Dynamic games in non-renewable resource markets

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Introduction

Attention for non-renewable resources:

1. Growth debate. Club of Rome. Sustainability feasible in the presence non-renewable resources (oil, coal,..)?

2. Oil crisis. OPEC versus Rest of the World. How to get an idea of future prices?

3. Climate change. Main greenhouse gas, CO2 from burning fossil fuel. How much to leave in the ground?

Outline

Lecture 1
Markets for non-renewable resource
1. Perfect competition
2. Monopoly
3. Social optimum
4. Order of extraction
5. Open loop Nash oligopoly
6. Open loop Nash cartel versus fringe
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Lecture 2
1. Closed loop Nash
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3. Open loop Nash
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5.Open loop Stackelberg
6. Closed loop Stackelberg

Lecture 3 Green paradoxes
1. Introduction
2. Battle between oil produces and oil consumers.

Markets for non-renewable resources

What is the market price of the extracted non-renewable resource in different market structures?

What is the order of extraction?

What are the (welfare) effects of changes in
-number of players?
-initial resource stocks?
-marginal extraction costs?

1.Perfect competition

Mine owner with initial stock $S_0 > 0$. Price taking. Price path $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$.
Constant interest rate r.Extraction costs $C(q, S)$ depend on the amount extracted q and the remaining resource S.
Assume $C_q > 0,\ C_S < 0$

Definition

An extraction path $q : \mathbb{R}_+ \to \mathbb{R}_+$ is *feasible* if $\int_0^\infty q(s)ds \leq S_0$.

Alternatively: $\dot{S}(t) = -q(t) \leq 0, S(t) \geq 0$ all $t \geq 0$

Definition.

The extraction path q is *profit maximizing for price path* p if q is feasible and

$$\int_0^\infty e^{-rs}[p(s)q(s) - C(q(s), S(s)]ds \geq \int_0^\infty e^{-rs}[p(s)\hat{q}(s) - C(q(s), \hat{S}(s))]ds$$

for all feasible $\hat{q}$ and corresponding $\hat{S}$.

Problem statement for firm

$$\max_q \int_0^\infty e^{-rs}[p(s)q(s) - C(q(s), S(s))]ds$$

subject to feasibility.
Hamiltonian

$$H(S, q, \lambda, t) = e^{-rt}[p(t)q - C(q, S)] + \lambda[-q]$$

Necessary conditions

$$\text{If } q(t) > 0 \text{ then } \frac{\partial H(S, q, \lambda, t)}{\partial q} = 0 : \; e^{-rt}[p(t) - C_q(q(t), S(t))] = \lambda(t)$$

$$\frac{\partial H(S, q, \lambda, t)}{\partial S} = -\dot{\lambda}(t) : \; -e^{-rt}C_S(q(t), S(t)) = -\dot{\lambda}(t)$$

Interpretation

Equilibrium

Inverse demand function f.

Definition

$(p, q) : \mathbb{R}_+ \to \mathbb{R}_+^{m+1}$ is a *competitive equilibrium* if

i. $p(t) = f(q(t))$ for all $t \geq 0$

ii. q is profit maximizing at p

Constant marginal extraction costs $k < f(0)$.
In equilibrium with $q > 0$

$$p(t) = k + e^{rt}\lambda$$

Hotelling's rule

Example 1.

Isoelastic demand: $f(q) = q^{\eta}$ with $\eta < -1$.

Zero extraction costs.

In equilibrium $q(t) > 0$ for all $t \geq 0$

Hence, $\dot{q}(t) = \frac{r}{\eta} q(t)$. In view of full exhaustion we find

$$q(t) = -\frac{r}{\eta} S_0 e^{\frac{r}{\eta} t}$$

-Higher initial stock uniformly increases extraction over time.

-Higher rate of interest has a positive effect on the initial extraction rate.

Example 2.

Linear demand $f(q) = \bar{p} - q$. Choke price $\bar{p}$. Marginal extraction costs $k < \bar{p}$.

Hence $q(t) = \bar{p} - k - e^{rt}\lambda$, as long as $q(t) > 0$. This holds for initial interval of time, ending at, say, T. Since

$$\int_0^T q(s)ds = S_0$$

we have

$$[\bar{p} - k]T - \frac{1}{r}[e^{rT} - 1]\lambda = S_0$$

Moreover, $q(T) = 0$, implying $\bar{p} - k = e^{rT}\lambda$. Final time T can be solved from

$$[\bar{p} - k][rT - 1 + e^{-rT}] = rS_0$$

A higher initial stock calls for a later final time.
Extraction rises over the extraction period.
Higher extraction costs lead to a longer time before depletion.
A lower choke price (backstop) has the same effect.

Definition

$(p, q) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^{m+1}$ is a *monopolistic equilibrium* if

$$i.\ p(t) = f(q(t)) \text{ for all } t \geq 0$$

$$ii.\ \int_0^\infty e^{-rs}[f(q(s))q(s) - C(q(s), S(s)]ds \geq \int_0^\infty e^{-rs}[f(\hat{q}(s))\hat{q}(s) - C(\hat{q}(s), \hat{S}(s))]ds$$

for all feasible $\hat{q}$ and corresponding $\hat{S}$.

Constant marginal extraction cost k. Then the problem statement for the monopolist reads:

$$\max_q \int_0^\infty e^{-rs}[f(q(s)) - k]q(s)ds$$

subject to feasibility.

Necessary conditions

Hamiltonian

$$H(S, q, \lambda, t) = e^{-rt}[f(q) - k]q + \lambda[-q]$$

Necessary conditions (interior solution)

$$\frac{\partial H(S, q, \lambda, t)}{\partial q} = 0 : \quad e^{-rt}[f'(q(t))q(t) + f(q(t)) - k] = \lambda(t)$$

$$\frac{\partial H(S, q, \lambda, t)}{\partial S} = -\dot{\lambda}(t) : \ \dot{\lambda}(t) = 0$$

Marginal revenues are not necessarily constant.

Example 1

$f(q) = q^{\eta},\ \eta < 0$

Perfect competition

$$\frac{\dot{p}}{p} = \eta \frac{\dot{q}}{q} = r$$

Monopoly case identical (Stiglitz, 1976). Intuition: also monopolist is constrained by the existing resource stock.

Example 2

$f(q) = \bar{p} - q$;
Hamiltonian:

$$H(S, q, \lambda, t) = e^{-rt}(\bar{p} - q - k)q + \lambda[-q]$$

Maximization with respect to q :

$$q(t) = \frac{1}{2}(\bar{p} - k) - \frac{1}{2}\lambda e^{rt}$$

Final time T. Then $[\bar{p} - k] = \lambda e^{rT}$. From resource constraint:

$$(\bar{p} - k)(e^{-rT} - 1 + rT) = 2rS_0$$

Perfect competition:

$$(\bar{p} - k)(rT - 1 + e^{-rT}) = rS_0$$

Exhaustion under monopoly takes place later than under perfect competition.
"The monopolist is the conservationist's best friend".

3. Social optimum

No externalities.
Perfect competition leads to social optium

4. The order of extraction

The Herfindahl rule
The cheaper resource must be depleted before the more expensive one is taken into exploitation. Herfindahl (1967).

