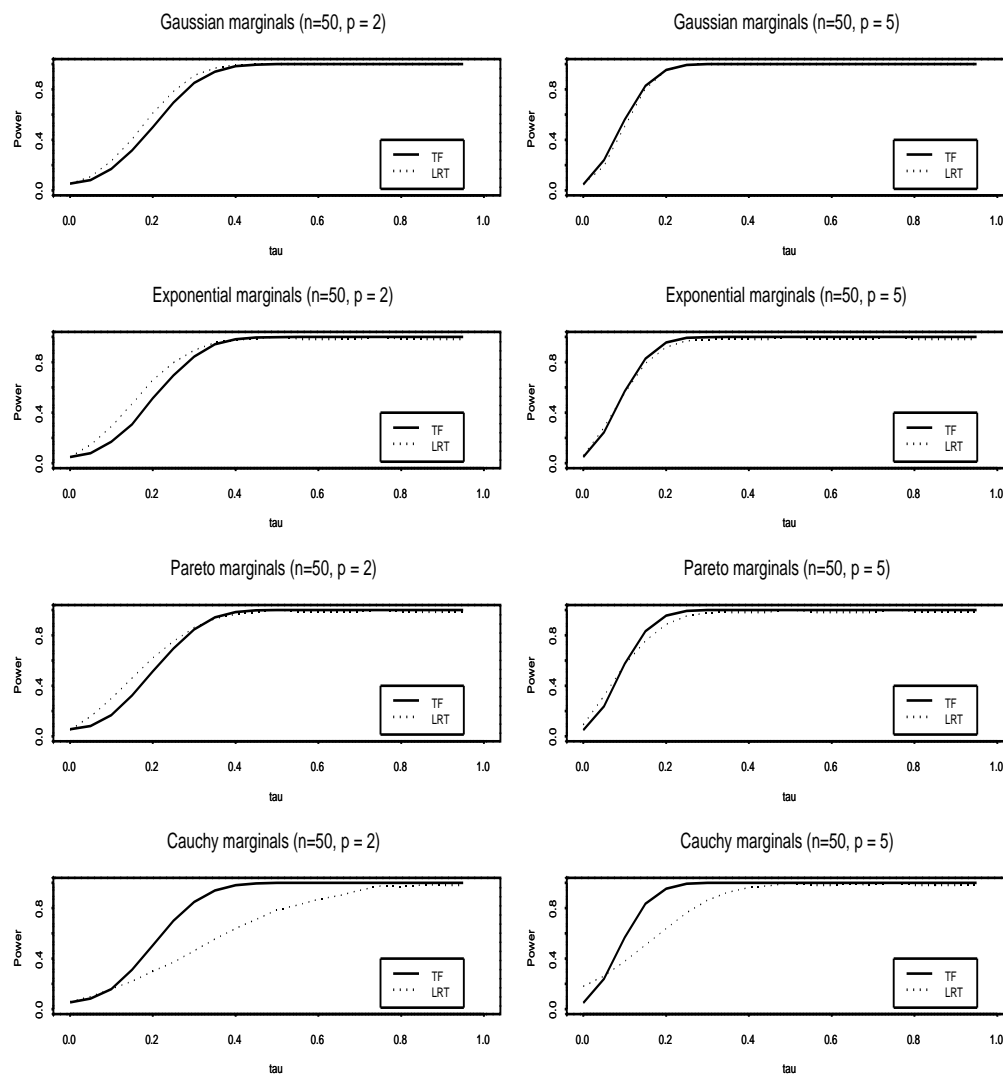


FIGURE 9. Power curves of the LRT and Fisher's $T_{F,n}$ test at the 5% level, based on 10,000 independent pseudo-random samples of size $n = 50$ from a p -variate distribution, $p = 2, 5$, with **Gumbel** copula and marginal distributions that are either Gaussian, exponential, Pareto or Cauchy.

