The complexity of pricing for the two-stage vehicle routing problem

Ricardo Fukasawa

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University of Waterloo

Column Generation Workshop 2023
May 18, 2023
Joint work with J. Gunter and M. Ota

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FACULTY OF MATHEMATICS
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and Optimization

Why I couldn't do pricing for the two-stage vehicle routing problem

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Outline

Takeaway 1:

- Spliet (2023):
  "Complexity result for pricing not enough for a CG talk"
  (paraphrased)

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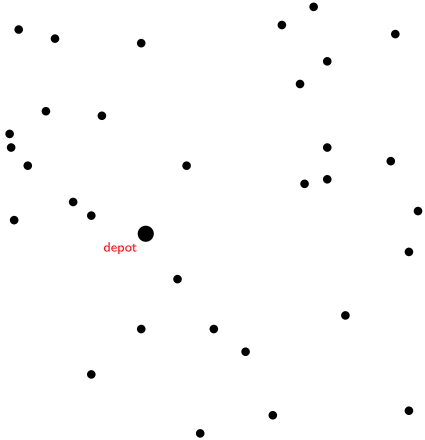
- Spliet (2023):
  "Complexity result for pricing not enough for a CG talk"
  (paraphrased)
- Conclusion:
  I shouldn't be giving this talk!

The two-stage vehicle routing problem



- $G = (V, E)$
- $V = \{0\} \cup V_+$
- Edge lengths $\ell_e,\ e \in E$
- K vehicles, capacity C
- Client demands $D_i, \forall i \in V_+$ are random variables.
- Let S_j be the set of clients served by route j.
  Then
  $\mathbb{E}[D(S_j)] := \sum_{u \in S_j} \mathbb{E}[D_u] \leq C$

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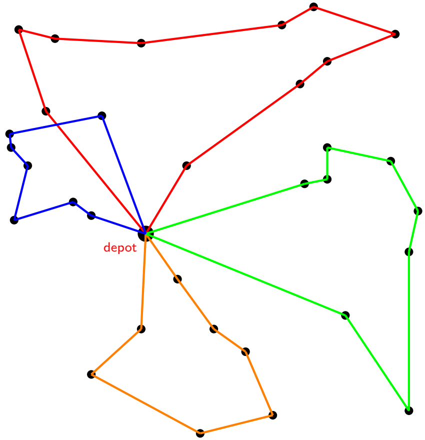

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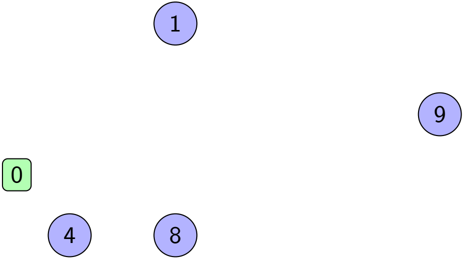
Two-stage stochastic VRP: Costs

- Routes are decided a-priori (first stage): Incur a First stage cost
- After demand realization, truck will follow route and pay second stage cost if capacity is exceeded
- Goal: Minimize expected cost (=first + expected second stage)

Recourse actions

Simple recourse: $C = 10$

Cap used:



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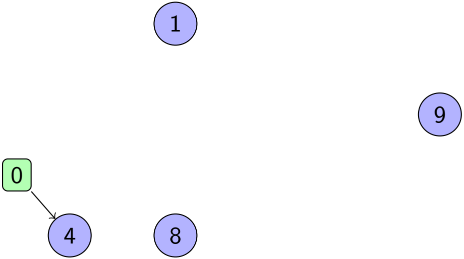
Cap used: 4



