Les Cahiers du GERAD ISSN: 0711–2440

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G–2007–45

July 2007

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Abstract

A stochastic control model is proposed as a paradigm for the design of optimal timing of greenhouse gases (GHG) emissions abatement. The resolution of uncertainty concerning climate sensitivity and the technological breakthrough providing access to a carbon-free production economy are modeled as controlled stochastic jump processes. The optimal policy is characterized using the dynamic programming solution to a piecewise deterministic optimal control problem. A numerical illustration is developed with a set of parameters calibrated on recently proposed models for integrated assessment of climate policies. The results are interpreted and the insights they provide on the timing issue of climate policy are discussed.

Résumé

Un modèle de contrôle stochastique est proposé comme cadre d'analyse du tempo optimal de la réduction des émissions de gaz à effet de serre (GES) par l'économie mondiale. La résolution de l'incertitude concernant la sensibilité du climat et une percée technologique donnant accès à une production économique sans émissions de carbone est modélisée par contrôle stochastique optimal avec processus de sauts markoviens. La politique optimale est caractérisée à partir des équations de la programmation dynamique associées à un problème de contrôle optimal déterministe par morceaux. Une illustration numérique, dont l'ensemble des paramètres est calibré à l'aide de modèles récemment proposés pour l'évaluation intégrée de politiques climatiques, est développée. Les résultats sont interprétés et les enseignements qu'ils fournissent concernant la question du timing des politiques climatiques sont discutés.

1 Introduction

While climate change has become an issue of primary concern for virtually all countries around the world, highly industrialized nations rightly feel particularly concerned. Indeed, it is the strong correlation between wealth production and sustainment of wealthy lifestyles on the one hand, and energy consumption and consequent pollution on the other hand, that makes them the historical culprits for the current state of the planet and suggests they be considered somewhat more responsible in insuring its survival, or rather our collective survival in it. Industrial developing giants, or immense consumer pools to be, such as China and India, will no doubt also have to play their part in helping stem the tide, but they can argue that they are relatively recent, albeit potentially very significant, polluters in this game. At the heart of the matter however, lie economic considerations. How can governments then best trade the line between wealth producing activities (typically correlated with energy voraciousness either as a process or as a consequence), and the carrying of their fair share of controlling greenhouse gas (GHG) emissions?

Rational answers to the above question critically depend on both an understanding of the impact of human activity on climate, in particular the impact of GHG concentrations in the earth atmosphere on earth mean air surface temperature (SAT), and on an ability to limit release of such gases in the atmosphere either via improved technologies, or a switch to less polluting, non fossil, renewable forms of energy, the latter being typically much more expensive to produce, or difficult to come by. However, precise climate sensitivity information, critical for decision making, is currently clouded with a great degree of uncertainty, and the same applies to the future of our pollution restricting technological know-how, both for intrinsic reasons and because it depends on yet to come economic decisions as to research investment levels.

The aim of this paper is to contribute an elementary model for environmentally conscious, rational economic decision making under uncertainty, with the long term goal of maintaining as high a level of welfare as permitted by the state of our planet and our technological know-how. A global infinite horizon discounted welfare criterion is specified, and decision making is formulated as an optimal stochastic control problem on a jump Markov non diffusion model. Such models include continuous evolutions labeled as modes, punctuated by mode jumps at random times. The model is construed as representing, albeit in a highly aggregated manner, the evolution of the critical decision parameters as they depend on time and the dynamic levels of capital investment. This model extends in some way the preliminary analysis presented by Manne [12], where a stochastic programming approach is implemented using the MERGE integrated assessment model [15] with an event-tree representing the possible resolution of uncertainty on climate sensitivity in a future date.

The use of optimal economic growth models, in line with the original model of Ramsey [22]—very much akin to optimal control models—, to build integrated assessment models for climate policy has been initiated by Manne [13], [14], [15] and Nordhaus [19], [20], [21]. More recently, the use of stochastic control models to develop climate-economy models has been advocated by P. Ambrosi *et al.* [1] and by Haurie [9].

Jump Markov models are associated by now with a rich control theoretic literature particularly starting with the works of Wonham [27] and Rishel [23], followed by Sworder [24], Mariton [16] and Dufour and Costa [7], for example. A generator theory for their non diffusive version, including so-called boundary induced or (equivalently) forced jumps, has been developed by Davis in his definitive monograph [6], and leads directly to related dynamic programming equations (see Vermes [26], for example).

The rest of the paper is organized as follows. In Section 2, we present our integrated assessment model. It includes a continuous part capturing the production process, as well as two distinct types of one-time random mode jumps. The first type of random jump is aimed at modeling the potentially abrupt move from current uncertainty in climatological knowledge, to a state of more thorough knowledge as data is recorded over time (in particular a definitive assessment of the earth mean SAT sensitivity to GHG concentration levels); the second type of jump is aimed at capturing the randomly timed occurrence of the widely anticipated move from the current state of limited environment related technological know-how, to technological breakthroughs leading to carbon free or lowered carbon production economies. Note that the timing of the occurrence of the second type of jump is strongly affected by cumulative capital investment into research programs. Further building on the model, in Section 3, a welfare criterion is introduced, the nature of control actions is detailed and an optimal control problem with penalty based enforcement of a precautionary principle is proposed. Section 4 is dedicated to the presentation of the associated dynamic programming equations and a discussion relating their solution to that of an intermediary sequence of deterministic infinite horizon optimal control problems. It provides also economic interpretations of the various rewards/decisions in the model. In Section 5, a numerical scheme based on solving a sequence of interlinked infinite horizon optimal control problems is presented and applied on a test case. Numerical results are discussed. Conclusions are drawn in Section 6.

2 The integrated assessment model

2.1 Economic modeling

We use an economic growth model a la Ramsey [22] where the economic good is produced by three factors, labor, physical capital and fossil energy which generates carbon emissions. We distinguish between two types of economy: the "carbon economy" where a high level of carbon emissions is necessary to obtain output and the so-called "carbon-free economy" where a much lower level of emissions is necessary to produce the economic good. As the associated technologies are completely different, we use two distinct types of capital for the two economies. The carbon-free capital cannot initially contribute to the production of economic output. For that to happen, the associated technology must first become available, i.e. a "breakthrough" is a precondition. However, investment into the carbon-free technology will increase the probability that such a breakthrough takes place. So the carbon free related physical capital is treated as cumulative research and development

(R&D) investment until the anticipated breakthrough occurs, after which it is treated as a productive capital.

2.1.1 Variables

The following variables enter in the description of the economic model:

$C(t) \geq 0$:	total consumption at time t, in trillions (10^{12}) of dollars;
$c(t)$:	per capita consumption, $c(t) = \frac{C(t)}{L(t)}$;
$E(t) \geq 0$:	global yearly emissions of GHG (in Gt–10^9 tons–carbon equivalent);
$E_1(t) \geq 0$:	part of the emissions attributed to the carbon economy;
$E_2(t) \geq 0$:	part of the emissions attributed to the carbon-free economy;
$I_i(t) \geq 0$:	investment at time t in capital $i = 1, 2$, in trillions of dollars;
$K_1(t) \geq 0$:	physical stock of productive capital in the carbon economy at time t, in trillions of dollars;
$K_2(t) \geq 0$:	physical stock of productive capital in the carbon-free economy at time t, in trillions of dollars;
$L(t) \geq 0$:	labor (world population), in millions (10^6) persons;
$L_1(t) \geq 0$:	part of the labor force allocated to the carbon economy;
$L_2(t) \geq 0$:	part of the labor force allocated to the carbon-free economy;
t:	running time, unit = 10 years;
$\xi(t) \in \{0, 1\}$:	indicator variable for the carbon-free economy, where $\xi(t) = 0$ indicates that $K_2(t)$ is an R&D capital;
W:	total discounted welfare;
$Y(t)$:	economic output at time t, in trillions of dollars.

2.1.2 Economic dynamics

A social planner is assumed to maximize social welfare (W), given by the integral over an infinite time horizon of discounted utility from per capita consumption with discount rate ρ (set typically to 3% per year or 30% per decade):

$$W = \int_0^\infty e^{-\rho t} L(t) \log[c(t)]\, dt. \tag{1}$$

Total labor (L) is divided between labor allocated to the carbon economy (L_1) and labor allocated to the carbon-free economy (L_2):

$$L(t) = L_1(t) + L_2(t). \tag{2}$$

Total labor evolves according to the following law of motion:

$$\dot{L}(t) = g_L(t) L(t), \tag{3}$$

where the growth rate g_L decays exponentially with coefficient δ_{g_L} (set to 0.3):

$$\dot{g}_L(t) = -\delta_{g_L} g_L(t). \tag{4}$$

Yearly GHG emissions come from the carbon economy (emission level E_1) and the carbon-free economy (emission level E_2):

$$E(t) = E_1(t) + E_2(t). \tag{5}$$

Economic output (Y) occurs in the two economies according to an extended Cobb-Douglas production function in three inputs, capital (K), labor (L) and energy (which use is measured through emission level E):

$$\begin{aligned} Y(t) &= A_1(t)K_1(t)^{\alpha_1}(\phi_1(t)E_1(t))^{\theta_1(t)}L_1(t)^{1-\alpha_1-\theta_1(t)} \\ &\quad +\xi(t)A_2(t)K_2(t)^{\alpha_2}(\phi_2(t)E_2(t))^{\theta_2(t)}L_2(t)^{1-\alpha_2-\theta_2(t)}. \end{aligned} \tag{6}$$

Notice that in Eq. (6), production from the carbon-free economy occurs only when $\xi(t) = 1$, namely when the technological breakthrough has occurred. The different terms in Eq. (6) are detailed below. Total factor productivity (A_i) evolves according to the following law of motion:

$$\dot{A}_i(t) = g_{A_i}(t)A_i(t) \quad i = 1, 2, \tag{7}$$

where the growth rate g_{A_i} decays exponentially with coefficient $\delta_{g_{A_i}}$ (set to 0.005):

$$\dot{g}_{A_i}(t) = -\delta_{g_{A_i}} g_{A_i}(t) \quad i = 1, 2. \tag{8}$$

Capital stock (K_i) evolves according to the choice of investment (I_i) and depreciation rate δ_{K_i} (set to 0.1) through the standard relationship given by:

$$\dot{K}_i(t) = I_i(t) - \delta_{K_i} K_i(t) \quad i = 1, 2. \tag{9}$$

The elasticity of output α_i with respect to capital K_i is set to 0.3. The evolution of energy efficiency (ϕ_i) is given by:

$$\dot{\phi}_i(t) = g_{\phi_i}(t)\phi_i(t) \quad i = 1, 2, \tag{10}$$

where the growth rate g_{ϕ_i} decays exponentially with coefficient $\delta_{g_{\phi_i}}$ (set to 0.2):

$$\dot{g}_{\phi_i}(t) = -\delta_{g_{\phi_i}} g_{\phi_i}(t) \quad i = 1, 2. \tag{11}$$