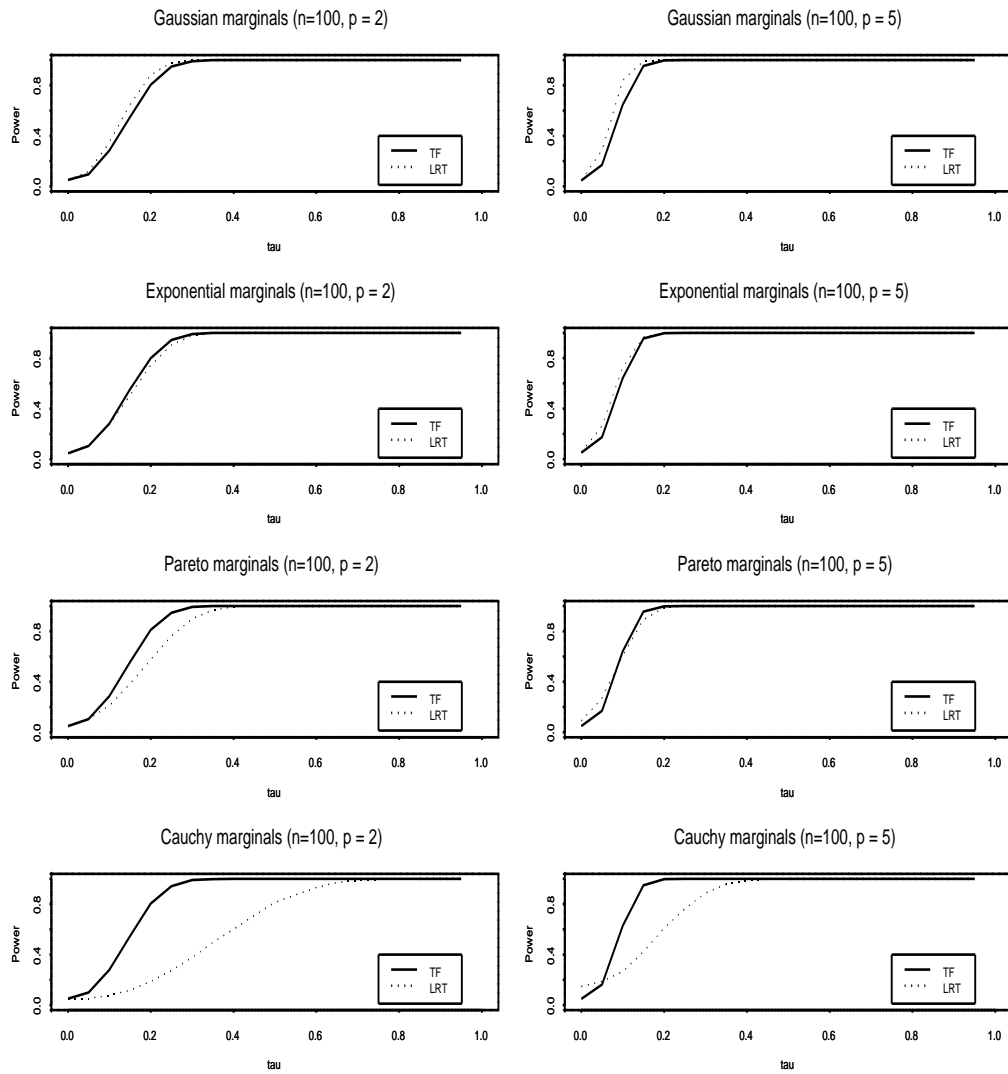


FIGURE 10. Power curves of the LRT and Fisher's $T_{F,n}$ test at the 5% level, based on 10,000 independent pseudo-random samples of size $n = 100$ from a p -variate distribution, $p = 2, 5$, with **normal** copula and marginal distributions that are either Gaussian, exponential, Pareto or Cauchy.

