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## Strengthened route-based formulations and a branch-cut-andprice algorithm for split delivery vehicle routing problems

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Joint work with:
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## Outline

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- Computational results;
- Conclusions.


## Split delivery vehicle routing problems

$\mid d_{i}$ : demand $\mid s_{i}$ : service time $\mid\left[\mathrm{e}_{i} l_{i}\right]$ : time window $\mid t_{i j}$ : travel time $\mid c_{i j}$ : travel cost $\mid$


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- A customer can be served by more than one vehicle, if necessary or beneficial (reductions of up to $50 \%$ in the cost);


## Split delivery vehicle routing problems



- Additionally to defining the routes, we have to decide how much each vehicle delivers to each customer.


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- Multiple visits to customers create modeling and algorithmic challenges: What are the actual routes here? (NP-hard!)


## Split delivery vehicle routing problems <br> $\triangleright$ Literature

- Compact formulations:
- Two-index vehicle flow formulations (arc-based variables only), can only provide relaxations: load transfer between vehicles;
- The best performing compact models are based on three-index variables, in which the indices represent two subsequent arcs (not vehicle indexed) [Munari and Savelsbergh, 2022];


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- The best performing compact models are based on three-index variables, in which the indices represent two subsequent arcs (not vehicle indexed) [Munari and Savelsbergh, 2022];
- Branch-and-cut methods (state-of-the-art so far)
- They are based on two-index (relaxed) formulations [Archetti et al., 2014; Bianchessi and Irnich, 2019; Gouveia et al., 2023] and combinations of two- and three-index formulations [Munari and Savelsbergh, 2022];


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- Desaulniers (2010), Archetti, Bouchard, and Desaulniers (2011):
- routes and extreme delivery patterns in the subproblem, delivery quantities as convex combinations in the master problems;
- extreme delivery pattern: a visit delivers nothing, the full demand, or a fraction of the demand (but for at most one customer in the route);


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- Archetti, Bianchessi and Speranza (2011): addressed the SDVRP replicating each customer node $d_{i}$ times;
- Ceselli et al. (2009) and Moreno et al. (2010): path-based formulations for the SDVRP.


## Properties

For a feasible instance with integer demands and vehicle capacity as well as costs and travel times that satisfy the triangle inequality, there exists an optimal solution with the following properties [(Dror and Trudeau, 1990; Feillet et al. 2006; Desaulniers 2010; Archetti, Bouchard and Desaulniers, 2011]:

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Strengthened route-based formulations and a branch-cut-and-price algorithm for SDVRPs

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- $\mathcal{F}(\tilde{\mathcal{R}})=\{0\} \cup \tilde{\mathcal{R}} \cup \mathcal{C} \cup\{n+1\} ;$
- $\mathcal{A}_{1}(\tilde{\mathcal{R}})$ : connects the source with each of the route nodes in $\tilde{\mathcal{R}}$. The capacity of these arcs is $Q$.
- $\mathcal{A}_{2}(\tilde{\mathcal{R}})$ : connects the route nodes in $\tilde{\mathcal{R}}$ with customer nodes in $\mathrm{C}: \operatorname{arc}$ $(r, i)$ belongs to $\mathcal{A}_{2}(\tilde{\mathcal{R}})$ if and only if route $r \in \tilde{\mathcal{R}}$ visits customer $i \in \mathcal{C}$. The capacity of these arcs is $\infty$.
- $\mathcal{A}_{3}(\tilde{\mathcal{R}})$ : connects each customer node to the sink. The capacity of an arc $(i, n+1) \in \mathcal{A}_{3}(\tilde{\mathcal{R}})$ is $d_{i}$.


## The flow graph $\mathcal{F}(\tilde{\mathcal{R}})$



## The flow graph $\mathcal{F}(\tilde{\mathcal{R}})$ - An example

- $\tilde{\mathcal{R}}=\left\{r_{1}=\{0,1,2,3,6\}, r_{2}=\{0,2,3,6\}, r_{3}=\{0,4,5,6\}\right\} ;$
- $\mathcal{C}=\{1,2,3,4,5\}, d=\{10,20,30,40,10\}, Q=30$.