Elasticity of output w.r.t. emissions (θ_i) evolves according to the following law of motion:

$$\dot{\theta}_i(t) = g_{\theta_i}(t)\theta_i(t) \quad i = 1, 2, \tag{12}$$

where the growth rate g_{θ_i} decays exponentially with coefficient $\delta_{g_{\theta_i}}$ (set to 0.008):

$$\dot{g}_{\theta_i}(t) = -\delta_{g_{\theta_i}} g_{\theta_i}(t) \quad i = 1, 2. \tag{13}$$

Economic output is used for consumption (C), investment (I) and the payment of energy costs:

$$Y(t) = C(t) + I_1(t) + I_2(t) + p_{E_1}(t)\phi_1(t)E_1(t) + p_{E_2}(t)\phi_2(t)E_2(t). \tag{14}$$

Energy prices (p_{E_i}) are inflated over-time using the discount rate (ρ) as follows:

$$\dot{p}_{E_i}(t) = e^{\rho t} p_{E_i}(t) \quad i = 1, 2. \tag{15}$$

One also imposes a limit on the fraction of economic output that can be used to invest on the clean capital (K_2)[1]:

$$I_2(t) \leq \kappa Y(t). \tag{16}$$

2.1.3 Initial values

In the above equations, parameters assume the following initial values[2]:

- $L(0)$: initial value for population level; $L(0) = 6409$;
- $g_L(0)$: initial value for growth rate of population; $g_L(0) = 0.08$;
- $A_i(0)$: initial value for total factor productivity; $A_1(0) = A_2(0) = 0.0302$;
- $g_{A_i}(0)$: initial value for growth rate of total factor productivity; $g_{A_1}(0) = g_{A_2}(0) = 0.1$;
- $\phi_i(0)$: initial value for energy efficiency; $\phi_1(0)$=1.3; $\phi_2(0)$=5.0;
- $g_{\phi_i}(0)$: initial value for growth rate of energy efficiency; $g_{\phi_1}(0) = g_{\phi_2}(0) = 0.15$ per decade;
- $\theta_i(0)$: initial value for elasticity of output w.r.t. emissions; $\theta_1(0) = \theta_2(0) = 0.05$;
- $g_{\theta_i}(0)$: initial value for growth rate of elasticity of output w.r.t. emissions; $g_{\theta_1}(0) = g_{\theta_2}(0) = 0.0116$;
- $p_{E_i}(0)$: initial price of energy input; $p_{E_1}(0) = 0.35$; $p_{E_2}(0) = 0.6$;

2.1.4 Asymptotic values

Asymptotic values of the parameters are given below:

- $L(\infty)$: asymptotic value for population level; $L(\infty) = 8368$;
- $A_i(\infty)$: asymptotic value for total factor productivity; $A_1(\infty) = A_2(\infty) = 0.3440$;
- $\phi_i(\infty)$: asymptotic value for energy efficiency; $\phi_1(\infty) = 2.75$; $\phi_2(\infty) = 10.59$;
- $\theta_i(\infty)$: asymptotic value for elasticity of output w.r.t. emissions; $\theta_1(\infty) = \theta_2(\infty) = 0.2132$.

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[1] This limit is introduced to reduce the spurious effect of very high investment in R&D capital to accelerate the occurrence of the breakthrough. Our numerical experiments (in Section 5) will show that this constraint may only be binding when the technological breakthrough has not yet occurred, namely when K_2 represents an R&D capital. We have selected a value for κ set to 7%.

[2] Mostly following the DICE model [20] and more precisely the DICE-2007 version; see: http://www.econ.yale.edu/∼nordhaus/DICEGAMS/DICE2007_d_v7.htm

2.2 Technological breakthrough dynamics

Let us recall that possibility of access to a cleaner economy at time t is represented by a binary variable $\xi(t) \in \{0, 1\}$ whereby the clean technology starts being available only when ξ switches from an initial value of 0 to 1. The initial value $\xi(0) = 0$ indicates that there is no access to the clean technology at the initial time. The switch to the value 1 occurs at a random time which is controlled through the global accumulation of R&D capital K_2. More precisely one introduces a jump rate function[3] $q_b(t, K_2(t))$ which will serve to determine the elementary probability of a switch around time t given that no switch has occurred up to that time:

$$\mathrm{P}[\xi(t + dt) = 1|\xi(t) = 0, K_2(t)] = q_b(t, K_2(t))\, dt + o(dt). \tag{17}$$

One uses an affine function form for $q_b(t, K_2(t))$ which is defined as:

$$q_b(t, K_2(t)) = \omega_b + \upsilon_b K_2(t), \tag{18}$$

where the parameters (ω_b, υ_b) are defined as follows:

ω_b: initial probability rate of discovery; $\omega_b = 0.05$;
υ_b: slope w.r.t $K_2(t)$ of the probability rate of discovery; $\upsilon_b = 0.0019$.

2.3 GHG concentration dynamics

Let $M(t)$ define atmospheric concentration of GHG at time t, in GtC equivalent. The accumulation of GHG in the atmosphere is described by the following equation:

$$\dot{M}(t) = \beta E(t) - \delta_M (M(t) - M_b), \tag{19}$$

where the initial value of atmospheric concentration is $M(0) = 808.8$ GtC and where the different parameters are defined as follows:

β: marginal atmospheric retention rate; $\beta = 0.64$;
δ_M: natural atmospheric elimination rate; $\delta_M = 0.036$;
M_b: preindustrial level of atmospheric concentration; $M_b = 588.2$ GtC.

2.4 Capturing uncertainty on global targets

The social planner is assumed to pursue the objective of the United Nations Framework Convention on Climate Change [18] that is the "stabilization of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system". The value of such a "safe" level depends basically on the climate sensitivity whose value remains uncertain. However, according to the Fourth Assessment Report of the International Panel on Climate Change [10], the latter "is likely to lie between 2°C and 4.5°C". Depending on the climate sensitivity ω, we assume for

[3] We always assume the required regularity, e.g. continuous differentiability w.r.t. time and state.

illustrative purposes the following values for an upper bound ($\bar{M}$) on GHG concentrations:
$\omega = 2°$C: $M(t) \leq \bar{M} = 1176.5$ GtC (560 ppmv);
$\omega = 4.5°$C: $M(t) \leq \bar{M} = 882.2$ GtC (420 ppmv).

Assume that these two possible values are considered at initial time ($t = 0$) with a priori probabilities $\pi_1 = \pi_2 = 0.5$. The time θ at which the true climate sensitivity is known is a random (Markov) time with intensity depending on the accumulated GHG stock:

$$\mathrm{P}[\theta \in (t, t + dt)|\theta \geq t, M(t)] = q_c(t, M(t))dt + o(dt), \tag{20}$$

where the jump rate function $q_c(t, M(t))$ is supposed to be known. Again, we assume an affine form for the jump rate function:

$$q_c(t, M(t)) = \omega_c + \upsilon_c M(t). \tag{21}$$

In the above equation the parameters (ω_c, υ_c) are defined as follows:

ω_c: initial probability rate of knowing; $\omega_c = 0.1$;
υ_c: slope w.r.t $M(t)$ of the probability rate of knowing; $\upsilon_c = 0.00006$.

3 The controlled stochastic processes

3.1 Stochastic control model

Define the state variable $s = (\mathbf{K}, M, \xi, \zeta)$ where $\mathbf{K} \in \mathbb{R}^{+2}$ represents the capital stocks, $M \in \mathbb{R}^+$ is the atmospheric GHG concentration, $\xi \in \{0, 1\}$ indicates the eventual availability of the advanced (clean) technology and $\zeta \in \{0, 1, 2\}$ represents the state of knowledge concerning climate sensitivity values (0 indicates initial uncertainty; 1,2 indicate knowledge that one of the two possible sensitivity values is the true one). At initial time $t^0 = 0$, the state $s^0 = (\mathbf{K}^0, M^0, \xi^0, \zeta^0)$ is such that $K_2^0 \simeq 0$ and $\xi^0 = 0$ since the advanced technology is not yet available, while $\zeta^0 = 0$ as one does not know the exact climate sensitivity.

It will be convenient to introduce a special notation for the continuous state variable $x = (\mathbf{K}, M)$ and for the discrete variables $\eta = (\xi, \zeta)$. Control variables are the emission rates $E_1(t), E_2(t)$, the investment rates $I_1(t), I_2(t)$ in the different types of capital and the labor allocation $L_1(t), L_2(t) = L(t) - L_1(t)$. Control variables are denoted $u = (\mathbf{E}, \mathbf{I}, \mathbf{L})$ where $\mathbf{E}(t) = (E_i(t))_{i=1,2}$, $\mathbf{I}(t) = (I_i(t))_{i=1,2}$, $\mathbf{L}(t) = (L_i(t))_{i=1,2}$. The dynamics of the state variable can be described by the following equation:

$$\dot{x}(t) = f^{\eta(t)}(t, x(t), u(t)). \tag{22}$$

Given a state variable x and a control variable u, the instantaneous utility of consumption is determined. Therefore one can introduce the reward function[4]:

$$\tilde{L}^{\xi}(t, x, u) = L \log\left(\frac{1}{L}\left[F^{\xi}(t, \mathbf{E}, \mathbf{K}, \mathbf{L}) - \sum_{i=1}^{2}(I_i + p_{E_i}\phi_i E_i)\right]\right), \tag{23}$$

[4] To simplify the notations, $\tilde{L}^{\xi(t)}(t, x(t), u(t))$ is denoted here $\tilde{L}^{\xi}(t, x, u)$.

where $F^{\xi}(t, \mathbf{E}, \mathbf{K}, \mathbf{L}) = Y$. The controls are subject to the constraints: $E_i(t) \geq 0$, $I_i(t) \geq 0$, $L_i(t) \geq 0$, $i = 1, 2$ and $L_1(t) + L_2(t) \leq L(t)$. This is summarized in general notations by: $u(t) \in U(t)$.