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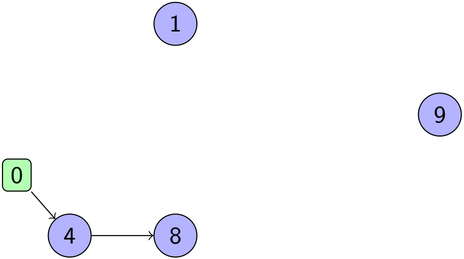
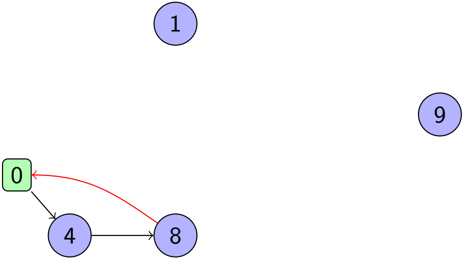
Cap used: 12



Recourse actions

Simple recourse: $C = 10$

Cap used: 2



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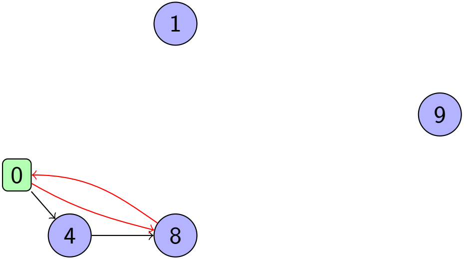

Recourse actions

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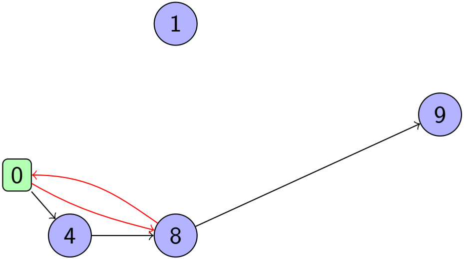
Cap used: 11



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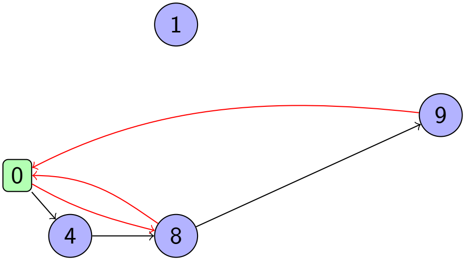

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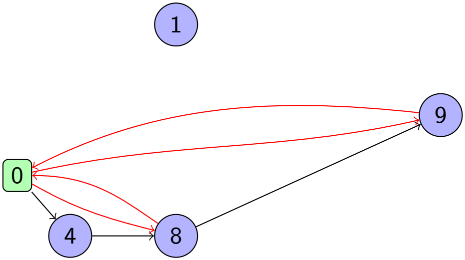

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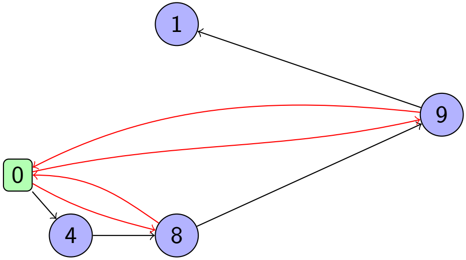

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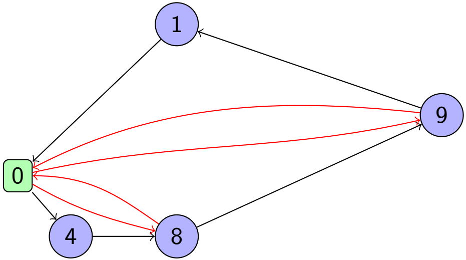

The 2-stage VRPSD: Problem definition

Given:

- $G = (V, E)$, $V = \{0\} \cup V_+$ (assume complete graph)
- Edge lengths ℓ_e, $e \in E$
- K vehicles, capacity C
- Random demands $D_j, \forall j \in V_+$

Given a route $r = (0, v_1, \ldots, v_k, 0)$, with $v_0 = v_{k+1} = 0$.

