Graph Master + Local Area Routes

Udayan Mandal, Amelia Regan, Louis-Martin Rousseau, David Pepper CEO NGR ,Julian Yarkony

Two New Technologies for VRP

Graph Master

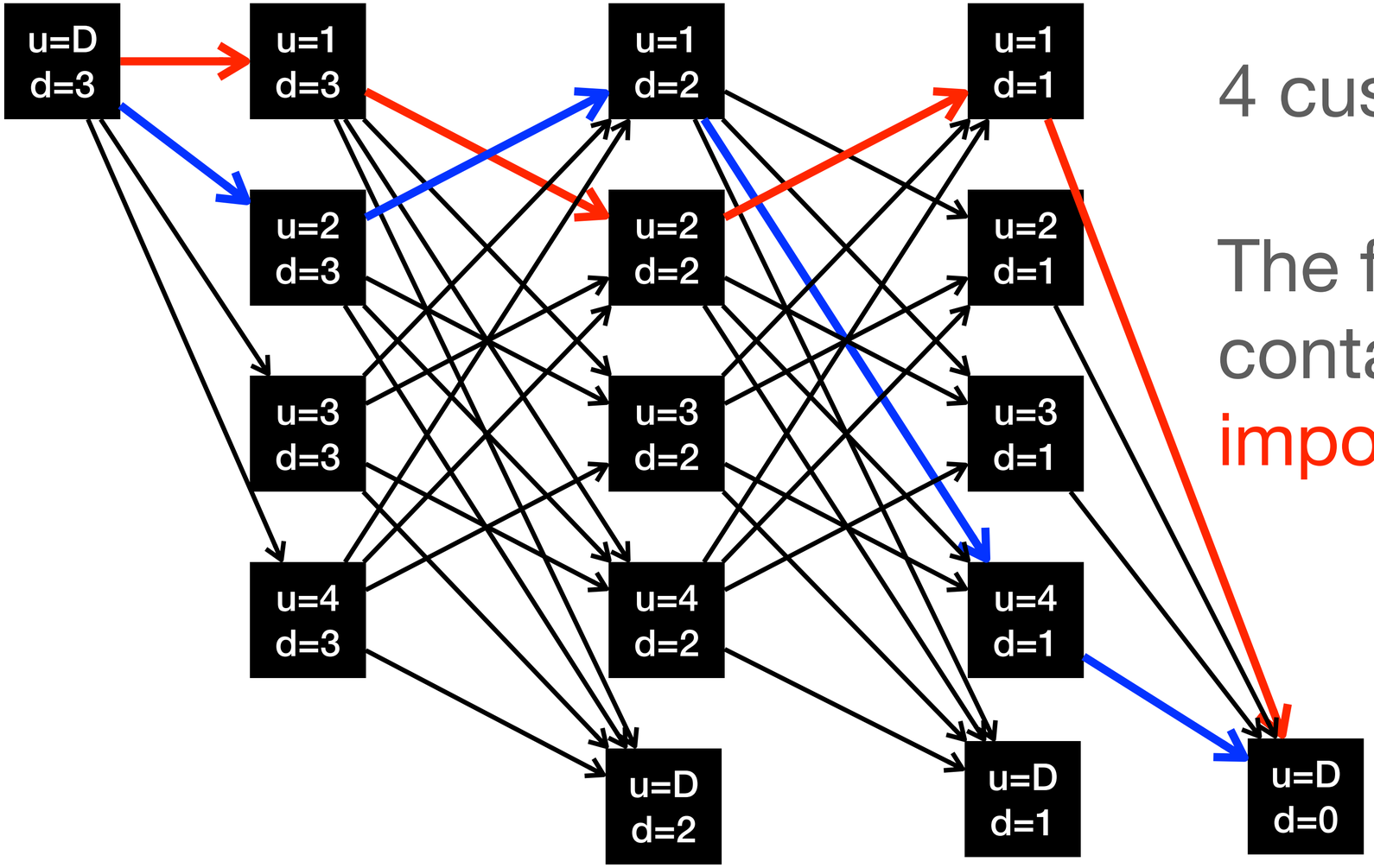
- Insight: The geometry of the VRP induces a graph structure on its columns, derived from a total ordering of the customers
- Every round we add in this entire graph to greatly reduce time to solution. The specific LP structure provides for efficient solution.

Local Area Routes

- Insight: Optimal routes over clusters of customers localized in space can be pre-computed.
- Integrate these optimized sub-routes into GM to tighten the underlying set cover relaxation without altering structure of pricing

Together, these technologies can provide speedups of 100x+

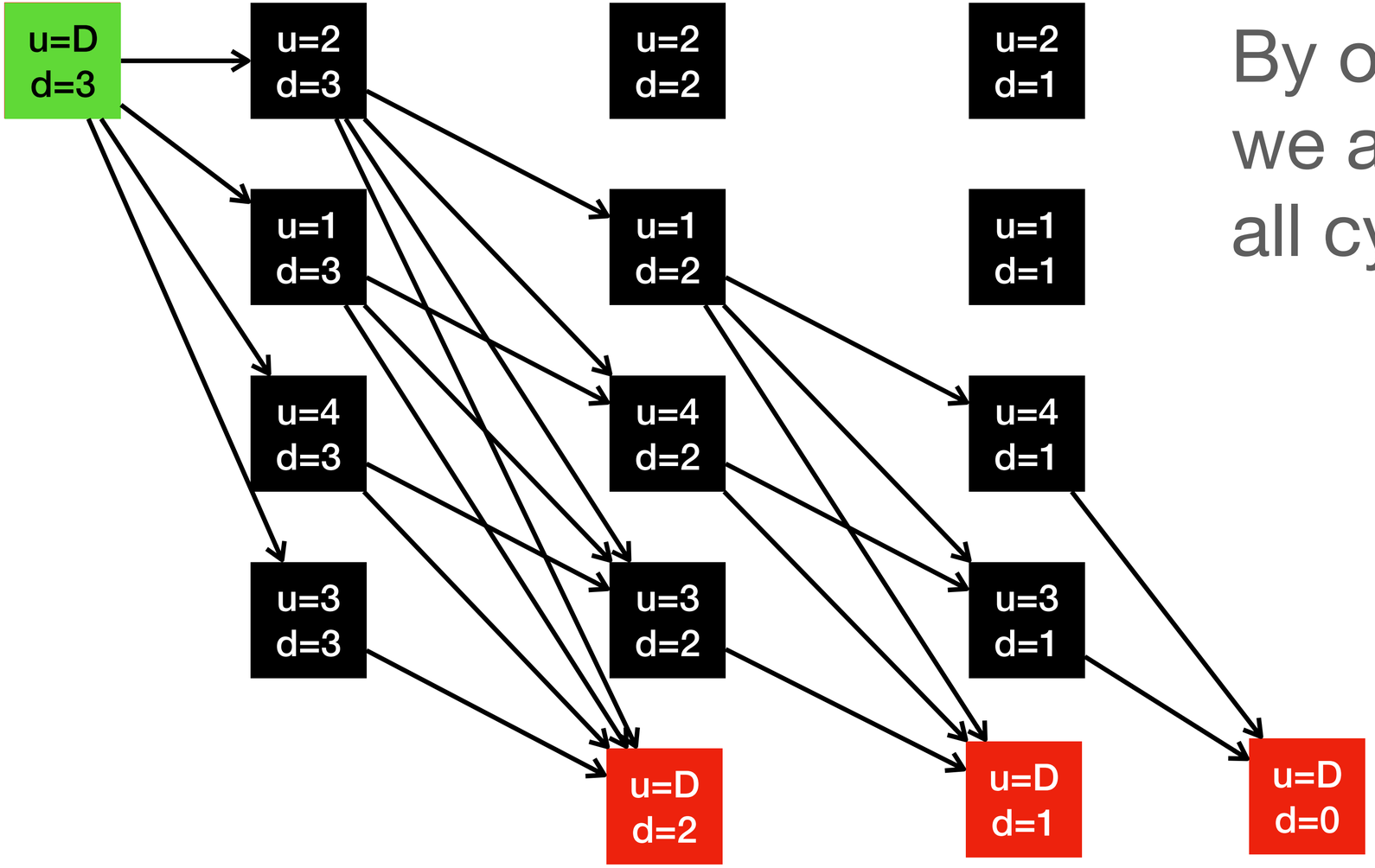
Graph Description of Routes



4 customers, capacity 3

The fully-connected graph contains both possible and impossible routes

Routes Consistent with Ordering [2, 1, 4, 3]



By ordering the customers, we automatically eliminate all cyclic routes

Generating Routes Ω_l

$Route : l =$ NYU UCLA DUKE

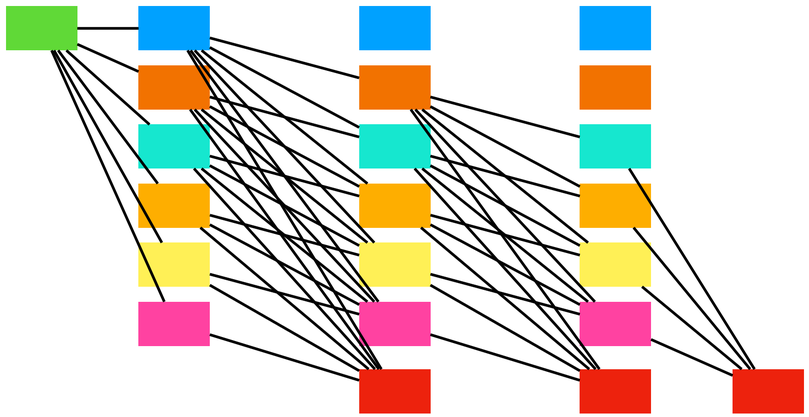
Other customers: MIT USC BU

PRESERVE Original Sequence: Approx Preserve Locality.