Two types of suppliers, differing only in marginal extraction costs, denoted by $k^l < k^h$.
Initial stocks S_0^l and S_0^h. Extraction rates q^l and q^h.Assume $\sup_{q \to 0} f(q) > k^h$

Definition
$(p, q) : \mathbb{R}_+ \to \mathbb{R}_+^{m+1}$ is a *competitive equilibrium* if
i) $p(t) = f(q^l(t) + q^h(t))$ for all $t \geq 0$
ii) For $i = l, f$ extraction q^i is maximizing total discounted profits at p subject to the resource constraint

Equilibrium

Intervals of time where $q^i > 0$ $(i = l, h)$

$$p(t) = k^i + e^{rt}\lambda^i$$

λ^i : constant costate for type i $(i = l, h)$. Two curves in (p, t) space:

$$P^l(t) = k^l + e^{rt}\lambda^l$$

$$P^h(t) = k^h + e^{rt}\lambda^h$$

The asymptotes for $t \to -\infty$ are k^l and k^h. The curves intersect once. If not the high cost mine is never operated, because if it would the low cost mine could make more profits than $\lambda^l S_0^l$. Intersection at t_1.
Simultaneous supply cannot occur.
Also $P^l(t) < P^h(t)$ for all $0 \leq t \leq t_1$, implying $p(t) = P^l(t) < P^h(t)$ for all $0 \leq t \leq t_1$.

Extensions

Generalization w.r.t. the number of types no problem.
Modifications
-General equilibrium (Kemp and Long, 1980a)
-$C_i(q_i) = fq_i + \frac{1}{2}g_i q_i^2$ ⇒simultaneous extraction in some interval of time (Kemp and Long, 1980b). Also in Nash.
-capacity constraints on backstop (Amigues et al., 1998)
-capacity constraint on non-renewable resource (Holland, 2003)

5 Open loop Nash equilibrium. Oligopoly

Two types of stocks: Alberta: up to \$30/b, Saudi Arabia $\sim$ \$1/b, Iraq: less than \$1/b

Lewis and Schmalensee (1980) only one player per type,

Loury (1986) identical players.

"Wisdoms":

Exploit the low cost oil stocks first: Herfindahl rule.

More competition is better (for social welfare), Loury (1986).

The model

Two types of mines l and h

Constant marginal extraction costs: $k^l < k^h$

n^l firms exploit the mines of type l

n^h firms exploit the mines of type h

Firm i of type j is endowed with an initial stock $S_{0i}^j = \frac{S_0^j}{n^j}$

Linear demand for the resource is $p(t) = \bar{p} - x(t)$ with $\bar{p} > k^h$.

Feasibility and objective

Extraction rates : $q_i^l(t)$ and $q_i^h(t)$ with $q^j(t) = \sum_{i=1}^{n^j} q_i^j(t)$

An extraction path q_i^j $(i = 1, 2, ..., n^j, j = l, h)$ is *feasible* if

$$\int_0^\infty q_i^j(t)dt \leq S_{i0}^j \text{ for all } i = 1, 2, ..., n^j \text{ with } j = l, h.$$

The objective of each firm i is

$$Max \int_0^\infty e^{-rt} \left[\{\bar{p} - q^l(t) - q^h(t)\} q_i^j(t) - k^j q_i^j(t) \right] dt$$

subject to feasibility.

Equilibrium

A vector of functions $q \equiv (q_1^l, .., q_{n^l}^l, q_1^h, .., q_{n^h}^h)$ with $q(t) \geq 0$ is an *open-loop Nash equilibrium* if
for all $i = 1, 2, ..., n^l$

$$\int_0^\infty e^{-rs}[\bar{p} - q^l - q^h - k^l]q_i^l dt \geq \int_0^\infty e^{-rs}[\bar{p} - \sum_{j \neq i} q_j^l - \hat{q}_i^l - q^h - k^l]\hat{q}_i^l dt$$

for all feasible $\hat{q}_i^l$,

for all $i = 1, .., n^h$

$$\int_0^\infty e^{-rs}[\bar{p} - q^l - q^h - k^h]q_i^h dt \geq \int_0^\infty e^{-rs}[\bar{p} - \sum_{j \neq i} q_j^h - \hat{q}_i^h - q^l - k^h]\hat{q}_i^h dt$$

for all feasible $\hat{q}_i^f$.

Approach to characterize OLNE

At each moment there are three different possibilities:

both types of firms are producing	S
only high cost firms are producing	C^h
only low cost firms are producing	C^l

Proceed in 3 steps

STEP 1: Write the necessary conditions that must be satisfied by an equilibrium path in each case

STEP 2: Then use:

Continuity of the price at any transition from one phase to another

Exhaustion of the resource

STEP 3: “Play” with that and get for each set of parameters the “equilibrium sequence”

Equilibrium candidates

Hamiltonian for firm i :

$$H_i^j(q_i^j, \lambda^j, t) = e^{-rt}\left(\bar{p} - q^l - q^h - k^j\right) q_i^j + \lambda_i^j\left(-q_i^j\right)$$

Use symmetry, i.e. $q_i^j = q^j / n^j$ and $\lambda_i^j = \lambda^j$ for $j = l, h$ and all $i = 1, .., n^j$.
Along S

$$e^{-rt}\left(\bar{p} - q^l(t) - q^h(t) - \frac{1}{n^l} q^l(t) - k^l\right) = \lambda^l \text{ MR=MC}$$

$$e^{-rt}\left(\bar{p} - q^h(t) - q^l(t) - \frac{1}{n^h} q^h(t) - k^h\right) = \lambda^h, \text{MR=MC}$$

Equilibrium candidates

Along C^l

$$e^{-rt}\left(\bar{p} - q^l(t) - \frac{1}{n^l}q^l - k^l\right) = \lambda^l, \text{ MR=MC}$$

$$p(t) = \bar{p} - q^l(t) \leq k^h + e^{rt}\lambda^h, \text{Price too low for } h$$

Along C^h

$$e^{-rt}\left(\bar{p} - q^h(t) - \frac{1}{n^h}q^h(t) - k^h\right) = \lambda^h, \text{MR=MC}$$

$$p(t) = \bar{p} - q^h(t) \leq k^l + e^{rt}\lambda^l, \text{ Price too low for } l$$

Continuous price path

- transition at t from S to C^l or vice versa requires

$$p(t) = \frac{1}{n^l + 1}(\bar{p} + n^l \left(k^l + \lambda^l e^{rt}\right)) = k^h + \lambda^h e^{rt}$$

- transition at t from S to C^h or vice versa requires

$$p(t) = \frac{1}{n^h + 1}(\bar{p} + n^h \left(k^h + \lambda^h e^{rt}\right)) = k^l + \lambda^l e^{rt}$$

- transition at t from C^l to C^h or vice versa requires

$$p(t) = \frac{1}{n^l + 1}(\bar{p} + n^l \left(k^l + \lambda^l e^{rt}\right)) = \frac{1}{n^l + 1}(\bar{p} + n^h \left(k^h + \lambda^h e^{rt}\right))$$

Lemma 1

i. No transition from C^l to C^h or vice versa.
ii. There exists a phase S.
iii. C^h cannot precede S.
Proof:
i.Transition from C^l to C^h.

$$C^l: \; p(t) = \frac{1}{n^l + 1}\left(\bar{p} + n^l\left(k^l + \lambda^l e^{rt}\right)\right) \leq k^h + \lambda^h e^{rt}$$

$$C^h: \; p(t) = \frac{1}{n^h + 1}\left(\bar{p} + n^h\left(k^h + \lambda^h e^{rt}\right)\right) \leq k^l + \lambda^l e^{rt}$$

Then $\bar{p} \leq k^h + \lambda^h e^{rt}$.We have

$$p(t) = \frac{1}{n^h + 1}\left(\bar{p} + n^h\left(k^h + \lambda^h e^{rt}\right)\right) \geq \frac{1}{n^h + 1}\left(\bar{p} + n^h \bar{p}\right) = \bar{p}$$

which implies a price that is too high to have a positive quantity demanded after the transition.

ii. This follows immediately from statement (i) of the lemma.
iii. Along C^h

$$\bar{p} + n^h k^h - \left(n^h + 1\right) k^l \leq \left(\left(n^h + 1\right) \lambda^l - n^h \lambda^h\right) e^{rt}$$

A transition from C^h to S or vice versa requires

$$\bar{p} + n^h k^h - \left(n^h + 1\right) k^l = \left(\left(n^h + 1\right) \lambda^l - n^h \lambda^h\right) e^{rt_1}$$

Since $\bar{p} > k^h > k^l$ we have $\bar{p} + n^h k^h - \left(n^h + 1\right) k^l > 0$. Note that if C^h precedes S, $\left(\left(n^h + 1\right) \lambda^l - n^h \lambda^h\right) e^{rt}$ is decreasing over time since it is a monotonic function of time and it is larger than $\bar{p} + n^h k^h - \left(n^h + 1\right) k^l$ before t_1 and equal to $\bar{p} + n^h k^h - \left(n^h + 1\right) k^l$ at t_1. This implies that $\left(n^h + 1\right) \lambda^l - n^h \lambda^h < 0$, which along with $\bar{p} + n^h k^h - \left(n^h + 1\right) k^l > 0$, yields contradiction.