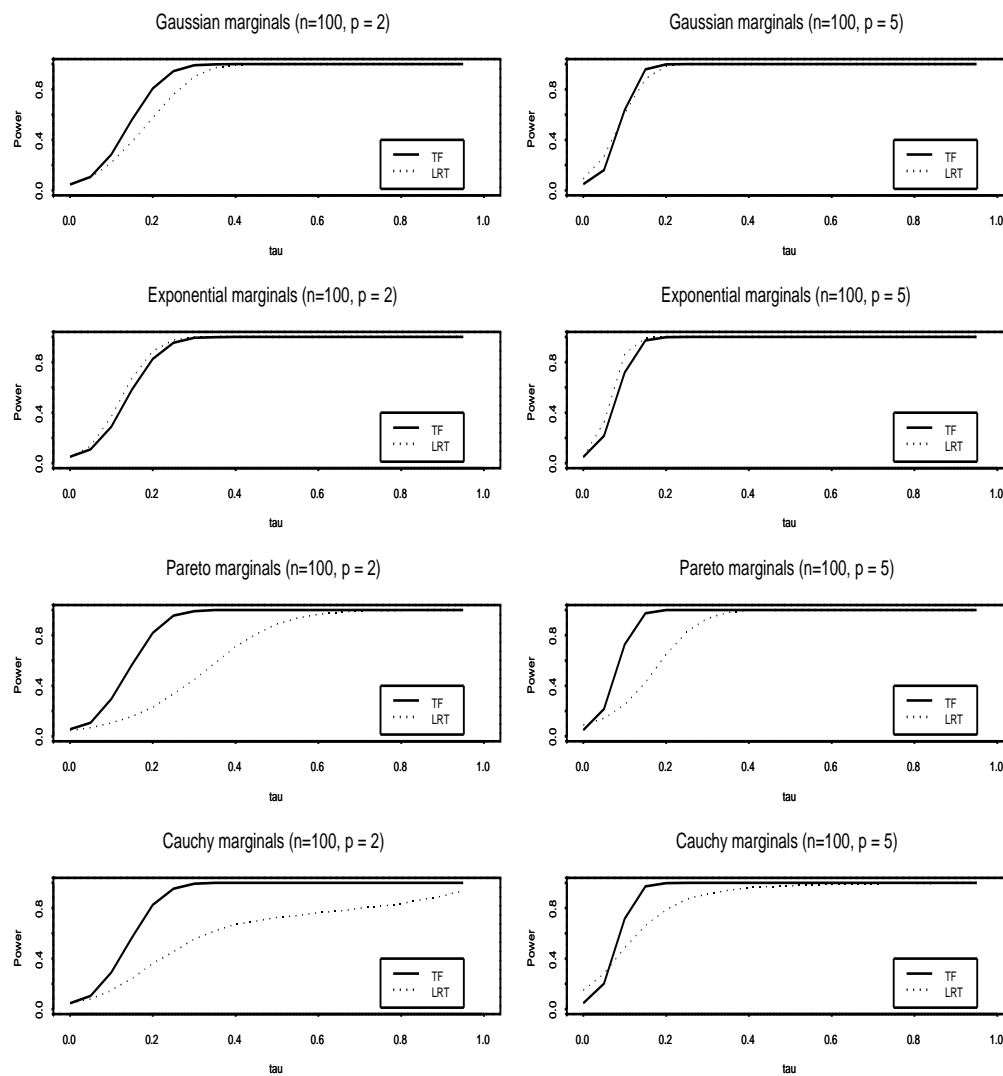


FIGURE 11. Power curves of the LRT and Fisher's $T_{F,n}$ test at the 5% level, based on 10,000 independent pseudo-random samples of size $n = 100$ from a p -variate distribution, $p = 2, 5$, with **Clayton** copula and marginal distributions that are either Gaussian, exponential, Pareto or Cauchy.