$Ordering : \beta^l =$ NYU BU MIT UCLA USC DUKE

All valid paths consistent with ordering

$Graph : G(\beta^l) =$

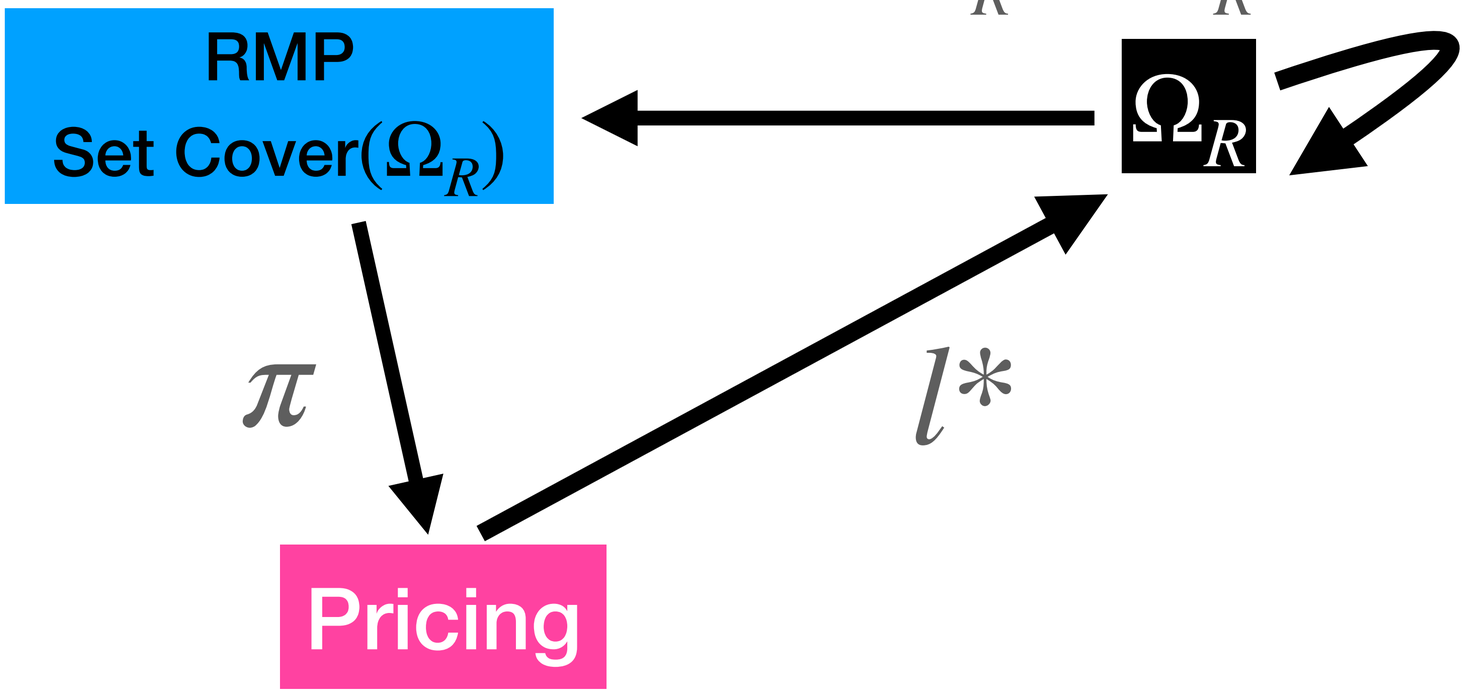


$Routes : \Omega_l$

Vanilla Column Generation

Ω_R Routes Generated so far

$\Omega_R \leftarrow \Omega_R + l^*$



Lots of Iterations needed

RMP Set Cover(Ω_R)

$$\min_{\theta \geq 0} \sum_{l \in \Omega_R} c_l \theta_l$$

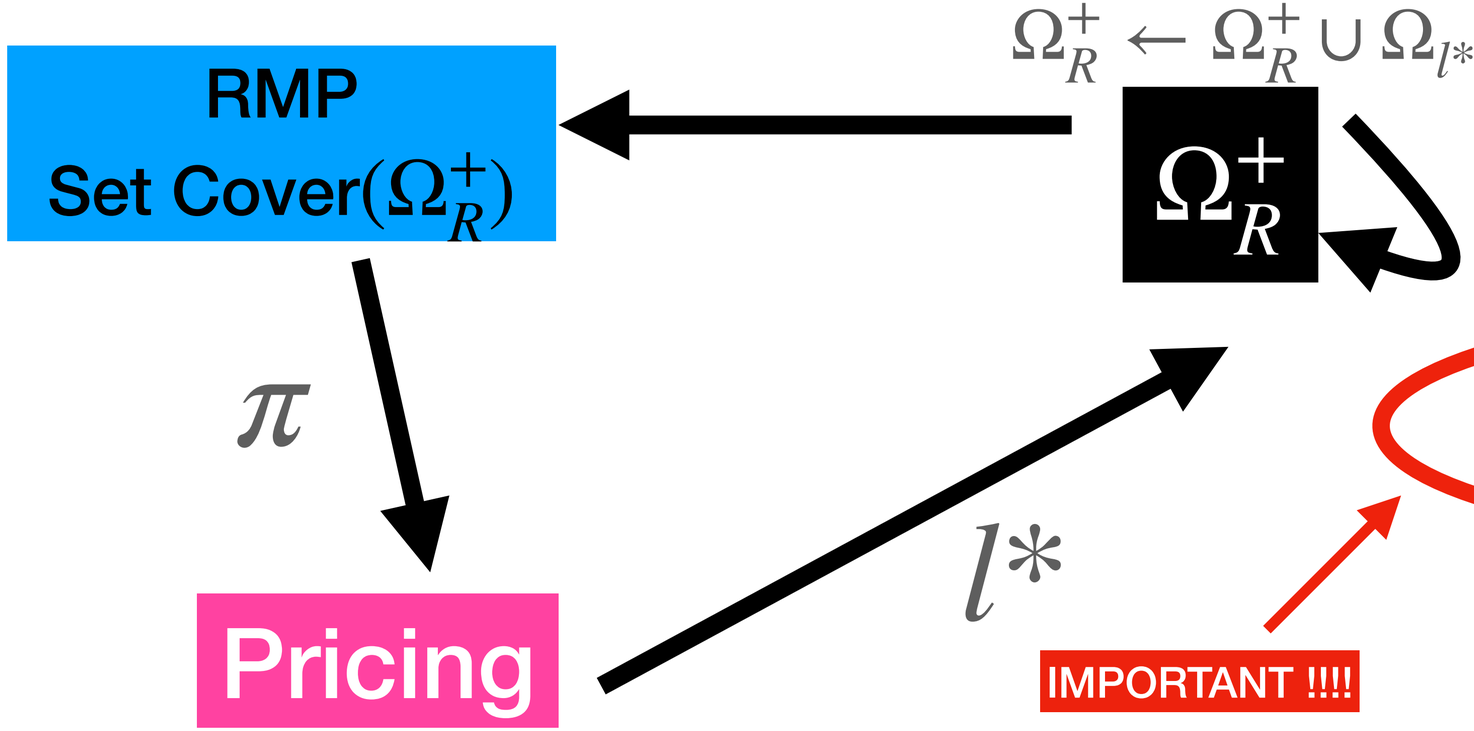
$$\sum_{l \in \Omega_R} a_{ul} \theta_l \geq 1 \, \forall u \qquad [\pi_u]$$

Pricing. Resource Constrained Shortest Path

$$l^* \leftarrow \min_{l \in \Omega} \bar{c}_l$$

$$\bar{c}_l = \sum_{u,v} (c_{uv} - \pi_v) a_{uvl}$$

Graph Master



Fewer Iterations Needed

$$|\Omega_R^+| >> |\Omega_R|$$

RMP Set Cover(Ω_R^+)

$$\min_{\theta \geq 0} \sum_{l \in \Omega_R^+} c_l \theta_l$$

$$\sum_{l \in \Omega_R^+} a_{ul} \theta_l \geq 1 \, \forall u \quad [\pi_u]$$

RMP. WAIT THIS LOOKS HARD

CAN'T ENUMERATE Ω_R^+

Pricing. Resource Constrained Shortest Path

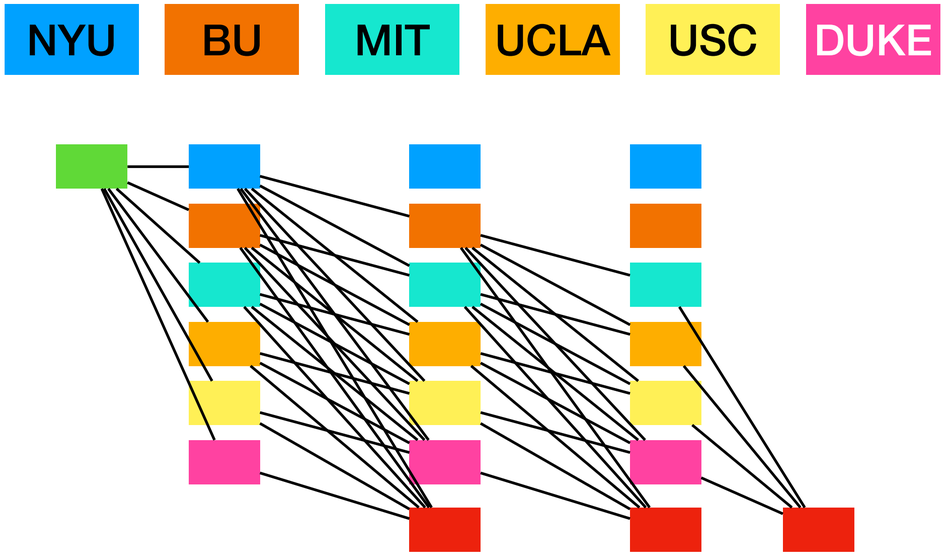
$$l^* \leftarrow \min_{l \in \Omega} \bar{c}_l$$