Lemma 2

Define

$$\bar{k}^l \equiv \frac{1}{n^l + 1}(\bar{p} + n^l k^l).$$

Since $\bar{p} > k^l$ we have $\bar{k}^l > k^l$.

Lemma 2

i. Suppose $k^h < \bar{k}^l$, then C^l cannot precede S.
ii. Suppose $k^h > \bar{k}^l$, then S cannot precede C^l.

Lemma 3

i. There is simultaneous supply throughout if only if

$$\frac{S_0^l / n^l}{S_0^h / n^h} = \frac{\bar{p} + n^h k^h - \left(n^h + 1\right) k^l}{\bar{p} + n^l k^l - \left(n^l + 1\right) k^h}$$

ii. If

$$k^h < \bar{k}^l$$

there is no simultaneous supply just before total exhaustion.

Lemma 4

Consider the sequence $S \rightarrow C^l$, with C^l the final phase before exhaustion and where the transition takes place at instant of time t_1 and exhaustion at T. Then

$$\frac{n^l + n^h + 1}{n^h} r S_0^h = \left(\bar{p} + n^l k^l - \left(n^l + 1\right) k^h\right) \left(rt_1 - 1 + e^{-rt_1}\right)$$

$$\begin{aligned} \frac{n^l + n^h + 1}{n^l} r S_0^l &= -\frac{n^h}{n^l + 1} \left(\bar{p} + n^l k^l - \left(n^l + 1\right) k^h\right) \left(rt_1 - 1 + e^{-rt_1}\right) \\ &+ \frac{n^l + n^h + 1}{n^l + 1} \left(\bar{p} - k^l\right) \left(rT - 1 + e^{-rT}\right) \end{aligned}$$

Lemma 5

Consider the sequence $S \rightarrow C^h$, with C^h the final phase before exhaustion and with the transition taking place at instant of time t_1 and exhaustion at T. Then

$$\frac{n^l + n^h + 1}{n^l} r S_0^l = \left(\bar{p} + n^h k^h - \left(n^h + 1\right) k^l\right) \left(rt_1 - 1 + e^{-rt_1}\right)$$

$$\begin{aligned} \frac{n^l + n^h + 1}{n^h} r S_0^h &= -\frac{n^l}{n^h + 1}\left(\bar{p} + n^h k^h - \left(n^h + 1\right) k^l\right) \left(rt_1 - 1 + e^{-rt_1}\right) + \\ & \frac{n^l + n^h + 1}{n^h + 1}\left(\bar{p} - k^h\right)\left(rT - 1 + e^{-rT}\right) \end{aligned}$$

$$\begin{aligned} \frac{n^l + n^h + 1}{n^h} r S_0^h &= -\frac{n^l}{n^h + 1}\left(\bar{p} + n^h k^h - \left(n^h + 1\right) k^l\right) \left(rt_1 - 1 + e^{-rt_1}\right) + \\ & \frac{n^l + n^h + 1}{n^h + 1}\left(\bar{p} - k^h\right)\left(rT - 1 + e^{-rT}\right) \end{aligned}$$

Proposition 1

1. *Suppose* $k^h > \bar{k}^l$. *For a given* S_0^h, *there exists* $\tilde{S}_0^l > 0$ *such that the equilibrium sequence reads* $C^l \rightarrow S \rightarrow C^h$ *if* $S_0^l > \tilde{S}_0^l$ *and* $S \rightarrow C^h$ *if* $S_0^l \leq \tilde{S}_0^l$.
2. *Suppose* $k^h \in \left(k^l, \bar{k}^l\right)$

If $\frac{S_0^l / n^l}{S_0^h / n^h} = \frac{\bar{p} + n^h k^h - \left(n^h + 1\right) k^l}{\bar{p} + n^l k^l - \left(n^l + 1\right) k^h}$ *then the equilibrium reads* S

If $\frac{S_0^l / n^l}{S_0^h / n^h} < \frac{\bar{p} + n^h k^h - \left(n^h + 1\right) k^l}{\bar{p} + n^l k^l - \left(n^l + 1\right) k^h}$ *then the equilibrium reads* $S \rightarrow C^h$

If $\frac{S_0^l / n^l}{S_0^h / n^h} > \frac{\bar{p} + n^h k^h - \left(n^h + 1\right) k^l}{\bar{p} + n^l k^l - \left(n^l + 1\right) k^h}$ *then the equilibriun reads* $S \rightarrow C^l$

Proposition (con't)

Knife-edge cases

3i. Suppose $k^h = \bar{k}^l$. Then the result of proposition 1 holds with C^l collapsing.
3ii. Suppose $k^l = k^h$. Then the results of proposition 2 hold with $k^l = k^h$.

Interpretation: role of relative extraction cost and abundance of initial resources
Easier in Cartel versus Fringe model.

6 Open-loop Nash equilibrium. Cartel versus Fringe

One cartel with k^c ; "large" number ($n^f \to \infty$) of fringe members with k^f
F ,C^m, S: Fringe is sole supplier, Cartel is acting as a monopolist, Simultaneous extraction

Proposition 2
The open loop Nash equilibrium in the cartel versus fringe model can be characterized as follows:
i. Suppose $\frac{1}{2}(\bar{p} + k^c) < k^f$. *Then* $C^m \to S \to F$, *with the* C^m *phase collapsing if* S_0^c *is smaller than a certain threshold.*
ii. Suppose $\frac{1}{2}(\bar{p} + k^c) = k^f$. *Then* $S \to F$
iii. Suppose $\frac{1}{2}(\bar{p} + k^c) > k^f$. *Then*

$$S \text{ if } \frac{S_0^c}{S_0^f} = \frac{k^f - k^c}{\bar{p} + k^l - 2k^h}$$

$$S \to F \text{ if } \frac{S_0^c}{S_0^f} < \frac{k^f - k^c}{\bar{p} + k^c - 2k^f}$$

$$S \to C^m \text{ if } \frac{S_0^c}{S_0^f} > \frac{k^f - k^c}{\bar{p} + k^c - 2k^f}$$

Interpretation

Fringe very expensive: cartel can exert monopoly power, especially with high resource stock

Fringe moderately more expensive:
With low cartel stock no monopoly phase.
With high cartel stock a monoply phase but only at the end, after exhaustion of the fringe's reserves, in order to prevent the fringe to supply at the monopoly phase

7 Sensitivity

Increasing n^h (Oligopoly)
Loury (1986) (firms have identical costs but different stocks):
"The deadweight loss from oil'igopoly is non decreasing in the number of firms"

Intuition In our model. Suppose

$$\frac{S_0^l}{S_0^h} = \frac{\left(\bar{p} + n^h k^h - \left(n^h + 1\right) k^l\right)/n^h}{\left(\bar{p} + n^l k^l - \left(n^l + 1\right) k^h\right)/n^l}$$

Keep $\frac{S_0^l}{S_0^h}$ fixed and increase n^h.Then RHS $\downarrow$ and the new equilibrium becomes $S \to C^l$ (Proposition)

Transition and terminal time

$$\frac{n^l + n^h + 1}{n^h} r S_0^h = \left(\bar{p} + n^l k^l - \left(n^l + 1\right) k^h\right) \left(rt_1 - 1 + e^{-rt_1}\right)$$

$$r \left(\frac{n^l + 1}{n^l} S_0^l + S_0^h\right) = \left(\bar{p} - k^l\right) \left(rT - 1 + e^{-rT}\right)$$

Increase n^h .Then T does not change and t_1 decreases. Therefore increasing n^h results in the low cost mines being exhausted last, a source of inefficiency.