3.2 Piecewise deterministic control model

Consider the sequence of random times τ^0, τ^1, τ^2 where $\tau^0 = 0$ is the initial time and τ^1, τ^2 are the jump times of the $\eta(\cdot) = (\xi(\cdot), \zeta(\cdot))$ processes. Denote s^0, s^1, s^2 the state observed at jump times τ^0, τ^1, τ^2. A policy γ is a mapping that associates with a jump time value τ and an observed state s at that time a control $u_{\tau,s}(\cdot)$[5]: $(\tau, \infty[\rightarrow U(\cdot)$ that will be used until the next jump occurs. This corresponds to the concept of piecewise deterministic control. Associated with a policy γ and an initial state s^0 one obtains an expected reward (welfare gain) defined by:

$$J(\gamma, s^0) = \mathrm{E}_\gamma \left[\int_0^\infty e^{-\rho t} \tilde{L}^{\xi(t)}(t, x(t), u(t))\, dt \right], \tag{24}$$

where the expectation is taken w.r.t. the probability measure induced by the policy γ.

A policy γ is admissible if it generates a control which satisfies almost surely all the constraints. An optimal policy, if it exists, maximizes the expected reward among all the admissible policies. When $\zeta(t) = 1, 2$ the climate sensitivity is known and the constraint on GHG concentration is a standard state constraint. When $\zeta(t) = 0$ there is uncertainty on climate sensitivity. Because the different constraints must be satisfied almost surely, one must therefore impose that $M(t) \leq \min_{\ell=1,2} \bar{M}^\ell$ as long as $\zeta(t) = 0$. An elegant way to take into account this latter constraint is to define an extended reward function:

$$\mathcal{L}^{\eta(t)}(t, x(t), u(t)) = \begin{cases} \tilde{L}^{\xi(t)}(t, x(t), u(t)) & \text{if } M(t) \leq \bar{M}^{\zeta(t)} \\ -\infty & \text{otherwise,} \end{cases}$$

when $\zeta(t) = 1, 2$. Thus one introduces, by way of a non differentiable reward, an infinite penalty for possible over-emissions once the climate sensitivity becomes known. This in fact corresponds to the enforcement of a precautionary principle as long as climate sensitivity, and thus the true target on GHG concentrations to be respected, are unknown.

4 Dynamic programming

4.1 After the last jump

Assume that the last jump occurs at time τ^2. From time τ^2 onwards, the carbon-free technology is available and there is no more uncertainty concerning climate. Indeed, one knows the true climate sensitivity and the state constraint on GHG concentrations is:

$$M(t) \leq \bar{M}^\ell. \tag{25}$$

[5] For simplicity, in the reminder of the paper, we shall drop the indices τ and s and write $u(\cdot)$ instead of $u_{\tau,s}(\cdot)$.

At time τ^2, for a given state $s^2 = (x^2, \eta^2) = (\mathbf{K}^2, M^2, 1, \ell)$, let us define the value function $V^2_{1,\ell}(\tau^2, x^2)$ as the solution to the optimization problem:

$$V^2_{1,\ell}(\tau^2, x^2) = \max_{u(\cdot)} e^{\rho\tau^2} \int_{\tau^2}^{\infty} e^{-\rho t} \mathcal{L}^{(1,\ell)}(t, x(t), u(t))\, dt \tag{26}$$

subject to the state equations:

$$\dot{x}(t) = f^{(1,\ell)}(t, x(t), u(t)); \quad u(t) \in U(t); \quad t \geq \tau^2; \quad x(\tau^2) = x^2. \tag{27}$$

Note that $V^2_{1,\ell}(\tau^2, x^2) \in \mathbb{R} \cup -\infty$, with $V^2_{1,\ell}(\tau^2, x^2) = -\infty$ if $M^2 > \bar{M}^\ell$.

4.2 After the first jump

The first jump occurs at time τ^1. At this jump time, two scenarios are possible: either the discrete state can switch from $(0, 0)$ to $(1, 0)$, which means that the cleaner technology becomes available before one knows exactly what the true climate sensitivity is; or it can switch from $(0, 0)$ to $(0, \ell)$, where $\ell = 1, 2$, which means that one learns about the true climate sensitivity before the cleaner technology becomes available.

4.2.1 Earlier availability of clean technology

In the first type of transition, let $s^1 = (\mathbf{K}^1, M, 1, 0)$ be the system state right after the jump time. Recalling Eq. (20), the stochastic control problem to solve can be described as follows:

$$\begin{aligned} V^1_{1,0}(\tau^1, x^1) &= \max_{u(\cdot)} \mathrm{E}_{M(\cdot)} e^{\rho\tau^1} \left[\int_{\tau^1}^{\tau^2} e^{-\rho t} \mathcal{L}^{(1,0)}(t, x(t), u(t))\, dt \right. \\ &\quad \left. + e^{-\rho\tau^2} V^2_{1,\zeta(\tau^2)}(\tau^2, x(\tau^2)) \right] \\ & s.t. \\ \dot{x}(t) &= f^{(1,0)}(t, x(t), u(t)); \quad u(t) \in U(t); \quad t \geq \tau^1; \quad x(\tau^1) = x^1. \end{aligned}$$

Here the second jump time τ^2 is stochastic. The associated jump rate is $q_c(t, M(t))$ at any time $t \geq \tau^1$. One has denoted $s^2(\tau^2)$ the random state reached after the second jump time. Using standard probability reasoning for this type of problem (see for instance Boukas, Haurie and Michel [3] or Carlson, Haurie and Leizarowitz [5]) one obtains the equivalent infinite horizon deterministic control problem:

$$\begin{aligned} V^1_{1,0}(\tau^1, x^1) &= \max_{u(\cdot)} e^{\rho\tau^1} \int_{\tau^1}^{\infty} e^{-(\rho t + \int_{\tau^1}^{t} q_c(s, M(s))\, ds)} \left[\mathcal{L}^{(1,0)}(t, x(t), u(t)) \right. \\ &\quad \left. + \quad q_c(t, M(t)) \sum_{\ell=1}^{2} \pi_\ell V^2_{1,\ell}(t, x(t)) \right] dt \\ & s.t. \end{aligned} \tag{28}$$

$$\dot{x}(t) \quad = \quad f^{(1,0)}(t, x(t), u(t)); \quad u(t) \in U(t); \quad t \geq \tau^1; \quad x(\tau^1) = x^1.$$

Since the value functions $V^2_{1,\ell}(t, x(t))$, $\ell = 1, 2$, take value in $\mathbb{R} \cup -\infty$, the function $V^1_{1,0}(\tau^1, x^1)$ is also defined on $\mathbb{R} \cup -\infty$ and this will enforce the observance of the concentration constraints.

4.2.2 Earlier knowledge of climate sensitivity

In the second type of transition, let $s^1 = (\mathbf{K}^1, M^1, 0, \ell)$ be the system state right after the jump time when the true climate sensitivity ($\ell = 1, 2$) has been revealed. Recalling Eq. (17), the stochastic control problem to solve can be described as follows:

$$\begin{aligned} V^1_{0,\ell}(\tau^1, x^1) \quad &= \quad \max_{u(\cdot)} \mathrm{E}_{K_2(\cdot)} e^{\rho\tau^1} \left[\int_{\tau^1}^{\tau^2} e^{-\rho t} \mathcal{L}^{(0,\ell)}(t, x(t), u(t))\, dt \right. \\ &\qquad \left. + e^{-\rho\tau^2} V^2_{1,l}(\tau^2, x^2(\tau^2)) \right] \\ s.t. \\ \dot{x}(t) \quad &= \quad f^{(0,\ell)}(t, x(t), u(t)); \quad u(t) \in U(t); \quad t \geq \tau^1; \quad x(\tau^1) = x^1. \end{aligned}$$

The second jump time τ^2 is still stochastic. The associated jump rate is $q_b(t, K_2(t))$ at any time $t \geq \tau^1$. One has again denoted $s^2(\tau^2)$ the random state reached after the second jump time. Using the same standard reasoning one obtains the equivalent infinite horizon deterministic control problem:

$$\begin{aligned} V^1_{0,\ell}(\tau^1, x^1) \quad &= \quad \max_{u(\cdot)} e^{\rho\tau^1} \int_{\tau^1}^{\infty} e^{-(\rho t + \int_{\tau^1}^{t} q_b(s, K_2(s))\, ds)} \left[\mathcal{L}^{(0,\ell)}(t, x(t), u(t)) \right. \\ &\qquad \left. + \quad q_b(t, K_2(t)) V^2_{1,\ell}(t, x(t)) \right] dt \qquad (29) \\ s.t. \\ \dot{x}(t) \quad &= \quad f^{(0,\ell)}(t, x(t), u(t)); \quad u(t) \in U(t); \quad t \geq \tau^1; \quad x(\tau^1) = x^1. \end{aligned}$$

Again, since the functions $V^2_{1,\ell}(t, x(t))$, $\ell = 1, 2$, take value in $\mathbb{R} \cup -\infty$, the functions $V^1_{0,\ell}(\tau^1, x^1)$ are also defined on $\mathbb{R} \cup -\infty$ and this will enforce the observance of the concentration constraints.