SAME AS BEFORE

$$\bar{c}_l = \sum_{u,v} (c_{uv} - \pi_v) a_{uvl}$$

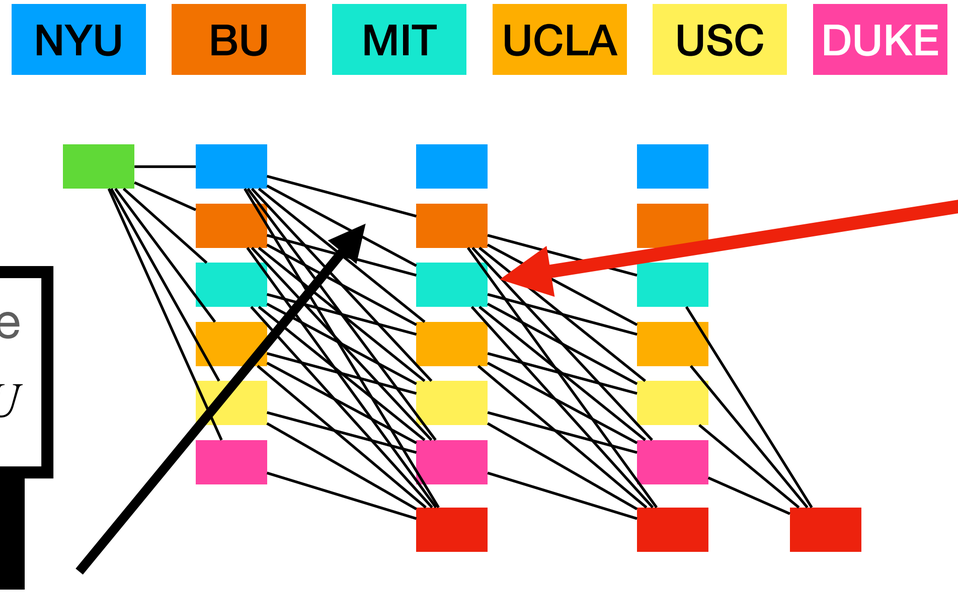
Solving Set Cover(Ω_R^+) by Exploiting Graph Structure

$\beta^{l=1} =$



Solving Set Cover(Ω_R^+) by Exploiting Graph Structure

$$\beta^{l=1} =$$



Flow in = Flow Out

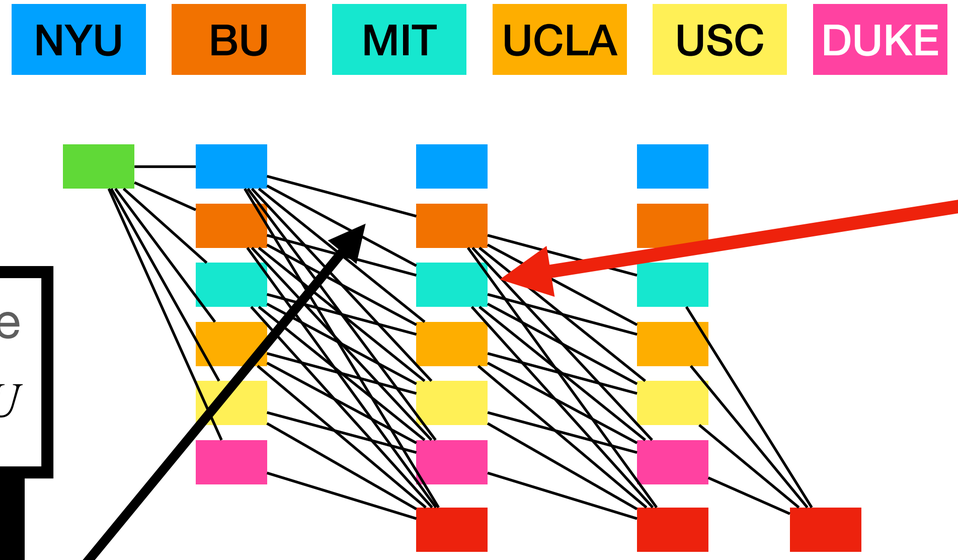
Contributes $c_{NYU,MIT}$ to Objective

Covers: NYU NOT BU

$x^{l=1}_{(NYU,3),(BU,2)}$

Solving Set Cover(Ω_R^+) by Exploiting Graph Structure

$\beta^{l=1} =$



Flow in = Flow Out

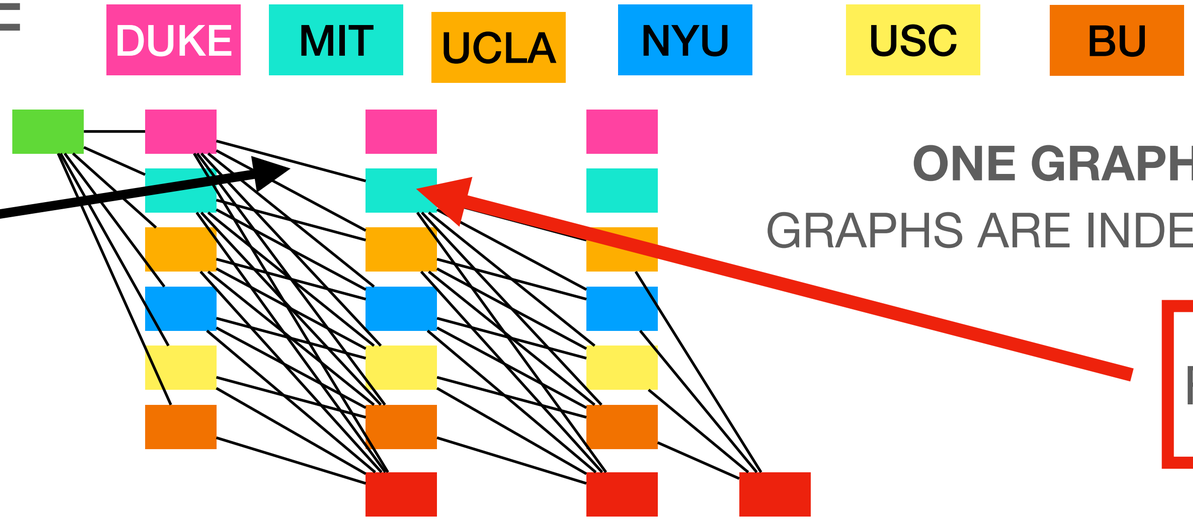
Contributes $c_{NYU,MIT}$ to Objective

Covers: NYU NOT BU

$x^{l=1}_{(NYU,3),(BU,2)}$

LP easy via specialized solver

$\beta^{l=2} =$



$x^{l=1}_{(Duke,3),(MIT,2)}$

ONE GRAPH for each $l \in \Omega_R$ **NOT** Ω_R^+

GRAPHS ARE INDEPENDENT AND DON'T SHARE FLOW

Flow in = Flow Out

Solving Set Cover(Ω_R^+) by Exploiting Graph Structure

FOR REFERENCE

$$\min_{\theta \geq 0} \sum_{l \in \Omega_R} \sum_{u,v;\beta_u^l < \beta_v^l} \sum_{d} c_{uv} x^l_{(u,d),(v,d-d_u)}$$

Objective

$$\sum_{l \in \Omega_R} \sum_{v;\beta_u^l < \beta_v^l} \sum_{d} x^l_{(u,d),(v,d-d_u)} \geq 1 \quad \forall u \quad [\pi_u]$$

Cover Constraint

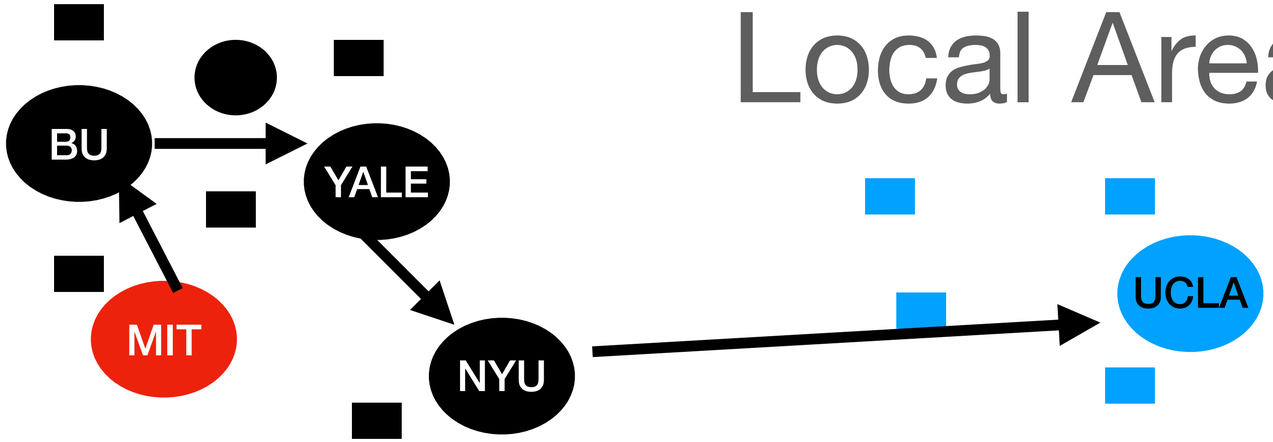
$$\sum_{v;\beta_u^l < \beta_v^l} x^l_{(u,d),(v,d-d_u)} = \sum_{v;\beta_u^l < \beta_v^l} x^l_{(v,d+d_v),(u,d)} \quad \forall u, d, l \quad [\text{NO } \pi]$$

FLOW CONSTRAINT

Separated by l

LP easy via specialized solver

Local Area Arcs:

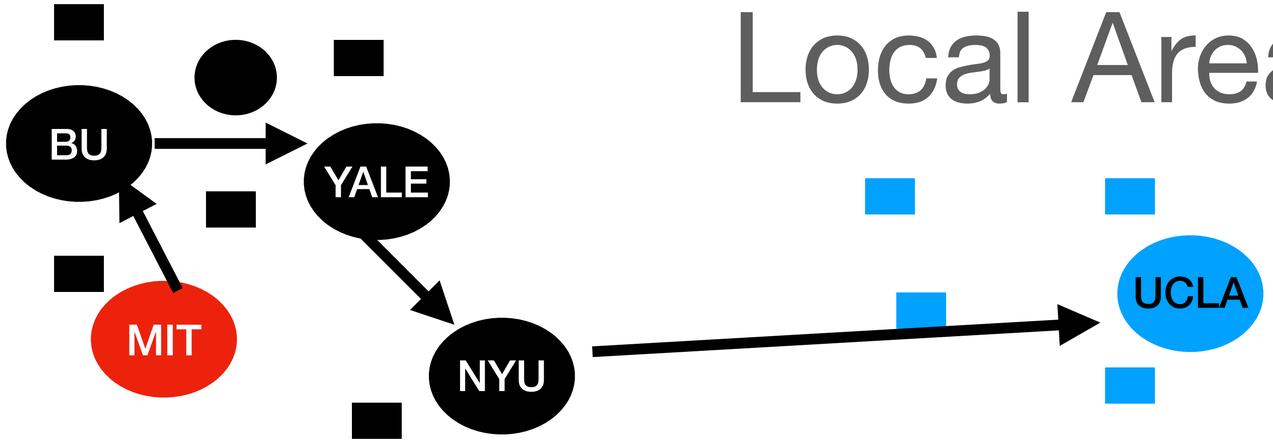


N_u = LA Neighbors

ex: $|N_u| = 10$

10 nearest

Local Area Arcs:



N_u = LA Neighbors

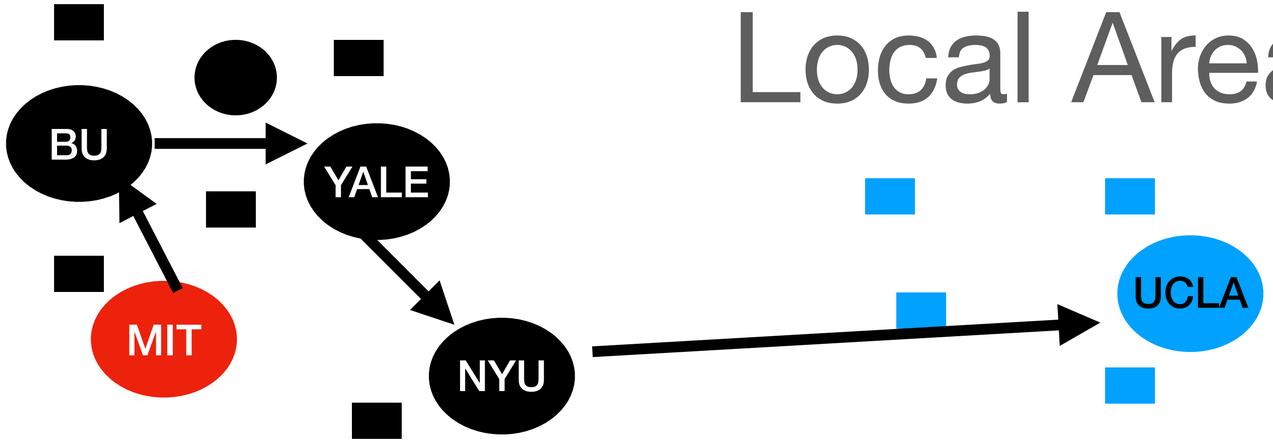
ex: $|N_u| = 10$

10 nearest

LA ARCs of route $l =$ MIT BU YALE NYU UCLA USC DUKE

$l =$ [Depot, MIT] [MIT, BU, YALE, NYU, UCLA] [UCLA, USC, DUKE] [DUKE, Depot]

Local Area Arcs:



N_u = LA Neighbors

ex: $|N_u| = 10$

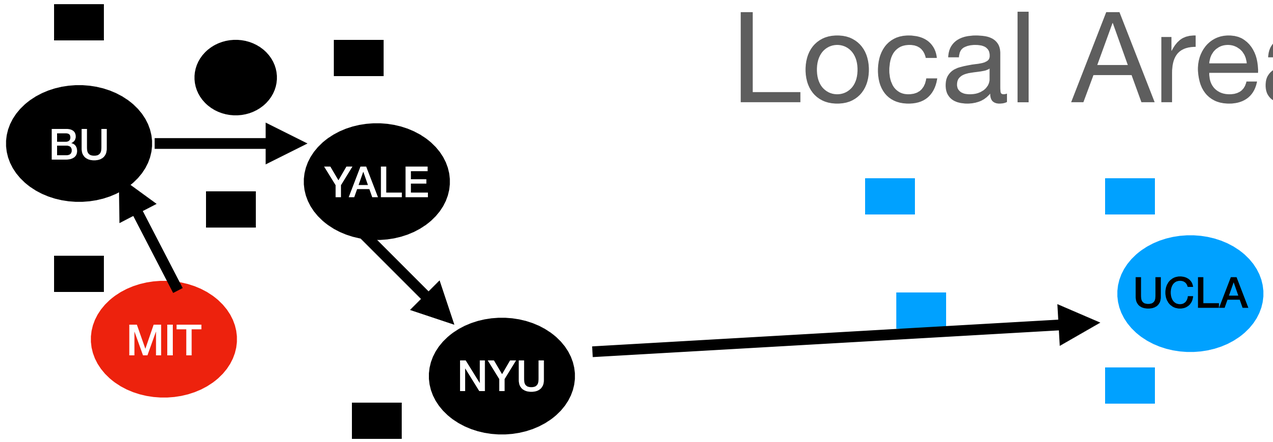
10 nearest

LA ARCs of route $l =$ MIT BU YALE NYU UCLA USC DUKE

$l =$ [Depot MIT] [MIT BU YALE NYU UCLA] [UCLA USC DUKE] [DUKE Depot]

$$c_l = c_{DepotMIT} + c_{MIT[BU,YALE,NYU]DUKE} + c_{DUKE,[],Depot}$$

Local Area Arcs:



N_u = LA Neighbors

ex: $|N_u| = 10$

10 nearest

LA ARCs of route $l =$ MIT BU YALE NYU UCLA USC DUKE

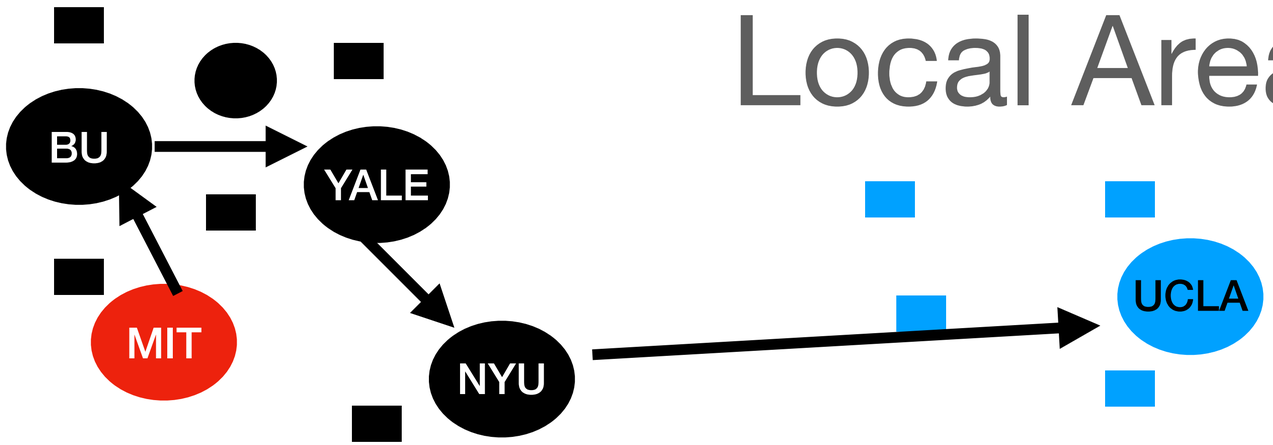
$l =$ [Depot MIT] [MIT BU YALE NYU UCLA] [UCLA USC DUKE] [DUKE Depot]

$$c_l = c_{DepotMIT} + c_{MIT[BU,YALE,NYU]DUKE} + c_{DUKE,[],Depot}$$

DYNAMIC PROGRAMMING SOLUTION

$$c_{MIT,[Yale,NYU,BU],UCLA} = \min \begin{pmatrix} c_{MIT,BU} + c_{BU,[Yale,NYU],UCLA} \\ c_{MIT,Yale} + c_{Yale,[NYU,BU],UCLA} \\ c_{MIT,NYU} + c_{NYU,[Yale,BU],UCLA} \end{pmatrix}$$

Local Area Arcs:



N_u = LA Neighbors

ex: $|N_u| = 10$

10 nearest

LA ARCs of route $l =$ MIT BU YALE NYU UCLA USC DUKE

$l =$ [Depot MIT] [MIT BU YALE NYU UCLA] [UCLA USC DUKE] [DUKE Depot]

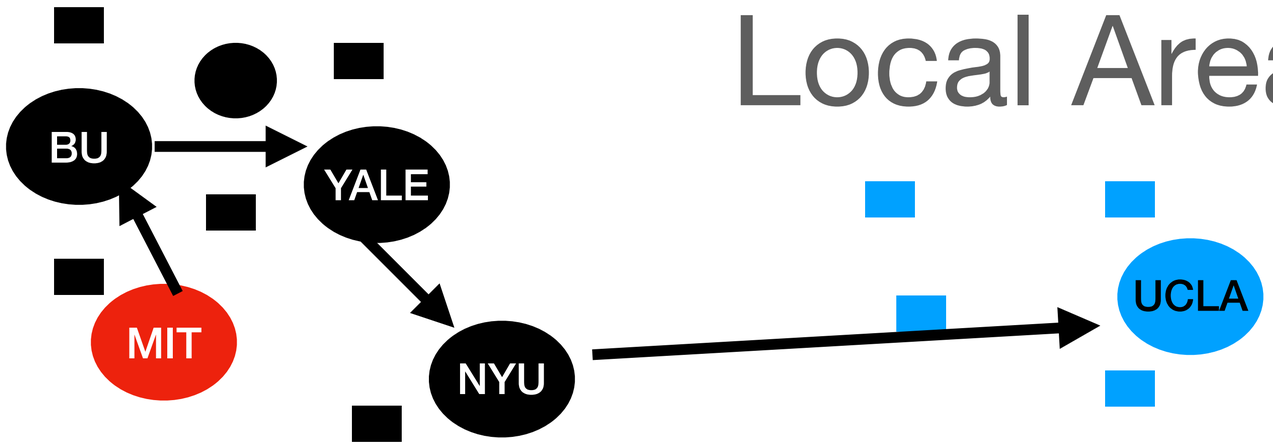
$$c_l = c_{DepotMIT} + c_{MIT[BU,YALE,NYU]DUKE} + c_{DUKE,[],Depot}$$