Sensitivity

Increasing fringe's stock (cartel versus fringe**)**
Suppose $k^f < \frac{1}{2}(\bar{p} + k^c)$. The equilibrium sequence of extraction is given by

$$S \rightarrow C^m \text{ if } \frac{S_0^c}{S_0^f} > \frac{k^f - k^c}{\bar{p} + k^c - 2k^f}$$

$$S \rightarrow F \text{ if } \frac{S_0^c}{S_0^f} < \frac{k^f - k^c}{\bar{p} + k^c - 2k^f}$$

$$S \text{ if } \frac{S_0^c}{S_0^f} = \frac{k^f - k^c}{\bar{p} + k^c - 2k^f}$$

Social welfare

Social welfare W: the sum of consumer surplus and producer surplus generated by the exploitation of the resource.

$$W = \int_0^{\infty} e^{-rt}[\frac{1}{2}(\bar{p} - p(t))(q^c(t) + q^f(t)) + (p(t) - k^c)q^c(t) + (p(t) - k^f)q^f(t)]dt$$

Suppose $k^f < \frac{1}{2}(\bar{p} + k^c)$ and equilibrium reads $S \rightarrow C^m$ (semi-reversal of the Herfindahl rule).
Result: marginal *welfare* value of an additional resource stock of the fringe can be negative.
Take $k^c = 0$. Final time T , transition time t_1. For $0 \leq t \leq t_1$

$$\begin{aligned} q^f(t) &= (\bar{p} - 2k^f) - (2\lambda^f - \lambda^c)e^{rt} \\ q^c(t) &= k^f + (\lambda^f - \lambda^c)e^{rt} \end{aligned}$$

For $t_1 \leq t \leq T$ only the dominant firm supplies:

$$q^c(t) = \frac{1}{2}\bar{p} - \frac{1}{2}\lambda^c e^{rt}$$

Extraction paths

Continuity of the price path at t_1 and $p(T) = \bar{p}$ imply

$$\begin{aligned} \lambda^c &= \bar{p}e^{-rT} \\ \lambda^f &= \frac{1}{2}(\bar{p} - 2k^f)e^{-rt_1} + \frac{1}{2}\bar{p}e^{-rT} \end{aligned}$$

Therefore, for $0 \leq t \leq t_1$

$$\begin{aligned} q^f(t) &= (\bar{p} - 2k^f)(1 - e^{rt - rt_1}) \\ q^c(t) &= k^f(1 - e^{rt - rt_1}) + \frac{1}{2}\bar{p}(e^{rt - rt_1} - e^{rt - rT}) \end{aligned}$$

and for $t_1 \leq t \leq T$

$$q^c(t) = \frac{1}{2}\bar{p}(1 - e^{rt - rT})$$

Resource constraints

$$rS^f = (\bar{p} - 2k^f)(rt_1 - 1 + e^{-rt_1})$$

$$rS^c + \frac{1}{2}rS^f = \frac{1}{2}\bar{p}(rT - 1 + e^{-rT})$$

Sign of $\frac{\partial W}{\partial S_0^f}$ in the limit case where $S_0^c \to \infty$ and $S_0^f \to 0$.
In this limit case compare welfare in a situation where the fringe is initially nonexistent to the case where it acquires an arbitrarily small amount of stock. The sign of $\frac{\partial W}{\partial S_0^f}$ is ambiguous and depends of the extraction costs and the choke price. Set $k^c = 0$.

Proposition 3

For $\frac{1}{4}\bar{p} < k^f < \frac{1}{2}\bar{p}$, there exists $\tilde{S}^f$ and $\tilde{S}^c$ such that for any S_0^f and S_0^c such that $S_0^f < \tilde{S}^f$ and $S_0^c > \tilde{S}^c$ we have

$$\frac{\partial W}{\partial S^f} < 0.$$

Suppose $k^f < \frac{1}{2}\bar{p}$. Then

$$\lim_{S_0^f \to 0} \lim_{S_0^c \to \infty} \frac{\partial W}{\partial S^f} = \frac{1}{4}(\bar{p} - 4k^f)$$

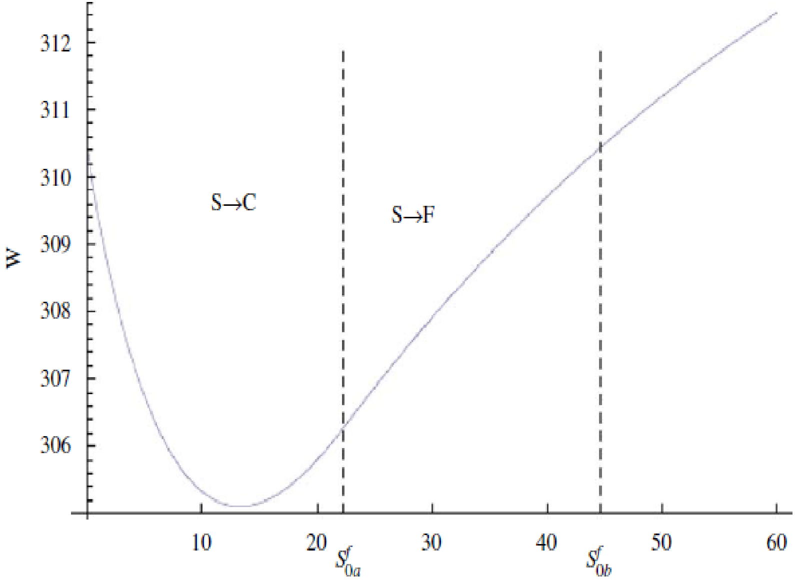
Proof omitted

Interpretation welfare loss

Liberalization of access to existing resource reserves located in preserved areas or development of technologies to increase the 'economically viable' stocks of resource may be bad
Due to the fact that exploitation of an expensive stock delays the extraction from an inexpensive stock. Consumers enjoy more surplus from lower prices, this gain comes at the expense of an increased production cost. The latter cost may outweigh all the benefits from the increase in available resource.

Not only with $k^f < \frac{1}{2}\bar{p}$. Presence of the fringe slows the dominant firm's (low cost) initial production which constitutes in itself a source of welfare loss that can outweigh any gain from the fringe's presence.

$k^c = 0, k^f = 4.5, \bar{p} = 10, r = 0.1, S_0^c = 100$



Drop in the extraction costs of the fringe

.
Suppose $k^f < k^c$ but they are close to each other.
Suppose also we are in regime S (possibly followed by C).
Consider a marginal drop in k^f. In the new situation we are in $S \rightarrow C$. Final time unaffected.
Then, social welfare is decreased.

1. Closed-loop Nash equilibrium

Change of notation:
$n^l = 1$: Cartel

$n^f = n^h$: Fringe

S, C^m, F intervals of time with Simultaneous supply, sole supply by the Cartel and sole supply by the Fringe.

A closed-loop strategy for a firm is a decision rule that gives the extraction rate at t as a function of t and the vector of stocks at time t,
$S(t) = (S^c(t), S_1^f(t), S_2^f(t), ..., S_n^f(t))$.

$$\phi_j^f(t, S(t)) \text{ and } \phi^c(t, S(t))$$

Do open loop and closed loop coincide?

Eswaran and Lewis (1985): OLNE and CLNE outcomes coincide when firms are symmetric.