4.3 At the initial time

At initial time the discrete state is $(0, 0)$, i.e. one does not know the true climate sensitivity and one does not have access to the clean technology. The stochastic control problem to solve can be described as follows:

$$\begin{aligned} V^0_{0,0}(x^0) \quad &= \quad \max_{u(\cdot)} \mathrm{E}_{K_2(\cdot), M(\cdot)} \left[\int_0^{\tau^1} e^{-\rho t} \mathcal{L}^{(0,0)}(t, x(t), u(t))\, dt \right. \\ &\qquad \left. + e^{-\rho\tau^1} V^1_{\xi(\tau^1), \zeta(\tau^1)}(\tau^1, x^1(\tau^1)) \right] \end{aligned}$$

$$
\begin{aligned}
& s.t. \\
\dot{x}(t) \quad &= \quad f^{(0,0)}(t, x(t), u(t)); \quad u(t) \in U(t); \quad t \geq 0; \quad x(0) = x^0.
\end{aligned}
$$

Still the same classical probability reasoning yields the associated infinite horizon control problem:

$$
\begin{aligned}
V^0_{0,0}(x^0) \quad &= \quad \max_{u(\cdot)} \int_0^\infty e^{-\left(\rho t + \int_0^t (q_b(s,K_2(s)) + q_c(s,M(s)))\, ds\right)} \Big[\mathcal{L}^{(0,0)}(t, x(t), u(t)) \\
&+ \quad q_b(t, K_2(t)) V^1_{1,0}(t, x(t)) \\
&+ \quad q_c(t, M(t)) \sum_{\ell=1}^{2} \pi_\ell V^1_{0,\ell}(t, x(t)) \Big] \, dt \qquad (30) \\
& s.t. \\
\dot{x}(t) \quad &= \quad f^{(0,0)}(t, x(t), u(t)); \quad u(t) \in U(t); \quad t \geq 0; \quad x(0) = x^0.
\end{aligned}
$$

Since the functions $V^1_{1,0}(t, x(t))$ and $V^1_{0,\ell}(t, x(t))$ take value in $\mathbb{R} \cup -\infty$, the function $V^0_{0,0}(x^0)$ is also defined on $\mathbb{R} \cup -\infty$ and this will enforce the observance of the concentration constraints.

4.4 Economic interpretation

Eq. (30) corresponds to the formulation of stochastic problem (24) as an equivalent open-loop, infinite horizon, deterministic optimal control problem. The economic interpretation of the various terms in Eq. (30) follows. There is first an endogenous, state dependent, discount factor:

$$
e^{-\left(\rho t + \int_0^t (q_b(s,K_2(s)) + q_c(s,M(s)))\, ds\right)} .
$$

The extended deterministic problem reward function is:

$$
\mathcal{L}^{(0,0)}(t, x(t), u(t)) + q_b(t, K_2(t)) V^1_{1,0}(t, x(t)) + q_c(t, M(t)) \sum_{\ell=1}^{2} \pi_\ell V^1_{0,\ell}(t, x(t)) .
$$

It takes into account the possible future switches to a better knowledge of the climate sensitivity or to an access to an improved technology. Finally, the decisions to abate (choice of $E_1(t), E_2(t)$) or to invest in dirty and clean technology (choice of $I_1(t), I_2(t)$) and labor allocation $L_1(t)$ are then the result of a tradeoff between these different contributions to the reward.

5 Simulation results

5.1 Numerical experimentation setting

Let us first make an important remark concerning the numerical approximation of the optimal control. Our model is basically a qualitative model which is meant to represent

schematically the interplay among large aggregates (capital, labor and energy) of two types of economy (carbon and carbon-free). The numerical solution that we provide here aims at capturing only qualitative aspects of the optimal control solution. Therefore we are in a situation where the precise evaluation of the convergence of the numerical approximation is not crucial. For example we shall replace the differential equations by difference equations, using a time step of 10 years, and we will not try to improve accuracy using smaller time steps. Our numerical solution involves the following steps:

Step-1; expressing $V^2_{1,\ell}$: For a given grid of initial values (state space and last jump time), solve the deterministic control problem which corresponds to Eqs. (26)-(27), using a discrete time approximation, with time steps of 10 years. The problem is formulated using the GAMS modeling language [4] and the nonlinear optimization problem is solved using CONOPT3 [25]. Then, based on a sufficient set of grid point evaluations, use a log-linear least squares algorithm to fit a product form $(a{K_1}^b{K_2}^c M^d)$ for the functions $V^2_{1,\ell}(\tau^2, \cdot)$.

Step-2; expressing $V^1_{1,0}$ and $V^1_{0,\ell}$: For a given grid of initial values, solve the deterministic control problems which correspond to Eq. (28) and to Eq. (29), respectively, using discrete time approximations with time steps of 10 years. The product form approximation obtained at Step-1 is used to obtain the reward functions (28) and (29). The deterministic problems are again formulated using the GAMS modeling language and the nonlinear optimization problems are solved using CONOPT3. Then, based on a sufficient set of grid point evaluations, use a log-linear least squares algorithm to fit a product form $(a{K_1}^b{K_2}^c M^d)$ for the functions $V^1_{1,0}(\tau^1, \cdot)$ and $V^1_{0,\ell}(\tau^1, \cdot)$.

Step-3; expressing $V^0_{0,0}$: For the given initial state, solve the deterministic optimal control problem which corresponds to Eq. (30), where the approximations of $V^1_{1,0}(t, x(t))$ and $V^1_{0,\ell}(t, x(t))$ obtained at Step-2 are used in the formulation of the reward function.

As an illustration, the deterministic control problem solved at Step-3 is formulated in GAMS in the Appendix.

5.2 Analytical approximations of V^2 and V^1

5.2.1 Class of approximating functions

The value functions $V^2_{1,\ell}(t, x(t))$, $V^1_{1,0}(t, x(t))$ and $V^1_{0,\ell}(t, x(t))$ are approximated via a class of functions of the form:

$$a(\tau){K_1(t)}^{b(\tau)}{K_2(t)}^{c(\tau)}{M(t)}^{d(\tau)}$$

which closely resemble extended Cobb-Douglas functions, standard in economics. They should be suitable for describing the interplay of economics aggregate at work here. Note that coefficient a, b, c and d are functions of the jump time τ at which the value function must be evaluated. This is consistent with the non-stationary characteristics of process dynamics (Eq. (1) to Eq. (21)).

Note that other classes of approximating functions are possible, including quadratic forms of the main variables and will be explored in future numerical experiments.

5.2.2 Grid specification

The goal in 5.2.1 is to produce approximate analytical expressions for value functions V^2 and V^1, which are to serve as building blocks in the complete infinite horizon optimal control computation in Eq. (30). Therefore, it is crucial that the coefficients in these approximate functions be computed on the basis of grid point evaluations which are likely to lie in the neighborhood of the optimal control trajectory, because this would increase the accuracy of computations where it is most needed. Unfortunately, the optimal trajectories are not available when carrying out these choices.

In the following, we describe the procedure for selecting grid points that was retained. Starting with present economic and GHG concentration conditions, two contrasted scenarios were considered on the basis of climate sensitivity, respectively a "low" sensitivity (2°C, corresponding to $l = 1$) and a "high" sensitivity (4.5°C, corresponding to $l = 2$). In both cases, the clean technology is assumed to be available. Note that this corresponds to removing all sources of uncertainty at the outset, and thus by solving the associated deterministic optimal control problems, one obtains the optimal trajectories $\mathbf{K}^*$ and M^*. For variable $\mathbf{K}$, at each discrete 10 year time interval, the range $R_{\mathbf{K}}$ of the grid is defined by:

$$R_{\mathbf{K}}(t) = \left[(1-\delta_{\mathbf{K}})\min(\mathbf{K}^*_{low}(t), \mathbf{K}^*_{high}(t));\ (1+\delta_{\mathbf{K}})\max(\mathbf{K}^*_{low}(t), \mathbf{K}^*_{high}(t))\right]$$

The coefficient $\delta_{\mathbf{K}}$ (set here to 0.2) accounts for uncertainty in the real optimal control trajectory, in particular the risk for over-shooting of one realizes that the climate sensitivity is lower than the one assumed during the early phase of the control (where a precautionary principle is followed). Five equidistant points are then selected on $R_{\mathbf{K}}(t)$ including the two extreme points. For variable M, the range $R_{M,l}$ of the grid (chosen invariant over time) is simply defined by $R_{M,l} = [M(0);\ \bar{M}^l]$. Five equidistant points are then selected on $R_{M,l}$ including the two extreme points. In total, each value function is then evaluated on 125 grid points.

The above choice of grid points tends to stir these points towards the states associated with optimal trajectories at jump times. As a result, jump times become indicators of the actual state of the optimally controlled system when value function evaluations are carried out. This observation will allow us to interpret the results of the analytical approximation of value functions in the next section based on the behavior of optimal trajectories.

5.2.3 Results of analytical approximation

In the following, we focus on analytical results for value functions $V^2_{1,\ell}$ as an illustration.

Tables 1 and 2 summarized the results of the approximation (parameters a, b, c and d) for 10 possible values of jump times τ^2.

From Tables 1 and 2, we can make the following observations:

a: which corresponds to a scaling factor, is rather stable;

b: which measures the relative contribution of capital K_1 in the welfare, decreases monotonically over-time, and at equal jump time its value is always lower in the high climate sensitivity case ($l = 2$) indicating reluctance to use this GHG intensive capital over-time, especially under a more stringent GHG concentration limit;

c: which measures the relative contribution of clean capital K_2 in the welfare, behaves in a somehow complementarity manner to coefficient b. It exhibits an increasing trend over-time, and at equal jump time its value is always higher in the high climate sensitivity case indicating a propensity to use this clean capital over-time, especially under a more stringent GHG concentration limit;

d: is a negative quantity indicating the deleterious impact of "pollution" (measured by GHG concentration levels) on welfare. In addition, this coefficient tends to increase over-time. This is both consistent with the expected state of the system on the evaluation grid at the jump time (less K_1 capital and more K_2 capital over-time) and with the trend (in the optimal control) to keep increasing the clean capital K_2, thus reducing over-time "pollution costs" measured as the economic burden of meeting the GHG concentration limit.