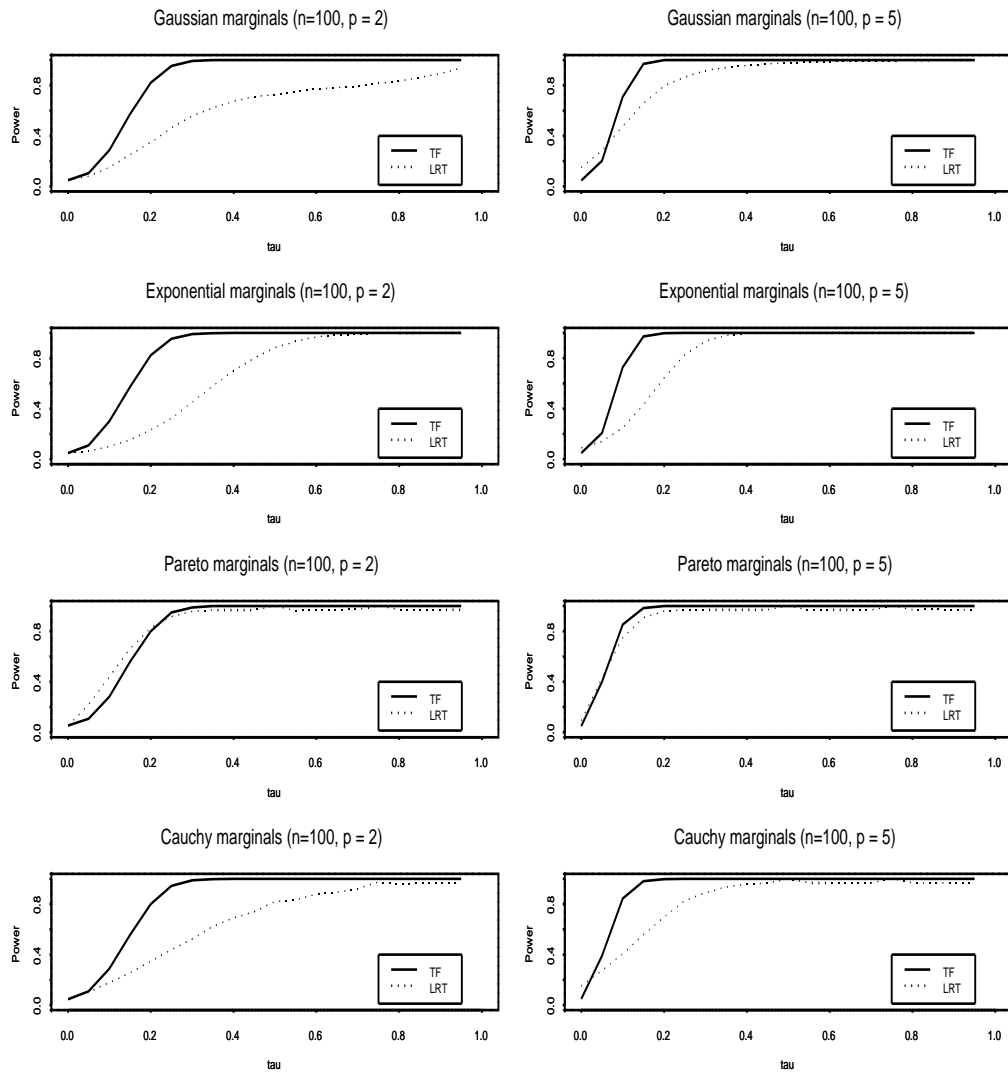


FIGURE 12. Power curves of the LRT and Fisher's $T_{F,n}$ test at the 5% level, based on 10,000 independent pseudo-random samples of size $n = 100$ from a p -variate distribution, $p = 2, 5$, with **Gumbel** copula and marginal distributions that are either Gaussian, exponential, Pareto or Cauchy.