What does "OLNE and CLNE outcomes coincide" mean?

Suppose q is an OLNE and the extraction path is $\{q(t)\}_t$

Suppose $\left(\phi^c(t, S(t)), \phi_j^f(t, S(t))\right)$ is a CLNE and, given $S(0) = S_0$, the extraction path is $\left\{\left(\phi^c(t, S(t)), \phi_j^f(t, S(t))\right)\right\}_{t, S(0)=S_0}$

For any given $S(0) = S_0$, is there a CLNE $\left(\phi^c(t, S(t)), \phi_j^f(t, S(t))\right)$ such that

$$\left\{\left(\phi^c(t, S(t)), \phi_j^f(t, S(t))\right)\right\}_{t, S(0)=S_0} = \{q(t)\}_{t, S(0)=S_0} \text{ ?}$$

Approach

Determine necessary conditions (*CondOLNE*) that must be satisfied by an OLNE path at each moment.

Determine necessary conditions (*CondCLNE*♠♡*OLNE*) for a CLNE path to satisfy (*CondOLNE*)

Show that (*CondCLNE*♠♡*OLNE*) is never satisfied

The case

$S \rightarrow F$

Cartel's problem:

$$\int_t^\infty e^{-rs}\left(\bar{p} - q^c(s) - \phi^f\left(S(s), s\right) - k^c\right) q^c(s)ds$$

subject to

$$\int_t^\infty q^c(s)ds \leq S^c$$

and

$$\int_t^\infty \phi_i^f\left(S\left(s\right), s\right)ds \leq S_i^f,\ i = 1, 2, ..., n$$

for all non-negative couples (S, t), with $q^c(s) = \phi^c\left(S(s), s\right)$.

Necessary conditions cartel

Hamiltonian cartel:

$$H^c(q^c, S, \mu_c^c, \mu_f^c, t) = e^{-rt}\left(\bar{p} - q^c - \sum_{i=1}^{n} \phi_i^f (S, t) - k^c\right) q^c - \mu_c^c q^c - \sum_{i=1}^{n} \mu_{fi}^c \phi_i^f (S$$

$$e^{-rt}\left(\bar{p} - 2q^c(t) - \phi^f (S(t), t) - k^c\right) - \mu_c^c(t) = 0$$

$$\dot{\mu}_c^c(t) = -\frac{\partial H^c}{\partial S^c} = \sum_{i=1}^{n} \left(e^{-rt} q^c(t) + \mu_{fi}^c(t)\right) \frac{\partial \phi_i^f (S(t), t)}{\partial S^c}$$

$$\dot{\mu}_{fi}^c(t) = -\frac{\partial H^c}{\partial S_i^f} = \sum_{i=1}^{n} \left(e^{-rt} q^c(t) + \mu_{fi}^c(t)\right) \frac{\partial \phi_i^f (S(t), t)}{\partial S_i^f}$$

where

$$\phi^f (S(t), t) = \sum_{i=1}^{n} \phi_i^f (S(t), t)$$

CLNE= OLNE along simultaneous regime?

$\mu_c^c(s) = \lambda^c$, for all instants $s \geq t$ for all $t \geq 0$:

$$e^{-rt}\left(\bar{p} - 2q^c(t) - q^f(t) - k^c\right) - \lambda^c = 0 \text{ for the OLNE}$$

$$e^{-rt}\left(\bar{p} - 2q^c(t) - \phi^f(S(t), t) - k^c\right) - \mu_c^c(t) = 0 \text{ for the CLNE}$$

From

$$\dot{\mu}_c^c(t) = -\frac{\partial H^c}{\partial S^c} = \sum_{i=1}^{n}\left(e^{-rt}q^c(t) + \mu_{fi}^c(t)\right)\frac{\partial \phi_i^f(S(t), t)}{\partial S^c}$$

it follows that

$$\sum_{i=1}^{n}\left(e^{-rt}q^c(t) + \mu_{fi}^c(t)\right)\frac{\partial \phi_i^f(S(t), t)}{\partial S^c} = 0$$

Given the symmetry of fringe firms

$$\mu_{fi}^c(t) = -e^{-rt}q^c(t) \text{ or } \frac{\partial \phi_i^f(S(t), t)}{\partial S^c} = 0$$

First possibility in contradiction with the necessary conditions since it implies that $\dot{\mu}_{fi}^{c} = 0$

$$\dot{\mu}_{fi}^{c}(t) = -\frac{\partial H^{c}}{\partial S_{i}^{f}} = \sum_{i=1}^{n} \left(e^{-rt} q^{c}(t) + \mu_{fi}^{c}(t) \right) \frac{\partial \phi_{i}^{f}\left(S\left(t\right), t\right)}{\partial S_{i}^{f}} = 0$$

but $e^{-rs} q^{c}\left(s\right)$ is not constant along the OLNE.

For CLNE to result in the extraction path of the OLNE, we must then have

$$\frac{\partial \phi_{i}^{f}\left(S\left(t\right), t\right)}{\partial S^{c}} = 0$$

Note. This is different (weaker) from

$$\frac{\partial \phi_{i}^{f}\left(S, t\right)}{\partial S^{c}} = 0$$

Proposition 1

Suppose the OLNE yields the sequence $S->F$, i.e.

$$\frac{1}{2}(\bar{p}+k^{c})<k^{f} \text{ or } \frac{S_0^c}{S_0^f/n}<\frac{\bar{p}+nk^{f}-(n+1)\,k^{c}}{\bar{p}+k^{c}-2k^{f}}$$

orthen the OLNE extraction path cannot be obtained as the extraction path of a CLNE.

REMARKS

1. Due to fact that every firm can affect actrions by all other firms
2. True even in the limit case where $n=\infty$ since for $n\rightarrow\infty$ we have

$$\frac{\partial\left(\lambda^{c}-2\lambda^{f}\right)}{\partial S^{c}}=r\left(\frac{1}{\left(Te^{rT}-e^{rt}\right)}-\frac{1}{\left(t_1e^{rt_1}-e^{rt}\right)}\right)\neq 0$$

Case

$S \rightarrow C$

The analysis requires

$$\frac{\partial \phi_i^f \left(S, t\right)}{\partial S_i^f}$$

Difficulty: we have solved the OLNE for the case of asymmetric fringe members

Intuition:
If one fringe member is given an additional reserve.
All other fringe members will exhaust their resource before this fringe member under consideration does.
Then it is left with the cartel as sole competitor.

We are therefore done if we can show that the OLNE and the CLNE do not coincide for the case of a single cartel and a single fringe member. But for that case repeat the same step as the case $S- > F$.

Closed-loop Cartel-Fringe equilibrium (CL-CFE)

A vector $\left(\pi, \phi^f, \phi^c, \phi^f, \phi_1^f, ..., \phi_n^f\right)$ with a price path $\pi = \pi(t)$ and extraction rules $\phi^c = \phi^c(t, S)$, $\phi^f = \phi^f(t, S)$ and $\phi_i^f = \phi_i^f\left(t, S_i^f\right)$ $(i = 1, 2, ..., n)$ is a *closed-loop Cartel-Fringe equilibrium* (CL-CFE) if

i. the resource constraint is satisfied for all firms, where $q^c(t) = \phi^c(t, S(t))$ and $q_i^f(t) = \phi_i^f\left(t, S_i^f(t)\right)$ $(i = 1, 2, ..., n)$

ii. given ϕ^f,

$$\begin{aligned} & \int_0^\infty e^{-rs}[\bar{p} - \phi^f(s, S(s)) - \phi^c(t, S(t)) - k^c]\phi^c(t, S(t)))ds \\ \geq & \int_0^\infty e^{-rs}[\bar{p} - \phi^f(s, S(s)) - \hat{\phi}^c(t, S(t)) - k^c]\hat{\phi}^c(t, S(t))\, ds \end{aligned}$$

for all feasible strategies $\hat{\phi}^c$.