5.3 Optimal control results

In this section, we compare the results of the initial optimal control policy (before the occurrence of the first jump) against several deterministic scenarios. "BaU" corresponds to a baseline scenario, where economic development proceeds without any GHG limitation. Scenarios denoted "*-560" (respectively "*-420") correspond to the case where the social planner enforces a limit of 560 ppmv (resp. 420 ppmv) on atmospheric GHG concentration levels. Scenarios denoted "K2*" (respectively "K1*") correspond to the case where the clean technology is available from the initial time (resp. never available at any time). So

Table 1: Approximation parameters for value functions $V^2_{1,1}(t, x(t))$

τ^2	$\ln a$	b	c	d
2005	16.01339431	0.012387505	0.000368663	-0.002499986
2015	16.02724273	0.011063475	0.000329844	-0.002555217
2025	16.03973220	0.009808449	0.000292801	-0.002542447
2035	16.05763282	0.006999073	0.000935996	-0.002163372
2045	16.06977235	0.005448058	0.000928692	-0.001977395
2055	16.07679189	0.004822908	0.000758673	-0.001931493
2065	16.08401863	0.003929632	0.000722954	-0.001748588
2075	16.08728702	0.003672447	0.000749241	-0.001656366
2085	16.08720689	0.003762964	0.001086648	-0.001482625
2095	16.09488809	0.000618022	0.002439739	-0.00126602
2105	16.09203061	0.00028236	0.003006358	-0.001021709

Table 2: Approximation parameters for value functions $V^2_{1,2}(t, x(t))$

τ^2	$\ln a$	b	c	d
2005	16.12693572	0.012297028	0.000384087	-0.019518188
2015	16.12936136	0.010994551	0.000343838	-0.017880967
2025	16.13103712	0.009750132	0.000305636	-0.016256709
2035	16.12531141	0.006945695	0.000953349	-0.012345226
2045	16.12491853	0.005407825	0.000947166	-0.010290004
2055	16.12687843	0.00478852	0.000774460	-0.009487311
2065	16.12692864	0.003902353	0.000739152	-0.008230716
2075	16.12748211	0.003649373	0.000763393	-0.007730267
2085	16.12288415	0.003745264	0.001100322	-0.006879109
2095	16.12242422	0.000603685	0.002509098	-0.005495424
2105	16.11226166	0.000272742	0.003074565	-0.004157042

we shall explore the five deterministic scenarios BaU, K1-560, K1-420, K2-560 and K2-420 and compare them to the stochastic control case, a scenario simply called "control".

5.3.1 GHG emission paths

Figure 1 reports on GHG emission trajectories. In the BaU scenario emissions reach 16.3 GtC/Y around 2100; this is well in the range of the SRES B family of scenarios of the IPCC [17]. In the *-560 scenarios, GHG emissions peak around 2065 and are then curbed in order to respect the concentration target. In the *-420 scenarios, GHG emissions must be reduced from 2005 on, down to 1.6 GtC (a reduction of 75% from present level), in order to meet the more stringent concentration target. Note that in both scenarios K2-560 and K2-420 emissions are in a first phase slightly higher than in the K1* scenarios. This can be explained by the availability of a cleaner technology that will permit in the future to maintain a much lower emission rate.

If we consider the stochastic control scenario we observe that the GHG emissions path in the initial phase (i.e. before the first jump τ^1) follows closely the K1-420 one; this is explained by the fact that under uncertainty one must impose the most stringent concentration constraint (i.e. a 420 ppmv cap) as a precautionary measure until the true climate sensitivity is revealed; in addition since the technological breakthrough has not yet occurred and hence the clean technology is not yet available the situation is very close to the K1-420 case.

5.3.2 Capital K_1 accumulation paths

As reported in the previous section, respecting a cap on atmospheric GHG concentrations requires curbing GHG emissions from their baseline levels. As a consequence, and since the (carbon) economy has to use less (fossil) energy, economic output is reduced (namely, GDP losses occur) in the K1* scenarios (where the clean technology is not available). Figure 2 indicates how the accumulation of capital K_1 is reduced to adjust to the lower production

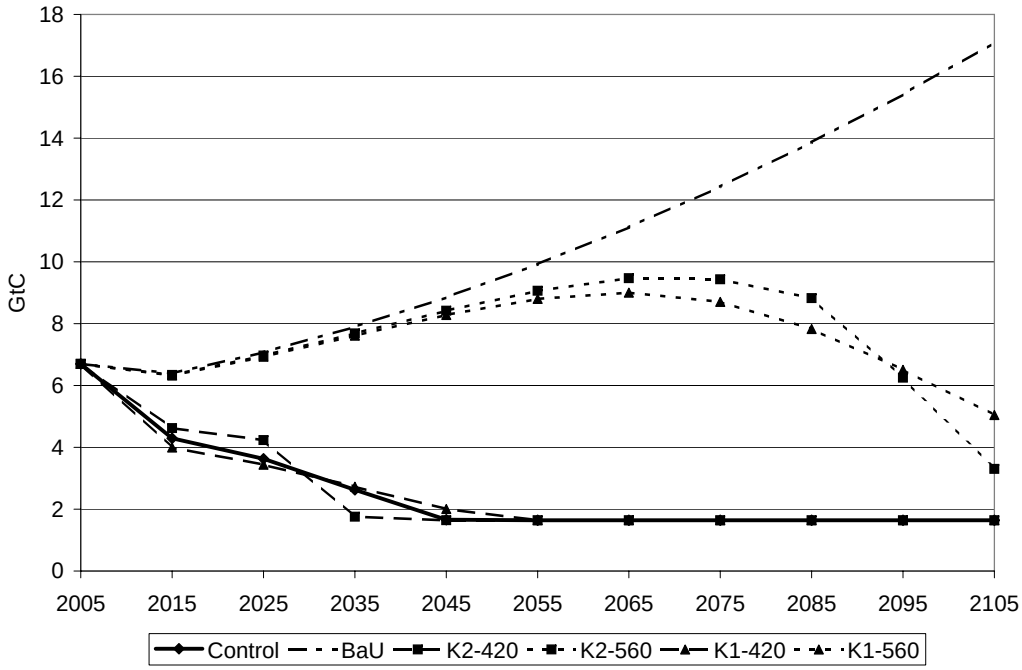

Figure 1: GHG emissions paths

levels. Notice that this trend is stronger under a more stringent GHG concentration limit, namely in the K1-420 scenario.

When the clean technology is available (K2* scenarios), the previously mentioned GDP losses are mitigated through the introduction of the clean technology (see Section 5.3.3 below) and a phasing out of capital K_1 occurs, starting in 2085 in the K2-560 scenario, or as early as 2025 in the K2-560 scenario.

In the initial phase of the "control" scenario the accumulation path of capital K_1 remains very close to the K1-420 scenario case. Notice however the larger accumulation by 2075, compared to K1-420, as the control strategy anticipates a massive R&D investment in the clean technology to come in subsequent years (see Figure 3) and consequently fewer investment possibilities in K_1.

5.3.3 Capital K_2 accumulation paths

Figure 3 reports on the accumulation of capital K_2.

Neither in the baseline, where it is not needed, nor in the K1* scenarios, where it is not available, does capital K_2 accumulate. In the K2* scenarios, the accumulation of K_2 simply mirrors the phasing out of K_1. Notice that the K_2 accumulation path shows a very steep growth starting in 2025 in the K2-420 case.

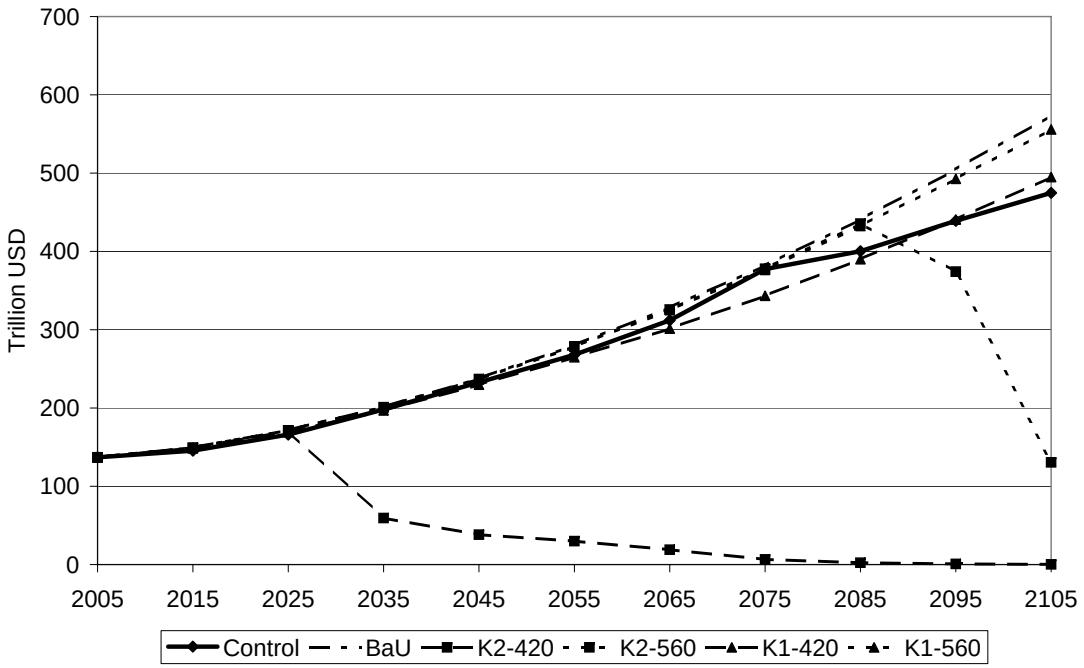

Figure 2: Capital K_1 accumulation paths

In the initial phase of the control scenario, K_2 represents the accumulation of R&D capital in the clean technology. It builds up slowly but gradually until 2075, after which a more important build up takes place to ensure a much higher probability of technological breakthrough before the end of the 21st century. Actually, the path from 2075 on is here directly related to the value of parameter κ in Eq. (16) that limits the fraction of economic output that can be invested in the R&D capital. Figure 4 below illustrates that point by relaxing progressively this limit. The optimal strategy is therefore to wait and at some point in the future start a very important investment in R&D to trigger the technology breakthrough and switch to a cleaner economy.

Notice finally that on the one hand R&D investments in the clean technology never exceeds one third of total economic output. On the other hand, this behavior of high investments (compared to historical R&D figures) is also found in energy models considering induced (namely, endogenous) technological changes and where consequently market penetration constraints must be imposed to avoid unrealistic investment patterns in prospective advanced clean energy technologies; see Bahn and Kypreos [2] for such an example.