DYNAMIC PROGRAMMING SOLUTION

$$c_{MIT,[Yale,NYU,BU],UCLA} = \min \begin{pmatrix} c_{MIT,BU} + c_{BU,[Yale,NYU],UCLA} \\ c_{MIT,Yale} + c_{Yale,[NYU,BU],UCLA} \\ c_{MIT,NYU} + c_{NYU,[Yale,BU],UCLA} \end{pmatrix}$$

Time: $O(N^2) < (N^2) * (10) * 2^{10}$

Local Area Arcs:



N_u = LA Neighbors

ex: $|N_u| = 10$

10 nearest

LA ARCs of route $l =$ MIT BU YALE NYU UCLA USC DUKE

$l =$ [Depot MIT] [MIT BU YALE NYU UCLA] [UCLA USC DUKE] [DUKE Depot]

$$c_l = c_{DepotMIT} + c_{MIT[BU,YALE,NYU]DUKE} + c_{DUKE,[],Depot}$$

DYNAMIC PROGRAMMING SOLUTION

$$c_{MIT,[Yale,NYU,BU],UCLA} = \min \begin{pmatrix} c_{MIT,BU} + c_{BU,[Yale,NYU],UCLA} \\ c_{MIT,Yale} + c_{Yale,[NYU,BU],UCLA} \\ c_{MIT,NYU} + c_{NYU,[Yale,BU],UCLA} \end{pmatrix}$$

Time: $O(N^2) < (N^2) * (10) * 2^{10}$

$$\bar{c}_l = \bar{c}_{DepotMIT} + \bar{c}_{MIT[BU,YALE,NYU]DUKE} + \bar{c}_{DUKE,[],Depot}$$

$$\bar{c}_{MIT[BU,YALE,NYU]DUKE} = c_{MIT[BU,YALE,NYU]DUKE} - \pi_{MIT} - \pi_{BU} - \pi_{Yale} - \pi_{NYU} - \cancel{\pi_{DUKE}}$$

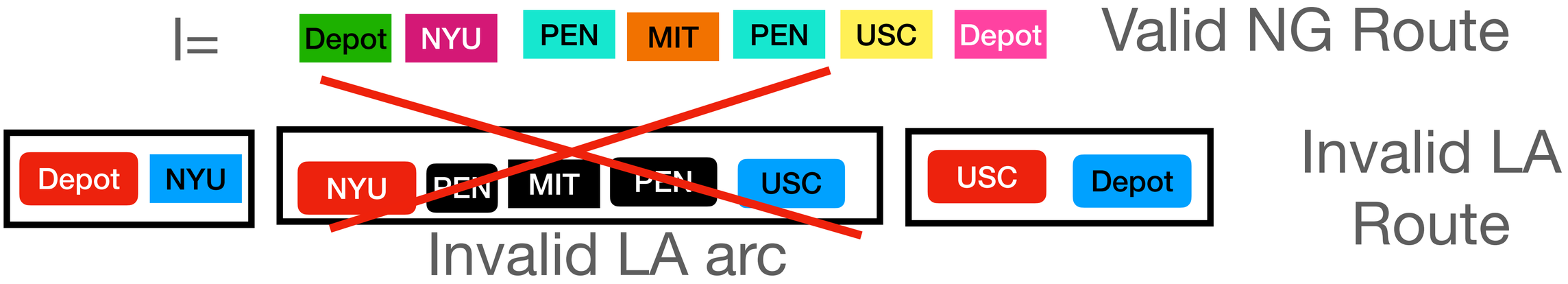
LA Routes VS NG Routes

Let: NG=LA ; PEN $\in$ LA(NYU) and PEN $\notin$ LA(MIT)

|= Depot NYU PEN MIT PEN USC Depot Valid NG Route

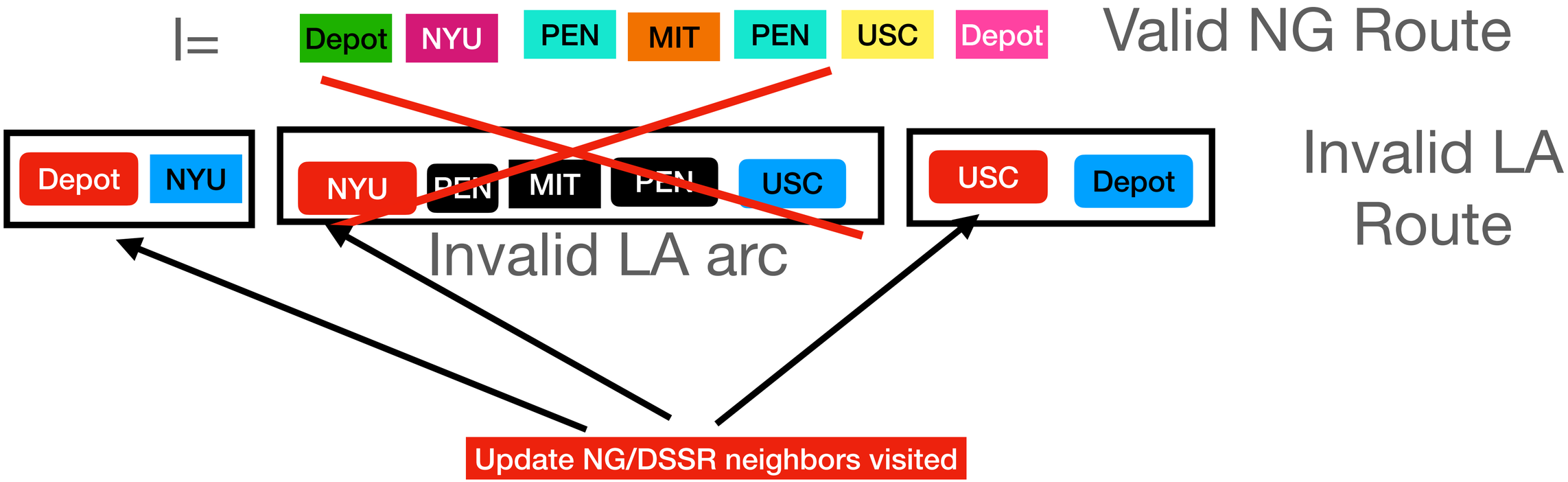
LA Routes VS NG Routes

Let: NG=LA ; PEN $\in$ LA(NYU) and PEN $\notin$ LA(MIT)



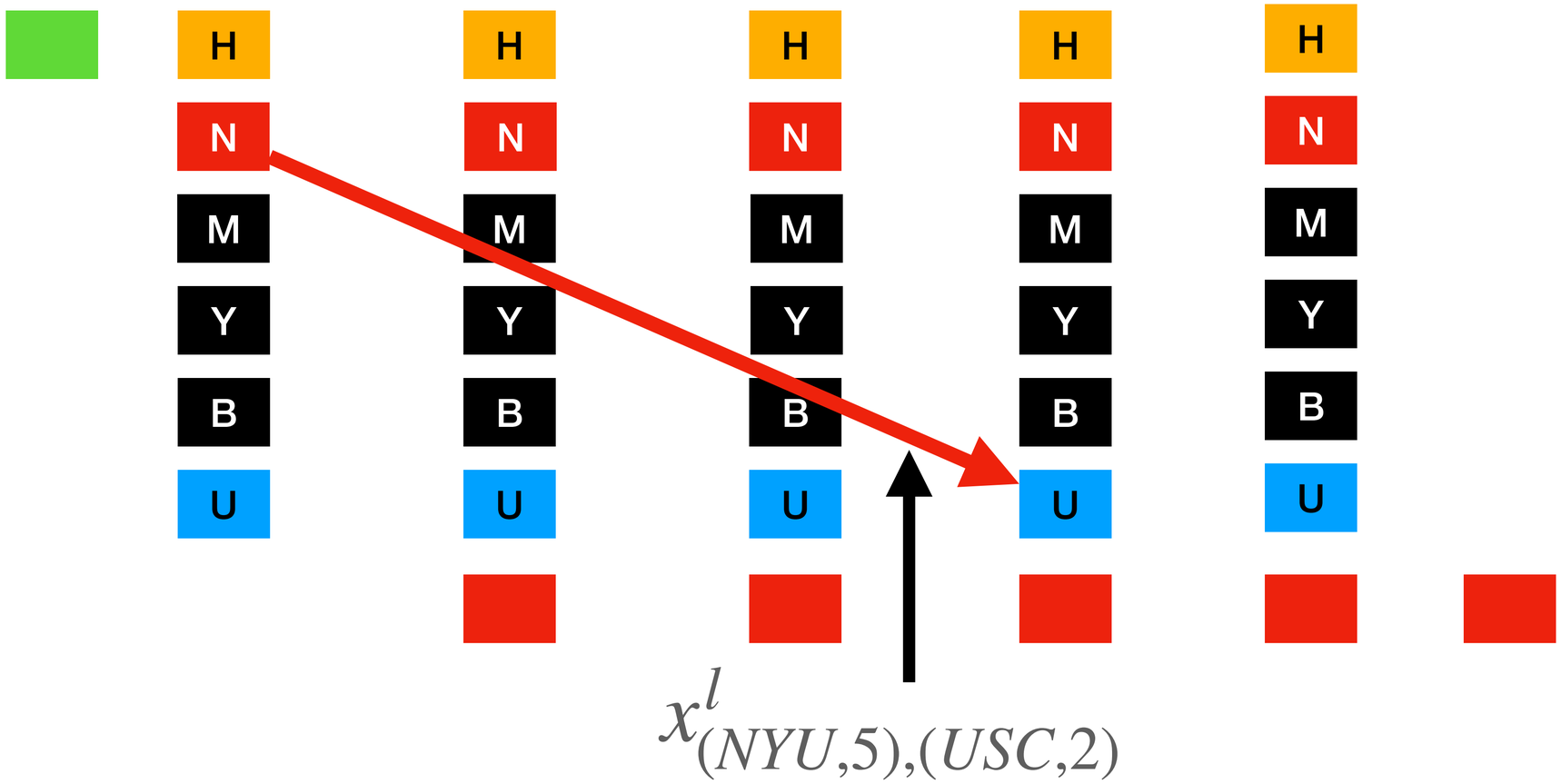
LA Routes VS NG Routes

Let: NG=LA ; PEN $\in$ LA(NYU) and PEN $\notin$ LA(MIT)