Closed loop Nash in cartel versus fringe model

Cartel: One firm (or coalition of firms) with market power

Fringe: large number of firms that take the price as given.

Closed-loop Cartel-Fringe equilibrium (CL-CFE)

iii. for all $i = 1, 2, ..., n$, given π

$$\int_0^\infty e^{-rs}[\pi(s) - k^f]\phi_i^f\left(s, S_i^f(s)\right) ds \geq \int_0^\infty e^{-rs}[\pi(s) - k^f]\hat{\phi}_i^f\left(s, S_i^f(s)\right) ds$$

for all feasible strategies $\hat{\phi}_i^f$.

iv. for all $t \geq 0 : \phi^f\left(t, S^f(t)\right) = \Sigma_{i=1}^n \phi_i^f\left(t, S_i^f(t)\right)$

v. for all $t \geq 0 : \pi(t) = Max\left\{\bar{p} - \phi^f\left(t, S^f(t)\right) - \phi^c(t, S(t)), 0\right\}$

Proposition 2

Identical fringe members.
For any $S^c, S^f \geq 0$, there exists a CL-CFE that yields the same outcome as the OL-CFE's outcome:

Approach:

Step 1: Build a CL formulation of the OL-CFE

Step 2: Check that conditions i-v in the Definition of a CL-CFE are satisfied

Step 2a For the fringe check that the CL strategy built in step 1 is a best response to the the OL-CFE price path

Step 2b For the cartel check that the CL strategy built in step 1 is a best response to the the CL fringe's strategy built in step 1

This is done by showing that the CL strategy built in step 1 satisfies the HJB equation associated with the cartel's problem

Conditions iv and v are then trivially satisfied

Outline step 1.

Assume OL CFE: $S \to C$.
Define $h(z) = \ln(1/z) + z - 1, 0 < z < 1$.
Define x and y by

$$h(x) = \frac{rS^f}{\bar{p} + k^c - 2k^f},\ h(y) = \frac{2rS^c + rS^f}{\bar{p} - k^c}$$

Take

$$\varphi^f(S^f) = (\bar{p} + k^c - 2k^f)(1 - x)$$

,

$$\varphi^c(S^c, S^f) = \frac{1}{2}(\bar{p} + k^c - 2k^f)(1 - x) - \frac{1}{2}(\bar{p} - k^c)(y - 1)$$

Then OL-CFE coincides with outcome of these closed loop strategies.
Moreove,HJB conditions are satisfied

Conclusions

"Wisdoms" revisited:

The OLNE extraction path cannot be obtained as the extraction path of a CLNE.

There exists a CL-CFE that yields the same outcome as the OL-CFE's outcome

Microfoundation of the dominant firm-fringe model

The open-loop cartel-fringe equilibrium corresponds to a limit case of an asymmetric OLNE
The closed-loop cartel-fringe equilibrium outcome DOES NOT coincide with the outcome of the limit case of the asymmetric oligopoly CLNE where the number of fringe firms tends to infinity.

Open-loop Stackelberg equlibrium in Cartel versus Fringe model

n fringe members and 1 coherent cartel.
Marginal extraction costs k^f and k^c.
Identical finge members $S_{i0}^f = S_0^f / n$, with S_0^f is the aggregate fringe's initial resource stock.
Each fringe member takes supply by other players as given
Fringe's objective

$$\max \int_0^\infty e^{-rs}(\bar{p} - k^f - q^c(s) - q^f(s))q_i^f(s)ds$$

subject to

$$\int_0^\infty q_i^f(s)ds \leq S_{0i}^f, \; q_i^f(t) \geq 0$$

where $q^f(t) = \sum_{i=1}^n q_i^f(t)$.

Fringe

Hamiltonian

$$H_i^f(q_i^f, \lambda_i^f, t) = e^{-rt}(\bar{p} - k^f - q^c - q^f)q_i^f + \lambda_i^f(-q_i^f)$$

Necessary conditions

$$\frac{\partial H_i^f}{\partial S_i^f} = -\dot{\lambda}_i^f : \dot{\lambda}_i^f(t) = 0$$

$$\frac{\partial H_i^f}{\partial q_i^f} = \bar{p} - k^f - q^c(t) - q^f(t) - q_i^f(t) - e^{rt}\lambda_i^f(t) \leq 0$$

$$(\bar{p} - k^f - q^c(t) - q^f(t) - q_i^f(t) - e^{rt}\lambda_i^f(t)]q_i^f(t) = 0$$

Fringe members identical: $\lambda_i^f = \lambda^f$ for all $i = 1, 2, ..., n$, and

$$q^f(t)(\bar{p} - k^f - q^c(t) - \frac{n+1}{n}q^f(t) - e^{rt}\lambda^f) = 0$$

Price taking

If $n \to \infty$ (with $S_{i0}^f = S_0^f / n \to 0$), the fringe acts as a price taker, meaning that $q^f(t) > 0$ implies $p(t) = k^f + e^{rt}\lambda^f$.

$$\bar{p} - k^f - q^c(t) - q^f(t) - e^{rt}\lambda^f \leq 0$$

$$q^f(t)(\bar{p} - k^f - q^c(t) - q^f(t) - e^{rt}\lambda^f] = 0$$

Cartel

$$\max \int_0^\infty e^{-rs}(\bar{p} - k^c - q^c(s) - q^f(s))q^c(s)ds$$

subject to

$$\int_0^\infty q^c(s)ds = S_0^c,\ q^c(t) \geq 0 \text{ for all } t \geq 0$$

and taking into account that the fringe also maximizes profits subject to its resource constraint.

$$\int_0^\infty q^f(s)ds = S_0^f,\ q^f(t) \geq 0 \text{ for all } t \geq 0$$

Constraint qualification

Define

$$g_1(q^c,q^f)=q^c, g_2(q^c,q^f)=q^f, g_3(q^c,q^f)=q^c+q^f+k^f+\lambda^f e^{rt}-\bar{p}$$

$$g_4(q^c,q^f)=q^f(q^c+q^f+k^f+\lambda^f e^{rt}-\bar{p}]$$

Then $\begin{bmatrix} \frac{\partial g_1}{\partial q^c} & \frac{\partial g_1}{\partial q^f} & g_1 & 0 & 0 & 0 \\ \frac{\partial g_2}{\partial q^c} & \frac{\partial g_2}{\partial q^f} & 0 & g_2 & 0 & 0 \\ \frac{\partial g_3}{\partial q^c} & \frac{\partial g_3}{\partial q^f} & 0 & 0 & g_3 & 0 \\ \frac{\partial g_4}{\partial q^c} & \frac{\partial g_4}{\partial q^f} & 0 & 0 & 0 & g_4 \end{bmatrix} =$

$$\begin{bmatrix} 1 & 0 & q^c & 0 & 0 & 0 \\ 0 & 1 & 0 & q^f & 0 & 0 \\ 1 & 1 & 0 & 0 & q^c+q^f+k^f+\lambda^f e^{rt}-\bar{p} & 0 \\ q^f & q^c+2q^e+k^f+\lambda^f e^{rt}-\bar{p} & 0 & 0 & 0 & 0 \end{bmatrix}$$

should have rank 4. Not so if for some instant of time $p(t)=k^f+\lambda^f e^{rt}$ and either $q^c(t)$ or $q^f(t)$ equals zero.

Multiplier associated with the objective function in the Hamiltonian is possibly zero.

Alternative approach needed

Suppose $q^c(t) > 0,\ q^f(t) = 0,\ \ p(t) \neq k^f + \lambda^f e^{rt}$ then $p(t) = \frac{1}{2}(\bar{p} + k^c) + \frac{1}{2}e^{rt}\lambda^c$. Define

$$P^1(t) := k^f + e^{rt}\lambda^f$$

$$P^2(t) := k^c + e^{rt}\lambda^c$$

$$P^3(t) := \frac{1}{2}(\bar{p} + k^c) + \frac{1}{2}e^{rt}\lambda^c$$

P^1 is *competitive* price, P^3 is *monopoly* price, P^2 auxiliary variable. λ^f and λ^c endogenous constants.