## The flow graph $\mathcal{F}(\tilde{\mathcal{R}})$ - An observation

- Observation 1. A set $\tilde{\mathcal{R}}$ of time-feasible routes forms a feasible SDVRPTW solution iff the maximum flow $f$ in graph $\mathcal{F}(\tilde{\mathcal{R}})$ has a value of $\sum_{i \in \mathrm{e}} d_{i}$. In such a case, values $f_{a}, a \in \mathcal{A}_{2}$, correspond to the delivery quantities for every route in $\tilde{\mathscr{R}}$ to each customer in $\mathcal{C}$.


## The flow graph $\mathcal{F}(\tilde{\mathcal{R}})$ - Checking feasibility

- $\tilde{\mathcal{R}}=\left\{r_{1}=\{0,1,2,3,6\}, r_{2}=\{0,2,3,6\}, r_{3}=\{0,4,5,6\}\right\} ;$
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- The max-flow in $\mathcal{F}(\tilde{\mathcal{R}})$ tells us if $\tilde{\mathcal{R}}$ forms a feasible solution.


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- $\tilde{\mathcal{R}}^{*}=\left\{r_{1}=\{0,1,2,3,6\}, r_{2}=\{0,2,3,6\}, r_{3}=\{0,4,6\}, r_{4}=\{0,4,5,6\}\right\} ;$
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The flow graph $\mathcal{F}(\tilde{\mathcal{R}})$ - Greatest common divisor

- All arc capacities are integer (other than $\infty$ ): there is an integer max-flow;
- Let $\bar{q}=\operatorname{gcd}\left(Q, d_{1}, d_{2}, \ldots, d_{n}\right)$ : we divide all arc capacities by this value



## A new property...

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4. If $\operatorname{arc}(i, j)$ is traversed, then $\operatorname{arc}(j, i)$ is not;
5. Delivery quantities are positive integers; and
6. Let $\bar{q}=\operatorname{gcd}\left(Q, d_{1}, d_{2}, \ldots, d_{n}\right)$. All delivery quantities are multiples of $\bar{q}$.

## Base formulation (F0)

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(F0): Min $\sum_{r \in \mathcal{R}} c^{r} \theta_{r}$,

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\text { s.t. } & \sum_{r \in \mathcal{R}} h_{r} \theta_{r} \geq\left\lceil\sum_{i \in S} d_{i} / Q\right],  \tag{2}\\
& \forall S \subseteq \mathcal{C}, \\
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- (2): strong $k$-path inequalities [Baldacci, Christofides and Mingozzi, 2008; Archetti, Bouchard and Desaulniers, 2011].


## Base formulation (FO)

- Constraints (2) suffice to define the set of feasible solutions;
- $\tilde{\mathcal{R}}(\overline{\boldsymbol{\theta}})=\left\{r_{1}, r_{2}, \ldots, r_{|\tilde{\mathcal{R}}(\overline{\boldsymbol{\theta}})|}\right\}$ : set of routes corresponding to an integer solution $\overline{\boldsymbol{\theta}}$;


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Theorem 1. If an integer solution $\overline{\boldsymbol{\theta}}$ satisfies Constraints (2), then set $\tilde{\mathcal{R}}(\overline{\boldsymbol{\theta}})$ of routes forms a feasible SDVRPTW solution.

Corolary 1. Constraints (2) can be exactly separated for integer solutions $\overline{\boldsymbol{\theta}}$ of Formulation (FO) in polynomial time by using the flow graph $\mathcal{F}(\tilde{\mathcal{R}}(\overline{\boldsymbol{\theta}})$ ) and finding a minimum cut in it.

## The flow graph $\mathcal{F}(\tilde{\mathcal{R}})$ - Checking feasibility

- $\tilde{\mathcal{R}}=\left\{r_{1}=\{0,1,2,3,6\}, r_{2}=\{0,2,3,6\}, r_{3}=\{0,4,5,6\}\right\} ;$
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Formulation (F1):

- We reduce the number of variables in Formulation (FO) by retaining only variables $\theta_{r}$ that correspond to routes delivering at least $\bar{q}$ at every visit.
- $D_{i}=\left\{\bar{q}, 2 \bar{q}, \ldots, d_{i}\right\}$;
- $\mathcal{R}^{1} \subseteq \mathcal{R}$ : set of routes that has length of at most $Q / \bar{q}$.