- It is feasible if $\sum_{j=1}^{k} \mathbb{E}[D_{v_j}] \leq C$
- Its first-stage cost c'_r is the sum of the edge lengths, i.e. $c'_r := \sum_{j=0}^{k} \ell_{v_j, v_{j+1}}$
- Its second-stage cost c''_r is the expected cost due to failures
  - Let $D(r, i) := \sum_{j=1}^{i} D_{v_j}$
  - The expected failure cost at the i-th vertex v_i is $EFC(r, i) := \sum_{u=1}^{\infty} 2\ell_{0,v_i}(\mathbb{P}[D(r, i-1) \leq uC \text{ and } D(r, i) > uC])$
  - $c''_r = \sum_{j=1}^{k} EFC(r, j)$
- Total expected cost: $\tilde{c}_r := c'_r + c''_r$

Literature review

VRPSD (2-stage)

- Heuristics: Stewart & Golden (1983), Dror & Trudeau (1986), Savelsbergh & Goetschalckx (1995), Novoa et al. (2006), Secomandi and Margot (2009), ...
- Integer L-Shaped: Gendreau et al. (1994), Laporte et al. (2002), ...
- Branch-and-cut: Laporte et al. (1989), ...
- Branch-and-price: Christiansen et al. (2007)
- Branch-and-cut-and-price: Gauvin et al. (2014)
- Complex recourse policies: Florio et al. (2020, 2021, 2022), Salavati-Khoshghalb et al. (2019), Louveaux and Salazar-González (2018).

Note: All approaches rely on strong assumptions on demands (independent random variables and/or particular distribution, like normal)

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Stochastic IP:

- Sample average approximation approach widely used (Shultz 1996, Ahmed and Shapiro 2002, Wang and Ahmed 2008, ...)
- Idea: Sample true distribution and use these samples as "proxy" for it
- Reduces problem to finite discrete distribution

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- Idea: Sample true distribution and use these samples as "proxy" for it
- Reduces problem to finite discrete distribution

Goal:

Develop models assuming that we are given a finite discrete distribution (scenarios).

Set partitioning with q-routes IP formulation (Balinsky and Quandt(1964), Christofides, Mingozzi and Toth, 1981)

Definition: A q-route is a walk that starts at the depot, traverses a sequence of customers with total demand at most C, and returns to the depot.
One binary variable (z_r) per possible q-route:

$$\begin{array}{lrcll} \min & \sum_{r \in \mathcal{R}} \tilde{c}_r z_r & & & \\ \text{s.t.} & \sum_{r \in \mathcal{R}} z_r & = & k & \\ & \sum_{r \in \mathcal{R}} a_{ir} z_r & = & 1 & , \forall i \in V \\ & z_r & \in & \{0,1\} & \end{array}$$

where:

- $\mathcal{R}$: set of all possible vehicle q-routes
- a_{ir} number of times q-route r goes through customer i.

Pricing problem

After incorporating dual variables in costs, pricing problem amounts to solving a problem of the form:

$$\min_{r \in \mathcal{R}} \tilde{c}_r \qquad \text{(2SQ)}$$

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Can (2SQ) be solved in pseudo-polynomial time?

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Observations:

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- Hardness of pricing q-routes implies hardness of pricing ng-routes

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Can (2SQ) be solved in pseudo-polynomial time? NO

Observations:

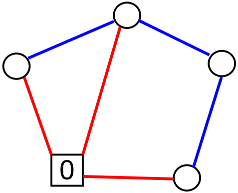
- Hardness of pricing q-routes implies hardness of pricing ng-routes
- Hardness of pricing with "simple recourse" indicates hardness of pricing with more complex recourse (though not necessarily implies it)

Outline

1. Introduction
2. Pricing complexity results
   - Scenarios
   - Independent normal
   - Revisiting scenarios
3. Conclusion

The scenarios case

Consider a graph $G = (V_+ \cup \{0\}, E')$.



and scenarios $s \in \{1, \ldots, n = |V_+|\}$ with $p_s = \frac{1}{n}$:

$$D_j^s = \begin{cases} n, & \text{if } j = s \\ 1, & \text{otherwise.} \end{cases}$$

with $C = 2n - 1$.