Proposition 3

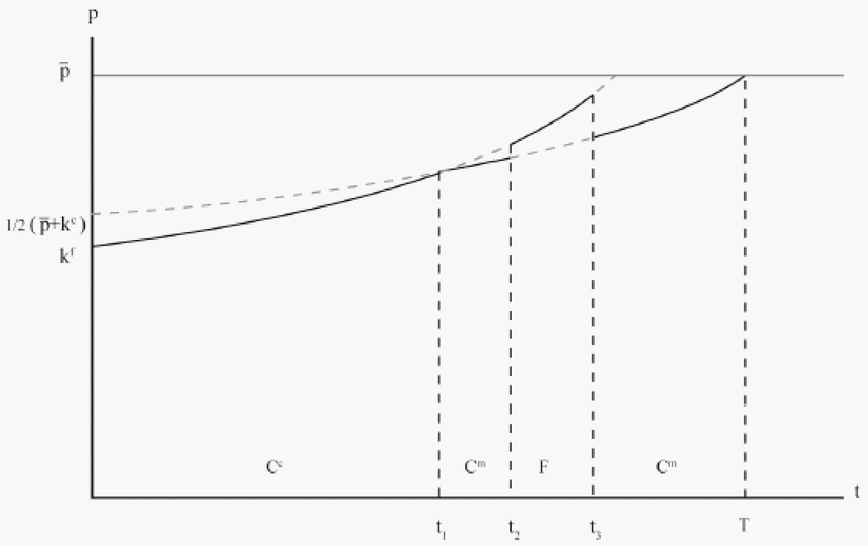
Suppose $k^f = \frac{1}{2}(\bar{p} + k^c)$.The equilibrium is $C^c \rightarrow F$

Suppose $k^c < k^f < \frac{1}{2}(\bar{p} + k^c)$. The equilibrium is $C^c \rightarrow C^m \rightarrow F \rightarrow C^m$ with the first C^m phase possibly collapsing.

Suppose $k^c = k^f$. The equilibrium is $S \rightarrow C^m$ with an arbitrary division of C^c and F also possible over the first interval.

Suppose $k^c > k^f$. The equilibrium is $F \rightarrow C^c \rightarrow C^m$

Suppose $k^c > k^f$. The equilibrium is $F \rightarrow C^c \rightarrow C^m$


p
p̄
1/2 (p̄+k^c)
k^f
C^c
C^m
F
C^m
t
t_1
t_2
t_3
T

Price discontinuit

y

Equilibrium price trajectory corresponding with $C^c \rightarrow C^m \rightarrow F \rightarrow C^m$, is discontinuous at switches from the cartel producing at the monopoly price to a period of fringe supply and vice versa.

Intuition

$$[e^{-rt_i}(P^3(t_i) - k^c) - \lambda^c](\bar{p} - P^3(t_i)) = \mu^f(\bar{p} - P^1(t_i)), \; i = 2, 3$$

At t_2 transition from the cartel supplying at the monopoly price to the fringe supplying.

At t_3 there is a transition just the other way around.

Suppose the cartel contemplates to extend the first monopoly phase.
On the one hand, it would supply $\bar{p} - P^3(t_2+)$ at t_2+ making a profit per unit of $e^{-rt_2}(P^3(t_2+) - k^c - \lambda^c)$
On the other hand it should compensate the fringe for not supplying at t_2. The compensation per unit is denoted by μ^f.
This is not taken into account in earlier studies except in Groot et al. (o.c.). The same argument applies to instant of time t_3.

Dynamic inconsistency

1. Large stock of the cartel with $k^c > k^f$. Then $F \rightarrow C^c \rightarrow C^m$. Let t_1 denote the transition time from F to C^c. At t_1 the fringe's stock is exhausted.
Then the cartel is a de facto monopolist and if, at that particular instant of time, it reconsiders its optimal extraction path, it will deviate from the path initially announced, because it will exercise it full monopoly power.
2. $k^c < k^f < \frac{1}{2}(\bar{p} + k^c)$. Whatever the initial stocks there is an initial interval of time along which the cartel supplies at the competitive price.

Feedback Stackelberg equilibrium

Remove time-inconsistency. Subgame perfectness.
HJB equation for cartel

$$\frac{\partial V^c}{\partial t} + \max_{q^c}[e^{-rt}(\bar{p} - q^f - q^c - k^c)q^c - \frac{\partial V^c}{\partial S^c}q^c - \frac{\partial V^c}{\partial S^f}q^f] = 0$$

Four possible regimes F, C^c, S and C^m.
In the first three the competitive price results $p(t) = k^f + e^{\rho t}\lambda^f$.
In regime C^m the cartel produces at monopoly price below competitive price.

$$p(t) = \overline{p} - q^c(t) - q^f(t) \leq k^f + e^{rt}\lambda^f, \text{ for all } t \geq 0$$

Moreover, if $\overline{p} \leq k^f + e^{\rho t_1}\lambda^f$ for some $t_1 > 0$, then $q^f(t) = 0$ for all $t > t_1$.

Preliminaries

$$L = e^{-rt}(\bar{p} - q^f - q^c - k^c)q^c - \frac{\partial V^c}{\partial S^c}q^c - \frac{\partial V^c}{\partial S^f}q^f + \psi(t)[k^f + e^{rt}\lambda^f - \overline{p} + q^c(t) +$$

If the cartel determines the extraction rates such that $p(t) < k^f + e^{\rho t}\lambda^f$, then $q^f(t) = 0$ and $\psi(t) = 0$ yielding

$$q^c(t) = \frac{1}{2}(\overline{p} - k^c - e^{rt}\frac{\partial V^c}{\partial S^c})$$

with a resulting price

$$p(t) = \frac{1}{2}(\overline{p} + k^c + e^{rt}\frac{\partial V^c}{\partial S^c})$$

So, we are in a C^m regime. Note that it should be the case that

$$\frac{1}{2}(\overline{p} + k^c + e^{rt}\frac{\partial V^c}{\partial S^c}) < k^f + e^{\rho t}\lambda^f$$

Competitive cartel

If the cartel determines the extraction rates such that $p(t) = k^f + e^{\rho t}\lambda^f$, the necessary conditions read

$$\frac{\partial L}{\partial q^f} \leq 0 : -e^{-rt}q^c - \frac{\partial V^c}{\partial S^f} - \psi(t) \leq 0,\ \frac{\partial L}{\partial q^f}q^f(t) = 0$$

$$\frac{\partial L}{\partial q^c} \leq 0 : e^{-rt}(\bar{p} - q^f - 2q^c - k^c) - \frac{\partial V^c}{\partial S^c} - \psi(t) \leq 0, \frac{\partial L}{\partial q^c}q^c(t) = 0$$

Three possibilities: S, C^c,F

$$\frac{1}{2}(\overline{p} + k^c + e^{rt}\frac{\partial V^c}{\partial S^c}) > k^f + e^{rt}\lambda^f$$

Three regimes possible

1. S. Then

$$-e^{-rt}q^c - \frac{\partial V^c}{\partial S^f} = \psi(t)$$

$$p(t) - k^c = e^{rt}\left(\frac{\partial V^c}{\partial S^c} - \frac{\partial V^c}{\partial S^f}\right)$$

2. C^c. Then

$$e^{-rt}(\bar{p} - 2q^c(t) - k^c) - \frac{\partial V^c}{\partial S^c} = \psi(t)$$

$$p(t) - k^c > e^{rt}\left(\frac{\partial V^c}{\partial S^c} - \frac{\partial V^c}{\partial S^f}\right)$$

3. F. Then,

$$-e^{-rt}q^c - \frac{\partial V^c}{\partial S^f} = \psi(t)$$

$$p(t) - k^c \leq e^{rt}\left(\frac{\partial V^c}{\partial S^c} - \frac{\partial V^c}{\partial S^f}\right)$$

The idea is to start from plausible extraction schedules and to verify that these optimality conditions and the resulting HJB differential equation are satisfied.