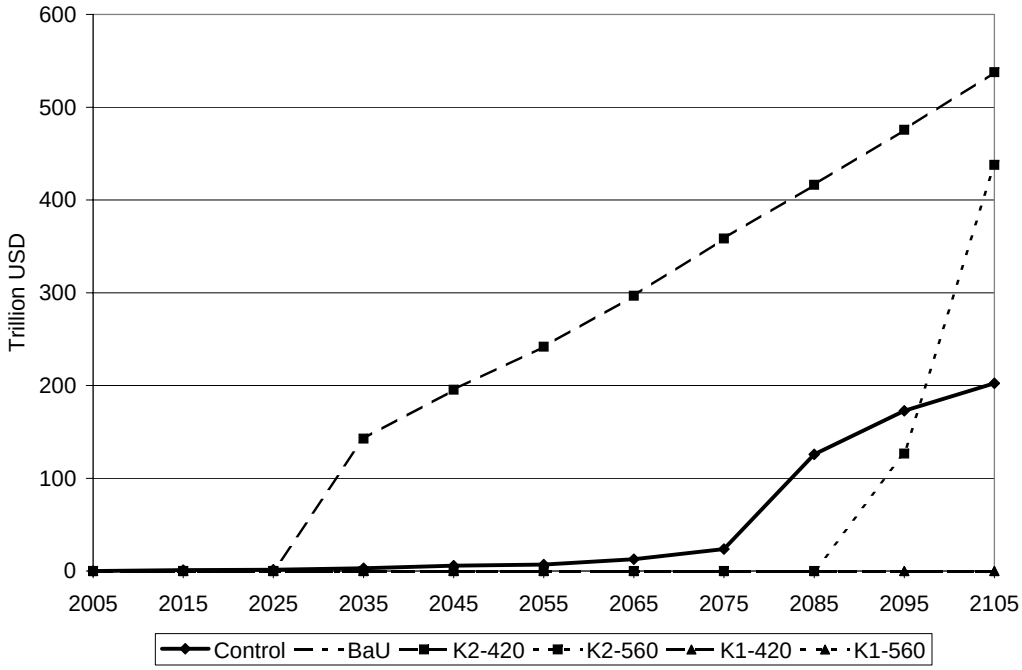

Figure 3: Capital K_2 accumulation paths

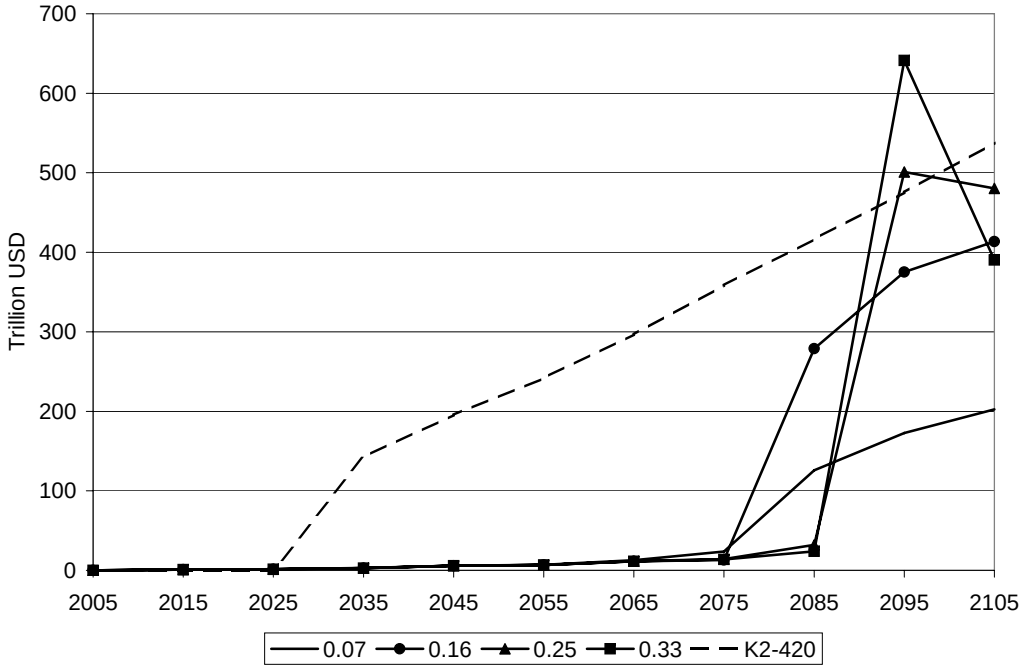

Figure 4: K_2 paths when relaxing the limit on R&D investments

6 Conclusion

In this paper we have shown how a piecewise deterministic stochastic control model could be used as a paradigm for the design of efficient climate policies. The model contains several elements that appear in the current debate about the implementation of a global climate policy. It recognizes the uncertainty concerning the true sensitivity of climate and the fact that the solution may reside in the introduction of new carbon free technologies that are not yet available, the date of availability of these technologies being also uncertain.

The proper timing of emissions abatement is therefore the result of a delicate trade-off. Abating too early may limit unduly the economic development, whereas not abating enough may cause immense damages if the climate sensitivity is high. The stochastic control model that we have formulated provides a formal and rigorous description of this problem of decision under uncertainty. The numerical illustration has shown how the optimal policy could be obtained for a calibration of the model parameters similar to well recognized deterministic assessment models like DICE or MERGE that represent the whole world economy.

The numerical solutions obtained give interesting insights on the delicate problem of timing in GHG abatement abatement policies and show the potential contribution of stochastic control methods to the understanding of an important societal and economic problem. In summary we see that the optimal strategy in the stochastic environment is to implement: (i) a precautionary control where one reduces the emissions almost at the same rate as in the case where one has no access to a clean technology and the most stringent concentration goal to achieve and, (ii) at some point in the future (and relatively late) a very active R&D activity to help the economy switch toward a cleaner production system. Indeed this strategy will be implemented as long as no modal change occurs. The behavior would adapt to the knowledge of a lower climate sensitivity or the occurrence of the technology advance. We let the reader compare this interpretation with the "European" attitude (arguing for an immediate implementation of a low concentration goal) and the "American" one (arguing for more patience and waiting for a technological solution), in the recent discussions concerning climate change policy. This simple model tends to show that a good strategy could be a mix of these two attitudes.

A natural extension of this formalism would be to introduce several groups of countries which would play strategically when deciding the timing of their respective abatement policies. A first step in this direction is presented in [8], using a deterministic differential game model. A piecewise deterministic differential game model is yet to be developed to represent the competition between industrialized and developing countries in the management of the global commons.

Another possible extension, that would also add more realism to our model, would be to consider a progressive learning, over time, of the true climate sensitivity with increased perturbations of the climate system; see for instance Leach [11].