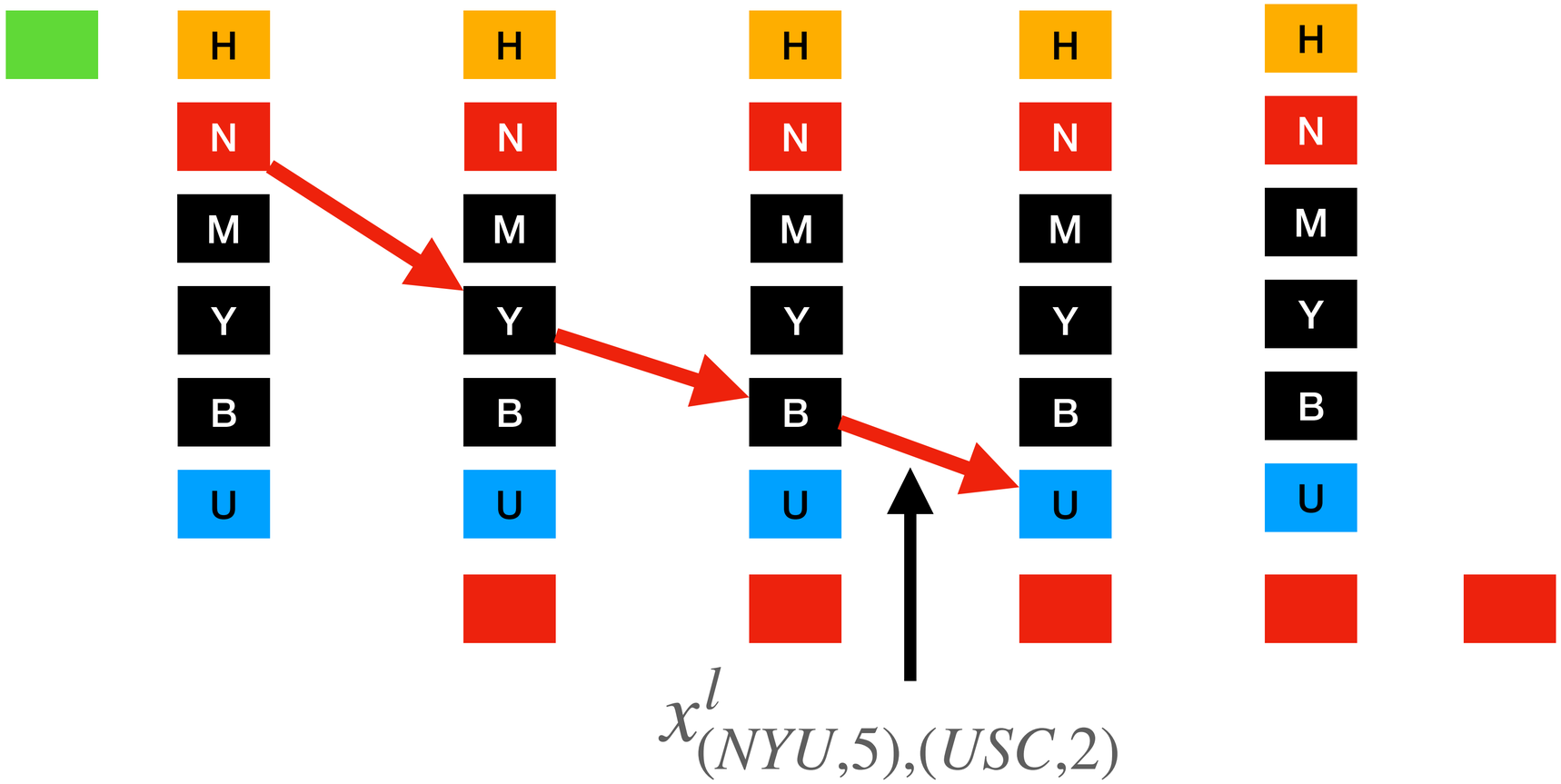


TIGHTER: NG/DSSR breaker must be non-intermediate.

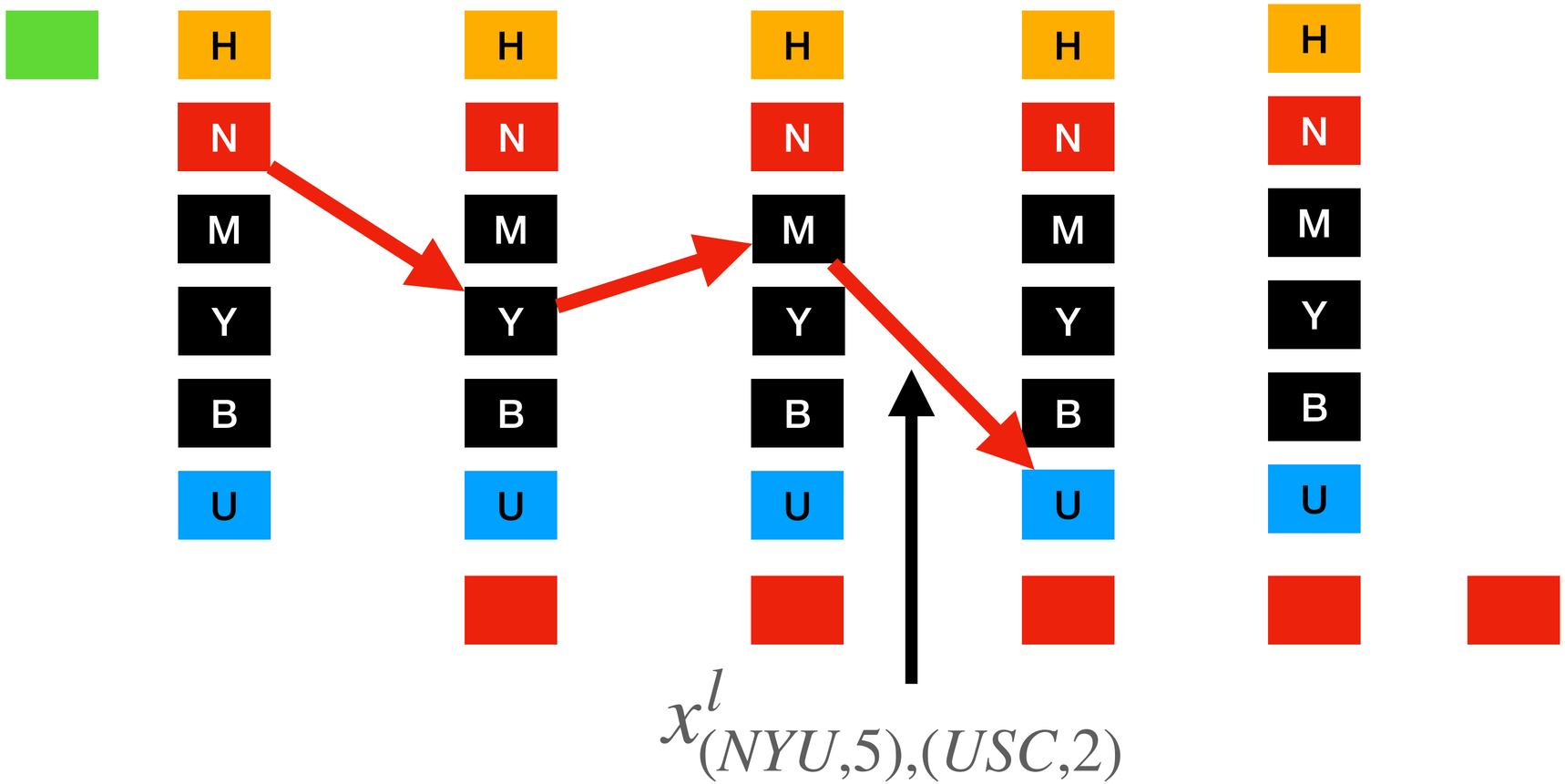
Set Cover(Ω_R^{LA})



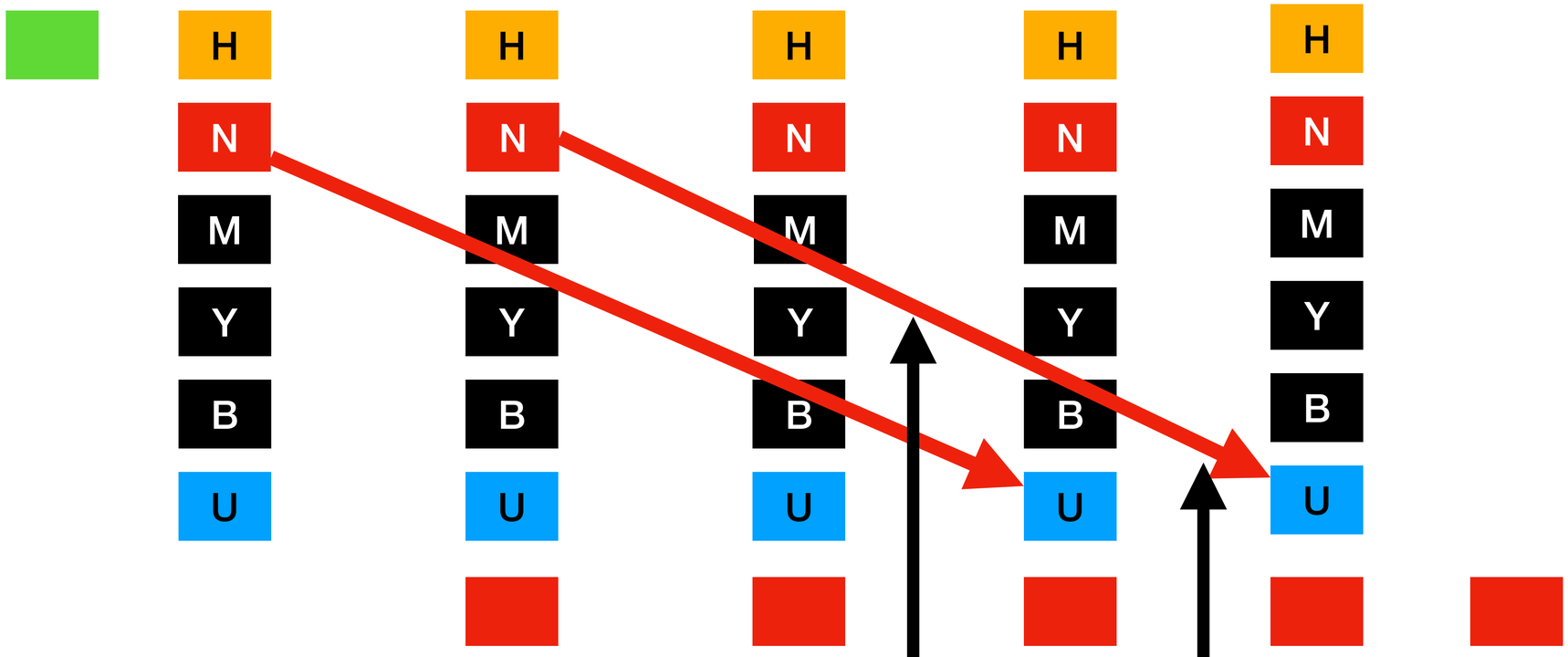
Set Cover(Ω_R^{LA})