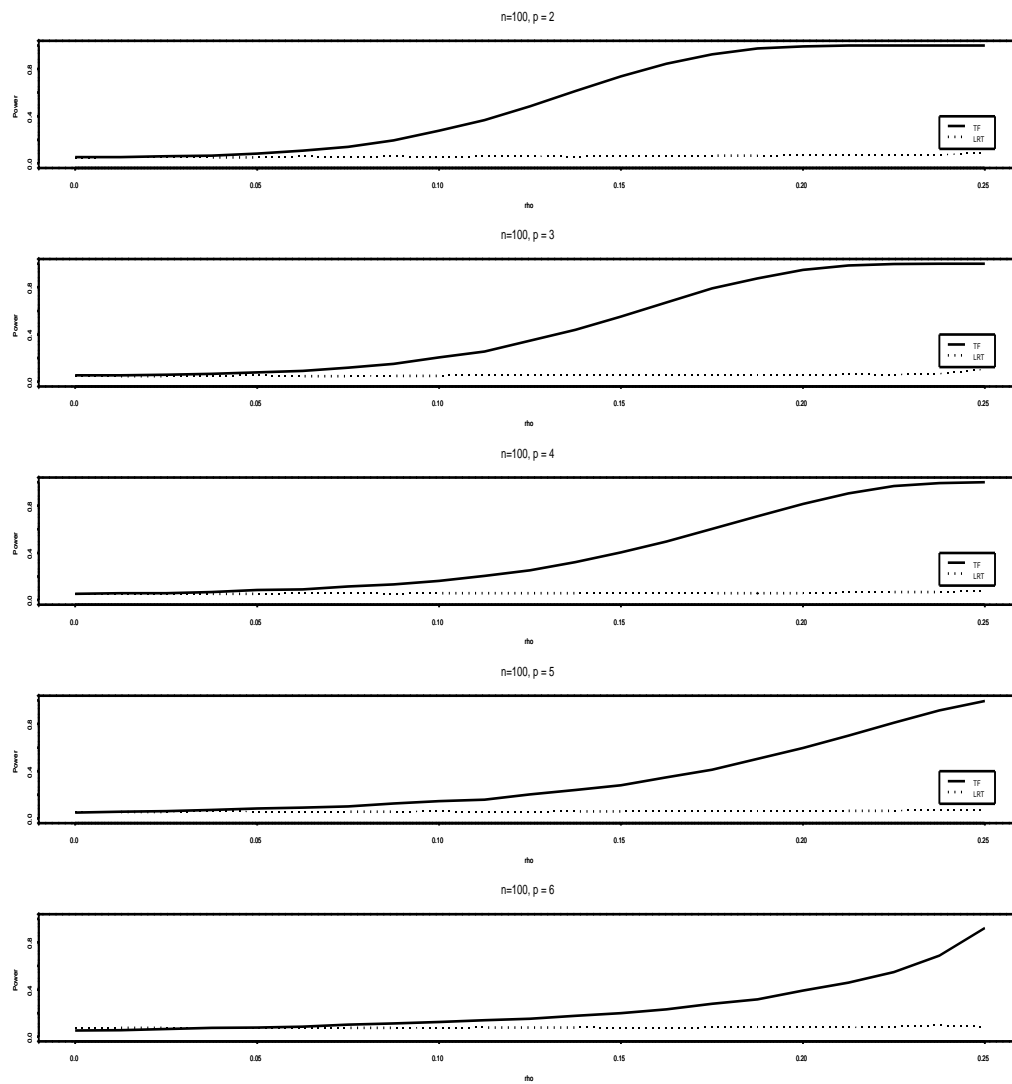


FIGURE 13. Power curves of the LRT and the serial version $T_{F,n}^s$ of Fisher's test at the 5% level, based on 10,000 independent tent map series of length $n = 50$.

7 Conclusion

This paper has shown how it is possible to exploit the asymptotic independence of a Möbius decomposition of the empirical copula process to devise powerful tests of independence based on Cramér-von Mises statistics. Following the work of Deheuvels (1979, 1980, 1981a,b,c), two different strategies were proposed. Fixing the overall level of the test first led to a graphical diagnostic tool for identifying failures of the independence hypothesis in subsets of variables. A second approach, based on Fisher's methods for the combination of P -values, led to an even better test that compares quite favorably to the standard likelihood ratio test statistic in a variety of settings. Both tests apply either to a multivariate or serial context, and the treatment provided herein allows for a unified study of the asymptotic behavior of linear rank statistics derived from copulas.

In future work, it would be of interest to calculate Pitman's asymptotic relative efficiency of the newly proposed tests with respect to the likelihood ratio statistic, and to extend numerical comparisons with alternative, more specialized, parametric tests. A greater challenge would be to try to characterize structures of serial and non-serial dependence for which Fisher's test is most likely to be powerful in small samples and for large dimensions.