7 Appendix: GAMS programs

The following GAMS code is given as an illustration. It computes the value function V^0.

```
Set t time periods
* As in DICE-2007, periods of 10 years from 2005
    /2005, 2015, 2025, 2035, 2045, 2055, 2065, 2075, 2085, 2095,
     2105, 2115, 2125, 2135, 2145, 2155, 2165, 2175, 2185, 2195, 2205 /;

Set ts(t) time periods
* To iterate on time t
    /2005, 2015, 2025, 2035, 2045, 2055, 2065, 2075, 2085, 2095,
     2105, 2115, 2125, 2135, 2145, 2155, 2165, 2175, 2185, 2195, 2205 /;

Set tfirst(t) first time period;
tfirst(t) = YES$(ord(t) eq 1);

Set tlast(t) last time period;
tlast(t) = YES$(ord(t) eq card(t));

Parameter nbp(t) number of periods since 2005
          / 2005 0, 2015 1, 2025 2, 2035 3, 2045 4, 2055 5, 2065 6, 2075 7,
            2085 8, 2095 9, 2105 10, 2115 11, 2125 12, 2135 13, 2145 14,
            2155 15, 2165 16, 2175 17, 2185 18, 2195 19, 2205 20/;

Parameter nby(t) number of years since 2005;
nby(t) = nbp(t)*10;

Parameter xi(t) indicator for the clean economy
          / 2005 0, 2015 0, 2025 0, 2035 0, 2045 0, 2055 0, 2065 0, 2075 0,
            2085 0, 2095 0, 2105 0, 2115 0, 2125 0, 2135 0, 2145 0, 2155 0,
            2165 0, 2175 0, 2185 0, 2195 0, 2205 0/;

Scalar  mtargM  target for carbon concentration  /882.2/;
* Target corresponds to 420 ppmv

* reads in regression coefficients of value functions V1-1,0 and V1-0-l
Parameter RegI0a(t)
  / 2005 10142871.692, 2015 10184819.557, 2025 10203329.829,
    2035 10165369.306, 2045 10189400.035, 2055 10206145.310,
    2065 10194693.615, 2075 10130157.452, 2085  9966987.005,
    2095  9981544.743, 2105  9889975.251, 2115  9889975.251,
    2125  9889975.251, 2135  9889975.251, 2145  9889975.251,
    2155  9889975.251, 2165  9889975.251, 2175  9889975.251,
    2185  9889975.251, 2195  9889975.251, 2205  9889975.251 /;
Parameter RegI0b(t)
  / 2005 0.011902180, 2015 0.010643992, 2025 0.009398880,
    2035 0.006693667, 2045 0.005218489, 2055 0.004624735,
    2065 0.003785861, 2075 0.003563049, 2085 0.003657250,
    2095 0.000598144, 2105 0.000273357, 2115 0.000273357,
    2125 0.000273357, 2135 0.000273357, 2145 0.000273357,
```

```
    2155 0.000273357, 2165 0.000273357, 2175 0.000273357,
    2185 0.000273357, 2195 0.000273357, 2205 0.000273357 /;
Parameter RegI0c(t)
  / 2005 0.000367779, 2015 0.000330369, 2025 0.000295219,
    2035 0.000920103, 2045 0.000915484, 2055 0.000749522,
    2065 0.000717963, 2075 0.000744075, 2085 0.001070689,
    2095 0.002470508, 2105 0.003044862, 2115 0.003044862,
    2125 0.003044862, 2135 0.003044862, 2145 0.003044862,
    2155 0.003044862, 2165 0.003044862, 2175 0.003044862,
    2185 0.003044862, 2195 0.003044862, 2205 0.003044862 /;
Parameter RegI0d(t)
  / 2005 -0.019679022, 2015 -0.018359028, 2025 -0.016765858,
    2035 -0.013263486, 2045 -0.011705760, 2055 -0.010899910,
    2065 -0.009533163, 2075 -0.008056254, 2085 -0.005528802,
    2095 -0.004518026, 2105 -0.003369844, 2115 -0.003369844,
    2125 -0.003369844, 2135 -0.003369844, 2145 -0.003369844,
    2155 -0.003369844, 2165 -0.003369844, 2175 -0.003369844,
    2185 -0.003369844, 2195 -0.003369844, 2205 -0.003369844 /;
Parameter RegI420a(t)
  / 2005 10509443.695, 2015 10567611.288, 2025 10611610.156,
    2035 10263925.572, 2045 10290375.969, 2055 10356887.354,
    2065 10338035.123, 2075 10259353.150, 2085 10084330.007,
    2095  8545391.236, 2105  8531200.197, 2115  8531200.197,
    2125  8531200.197, 2135  8531200.197, 2145  8531200.197,
    2155  8531200.197, 2165  8531200.197, 2175  8531200.197,
    2185  8531200.197, 2195  8531200.197, 2205  8531200.197 /;
Parameter RegI420b(t)
  / 2005 0.016135140, 2015 0.014466803, 2025 0.012840872,
    2035 0.015733300, 2045 0.013032076, 2055 0.011258448,
    2065 0.009469613, 2075 0.009529415, 2085 0.010528846,
    2095 0.020082639, 2105 0.018568719, 2115 0.018568719,
    2125 0.018568719, 2135 0.018568719, 2145 0.018568719,
    2155 0.018568719, 2165 0.018568719, 2175 0.018568719,
    2185 0.018568719, 2195 0.018568719, 2205 0.018568719 /;
Parameter RegI420c(t)
  / 2005 0.000140628, 2015 0.000126915, 2025 0.000120067,
    2035 0.000390680, 2045 0.000518685, 2055 0.000429605,
    2065 0.000505992, 2075 0.000494323, 2085 0.000744772,
    2095 0.017848451, 2105 0.021089232, 2115 0.021089232,
    2125 0.021089232, 2135 0.021089232, 2145 0.021089232,
    2155 0.021089232, 2165 0.021089232, 2175 0.021089232,
    2185 0.021089232, 2195 0.021089232, 2205 0.021089232 /;
Parameter RegI420d(t)
  / 2005 -0.028816812, 2015 -0.027331127, 2025 -0.025739578,
    2035 -0.022057913, 2045 -0.019440436, 2055 -0.018564217,
    2065 -0.016212735, 2075 -0.014814791, 2085 -0.012724072,
    2095 -0.010129006, 2105 -0.009664188, 2115 -0.009664188,
    2125 -0.009664188, 2135 -0.009664188, 2145 -0.009664188,
    2155 -0.009664188, 2165 -0.009664188, 2175 -0.009664188,
    2185 -0.009664188, 2195 -0.009664188, 2205 -0.009664188 /;
```

```
Parameter RegI560a(t)
  / 2005 8842835.212, 2015 8990467.058, 2025 9128144.570,
    2035 9025937.867, 2045 9199918.740, 2055 9308062.771,
    2065 9427225.826, 2075 9420710.400, 2085 9379598.291,
    2095 8077094.217, 2105 8095578.602, 2115 8095578.602,
    2125 8095578.602, 2135 8095578.602, 2145 8095578.602,
    2155 8095578.602, 2165 8095578.602, 2175 8095578.602,
    2185 8095578.602, 2195 8095578.602, 2205 8095578.602 /;
Parameter RegI560b(t)
  / 2005 0.016176244, 2015 0.014485883, 2025 0.012855401,
    2035 0.015766957, 2045 0.013020977, 2055 0.011300964,
    2065 0.009351525, 2075 0.009443633, 2085 0.010469508,
    2095 0.020075000, 2105 0.018517177, 2115 0.018517177,
    2125 0.018517177, 2135 0.018517177, 2145 0.018517177,
    2155 0.018517177, 2165 0.018517177, 2175 0.018517177,
    2185 0.018517177, 2195 0.018517177, 2205 0.018517177 /;
Parameter RegI560c(t)
  / 2005 0.000130395, 2015 0.000116856, 2025 0.000109863,
    2035 0.000374621, 2045 0.000491971, 2055 0.000405293,
    2065 0.000480969, 2075 0.000473933, 2085 0.000719813,
    2095 0.017660998, 2105 0.020744890, 2115 0.020744890,
    2125 0.020744890, 2135 0.020744890, 2145 0.020744890,
    2155 0.020744890, 2165 0.020744890, 2175 0.020744890,
    2185 0.020744890, 2195 0.020744890, 2205 0.020744890 /;
Parameter RegI560d(t)
  / 2005 -0.002892596, 2015 -0.003037218, 2025 -0.003090190,
    2035 -0.002750604, 2045 -0.002559810, 2055 -0.002527555,
    2065 -0.002244054, 2075 -0.001922636, 2085 -0.001778262,
    2095 -0.001495030, 2105 -0.001454043, 2115 -0.001454043,
    2125 -0.001454043, 2135 -0.001454043, 2145 -0.001454043,
    2155 -0.001454043, 2165 -0.001454043, 2175 -0.001454043,
    2185 -0.001454043, 2195 -0.001454043, 2205 -0.001454043 /;

Scalars

*** Breakthrough dynamics for clean capital
    vb       slope w.r.t K2(t) of the probability rate of discovery /0.0019/
    wb       initial probability rate of discovery                  /0.05/

*** Dynamics for the knowledge of the climate sensitivity
    vc       slope w.r.t M(t) of the probability rate of knowing    /0.00006/
    wc       initial probability rate of knowing                    /0.1/
    probhigh probabilty to have a high climate sensitivity          /0.5/

*** Carbon related, values from DICE-1999 and -2007
    beta     marginal atmospheric retention rate                    /0.64/
    delta_M  carbon removal rate per decade                         /0.036/

*** Energy related, calibrated to match early carbon emissions of DICE-2007
    piE1     energy price in dirty economy                          /0.35/
```

```
    piE2     energy price in clean economy                          /0.60/

*** Preferences, values from DICE-2007
    elasmu   elasticity of marginal utility of consumption          /2.5/
    prstp    initial rate of social time preference per year        /0.015/
    dr       decline rate of social time preference per year        /0.000001/
    rho      discount rate                                          /0.03/

*** Population and technology
    A0       initial value for total factor productivity (TFP)      /0.0302/
    gA0      initial value for growth rate of TFP                   /0.10/
    dgA      rate of decrease for growth rate of TFP                /0.005/
*   TFP values from DICE-2007
    alphad   dirty capital elasticity in production function        /0.3/
    alphac   clean capital elasticity in production function        /0.3/
*   alpha value from DICE-2007
    dk       annual capital depreciation rate                       /0.1/
*   dk value from DICE-2007
    L0       2005 world population (millions)                       /6409/
    gL0      initial value for growth rate of population            /0.08/
    dgL      rate of decrease for growth rate of population         /0.3/
*   population values from DICE-2007
    phid0    initial value for energy efficiency (EE) in dirty eco  /1.3/
    gphid0   initial value for growth rate of EE in dirty economy   /0.15/
    dgphid   rate of decrease for growth rate of EE in dirty eco    /0.2/
    phic0    initial value for EE in clean economy                  /5.0/
    gphic0   initial value for growth rate of EE in clean econony   /0.15/
    dgphic   rate of decrease for growth rate of EE in clean eco    /0.2/
    thetad0  initial value for dirty energy elas. in prod. function /.05/
    gthetad0 initial value for growth rate of dirty energy elas.    /-0.0116/
    dgthetad rate of decrease for growth rate of dirty energy elas. /.008/
    thetac0  initial value for clean energy elas. in prod. function /.05/
    gthetac0 initial value for growth rate of clean energy elas.    /-0.0116/
    dgthetac rate of decrease for growth rate of clean energy elas. /.008/

*** Scaling
    scale1 scaling coefficient in objective function                /300/
    scale2 scaling coefficient in objective function                /1.00E+07/
;

Parameter A(t) total factor productivity;
* Values to match DICE-2007 GDP by 2005
A(tfirst) = 1.38734*A0;
loop(t, A(t+1) = A(t) / (1-gA0*exp(-dgA*10*nbp(t))) );

Parameter L(t) labor - world population (millions);
* Values match those of DICE-2007
L(t) =  L0 * exp( (gl0/dgL)*(1-exp( -dgL*nbp(t) )) );

Parameter phid(t) energy efficiency in the dirty economy;
```

```
phid(t) = (phid0 * exp( (-gphid0/dgphid)*exp(-dgphid*(nbp(t)+1)) ) ) /
          exp(-gphid0/dgphid);

Parameter phic(t) energy efficiency in the clean economy;
phic(t) = (phic0 * exp( (-gphic0/dgphic)*exp(-dgphic*(nbp(t)+1)) ) ) /
          exp(-gphic0/dgphic);

Parameter r(t) instantaeous rate of social time preference;
* From DICE-2007
r(t)=prstp*EXP(-dr*10*(ord(t)-1));

Parameter rr(t) average utility social discount rate;
* From DICE-2007
rr(tfirst)=1;
loop(t, rr(t+1) = rr(t) / ((1+r(t))**10) );

Parameter thetad(t) dirty energy elasticity in production function;
thetad(t) = (thetad0 * exp( (gthetad0/dgthetad)*exp(-dgthetad*(nbp(t)+1)) ) ) /
            exp(gthetad0/dgthetad);

Parameter thetac(t) clean energy elasticity in production function;
thetac(t) = (thetac0 * exp( (gthetac0/dgthetac)*exp(-dgthetac*(nbp(t)+1)) ) ) /
            exp(gthetac0/dgthetac);

Positive variables
    C(t)  total consumption (trillion dollars)
    E1(t) carbon emissions from the dirty economy (GtC)
    E2(t) carbon emissions from the clean economy (GtC)
    I1(t) investment in dirty capital K1 (trillion dollars)
    I2(t) investment in clean capital K2 (trillion dollars)
    K1(t) physical stock of dirty capital (trillion dollars)
    K2(t) physical stock of clean capital (trillion dollars)
    L1(t) labor for the dirty economy (millions)
    L2(t) labor for the clean economy (millions)
    M(t)  atmospheric carbon concentration
    PK(t) probability of not accessing clean capital until t
    PL(t) probability of not knowing true climate sensitivity until t
    Y(t)  economic output (trillion dollars);

Variables
    PW(t)     welfare of current period
    VI0(t)    value fonction V_1-1-0
    VI420(t)  value fonction V_1-0-420
    VI560(t)  value fonction V_1-0-560
    W0        total discounted welfare for the initial time (V_0-0-0);

Equations
    ALOCLABOR(t)  compute allocation of total labor
    CARBDYNAM(t)  compute evolution of carbon concentrations
    E1FIRST(t)    initial value for carbon emissions from dirty capital
```

```
    E2FIRST(t)    initial value for carbon emissions from clean capital
    I2MAX(t)      bound for investment on clean capital
    K1ACCUMUL(t)  compute dirty capital accumulation
    K1FIRST(t)    initial value for dirty capital
    K1LAST(t)     final value for dirty capital
    K2ACCUMUL(t)  compute clean capital accumulation
    K2FIRST(t)    initial value for clean capital
    K2LAST(t)     final value for clean capital
    MFIRST(t)     initial value for carbon concentration
    TARGETM(t)    impose cap on atmospheric carbon concentrations
    TOTALCONS(t)  compute total consumption
    TOTALPROD(t)  compute total production
    PKEQ(t)       compute probability PK(t)
    PLEQ(t)       compute probability PL(t)
    PWEQ(t)       compute welfare of current period
    VALUEI0(t)    compute value function V_1-1-0
    VALUEI420(t)  compute value function V_1-0-420
    VALUEI560(t)  compute value function V_1-0-560
    WELFARE0      compute total welfare for the initial time;

ALOCLABOR(t)..
  L1(t) + L2(t) =l= L(t);

CARBDYNAM(t+1)..
  M(t+1) =e= beta*10*(E1(t)+E2(t)) + M(t)*(1-delta_M) + delta_M*590;

MFIRST(tfirst)..
* 2005 value around 385 ppmv = 808.8 GtC
  M(tfirst) =e= 808.8;

E1FIRST(tfirst)..
* DICE-2007 value
  E1(tfirst) =e= 6.7;

E2FIRST(tfirst)..
* 0.0001 of DICE-2007 value
  E2(tfirst) =e= 0.00067;

I2MAX(t)..
  I2(t) =l= 0.07*Y(t);

K1FIRST(tfirst)..
* DICE-2007 value (137.0) for total capital
  K1(tfirst) =e= 137.0;

K1ACCUMUL(t)..
  K1(t+1) =l= K1(t)*(1-dk)**10 + 10*I1(t);

K1LAST(tlast)..
  0.02*K1(tlast) =l= I1(tlast);
```

```
K2FIRST(tfirst)..
* 0.0001 of DICE-2007 value
  K2(tfirst) =e= 0.0137;

K2ACCUMUL(t)..
  K2(t+1) =l= K2(t)*(1-dk)**10 + 10*I2(t);

K2LAST(tlast)..
  0.02*K2(tlast) =l= I2(tlast);

TARGETM(t)..
  M(t) =l= mtargM;

TOTALPROD(t)..
  Y(t) =e= A(t) * (
  K1(t)**alphad * (phid(t)*E1(t))**thetad(t) * L1(t)**(1-alphad-thetad(t)) +
  xi(t)*K2(t)**alphac * (phic(t)*E2(t))**thetac(t) * L2(t)**(1-alphac-thetac(t))
                 );

TOTALCONS(t)..
  C(t) =e= Y(t) - I1(t) - I2(t) - piE1*phid(t)*E1(t) - piE2*phic(t)*E2(t);

PKEQ(t)..
  PK(t) =e= exp( -(0.1*nby(t)+1)*wb - vb*sum(ts$(ord(ts)<=ord(t)), K2(ts)) );

PLEQ(t)..
  PL(t) =e= exp( -(0.1*nby(t)+1)*wc - vc*sum(ts$(ord(ts)<=ord(t)), M(ts)) );

PWEQ(t)..
* Equation from DICE-2007
  PW(t) =e= ( (C(t)/L(t))**(1-elasmu) -1) / (1-elasmu);

VALUEI0(t)..
  VI0(t) =e= RegI0a(t) * K1(t)**RegI0b(t) * K2(t)**RegI0c(t) *  M(t)**RegI0d(t);

VALUEI420(t)..
  VI420(t) =e= RegI420a(t) * K1(t)**RegI420b(t) * K2(t)**RegI420c(t) *
               M(t)**RegI420d(t);

VALUEI560(t)..
  VI560(t) =e= RegI560a(t) * K1(t)**RegI560b(t) * K2(t)**RegI560c(t) *
               M(t)**RegI560d(t);

WELFARE0..
  W0 =e=
  sum(t, exp(-rho*nby(t)) * PK(t) * PL(t) *
         ( (10*L(t)*(PW(t))/scale1) +
           (wb+vb*K2(t))*(VI0(t)-scale2) +
           (wc+vc*M(t)) * (probhigh*(VI420(t)-scale2) + (1-probhigh)*(VI560(t)-scale2))
```

```
        )
    )
 + scale2 ;

**  Upper and Lower Bounds: General conditions imposed for stability
C.lo(t)     = 1; C.up(t)     = 1000; E1.lo(t)    = 0.0000001;
E1.up(t)    = 100; E2.lo(t)    = 0.00000000001; E2.up(t)    = 10;
I1.lo(t)    = 0.1; I1.up(t)    = 500; I2.lo(t)    = 0.1; I2.up(t) =
100; K1*lo(t)    = 0.1; K1*up(t)    = 3000; K2*lo(t)    = 0.001;
K2*up(t)    = 2000; L1.lo(t)    = 1; L1.up(t)    = 10000; L2.lo(t) =
0.000000001; L2.up(t)    = 1; M.lo(t)     = 800; M.up(t)     = 1000;
Y.lo(t)     = 5; Y.up(t)     = 1000; VI0.up(t)   = 1e+08; VI0.lo(t)
= 1e+05; VI420.up(t) = 1e+08; VI420.lo(t) = 1e+05; VI560.up(t) =
1e+08; VI560.lo(t) = 1e+05;

** Solution options
option iterlim = 99900;
option reslim = 99999;
option solprint = on;
option limrow = 0;
option limcol = 0;

Model ECM /all/ ;
* Use CONOPT option file
ECM.OPTFILE =1;

Solve ECM using nlp maximising W0;
Solve ECM using nlp maximising W0;
```