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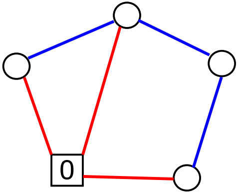

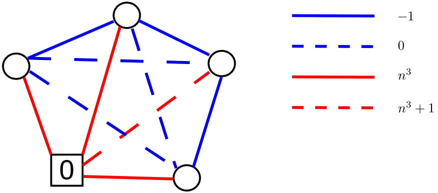

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Lemma (F. and Gunter '22)

Elementary routes r have $c''_r = 0$. Nonelementary routes r have $c''_r \geq n^2$.

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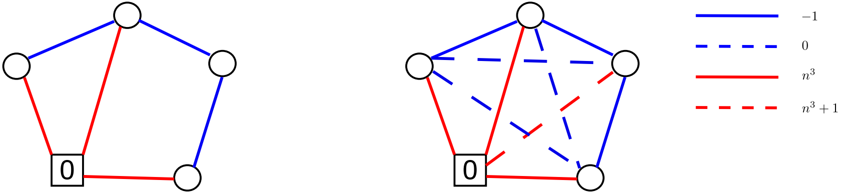

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Lemma (F. and Gunter '22)

A q-route is feasible if and only if it has length $\leq n$.

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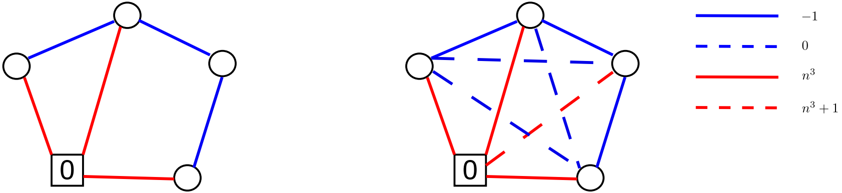

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A minimum 2-stage cost q-route must be elementary.

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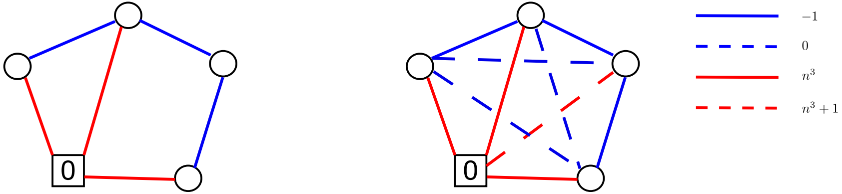

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with $C = 2n - 1$.

Theorem (F. and Gunter '22)

G has a Hamiltonian-cycle if and only if the minimum 2-stage cost q-route is a Hamiltonian cycle.

Outline

Independent normal

Theorem (F. and Gunter '22)

Suppose one can solve (2SQ) under the following assumptions:

1. Demands are independent and identically distributed normal $N(\mu, \sigma^2)$
2. μ and σ^2 are constant integers which do not grow in n.
3. C polynomially bounded in n.
4. All elementary operations can be performed in $O(1)$ time.
5. $RF(\mu, \sigma^2)$ is computable in polynomial time.

Then there exists an algorithm using polynomially many operations that solves the Hamiltonian cycle problem with polynomially many calls to this algorithm.

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Comments:

- Indicates strong NP-hardness but does not prove it
- Knapsack is polytime solvable with either small weights or small costs

P=NP?

Christiansen and Lysgaard (2007), Gauvin et al (2014):
Propose a pseudo-polynomial time algorithm for pricing for independent normal demands $N(\mu_i, \sigma_i^2)$, with μ_i, σ_i^2 integers.

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Issue:

What is the probability distribution of the total demand of r? (Assume $\mu_i = \sigma_i^2 = 1$)

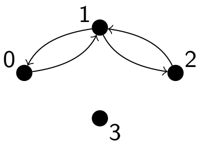

Figure: Route $r = (0, 1, 2, 1, 0)$.

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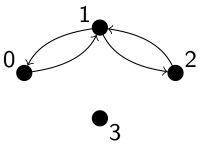

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  Calculates second stage cost $\tilde{c}$ based on $2D_1 + D_2 \sim N(3, 5)$

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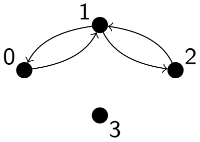

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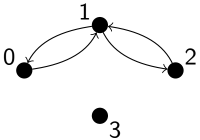

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  Calculates a different cost $\hat{c}$, based on $N(3, 3)$ - Sum of three independent $N(1, 1)$ random variables:
- On elementary routes, both approaches calculate the same expected cost.