Appendix A: Proof of Proposition 2

The derivation of representation (9) is based on the technique of pseudo-observations developed by Ghoudi & Rémillard (1998, 2000). To this end write \mathbb{C}_n^s in the alternative form

$$\mathbb{C}_n^s(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-p+1} \left[\prod_{j=1}^p \mathbb{I}\{F_n(Y_{i+j-1}) \leq t_j\} - \prod_{j=1}^p t_j \right]$$

in terms of pseudo-observations $e_{i,n} = (F_n(Y_i), \dots, F_n(Y_{i+p-1}))$, where F_n is the empirical distribution function of the sample Y_1, \dots, Y_n .

Now, as is well known, $\mathbb{F}_n(y) = \sqrt{n}\{F_n(y) - F(y)\}$ converges in law to a continuous Gaussian process $\mathbb{F} = \beta \circ F$, where β is a Brownian bridge having representation $\beta(u) = \mathbb{C}^s(u, 1, \dots, 1)$.

Since conditions R1–R3 of Theorem 2.3 of Ghoudi *et al.* (2001) are easily verified, one can conclude that the limit of \mathbb{C}_n^s in $D([0, 1]^p)$ can be written as

$$\tilde{\mathbb{C}}^s(t) = \mathbb{C}^s(t) - \sum_{j=1}^p \mu_j(t, \mathbb{H}_j),$$

where $\mathbb{H}_j = \mathbb{F}$. Here, the functional μ_j is defined by

$$\mu_j(t, \phi \circ F) = \phi(t_j) \prod_{k \neq j} t_k$$

for any continuous function ϕ vanishing at both ends of the interval $[0, 1]$. Therefore,

$$\mu_j(t, \mathbb{H}_j) = \mu_j(t, \beta \circ F) = \beta(t_j) \prod_{k \neq j} t_k,$$

as claimed.

Appendix B: Proof of Proposition 4

As the proof is similar in the serial and non-serial cases, only the latter is sketched here. Given $M > 1$, it is clear that the mapping

$$f \mapsto \int_{\mathbb{R}^A \cap [-M, M]^p} f \circ K(x) \, dx,$$

defined for all continuous functions f vanishing outside $(0, 1)^p$, is continuous. To prove the proposition, therefore, it suffices to show that the variance of

$$\int_{\mathbb{R}^A \cap (\mathbb{R}^p \setminus [-M, M]^p)} \mathbb{G}_{A,n}^* \{K(x)\} \, dx$$

is arbitrarily small, when M is large enough.

Since

$$1 - \mathbb{I}_{E_1 \times \dots \times E_p} = \sum_{B \subset S_p, B \neq \emptyset} \left(\prod_{j \in B^c} \mathbb{I}_{E_j} \right) \left(\prod_{j \in B} \mathbb{I}_{E_j^c} \right)$$

for arbitrary events E_1, \dots, E_p , then in particular, when $E_1 = \dots = E_p = [-M, M]$,

$$\text{var}^{1/2} \left[\int_{\mathbb{R}^A \cap (\mathbb{R}^p \setminus [-M, M]^p)} \mathbb{G}_{A,n}^* \{K(x)\} \, dx \right] \leq \sum_{B \subset S_p, B \neq \emptyset} \text{var}^{1/2} \left[\int_{E_{B,M}} \mathbb{G}_{A,n}^* \{K(x)\} \, dx \right]$$

by Minkowski's inequality, where the event

$$E_{B,M} = \mathbb{R}^A \cap \left(\bigotimes_{j \in B^c} E_j \right) \left(\bigotimes_{j \in B} E_j^c \right)$$

is either empty or has at least one component of the form $[-M, M]^c$. The latter occurs

only when $B \subset A$, in which case

$$\begin{aligned} \text{var} \left[\int_{E_{B,M}} \mathbb{G}_{A,n}^* \{K(x)\} dx \right] &= \left\{ 1 + \frac{(-1)^{|A|}}{(n-1)^{|A|-1}} \right\} \\ &\quad \times \prod_{j \in B} \left\{ \int_{[-M,M]^c \times [-M,M]^c} V_n(x_j, y_j) dx_j dy_j \right\} \\ &\quad \times \prod_{j \in A \setminus B} \left\{ \int_{[-M,M]^2} V_n(x_j, y_j) dx_j dy_j \right\} \\ &\leq 2 \left(\prod_{j \in A \setminus B} \sigma_j^2 \right) \\ &\quad \times \prod_{j \in B} \left\{ \int_{[-M,M]^c \times [-M,M]^c} V_n(x_j, y_j) dx_j dy_j \right\}, \end{aligned}$$

where $V_n(x_j, y_j) = U_n\{K_j(x_j \wedge y_j)\} - U_n\{K_j(x_j)\}U_n\{K_j(y_j)\} \geq 0$.