References

[1] P. Ambrosi, J.-C. Hourcade, S. Hallegatte, F. Lecocq, P. Dumas, and M. Ha Duong. Optimal control models and elicitation of attitudes towards climate damages. *Environmental Modeling and Assessment*, 8(3):133–148, 2003.

[2] O. Bahn and S. Kypreos. Incorporating different endogenous learning formulations in MERGE. *Int. J. of Global Energy Issues*, 19:333–358, 2003.

[3] K. Boukas, A. Haurie, and P. Michel. An optimal control model with a random stopping time. *Journal of Optimization Theory and Application*, 64:471–480, 1990.

[4] A. Brooke, D. Kendrick, A. Meeraus, and R. Raman. *GAMS, A Users Guide*. GAMS Development Corporation, 2005.

[5] D. Carlson, A. Haurie, and Leizarowitz. A. *Infinite horizon optimal control: Deterministic and stochastic systems*. Springer, Berlin, Germany, 1994.

[6] M. Davis. *Markov Models and Optimization*. Chapman and Hall, London, 1993.

[7] F. Dufour and O.L.V. Costa. Stability of piecewise deterministic markov processes. *SIAM Journal on Control and Optimization*, 37:1483–1502, 1999.

[8] A. Haurie. Piecewise deterministic differential games. In T. Başar and P. Bernhard, editors, *Differential Games and Applications*, Lecture Notes in Control and Information Sciences. Springer-Verlag, 1989.

[9] A. Haurie. Integrated assessment modeling for global climate change: an infinite horizon optimization viewpoint. *Environmental Modeling and Assessment*, 8(3):117–132, 2003.

[10] IPCC. *Climate Change 2007: The Physical Science Basis. Contribution of Working Group I to the Fourth Assessment Report of the Intergovernmental Panel on Climate Change*. Cambridge University Press, Cambridge, U.K., 2007.

[11] A. Leach. The climate change learning curve. *Journal of Economic Dynamics and Control*, 31:1728–1752, 2007.

[12] A.S. Manne. Global climate decisions under uncertainty, mimeo. In *International Energy Workshop, Kyoto*, August 2005.

[13] A.S. Manne, R. Mendelsohn, and R. Richels. MERGE - a model for evaluating regional and global effects of GHG reduction policies. *Energy Policy*, 23(1):17–34, 1995.

[14] A.S. Manne and R.G. Richels. *Buying Greenhouse Insurance: The Economic Costs of CO_2 Emission Limits*. The MIT Press, Cambridge, 1992.

[15] A.S. Manne and R.G. Richels. MERGE: an integrated assessment model for global climate change. In R. Loulou, J.-P. Waaub, and G. Zaccour, editors, *Energy and Environment*, GERAD 25th Anniversary Series, pages 175–189. Springer, 2005.

[16] M. Mariton. *Jump Linear Systems in Automatic Control*. Marcel Dekker, New York, 1990.

[17] N. Nakicenovic and R. Swart, editors. *Special report on emissions scenarios*. Cambridge University Press, Cambridge, U.K., 2000.

[18] United Nations. Framework convention on climate change. *International Legal Materials*, 31:849–873, 1992.

[19] W.D. Nordhaus. An optimal path for controlling greenhouses gases. *Science*, 258(5086):1315–1319, nov 1992.

[20] W.D. Nordhaus. *Managing the Global Commons*. The MIT Press, Cambridge, 1994.

[21] W.D. Nordhaus and Z. Yang. RICE - a regional dynamic general equilibrium model of alternative climate change strategies. *American Economic Review*, 86:741–765, 1996.

[22] F. Ramsey. A mathematical theory of saving. *Economic Journal*, 38:543–559, 1928.

[23] R. Rishel. Dynamic programming and minimum principles for systems with jump markov disturbances. *SIAM Journal on Control and Optimization*, 13:338–371, 1975.

[24] D.D. Sworder. Feedback control of a class of linear systems with jump parameters. *IEEE Transactions on Automatic Control*, 14:9–14, 1969.

[25] General Algebraic Modeling System. *GAMS, The Solver Manuals*. GAMS Development Corporation, 2005.

[26] D. Vermes. Optimal control of piecewise deterministic markov processes. *Stochastics*, 14:165–208, 1985.

[27] W.M. Wonham. Random differential equations in control theory. In A.T. Bharucha-Reid, editor, *Probabilistic Methods in Applied Mathematics*, volume 2, pages 131–212. New York, 1970.