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\text { s.t. } & \sum_{r \in \mathcal{R}} h_{r S} \theta_{r} \geq\left\lceil\sum_{i \in S} d_{i} / Q\right\rceil, & \forall S \subseteq \mathcal{C}, \\
& \sum_{r \in \mathcal{R}}\left(2 b_{i \mathrm{~F}}^{r}+b_{i \mathrm{P}}^{r}\right) \theta_{r} \geq 2, & \forall i \in \mathcal{C} .  \tag{4}\\
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- (4): special case of the strong minimum number of vehicles inequalities used by Archetti, Bouchard, and Desaulniers (2011).


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- $\mathcal{C}(K)=\left\{i \in \mathcal{C}: K \bar{q}<d_{i}\right\}:$ contains each customer $i \in \mathcal{C}$ to which $K$ deliveries of size $\bar{q}$ are not enough to satisfy demand $d_{i}$;


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- For $i \in \mathcal{C}(K)$, we multiply (5) by $(K-1) /\left(d_{i}-\epsilon\right)$, where $0<\epsilon \ll \bar{q}$;


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- Given $K$ and delivery quantity $q \in D_{i}$, the rounded-up coefficient is a step function that assumes integer values $k$ from 1 to $K$;


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\sum_{r \in \mathcal{R}} \sum_{q \in D_{i}}\left\lceil\frac{(K-1) q b_{i q}^{r}}{d_{i}-\epsilon}\right\rceil \theta_{r} \geq\left\lceil\frac{(K-1) d_{i}}{d_{i}-\epsilon}\right\rceil . \tag{6}
\end{equation*}
$$

- Since $d_{i} /\left(d_{i}-\epsilon\right) \approx 1$ and $b_{i q}^{r}$ is integer:

$$
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\sum_{r \in \mathfrak{R}} \sum_{q \in D_{i}} b_{i q}^{r}\left\lceil\frac{(K-1) q}{d_{i}-\epsilon}\right\rceil \theta_{r} \geq K . \tag{7}
\end{equation*}
$$

- Given $K$ and delivery quantity $q \in D_{i}$, the rounded-up coefficient is a step function that assumes integer values $k$ from 1 to $K$;
- For $K=2: 1$, if $q<d_{i}$; and 2, if $q=d_{i}$ [the same as (4) used in (F2)];


## A family of route-based formulations (FK)

- We apply Chvátal-Gomory rounding on both sides and obtain:

$$
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- For $K=2: 1$, if $q<d_{i}$; and 2, if $q=d_{i}$ [the same as (4) used in (F2)];
- For $K=3: 1$, if $q<d_{i} / 2 ; 2$, if $d_{i} / 2 \leq q<d_{i}$; and 3, if $q=d_{i}$.


## A family of route-based formulations (FK)

- For an arbitrary $K$ and delivery quantity $q \in D_{i}$ :

$$
\left\lceil\frac{(K-1) q}{\left(d_{i}-\varepsilon\right)}\right\rceil=k \text {, if } \frac{(k-1) d_{i}}{K-1} \leq q<\frac{k d_{i}}{K-1} \text {, for } k=1, \ldots, K \text {; }
$$

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$$

- Using this observation, we define the binary parameter $g_{i q}^{k}$ that assumes the value of 1 if and only if $\frac{(k-1) d_{i}}{K-1} \leq q<\frac{k d_{i}}{K-1}$ and obtain:

$$
\begin{equation*}
\sum_{r \in \mathcal{R}} \sum_{q \in D_{i}} \sum_{k=1}^{K} b_{i q}^{r} g_{i q}^{k} k \theta_{r} \geq K \tag{8}
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- We defined the sets of possible delivery quantities to each customer:

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D_{i}(K)=\bigcup_{k=1}^{K}\left\{\min \{l \bar{q}\}: l \bar{q} \in\left[\frac{(k-1) d_{i}}{K-1}, \frac{k d_{i}}{K-1}\right), l \in \mathbb{N}\right\} \quad \text { if } i \in \mathcal{C}(K),
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D_{i}(K)=\left\{l \bar{q}: \forall l=1, \ldots, d_{i} / \bar{q}\right\} \quad \text { if } i \in \mathcal{C} \backslash \mathcal{C}(K)
\end{gathered}
$$