Set Cover(Ω_R^{LA})

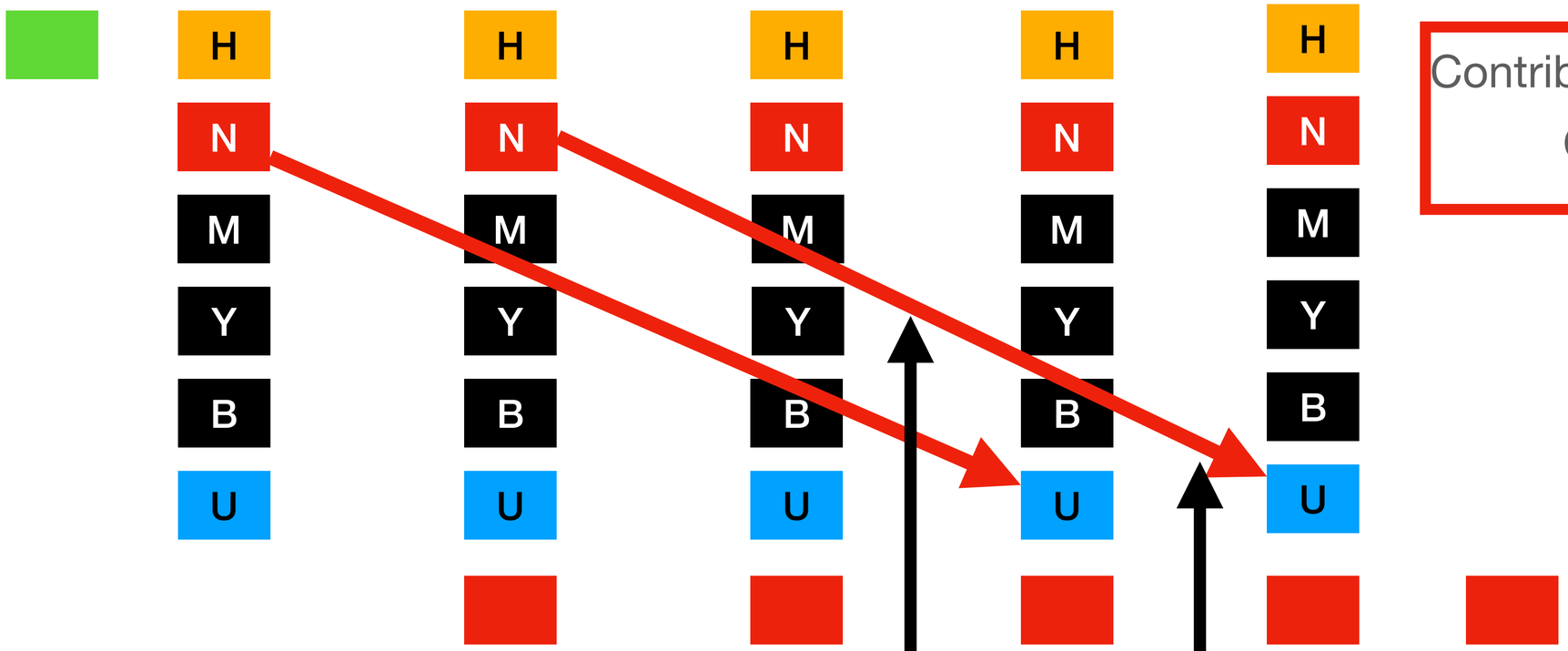


Set Cover(Ω_R^{LA})



$$x^l_{(NYU,5),(USC,2)} + x^l_{(NYU,4),(USC,1)}$$

Set Cover(Ω_R^{LA})



Contributes $c_{NYU,BU,MIT,USC}$ to Objective

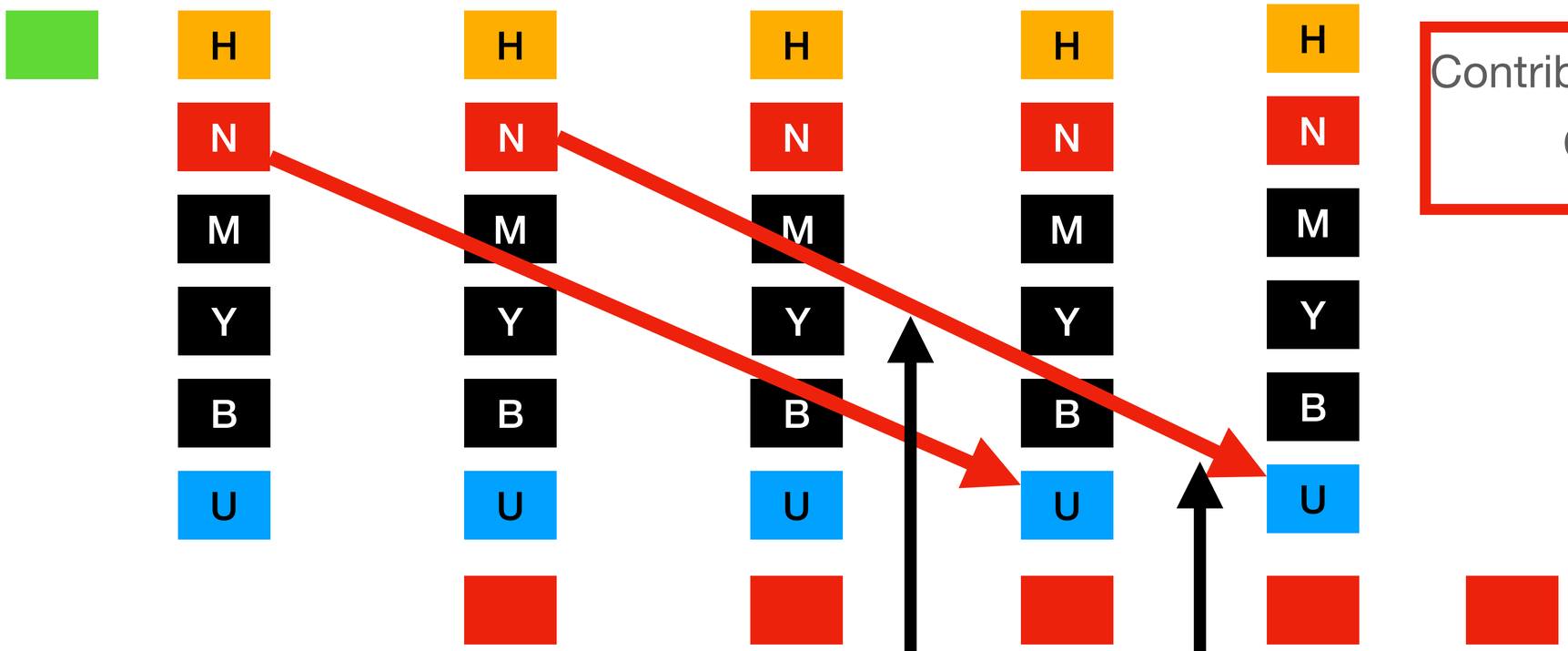
Covers: NYU, BU, MIT NOT USC

$$x^l_{(NYU,5),(USC,2)} + x^l_{(NYU,4),(USC,1)}$$

$$=$$

$$x^l_{NYU,MIT,YALE,USC} + x^l_{NYU,YALE,BU,USC} + \boxed{x^l_{NYU,BU,MIT,USC}}$$

Set Cover(Ω_R^{LA})