Since $B \neq \emptyset$, it only remains to show that for any $1 \leq j \leq p$,

$$\limsup_{n \rightarrow \infty} \int_{[-M,M]^c \times [-M,M]^c} V_n(x_j, y_j) dx_j dy_j \rightarrow 0$$

as M tends to infinity. The left-hand side is the sum of three positive terms, each of which involves $V(x_j, y_j) = K_j(x_j \wedge y_j) - K_j(x_j)K_j(y_j) \geq 0$ as an integrand. They are respectively

$$\begin{aligned} \int_{-\infty}^{-M} \int_{-\infty}^{-M} V(x_j, y_j) dx_j dy_j &\leq -2 \int_{-\infty}^{-M} y_j K_j(y_j) dy_j, \\ 2 \int_{-\infty}^{-M} \int_M^{\infty} V(x_j, y_j) dx_j dy_j &= \int_{-\infty}^{-M} K_j(y_j) dy_j \int_M^{\infty} \{1 - K_j(x_j)\} dx_j, \end{aligned}$$

and

$$\int_M^{\infty} \int_M^{\infty} V(x_j, y_j) dx_j dy_j \leq 2 \int_M^{\infty} y_j \{1 - K_j(y_j)\} dy_j.$$

In all cases, the upper bound approaches 0 as $M \rightarrow \infty$ because σ_j^2 is finite.

Finally, the limiting variance σ_A^2 of $J_{A,n}$ is given by

$$\begin{aligned} \sigma_A^2 &= E \left[\int_{\mathbb{R}^A} \int_{\mathbb{R}^A} \mathbb{G}_A \{K(x)\} \mathbb{G}_A \{K(y)\} dx dy \right] \\ &= \prod_{j \in A} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} \{K_j(x_j \wedge y_j) - K_j(x_j)K_j(y_j)\} dx_j dy_j \right] \\ &= \prod_{j \in A} \sigma_j^2, \end{aligned}$$

by Hoeffding's identity.

For the non-serial case,

$$\begin{aligned}
I_{ij,n} &= \int_{-\infty}^{\infty} \left[\mathbb{I} \left\{ \frac{R_{ij}}{n+1} \leq K_j(x_j) \right\} - U_n \circ K_j(x_j) \right] dx_j \\
&= \sum_{\ell=1}^{n-1} \left\{ K_j^{-1} \left(\frac{\ell+1}{n+1} \right) - K_j^{-1} \left(\frac{\ell}{n+1} \right) \right\} \left\{ \mathbb{I}(\ell \geq R_{ij}) - \frac{\ell}{n} \right\} \\
&= - \sum_{\ell=1}^{n-1} \frac{\ell}{n} \left\{ L_j \left(\frac{\ell+1}{n+1} \right) - L_j \left(\frac{\ell}{n+1} \right) \right\} \\
&\quad + \sum_{\ell=R_{ij}}^{n-1} \left\{ L_j \left(\frac{\ell+1}{n+1} \right) - L_j \left(\frac{\ell}{n+1} \right) \right\} \\
&= -L_j \left(\frac{R_{ij}}{n+1} \right) + \frac{1}{n} \sum_{\ell=1}^n L_j \left(\frac{\ell}{n+1} \right) \\
&= -L_j \left(\frac{R_{ij}}{n+1} \right) + \bar{L}_j.
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^A} \mathbb{G}_{A,n} \{K(x)\} dx = \frac{1}{\sqrt{n}} \sum_{i=1}^n \prod_{j \in A} I_{ij,n} = (-1)^{|A|} J_{A,n}.$$

The serial case is similar.

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