Outline

Can this idea help with scenarios?

What if we solve

$$\min_{r \in \mathcal{R}} \hat{c}_r \tag{2SQ'}$$

where $\hat{c}_r = \tilde{c}_r$ for all elementary r.

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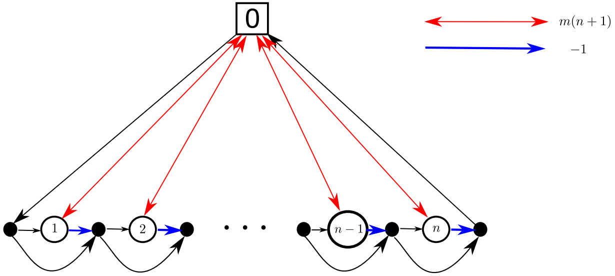
Theorem (F. and Ota '23)

(2SQ') is strongly NP-hard when demands are given as scenarios

Proof idea:

Given $G = (V, E)$ with n vertices, m edges, we wish to find maximum cardinality independent set.

Solve (2SQ') on the following graph:



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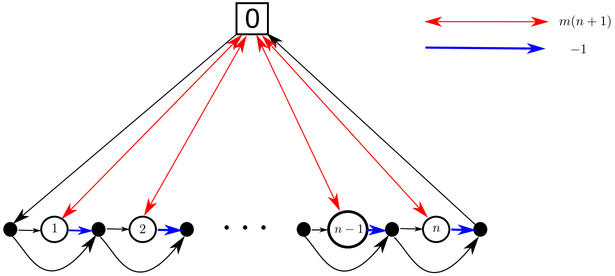

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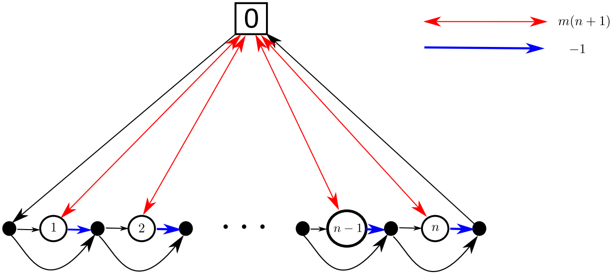


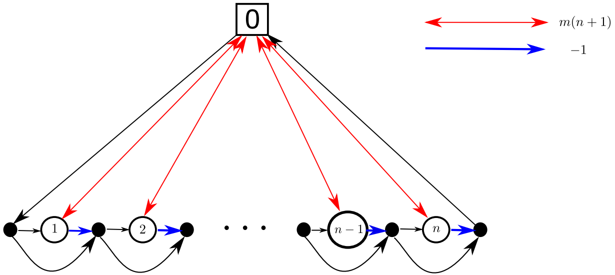
Observations:

- Graph above is independent of G
- All routes are elementary, thus (2SQ') and (2SQ) are the same

Proof idea:

$C = 2m - 1$, and create $m = |E|$ scenarios:

- Scenario for edge $e = ij$ is:
$$D^e(w) = \begin{cases} m, & \text{if } w = i \text{ or } j \\ 0, & \text{otherwise.} \end{cases}$$



- If a route goes through both i and j, then in scenario D^{ij}, capacity is exceeded, so total cost will be positive
- There exists a route of cost 0
- Negative cost route goes through a subset of vertices S that form an independent set in G

Conclusion

Main takeaway

Pricing with correlations is hard
(also pointed out in Gendreau, Jabali and Rey 2016)

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Pricing with correlations is hard
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Future work:

- Can specific reduced cost structure help?
- Do we need to solve (2SQ)?
- Branch-and-cut and branch-and-price for the problem (and other variants)

THANK YOU!