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(FK): Min $\quad \sum_{r \in \mathcal{R}} c^{r} \theta_{r}$,

$$
\begin{array}{lll}
\text { s.t. } & \sum_{r \in \mathcal{R}} h_{r S} \theta_{r} \geq\left\lceil\sum_{i \in S} d_{i} / Q\right\rceil, & \forall S \subseteq \mathcal{C}, \\
& \sum_{r \in \mathcal{R}} \sum_{q \in D_{i}} \sum_{k=1}^{K} b_{i q}^{r} g_{i q}^{k} k \theta_{r} \geq K, & \forall i \in \mathcal{C}(K), \\
& \sum_{r \in \mathcal{R}} \sum_{q \in D_{i}} q b_{i q}^{r} \theta_{r} \geq d_{i}, & \forall i \in \mathcal{C} \backslash \mathcal{C}(K), \\
& \theta_{r} \in \mathbb{Z}^{+}, & \forall r \in \mathcal{R} .
\end{array}
$$

- $\mathcal{R}^{K}=\left\{r \in \mathcal{R}: b_{i q}^{r}=0, \forall i \in \mathcal{C}, \forall q \notin D_{i}(K)\right\}$;
- There exists an optimal solution $\overline{\boldsymbol{\theta}}$ to the linear relaxation of Formulation (FK) such that $\bar{\theta}_{r}=0$ for all $r \notin \mathcal{R}^{K}$.


## A family of route-based formulations (FK)

- For $K_{\max }=\max _{i \in \mathcal{C}} d_{i} / \bar{q}$, Formulation ( $\mathrm{F} K_{\max }$ ) is fully discretized: all possible delivery quantities from Property 6 are considered $\left(\mathcal{R}^{K}=\mathcal{R}\right)$.


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- Formulations ( $\mathrm{F} K$ ) with $K>K_{\text {max }}$ are equivalent to ( $\mathrm{F} K_{\text {max }}$ ).
- Formulations ( FK ) with $K<K_{\text {max }}$ are partially discretized.
- The exact separation of strong $k$-path inequalities (2) is necessary to ensure the feasibility of integer solutions.


## Branch-cut-and-price: Main components

- Rounded capacity inequalities (RCI); Limited-memory subset-row packing inequalities [valid for formulations ( $\mathrm{F} K$ ) with $K \geq 1$ ]; Limited-memory subset-row covering inequalities; Limited-memory strong $k$-path inequalities;


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- Automatic dual price smoothing stabilization [Pessoa et al., 2018];
- Elementary route enumeration based on the primal-dual gap;
- Before launching our BCP algorithm, we run a matheuristic and the value of the best solution (plus a small epsilon) is then used as the initial upper bound in the BCP algorithm.


## Branch-cut-and-price: Pricing subproblem

- Resource constrained shortest path problem in multi-graph $\mathcal{G}^{\prime}(K)=\left(\mathcal{V}, \mathcal{A}^{\prime}(K)\right)$ (capacity and time-window constrained);
- Every arc $(i, j)$ in the original graph $\mathcal{G}$ is replaced by multiple $\operatorname{arcs}(i, j, q)$, $q \in D_{j}(K)$, between nodes $i$ and $j$ in $\mathcal{G}^{\prime}(K)$;
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- The resource-feasible paths in $\mathcal{G}^{\prime}(K)$ are the routes in $\mathcal{R}^{K}$;
- We use the bucket-graph based bidirectional labeling algorithm of Sadykov, Uchoa, and Pessoa (2021), with ng-path relaxation;
- Every label $L$ represents a partial path $\mathcal{G}^{\prime}(K)$, which is either forward (starting from node 0 ) or backward (starting from node $n+1$ ).