The strongly time-consistent solution

The idea is to start from plausible extraction schedules and to verify that these optimality conditions and the resulting HJB differential equation are satisfied.
Reminder Open loop
If $k^f > \frac{1}{2}(\overline{p} + k^c)$ and S_0^c is small: $C^c \longrightarrow F$. Otherwise $C^m \longrightarrow C^c \longrightarrow F$.
Both are weakly time-consistent.
If $k^c < k^f \leq \frac{1}{2}(\overline{p} + k^c)$ and S_0^c is small: $C^c \longrightarrow F$. Otherwise time inconsistency
If $k^f = k^c$, infinitely many solutions. One is the open-loop Nash equilibrium, given by the extraction schedule $S \longrightarrow C^m$ which is weakly time-consistent.
If $k^f < k^c$, time-inconsistency.

Weakly time consistent paths

Weakly time-consistent extraction schedules $C^c \longrightarrow F$, $C^m \longrightarrow C^c \longrightarrow F$ and $S \longrightarrow C^m$ are also strongly time-consistent.
For $C^m \longrightarrow C^c \longrightarrow F$ open-loop von Stackelberg solution is strongly time-consistent.
For $C^c \longrightarrow F$ and $S \longrightarrow C^m$ conditions for subgame perfectness hold for a wider range of parameter values than the range for which these schedules are the open-loop von Stackelberg solution.
For some parameters $C^c \longrightarrow F$ is the solution, for others it is $S \longrightarrow C^m$
$F \longrightarrow S \longrightarrow C^m$ is a possible solution to the problem.

Proposition 1

There exists k with $\frac{1}{2}(\overline{p} + k^c) < k < \overline{p}$ such that if $k^c < k^f < k$, the extraction schedule $C^c \longrightarrow F$ is a candidate for the strongly time-consistent solution.

Suppose $\frac{1}{2}(\overline{p} + k^c) \leq k^f$.
Proposition 1 states
1. for small k^f the extraction schedule $C^c \longrightarrow F$ is a candidate solution
2. for large k^f it is not.
For $\frac{1}{2}(\overline{p} + k^c) \leq k^f$, the extraction schedule $C^c \longrightarrow F$ is the open-loop von Stackelberg solution if the fringe's initial resource stock is large relative to the cartel's.
This determines a dividing curve in the $k^f - S_0^f$ plane, below which $C^c \longrightarrow F$ results.
Above this curve, the extraction schedule $C^m \longrightarrow C^c \longrightarrow F$ is the open-loop von Stackelberg solution, which is weakly time-consistent.
The next step is therefore to prove that this extraction schedule is also strongly time-consistent.

Proposition 2

The weakly time-consistent open-loop von Stackelberg equilibrium with the extraction schedule $C^m \longrightarrow C^c \longrightarrow F$ is strongly time-consistent.

The extraction schedule $S \longrightarrow C^m$ is an open-loop von Stackelberg solution in the case $k^f = k^c$ and it is also the open-loop Nash equilibrium. Newbery (1993) already conjectured that this form of extraction schedule could be the strongly time-consistent solution in certain cases. The next proposition shows that $S \longrightarrow C^m$ is indeed a candidate solution for a wide range of parameter values. The problem with the $S-$regime is to determine the division of supply over the cartel and the fringe. What we postulate is a specific division, which depends on the initial stocks of the cartel and the fringe.

Proposition 3
If $\frac{1}{2}(p + k^c) > k^f$ and if the cartel's initial resource stock of the cartel is large relative to the fringe's, the extraction schedule $S \longrightarrow C^m$, is a candidate for the strongly time-consistent solution.

Proof
Consider

$$q^c(t) = \frac{[\overline{p} - k^f)[k^f - k^c) + [e^{rt}\lambda^f)^2 - [\overline{p} - k^f)\lambda^c e^{rt}\left[\frac{e^{rt}[2\lambda^f - \lambda^c)}{\overline{p} + k^c - 2k^f}\right]^{\frac{\overline{p}+k^c-2k^f}{\overline{p}-k^c}}}{\sqrt{(k^f - k^c + q^{rt}\lambda^f)^2 - (\bar{p} - k^c)e^{rt}\lambda^c\left[\frac{e^{rt}(2\lambda^f - \lambda^c)}{\bar{p} + k^c - 2k^f}\right]^{\frac{\bar{p}+k^c-2k^f}{\bar{p}-k^c}}}}$$

Leading idea: there exist functions g^f and g^c and positive constants λ^f and λ^c constants such that the feedback equilibrium realization in part S of the $S \longrightarrow C^m$ regime can be described as $q^f(s) = g^f(e^{rs}\lambda^f, e^{rs}\lambda^f)$ and $q^c(s) = g^c(e^{rs}\lambda^f, e^{rs}\lambda^f)$ for all $s \geq 0$.

Theorem

If $\frac{1}{2}(\overline{p}+k^c) \leq k^f$, the extraction schedules are either $C^c \longrightarrow F$ or $C^m \longrightarrow C^c \longrightarrow F$. The last schedule occurs if k^f is high or, to put it differently, if the fringe's initial resource stock is relatively small. Fixing all the other parameters, a dividing curve can be drawn in the $k^f - S_0^f$ plane above which $C^m \longrightarrow C^c \longrightarrow F$ is the solution and below which $C^c \longrightarrow F$ is the solution.
If $k^c < k^f < \frac{1}{2}(\overline{p}+k^c)$, the extraction schedule $C^c \longrightarrow F$ is a candidate solution together with the extraction schedule $S \longrightarrow C^m$ below a dividing curve in the $k^f - S_0^f$ plane where it exists. Which one actually prevails depends on which one gives the highest profits to the cartel, since the cartel is the leader in the game. This yields another dividing curve in the $k^f - S_0^f$ plane above which $C^c \longrightarrow F$ is the solution and below which $S \longrightarrow C^m$ is the solution.
If $k^f = k^c$, the extraction schedule $S \longrightarrow C^m$ is the solution.
If $k^f < k^c$, the extraction schedules are either $S \longrightarrow C^m$ or $F \longrightarrow S \longrightarrow C^m$. The last schedule occurs if k^f is low or, to put it differently, if the initial resource stock of the fringe S_0^f is relatively high. Fixing all the other parameters, a dividing curve can again be drawn in the $k^f - S_0^f$ plane above which $S \longrightarrow C^m$ is the solution and below which $F \longrightarrow S \longrightarrow C^m$ is the solution.

Incomplete story?

Other candidate solutions may exist that give higher profits to the cartel. Suppose $k^f < k^c$
No C^c in the final stage: at the moment of exhaustion of the fringe, the cartel will start exploiting its monopoly power.
No S at the end. The formal proof was given above where we show that the sequence $S \longrightarrow C^m$ is a solution with the C^m holding on a non-degenerate interval of time.
What if F is final stage? No C^m before F because the price trajectories P^1 and P^3 can intersect only once. No S before F : better for cartel to split interval with simultaneous supply split into two subintervals: a first interval in which the fringe supplies alone, the second in which the cartel supplies. No C^c before F.
Also $F \longrightarrow C^m$ does not satisfy the HJB equation.
The candidate solutions are therefore $S \longrightarrow C^m$, $F \longrightarrow S \longrightarrow C^m$, and $C^c \longrightarrow S \longrightarrow C^m$. The latter can be excluded using a continuity argument with regard to the value function.

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