Contributes $c_{NYU,BU,MIT,USC}$ to Objective

Covers: NYU, BU, MIT NOT USC

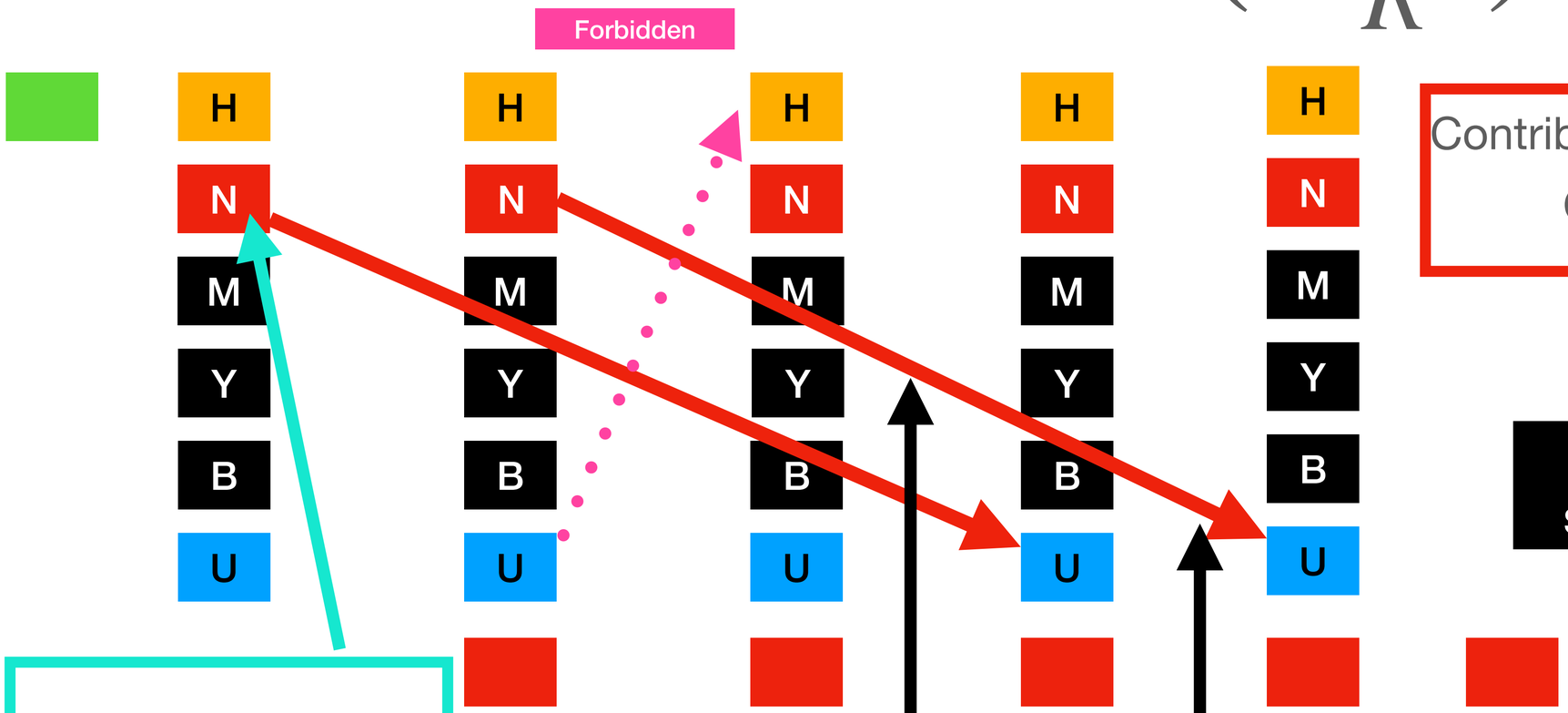
$$x^l_{(NYU,5),(USC,2)} + x^l_{(NYU,4),(USC,1)}$$

$$=$$

$$P^l_{NYU,USC,2}$$

$$\cancel{x^l_{NYU,Harvard,Yale,USC}} + x^l_{NYU,MIT,YALE,USC} + x^l_{NYU,YALE,BU,USC} + \boxed{x^l_{NYU,BU,MIT,USC}}$$

Set Cover(Ω_R^{LA})



Contributes $c_{NYU,BU,MIT,USC}$ to Objective

Covers: NYU, BU, MIT NOT USC

LP easy via specialized solver

Flow in =Flow Out

$$x^l_{(NYU,5),(USC,2)} + x^l_{(NYU,4),(USC,1)}$$

$$=$$

$$P^l_{NYU,USC,2}$$

$$\cancel{x^l_{NYU,Harvard,Yale,USC}} + x^l_{NYU,MIT,YALE,USC} + x^l_{NYU,YALE,BU,USC} + \boxed{x^l_{NYU,BU,MIT,USC}}$$

Set Cover(Ω_R^{LA})

$$\boxed{a_{wp} = w \in (N_p + u_p)}$$

FOR REFERENCE

LP easy via specialized solver

$$\min_{x \geq 0} \sum_{l \in \Omega_R} \sum_{p \in P^l} c_p x_p^l$$

TOTAL COST

$$\sum_{l \in \Omega_R} \sum_{p \in P^l} x_p^l \boxed{a_{wp}} \geq 1 \, \forall w$$

COVER

$$\sum_{v \notin N_u, d_1} x^l_{(v,d_1),(u,d)} = \sum_{v \notin N_u, d_1} x^l_{(u,d),(v,d_1)} \quad \forall l \in \Omega_R, u, d$$

FLOW

Separated by l

$$\sum_{d_1} x^l_{(u,d_1),(v,d_1-d-1)} = \sum_{p \in P^l_{u,v,d}} x_p^l \quad \forall l \in \Omega_R, u, v, d,$$

ARC CONSISTENCY

Separated by l

GM/LA RMP NEVER BOTTLECK

RMP over $\hat{I}, \hat{P}$

$\hat{I}$: Nodes (u,d) seen thus far

$\hat{P}$: LA arcs seen thus far

$\hat{I}, \hat{P}$

SHARE over Ω_R

$$\hat{P} \leftarrow \hat{P} \cup \hat{P}^{l_1^*} \cup \hat{P}^{l_2^*} \ldots$$

$$\hat{I} \leftarrow \hat{I} \cup \hat{I}^{l_1^*} \cup \hat{I}^{l_2^*} \ldots$$

π

$(\hat{I}^l, \hat{P}^l)$

Extract nodes, arcs of route l

Some Options

1. LA ARC Cover $\leq$ Demand
2. Add edges as needed
3. Warm start with last $\hat{I}, \hat{P}$

Price over Ω_{l1}^{LA}

Price over Ω_{l2}^{LA}

$\forall l \in \Omega_R$

EASY PRICING

Use Only $P^l \rightarrow$ Elementary

TEASER: LA- Subset Row Inequalities θ

2 cross country trips needed but 1.5 used

UCLA=DEPOT	NYU d=2	MIT d=1	UCLA=DEPOT d=0	$= 0.5$
UCLA=DEPOT	NYU d=2	Yale d=1	UCLA=DEPOT d=0	$= 0.5$
UCLA=DEPOT	NYU d=2	MIT d=1	UCLA=DEPOT d=0	$= 0.5$

SRI Fit Poorly in Pricing

$$\sum_{l \in \Omega} [a_{NYU,l} + a_{Yale,l} + a_{MIT,l} \geq 2]\theta_l \leq 1$$

LA-SRI FIT ELEGANTLY INTO PRICING

$$\sum_{l \in \Omega_R} \sum_{P \in P^l} [a_{NYU,p} + a_{Yale,p} + a_{MIT,p} \geq 2]x_p^l \leq 1 \quad [\pi_{\delta=NYU,MIT,YALE}]$$

$a_{p\delta}$

$$a_{wp} = w \in (N_p + u_p)$$

TEASER: Optimal Order Varies with Time?

R_p = Efficient Frontier (1) c_r = Cost, (2) Latest Start, (3). Earliest Non-Wait Start

DYNAMIC PROGRAM

Possible Successors

$$R_p \leftarrow \text{Efficient Frontier}\Big(\bigcup_{\substack{w \in N_p \\ q = (w, N_p - w, v_p)}} \bigcup_{r^- \in R_q} [u_p, r^-] \Big)$$

$$c_{p,t_1,t_2} = \min_{r \in R_p^*} c_r; \quad R_p^* \subseteq R_p \text{ s.t. depart } (u_p, v_p) \text{ no later/ no earlier } (t_1, t_2)$$

MASTER: Induces SPACE TIME GRAPH

1. Adjusts RMP Edges-LA-arc equivalence
2. Longer LA Arcs → Fewer nodes generated as no time relaxation in N_p .

TEASER: Pricing via Discretization Discovery

Discretize Nodes

$$J_g = \{i \in J, u_g = u_i, d_g = d_i$$
$$t_g^- \leq t_i \leq t_g^+,$$
$$M_g^- \subseteq M_i \subseteq M_g^+\}$$

Like Boland
Like NG/DSSR

Optimistic Edges

$$\bar{c}_{fg} = \min_{\substack{i \in J_f \\ j \in J_g}} \bar{c}_{ij}$$

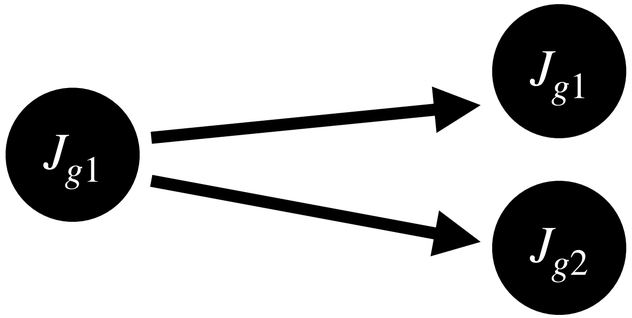
EASY TO COMPUTE

use times t_f^+, t_g^-

Algorithm:

1. Compute shortest path
2. Split violated cycle/time window

Time Split



$$t_{g_1}^+ = t_{g_2}^- - 1$$

Longer LA Arcs-Better since
No Relaxation of time in N_p .

See Paper Demand Discretization.

Experiments: Solomon CVRPTW

FACTOR SPEEDUP

Solution Time in Seconds

[———-——Ratio of Solution Time to CG———-—]

Instance	CG	GM	LA	GM+LA
C101	12.8	0.95	0.70	0.66
C106	22.8	1.42	0.67	1.08
C107	25.4	0.14	0.31	0.68
C105	26.0	0.64	0.57	0.85
C108	909	2.56	3.87	12.6
C109	2,815	7.6	4.3	34
C102	48,637	50	31	360
C104	86,400*	6.4	3.1	138
C103	129,600*	91	38	453

> 453 times faster. NOT percent

* Indicates that the instance timed out without a solution

Conclusions, Extensions, Future Work

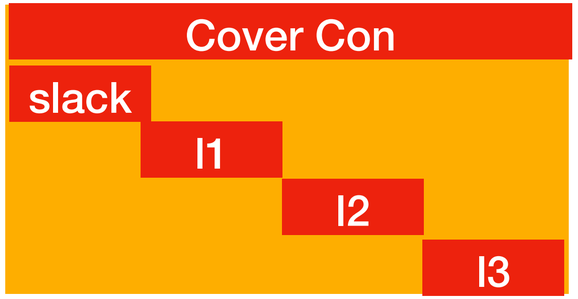
Conclusion: Faster, Tighter, Fewer Iterations

In Paper Extensions: APPROX Solve RMP as ILP,
LA rounded capacity inequalities

Future Work: Branching: Ryan/Adams. Should be tighter as branches removes LA-Arcs.

Future Work: Exploit Jordi Castro Interior Point Solver
Not needed for RMP but maybe for ILP

$A =$



$A(DIAG)A^\top =$

Easy Linear System for Interior Points