## Branch-cut-and-price: Pricing subproblem

- The reduced cost $\bar{c}_{(i, j, q)}$ of each $\operatorname{arc}(i, j, q) \in \mathcal{A}^{\prime}(K)$ is

$$
\bar{c}_{(i, j, q)}=c_{i j}-\sum_{\substack{o \in O_{j}^{\prime} \\(i, j) \in \delta\left(s^{0}\right)}} \bar{\rho}_{o}- \begin{cases}0, & \text { if } K<2, \\ \bar{k}(j, q, K) \cdot \bar{\pi}_{j}, & \text { if } K \geq 2 \text { and } j \in \mathcal{C}(K), \\ q \bar{\pi}_{j}, & \text { if } K \geq 2 \text { and } j \in \mathcal{C} \backslash \mathfrak{C}(K) .\end{cases}
$$

- $\bar{k}(j, q, K)$ is equal to the value $k$ that satisfies $\frac{(k-1) d_{j}}{K-1} \leq q<\frac{k d_{j}}{K-1}$;
- $O$ : set of active RCIs, with $S^{o} \subseteq \mathcal{C}$ defining the rounded capacity inequality $o \in O$ with dual value $\bar{\rho}_{o}>0$;
- Let $\delta\left(S^{o}\right)$ also be the set of arcs in $\mathcal{A}$ which have exactly one node in $S^{o}$.


## Computational results

- The BCP algorithm was coded in C++ on top of the generic BCP library BaPCod [Sadykov and Vanderbeck, 2021] with its VRPSolver extension [Pessoa et al., 2020];
- We use the IBM CPLEX Optimization Studio 20.1 as LP and MIP solver;
- Server with processor 2.6GHz Cascade Lake Intel Xeon Skylake Gold 6240 with 36 cores and 196 GB of RAM (up to 36 instances were run in parallel on each node, each run using a single core).


## Computational results: Benchmark instances

- SDVRPTW: Adapted Solomon's VRPTW instances, having $n=\{50,75,100\}$ and $Q=\{30,50,100\}$ [total: $56 \times 3 \times 3=504]$;


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- New instances based on those with $\bar{q}=10$ (all instances in class C, and instances in class RC with 50 customers). We add a random integer value in $[-3,3]$ to the demand of every customer $i \in \mathcal{C}$ such that $d_{i} \in\left[1.2 d_{\text {min }}, 0.8 d_{\text {max }}\right]$; We denote these new instances as 50P, 75P, and 100P [total: 201 instances].


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- Thus, the total number of SDVRPTW instances is 705;
- SDVRP: We perform 352 tests derived from 88 instances (S, SD, eil, p), limiting, or not, the size of the fleet (LF/UF) and rounding, or not, distances (LF-r/UF-r) [total: $88 \times 4=352]$.


## Computational results: Comparing formulations (FK)

Root node results for SDVRPTW instances with $n=50$
[C and RC: $K_{\max }=4$; R: $K_{\max }=36$ ].


## Computational results: Comparing valid inequalities

## SDVRPTW - F2 and F $K_{\max }$ (one-hour time limit)

| BCP variant | Opt | Geom. time(s) | Time(s) | Root gap(\%) | Final gap(\%) |
| :--- | :---: | :---: | :---: | :---: | :---: |
| BCPmax $_{\text {all }}$-SKPI-SRCI | 361 | 360.40 | 1837.16 | 0.87 | 0.72 |
| BCPmax $_{\text {all }}$-SKPI | 358 | 362.56 | 1865.11 | 0.86 | 0.72 |
| BCPmax $_{\text {all }}$ SRCI | 368 | 354.53 | 1820.09 | 0.84 | 0.70 |
| BCPmax $_{\text {all }}$ | 363 | 349.39 | 1828.03 | 0.82 | 0.69 |
| BCP2 $_{\text {all }}$ SRCI | 273 | 734.92 | 2243.48 | 2.42 | 2.00 |
| BCP2 $_{\text {all }}$ | 273 | 730.12 | 2238.13 | 2.39 | 1.93 |

- SKPI and SRCI contribute to reducing root and final gaps (on average).
- Strong $k$-path inequalities - SKPI increase the number of optimally solved instances.
- Subset-row covering inequalities - SRCI does not allow us to solve more instances to optimality.


## Computational results: SDVRPTW

## SDVRPTW $-F 2$ and $F K_{\max }$

| $n$ | Benchmark run - 3600 s |  |  |  | Long run - 18000s |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | ( $F K_{\text {max }}$ ) | MS22 | Bl19 | ABD11 | (F2) | $\left(F K_{\text {max }}\right)$ | $\operatorname{Best}\left(F 2, F K_{\text {max }}\right)$ |
| 25 | 168 | 168 | 168 | 168 | 168 | 168 | 168 (0) |
| 50 | 152 (27) | 123 | 104 | 86 | 136 | 168 | 168 (40) |
| 100 | 54 (48) | 4 | 5 | 8 | 24 | 55 | 56 (50) |
| \{50, 100 \} | 206 | 127 | 109 | 94 | 160 | 223 | 224 (90) |
| $\{25,50,100\}$ | 374 (75) | 295 | 277 | 262 | 328 | 391 | 392 (90) |

MS22: Munari and Savelsbergh (2022); BI19: Bianchessi and Irnich (2019); A11: Archetti, Bouchard and Desaulniers (2011).

- Formulation $\left(F K_{\max }\right)$ finds 374 optimal solutions, 75 for the first time, within one hour benchmark tests.
- Formulations (F2) and ( $F K_{\max }$ ) all together find 392 optimal solutions, 90 for the first time, within five hours.


## Computational results: SDVRP

- Three variants of our BCP algorithm on the full test set of SDVRP instances: $B C P 2_{\text {all-SRCI }}$, BCP $_{10}{ }_{\text {all-SRCI }}$ and BCPmax ${ }_{\text {all-SRCI }}$ (time limit of 2 hours);


## Computational results: SDVRP

- Three variants of our BCP algorithm on the full test set of SDVRP instances: BCP2 ${ }_{\text {all-SRCI }}$, BCP10 $_{\text {all-SRCI }}$ and BCPmax ${ }_{\text {all-SRCI }}$ (time limit of 2 hours);
- Best overall performance: BCP10 all-SRCI;

| Class | \# | LF-r |  |  | LF |  |  | UF-r |  |  | UF |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Opt | LB* | Opt* | Opt | LB* | Opt* | Opt | LB* | Opt* | Opt | LB* | Opt* |
| eil | 11 | 8 | 5 | 2 | 7 | 6 | 2 | 9 | 5 | 3 | 7 | 6 | 2 |
| p | 42 | 3 | 16 | 1 | 2 | 9 | 0 | 2 | 16 | 0 | 2 | 7 | 0 |
| S | 14 | 3 | 2 | 0 | 4 | 1 | 0 | 3 | 5 | 0 | 4 | 2 | 0 |
| SD | 21 | 12 | 9 | 1 | 13 | 9 | 1 | 12 | 9 | 1 | 11 | 9 | 1 |
| Total | 88 | 26 | 32 | 4 | 26 | 25 | 3 | 26 | 35 | 4 | 24 | 24 | 3 |

- 102 optimally solved, 14 for the first time.
- We improve 116 best known LB's (approximately 50\% of open instances).


## Conclusions

- We proposed a new family of partially discretized route-based formulations (FK) for split delivery vehicle routing problems;
- $K$ is the maximum number of different delivery quantities allowed when visiting a customer. In the fully discretized formulation ( $\mathrm{F} K_{\text {max }}$ ), all possible delivery quantities are considered;
- We propose a property that provides a minimum delivery quantity based on customer demand and vehicle capacity. This property is likely to benefit other formulations, as well as other exact and heuristic approaches in the literature;
- To effectively solve the formulations, we have designed a BCP algorithm that resorts to new and state-of-the-art algorithmic improvements.


## Conclusions

- On SDVRPTW instances: the BCP based on ( $\mathrm{F} K_{\text {max }}$ ) achieves best overall performance; (F2) is more efficient for instances with long routes and large value $K_{\text {max }}$;
- On SDVRP instances: (F10) is the best, and ( $\mathrm{F} K_{\text {max }}$ ) is the second best;
- The proposed formulations and BCP algorithm establish a new state-of-the-art for the SDVRPTW, and are highly competitive with the best approach in the literature for the SDVRP.
- Future research topics include trying different versions of the proposed formulations, possibly including delivery quantities explicitly in the master problem (e.g. using flow variables); using other families of valid inequalities; addressing related problem variants.


## Merci :)



Balster, I.; Bulhões, T.; Munari, P.; Sadykov, R. A new family of route formulations for split delivery vehicle routing problems. Technical Report 8918, Operations Research Group, Production Engineering Department, Federal University of São Carlos. 2022.

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http://www.optimization-online.org/DB_HTML/2022/05/8918.html
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